CS581 Theory of Computation: Homework #2

Due on February 1 2016 at 2:00pm

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Problem 2.6
Give context-free grammars generating the following languages.

Problem 2.6 b
The complement of the language $\{a^n b^n | n \geq 0\}$

$$S \to aSb \mid bY \mid Ya$$
$$Y \to bY \mid aY \mid \epsilon$$

Problem 2.6 d
$\{x_1 \# x_2 \# \cdots \# x_k | k \geq 1, \text{ each } x_i \in \{a, b\}^*, \text{ and for some } i \text{ and } j, x_i = {x_j}^R\}$

$$S \to ABC$$
$$A \to D\#A \mid \epsilon$$
$$B \to 0B0 \mid 1B1 \mid E$$
$$C \to \#DC \mid \epsilon$$
$$D \to 1D \mid 0D \mid \epsilon$$
$$E \to \#DE \mid \#$$

Problem 2.7
Give informal English description of PDAs for the languages in Exercise 2.6

Problem 2.7 b
The complement of the language $\{a^n b^n | n \geq 0\}$

The PDA would work as follows:

1. Push start symbol on the stack.
2. If the first symbol is a b then move to accept state, since $a^n b$ cannot start with b therefore this string is in compliment.
3. If the first symbol is a then push a on the stack, keep pushing a's for all consecutively following a's. When see first b after stream of a's, start consuming a's from the stack.
4. If stream is over and there is only start symbol - fail.
5. If start symbol is consumed when b is scanned, then go to accept $b > a$.
6. If followed by a, then go to accept, since string has a form of $a^i b^j a \cdots$.
7. Last option all bs are consumed, but there is stil a remains on the stack $\to$ accept state since $a > b$

Problem 2.7 d
$\{x_1\#x_2\#\cdots\#x_k | k \geq 1, \text{ each } x_i \in \{a,b\}^*, \text{ and for some } i \text{ and } j, x_i = {x_j}^R\}$

A PDA that recognizes this language will nondeterministically check all possible combinations of x_i and x_j, by pushing character of x_i onto the stack, and consuming them when checking x_j. And skipping all other correct string sequences. If a pair $x_i = x_j^R$ is found, and all the rest of the imput is in form of $\{x_1\#x_2\#\cdots\#x_k\}$ - Accept, fail otherwise.

Problem 2.9
Give a context-free grammar that generates the language

$$A = \{a^i b^j c^k | i = j \text{ or } j = k \text{ where } i, j, k \geq 0\}$$

Is your grammar ambiguous? Why or why not?

$$\begin{aligned} &S \to AB \mid CD \\ &A \to aAb \mid \epsilon \\ &B \to cB \mid \epsilon \\ &C \to aC \mid \epsilon \\ &D \to bDc \mid \epsilon \end{aligned}$$

The language is not ambiguous...

Problem 2.13

Let $G = (V, \sum, R, S)$ be the following grammar. $V = \{S < TU\}; \sum = \{0, \#\}$; and R is the set of rules:

S $\rightarrow TT \mid U$
T $\rightarrow 0T \mid T0 \mid \#$
U $\rightarrow 0U00 \mid \#$

Problem 2.13 a

Describe $L(G)$ in English.

Informally $L(G)$ is either two or more # separated by arbitrary number of 0's (zero or more) or zero or more zero followed by # and by twice as many zeros as in before #. More formally it is $\{0_1^{i_1}\#0_2^{i_2}\#0_3^{i_3}\#0_4^{i_4}\# \cdots \#0_k^{i_n} \mid \text{where } i_j \geq 0 \text{ and } k \geq 3\}$ or $\{0^n\#0^{2n} | n \geq 1\}$

Problem 2.13 b

Prove that $L(G)$ is not regular.

Case 1. Assume that $L(G)$ is regular.

Case 2. Consider the word $0^p\#0^{2p} \in L(G)$. By pumping lemma there eixsts a word $xyz \in L(G)$, and puping lenght p, such that $|xy| \leq p$ and for all $i \geq 0$, $xy^iz \in L$. In case of word $= 0^p\#0^{2p}$, xy can only be in 0^p, however for $i \geq 0$ xy^iz is $0^{p+i}\#0^{2p} \notin L(G)$ is a contradiction. Therefore $L(G)$ is no regular.

Problem 2.15

Give a counterexample to show that the following construction fails to prove that the class of context-free languages is closed under star. Let A be a CFL that is generated by th CFG $G = (V, \sum, R, S)$. Add the new rule $S \rightarrow SS$ and call the resulting grammar G'A. This grammar is supposed to generate $A*$.

This rule applied to grammar $G = (\{S\}, \{a, b\}, \{S \rightarrow aSb|\epsilon\}, S)$ + $S \rightarrow SS$ we get grammar G' that can produce a word: $aababb \notin A^*$

Problem 2.19
Let CFG G be the following grammar.

$$S \to aSb \mid bY \mid Ya$$
$$Y \to bY \mid aY \mid \epsilon$$

Give a simple description of L(G) in English. Use that description to give a CFG for $\overline{L(G)}$, the compliment of $L(G)$.

L(G) is the language that produces all strings not in $a^n b^n$, i.e. compliment of of $a^n b^n$.

CFG of $\overline{L(G)}$:

$$S \to aSb \mid \epsilon$$

Problem 2.28

Give unambiguous CFGs for the following languages.

Problem 2.28 a
$\{w|$ in every prefix of w the number of a's is at least the number of b's $\}$

$$S \to aS \mid BS \mid \epsilon$$
$$B \to aBBb \mid \epsilon$$

Problem 2.28 b
$\{w|$ the number of a's and the number of b's in w are equal $\}$

$$S \to aA \mid bB \mid \epsilon$$
$$A \to bS \mid aAA$$
$$B \to aS \mid bBB$$

Problem 2.28 c
$\{w|$ the number of a's is at least the number of b's in $w\}$

$$S \to aA \mid bB \mid a$$
$$A \to bS \mid aAA$$
$$B \to aS \mid bBB$$

Problem 2.30
Use the pumping lemma to show that the following languages are not context free.

Problem 2.30 a
$L = \{0^n 1^n 0^n 1^n | n \geq 0\}$

Let p be the pumping lentgth, $s = 0^p 1^p 0^p 1^p$. By pumping lemma there exists $s = uv^ixy^iz \in L$, where $|vxy| \leq p$ and $|vy| > 1$. There are two possible cases for v and y.

Case 1. v and y contain at most one type of symbols, then when we pump $i = p$ we will get string with member of unequal length: $s = 0^p 1^{2p} 0^p 1^p \notin L$ or $s = 0^{2p} 1^p 0^p 1^p \notin L$ or $s = 0^{2p} 1^p 0^{2p} 1^p \notin L$ or $s = 0^{2p} 1^p 0^{2p} 1^{2p} \notin L$

Case 2. v and y contain different types of symbols, then when i is pumped the resulting string will have members out of order, for example $0001110001010111 \notin L$. Therefore L is not Context free.

Problem 2.30 d
$L = \{t_1 \# t_2 \# \cdots \# t_k | k \geq 2, \text{ each } t_i \in \{a, b\}^*, \text{ and } t_i = t_j \text{ for some } i \neq j\}$

Let p be the pumping lentgth, $s = a^p b^p \# a^p b^p \in L$. By pumping lemma there exists $s = uv^ixy^iz \in L$, where $|vxy| \leq p$ and $|vy| > 1$. There are three cases

Case 1. v or y contain $\#$. Then for $i = 0$, $s = uxz$ and it doesn't contain $\# \notin L$

Case 2. v and y on the left side of $\#$. Then when we pump i left side becomes longer than the right side $\notin L$

Case 3. v and y on the right side of $\#$. Then when we pump i right side becomes longer than the left side $\notin L$

Therefore L is no Context free.

Problem 2.31
Let L be the language of all palindromes over $\{0, 1\}$ containing equal numbers of 0s and 1s. Show that B is not context free.

Let p be the pumping lentgth, $s = 0^p 1^2 p 0^p \in L$. By pumping lemma there exists $s = uv^ixy^iz \in L$, where $|vxy| \leq p$ and $|vy| > 1$. To prove that L is not context free we need to consider following cases

Case 1. vxy consits of only 1's. Then if we pump $i = p$, then $s = 0^p 1^4 p 0^p \notin L$, since number of 1's is greater than number of 0's.

Case 2. vxy consits of only 0's. Then if we pump $i = p$, then $s = 0^2 p 1^2 p 0^p \notin L$, since number of 0's is greater than number of 1's and s is no longer a palindrome.

Case 3. vxy consits of 0's and ones. Then if we pump $i = 2$, then $s = uv^2xy^2z \notin L$, since s is no longer a palindrome.

Therefore L is not context free.

Problem 2.33
Show that $F = \{a^i b^j \mid i = kj \text{ for some positive integer } k\}$ is not context free.

Let p be the pumping lentgth, $s = a^{q^2} b^q \in F$, where q is a prime, and $q > p$. By pumping lemma there exists $s = uv^ixy^iz \in L$, where $|vxy| \leq p$ and $|vy| > 1$. To prove that L is not context free we need to consider following cases:

Case 1. vxy consits of only a's, then $v = a^k$ and $y = a^l$ in this case it is easy to find such i that $t = \dfrac{q^2 + i \times (k + l)}{q}$ is not an integer

Case 2. vxy consits of only b's, then $v = b^k$ and $y = b^l$ in this case it is easy to find such i that $t = \dfrac{q^2}{q + i \times (k + l)}$ is not an integer

Case 3. vxy consits of a's and b's, then $v = a^k$ and $y = b^l$, i that $t = \dfrac{q^2 + i \times k}{q + i \times l}$, there are many possible combinations of i, k, l, that make t not intger.

Therefore F is not regular.

Problem 2.35
Let G be a CFG in Chomsky normal form that contains b variables. Show that if G generates some string with a derivation having at least 2^b steps, $L(G)$ is infinite.

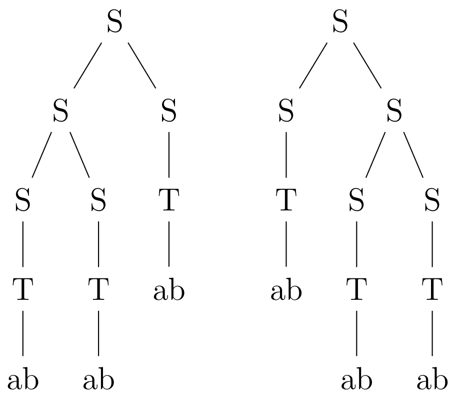
Since Chomsky form can have eiter two variables, or one terminal on the right, therefore it's derivation tree will be a binary tree. String w that derived with 2^b steps will have length $|w| \geq w^b$. Height of such a tree will be $b + 1$, since there is only b variables, then at least one or more variable is used twice, i.e. language contains recursion, hence $L(G)$ is infinite.

Problem 2.46
Consider the following CFG G: Describe $L(G)$ and show that G is ambiguous. Give an

$$S \to SS \mid T$$
$$T \to aTb \mid ab$$

unambiguous grammar H where $L(H) = L(G)$ and sketch a proof that H is unambiguous.

$L(G) = \{(a^n b^n)^i | \text{ where } i, n > 0\}$. $ababab \in L(G)$ and has two derivation trees



Therefore $L(G)$ is ambiguous.
Unambiguous grammar for this language is:

$$S \to TS \mid \epsilon$$
$$T \to aTb \mid ab$$