CS558 Programming Languages

Fall 2015

Lecture 3

IMPERATIVE LANGUAGES

Most commonly-used programming languages are **imperative**: they consist of a sequence of actions that alter the **state** of the world.

State includes the values of program variables and also the program's external environment (e.g. files the program reads or writes).

High-level imperative languages mimic the style of the underlying **Von Neumann** machine architecture, where programs are sequences of instructions that modify the contents of registers and memory locations.

This makes it relatively straightforward to compile imperative languages to efficient code:

- High-level variables are mapped to machine locations.
- High-level operations are mapped to (multiple) machine instructions.

Imperative languages are also natural for writing **reactive** programs that interact with the state of the "real world." Examples:

- Reading mouse clicks and modifying the contents of a display.
- Controlling a set of relays in an external device.

ASSIGNMENT

The basic primitive stateful operation is typically **assignment**, which alters a value stored in a **location**.

In the simplest form, the location is associated with a **variable**, e.g.,

```
a := a + 2
```

(Note: C/C++/Java popularized the use of plain = for assignment and == for relational equality: a bad idea, because **both** form expressions and are easy to confuse!)

In most languages, the variable name `a` means different things on the left-hand side (LHS) and right-hand side (RHS) of an assignment.

On the LHS, `a` denotes the **location** of the variable `a`, into which the value of the RHS expression is to be stored. Here we say `a` is an **l-value**.

On the RHS, `a` denotes the **value** currently contained in `a`, i.e., it indicates an implicit **dereference** operation. Here, `a` is an **r-value**.

If assignment is an expression, what is its value? Common choices are the value of the RHS, or a special "empty" value (e.g. `():Unit` in Scala).

INITIALIZATION VALUES

Most languages require variables to be **declared** before they are used: this gives them a scope, perhaps a type, and **optionally**, an initial value specified by an expression.

In fact, it is surely a **bug** to use any variable as an r-value unless it has previously assigned a value. But many languages permit us to write such code, resulting in runtime errors—either checked (e.g., as in Python) or unchecked (as in C).

The simplest fix is to **require** an initial value to be given for every declared variable (e.g. as in Scala).

Java takes a slightly more sophisticated approach:

- variables do not need to be initialized at the point of declaration; but
- they **must** be initialized before they are actually used.

DEFINITE ASSIGNMENT

Yet in any reasonably powerful language, checking initialization before use is an **uncomputable** problem. (Why?)

So the Java language reference manual carefully details a **conservative**, computable, set of conditions, which every program must meet, that guarantee there will be no uses before definition.

This is called the **definite assignment** property; just defining it takes 16 pages of the reference manual.

Some programs that **do** in fact initialize before use will be rejected because they violate the conditions.

Legal example:

```
int a;
if (b)  /* b is boolean */
  a = 3;
else
  a = 4;
a = a + 1;
```

Illegal example:

```
int a;
if (b)
  a = 3;
if (!b)
  a = 4;
a = a + 1;
```

SEQUENCING

Order of stateful operations obviously affects program semantics. Contrast:

```
x := 10;
x := 20;
y := x
```

```
x := 20;
x := 10;
y := x
```

Notice that we have introduced a **sequencing** operator, typically written as here with a semicolon(**;**).

For expressions, the meaning of $e_1; e_2$ is

- evaluate e_1
- throw away the result
- then evaluate e_2.

Obviously, this is only interesting if e_1 performs a stateful operation as a side-effect.

Problems with Order of Evaluation

In many langauges, order of expression evaluation is **under-specified** (precedence and associativity don't always fix order).

ANSI C example:

```
a = 0;
b = (a = a + 1) - (a = a + 2);
```

What's the result in `b`?

Answer: could be anything! In C, the order of evaluation of the operands to + is undefined. Consequently, the meaning of this program is also undefined, and a compiler is free to return any value it wants.

In fact, this is an **illegal** C program because of its ambiguous meaning, but many compilers will accept it without even giving a warning.

HIDDEN SIDE EFFECTS

The impact of evaluation order on program behavior is not always obvious, because side-effects can occur at a distance:

```
int a = 0;
int h (int x, int y) { return x; }
int f (int z) { a = z; return 0; }
h(a,f(2));  // = 0 or 2 ??
```

Answer: Depends on evaluation order for function actual parameters, which is language-dependent (and possibly unspecified).

Keeping expression evaluation order or argument evaluation order undefined sometimes lets compiler generate more efficient code.

But most modern languages (e.g., Java) have moved towards precise definition of evaluation order within expressions (typically left-to-right).

BOXES

In functional languages, ordinary "variables" are **immutable**, i.e., they are really just names for values (computed just once at runtime), not for locations.

Making variables mutable would fundamentally change the character of a functional language. A gentler approach is to add mutable **data structures**.

The simplest possible structure is a single mutable cell containing a value. PAPL calls this a **box**. You can think of a box as a degenerate record with one field or array with one element.

The mutable cell is allocated and initialized by a constructor call, e.g.

```
x = box(42)
```

binds `x` to a **location** containing (initially) the value 42.

At an abstract level, we can temporarily ignore the question of where new cells are created, and simply say that the program has an extendible, mutable **store**, which maps locations to values.

WORKING WITH BOXES

The two operations on boxes are **unbox**, which gets the current box contents, and **setbox**, which changes the contents.

Box values are l-values; they must be **explicitly** dereferenced to get their contents, e.g.

```
let x = box(40) in
  ( setbox x (2 + unbox x);  unbox x )   (* yields 42 *)
```

This approach is obviously more verbose than conventional imperative code, but at least it removes any confusion between l-value and r-value.

Moreover, boxes are also first-class values,e.g.

```
let incby2 (y:int box) =  setbox y (2 + unbox y) in
  let z = box(40) in
     ( incby2(z); unbox z )    (* yields 42 *)
```

The ML language provides boxes under the name “references.”

THE STORE

To model the semantics of a language with boxes, we add a `Store` component to the interpreter. A store is simply an (immutable) record containing

- A map from Locations to Values
- A counter recording the next free Location

```
object Store {
  type Location = Int
  val empty = Store(0,Map[Location,Value]())
}
case class Store(nextAddr : Store.Location,
                 contents : Map[Store.Location,Value]) {
  type Location = Int
  def allocate(n:Int) : (Location,Store) =
     (nextAddr,Store(nextAddr+n,contents))
  def apply(a:Location) = contents(a)  // exception if absent
  def +(av:(Location,Value)) = Store(nextAddr,contents + av)
}
```

Using the store:

```
case class BoxV(a:Store.Location) extends Value
def interpE(expr:Expr,env:Env,st:Store) : (Value,Store) = expr match {
  ...
  case Box(e) => {
    val (v,st1) = interpE(e,env,st)
    val (a,st2) = st1 allocate 1
    (BoxV(a),st2 + (a -> v))
  }
  case UnBox(b) => interpE(b,env,st) match {
    case (BoxV(a),st1) => (st1(a),st1)
    case _ => throw TypeError("Attempt to unbox a non-box")
  }
  case SetBox(b,e) => interpE(b,env,st) match {
    case (BoxV(a),st1) => {
      val (v,st2) = interpE(e,env,st1)
      (v,st2 + (a -> v))
    }
    case _ => throw TypeError("Attempt to set a non-box")
  }
  ...
}
```

Threading the store:

```
def interpE(expr:Expr,env:Env,st:Store) : (Value,Store) = expr match {
  case Num(n) => (NumV(n),st)
  case Add(l,r) => arithBinOp(l,r,env,st) { (lv,rv) => NumV(lv + rv) }
  case Mul(l,r) => arithBinOp(l,r,env,st) { (lv,rv) => NumV(lv * rv) }
  case Let(x,d,b) => {
    val (v,st1) = interpE(d,env,st)
    interpE(b,env + (x -> v),st1)
  }
  ...
}

def arithBinOp(l: Expr, r:Expr,env:Env,st:Store)
              (op: (Int,Int) => Value) = {
  val (lv,st1) = interpE(l,env,st)
  val (rv,st2) = interpE(r,env,st1)
  (lv,rv) match {
     case (NumV(ln),NumV(rn)) => (op(ln, rn),st2)
     case _ => throw TypeError("...")
  }
}
```

SEMANTICS OF VARIABLES

In most imperative programming languages, **variable** names are bound to **locations**, i.e. memory addresses, which in turn contain **values**. So processing a variable declaration (e.g. a parameter or local) conceptually involves two separate operations:

- allocating a new location in the store and perhaps initializing its contents;
- creating a new binding in the environment from the variable name to the location.

Assignment to the variable involves an implicit update to the store, and a use of the variable as an r-value involves an implicit fetch from the store.

The store threading already described works just the same for variables as for boxes!

Key idea: Once created, environment bindings don't change.

Modelling Mutable Variables:

```
type Env = Map[String,Store.Location]
def interpE(expr:Expr,env:Env,st:Store) : (Value,Store) = {
  ...
  case Let(x,d,b) => {
    val (v,st1) = interpE(d,env,st)
    val (a,st2) = st1 allocate 1
    interpE(b,env + (x -> a),st2 + (a -> v))
  }
  case Var(x) => env get x match {
    case Some(a) => (st(a),st)
    case None => throw UndefinedVariable("Undefined variable use:" + x)
  }
  case Assgn(x,e) => env get x match {
     case Some(a) => {
       val (v,st1) = interpE(e,env,st)
       (v,st1 + (a -> v))
     }
     case None => throw UndefinedVariable("...'')
  }
  ...
}
```

PAIRS

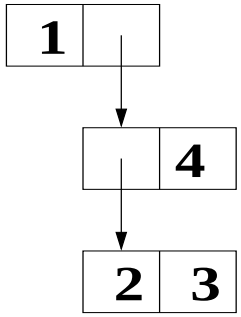
So far, our interpreted languages do not have a way to build interesting **data structures**. (Boxes can only store a single value.)

We can remedy this by adding in just one new kind of value, the **pair**. You can think of a pair as a record with two fields, each containing a value — which might be a base value such as integer or another pair.

We write pairs using "infix dot" notation. For example:

```
(1 . ((2 . 3) . 4))
```

corresponds to the structure:



We can build larger records of a fixed size just by nesting pairs.

LISTS

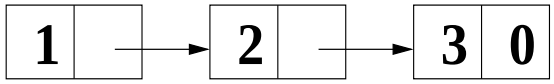
We can also build all kinds of interesting arbitrary-sized **recursive** structures using pairs.

For example, to represent **lists** we can use a pair for each link in the list. The left field contains an element; the right field points to the next link, or is 0 to indicate end-of-list.

Example:

```
[1,2,3]
```

```
(1.(2.(3.0)))
```



Note that for programs to detect when they've hit the end of a list, they'll need a way to distinguish the value 0 from any pair value.

Pairs and lists built from them are the entire basis of the LISP language and its descendents.

Pragmatics of Large Values

Real machines are very efficient at handling small, fixed-size chunks of data, especially those that fit in a single machine **word** (e.g. 16-64 bits), which usually includes:

- Numbers, characters, booleans, enumeration values, etc.
- Memory addresses (locations).

But often we want to manipulate larger pieces of data, such as records and arrays, which may occupy many words.

There are two basic approaches to representing value:

- The **unboxed** representation holds the actual contents of the value, using as many machine words as necessary.
- The **boxed** representation **implicitly** allocates storage (the "box") for the contents in memory, and then represents the value by the **location** of that storage.

(Note: this use of the term "boxed" applies to our interpreter's `box` values, but also to many other kinds of values, including much larger ones.)

BOXED VS. UNBOXED

For example, consider an array of 100 integers. In an **unboxed** representation, the array would be represented directly by 100 words holding the array contents. In a **boxed** representation, the array would be indirectly represented by an implicit 1-word pointer to 100 consecutive locations holding the array contents.

The language's choice of representation makes a big difference to the semantics of operations on the data, especially if values are mutable. Key questions include:

- What does assignment mean?
- How does parameter passing work?
- What do equality comparisons mean?

Unboxed Representation Assignment Semantics

Earlier languages often used unboxed representations for records and arrays. For example, in Pascal and related languages,

```
TYPE Employee =
RECORD
  name : ARRAY (1..80) OF CHAR;
  age : INTEGER;
END;
```

specifies an unboxed representation, in which value of type `Employee` will occupy 84 bytes (assuming 1 byte characters, 4 byte integers).

The semantics of assignment is to copy the entire representation. Hence

```
VAR e1,e2 : Employee;
e1.age := 91;
e2 := e1;
e1.age := 19;
WRITE(e1.age, e2.age);
```

prints 19 followed by 91.

UNBOXED REPRESENTATION PROBLEMS

Assignment using the unboxed representation has appealing semantics, but two significant problems:

- Assignment of a large value is expensive, since lots of words may need to be copied.

- Since compilers need to generate code to move values, and (often) allocate space to hold values temporarily, they need to know the **size** of the value.

These problems make the unboxed representation unsuitable for value of **arbitrary size**. For example, unboxed representation can work fine for pairs of integers, but not for pairs of arbitrary values that might themselves be pairs.

Boxed Representation Assignment Semantics

Most modern languages (e.g. Java, Python, Haskell) **implicitly** allocate memory for all objects (i.e. objects, records, constructed values) that are larger than a word, and represents values of these types by **references** (pointers, locations) into memory.

As a natural result, these languages use so-called **reference** semantics for assignment and argument passing. Scala example:

```
case class emp(var name:String, var age:Int)
object Bat {
  def main(argv:Array[String]) = {
    val e1 = emp("fred",91)
    val e2 = e1
    e1.age = 19
    println(e1.age +  " " + e2.age)
  }
}
```

prints 19 19

BOXED REPRESENTATION (2)

In these languages, if you want to copy the entire contents of record or object, you must do it yourself, element by element (though often there is a standard library method to do the job, e.g. `clone` in Java).

Notice that the difference between copy and reference assignment semantics really only matters for **mutable** values; for immutable data you can't observe the difference (except perhaps for efficiency).

EXPLICIT POINTERS

Many languages that use unboxed semantics also have separate **explicit pointer types** to enable programmers to construct recursive data structures, e.g. in C++

```
struct Intlist {
  int head;
  Intlist *tail;
};
Intlist *mylist = new Intlist;
mylist->head = 42;
mylist->tail = mylist;
delete mylist;
```

In C/C++, `struct` and `class` instances are fundamentally unboxed, i.e. they use unboxed assignment semantics. But the programmer usually boxes them explicitly (using `new` or `malloc`) and manipulates them via explicit pointers.

VARIETIES OF EQUALITY

Languages typically provide some form of built-in equality testing on values. So when are two (large) values equal?

• One possible definition is: when their contents are equal, field by field. This **structural equality** is the only sensible definition for unboxed values.

• For boxed values, we could instead say: when their locations are identical. This **reference equality** is more efficient, but might be less useful.

Note that reference equality implies structural equality, but not vice-versa.

Some languages provide both, under different names. They may also provide a standard way for the programmer to define equality on a given type in an ad-hoc way. For example, in Scala, the `eq` operator gives reference equality, whereas the == operator invokes a user-defined `equals` method; for case classes, the latter is pre-defined to be structural equality.

PROCEDURE PARAMETER PASSING

When we apply a function in an imperative language, the formal parameters get bound to locations containing values.

- How is this done and which locations are used?
- Do we pass addresses or contents of variables from the caller?
- How do we pass actual values that aren't variables?
- What does it mean to pass a large value like an array?

Two main approaches:

- call-by-value (CBV)
- call-by-reference (CBR)

CALL-BY-VALUE

- As we've seen already, each actual parameter is **evaluated** to a **value** before call.
- On entry to the function, each formal parameter is bound to a freshly-allocated location, and the actual parameter value is **copied** into that location. (This is much like processing the declaration and initialization of a local variable.)
- Updating the formal parameter doesn't affect actuals in **calling** procedure.

```
double hyp(double a,double b) {
  a = a * a;
  b = b * b;
  return sqrt(a+b);
}
```

- Semantics are just like for assignment.
- Simple; easy to understand!

PROBLEMS WITH CALL-BY-VALUE (1)

- Can be inefficient for large unboxed values:

Example (C): Calls to `dotp` copy 20 doubles

```
typedef struct {double a1,a2,...,a10;}
                                        vector;
double dotp(vector v, vector w) {
   return v.a1 * w.a1 + v.a2 * w.a2 + ...
          + v.a10 * w.a10;
}
vector v1,v2;
double d = dotp(v1,v2);
```

PROBLEMS WITH CALL-BY-VALUE (2)

- Cannot affect calling environment directly.

Example: calls to `swap` have no effect:

```
void swap(int i,int j) {
  int t;
  t = i ; i = j; j = t;
}
...
swap(a[p],a[q]);
```

(Of course, perhaps this is usually a **good** thing!)

- Can at best **return** only one result (as a value), though this might be a record.

CALL-BY-REFERENCE

- Pass the existing **location** of each actual parameter.
- On entry, the formal parameter is bound directly to this location. Thus, it can be dereferenced to get the value, but it can also be **updated**.
- If actual argument doesn't have a location (e.g., `(x + 3)`), either:

- Evaluate it into a temporary location and pass address of temporary, or

- Treat as an error.

- Now `swap`, etc., work fine!
- Lots of opportunity for **aliasing** problems, e.g.,

```
PROCEDURE matmult(a,b,c: MATRIX)
... (* sets c := a * b *)

matmult(a,b,a) (* oops! *)
```

- **Call-by-value-result** (a.k.a. **copy-restore**) addresses this problem, but has other drawbacks.

HYBRID METHODS; RECORDS AND ARRAYS

How might we combine the simplicity of call-by-value with the efficiency of call-by-reference, especially for large unboxed values?

• In Pascal, Ada, and similar languages, where records and arrays are both unboxed, the programmer can specify (in the procedure header) for each parameter whether to use call-by-value or call-by-reference.

• In C/C++, record (`struct` or `class`) values are unboxed, but arrays are boxed. C++ allows per-argument calling mode specification. C always uses call-by-value, but programmers can take the address of a variable explicitly, and pass that to obtain CBR-like behavior:

```
swap(int *a, int *b) {
  int t;
  t = *a; *a = *b; *b = t; }
swap (&a[p],&a[q]);
```

Of course, it is the programmer's responsibility to make sure that the address remains valid (especially when it is **returned** from a function).

VALUES CAN BE REFERENCES

In a language like Java or Python that boxes large values such as records or arrays, these values are already pointers (or references).

Thus, even if the language uses call-by-value, the values that are passed are actually references. Calls don't cause any actual copying of the large values.

But it is a mistake (which some otherwise good authors make) to say that these languages use "call-by-reference." (If they did, they would pass a reference to the reference!)

For example, the first interpreter in this week's homework uses call-by-value, but boxes (and pairs) are among the possible values.

INFORMAL SEMANTICS

• Grammars can be used to define the legal programs of a language, but not what they do! (Actually, most languages place further, non-grammatical restrictions on legal programs, e.g., type-correctness.)

• Language behavior is usually described, documented, and implemented on the basis of **natural-language** (e.g., English) descriptions.

• Descriptions are usually structured around the language's grammar, e.g., they describe what each nonterminal does.

• Natural-language descriptions tend to be **imprecise**, **incomplete**, and **inconsistent**.

EXAMPLE: FORTRAN DO-LOOPS

"DO n i = m_1,m_2,m_3

Repeat execution through statement n, beginning with $i = m_1$, incrementing by m_3, while i is less than or equal to m_2. If m_3 is omitted, it is assumed to be 1. m's and i's cannot be subscripted. m's can be either integer numbers or integer variables; i is an integer variable."

- from DEC Fortran-II manual, 1974.

Consider:

```
      DO 100 I = 10,9,1
...
100   CONTINUE
```

How many times is the body executed?

Experimental Semantics

Try it and see!

Implementation becomes standard of correctness.

This is certainly **precise**: compiler source code becomes specification.

But it is:

- difficult to understand;
- awkward to use;
- subject to accidental change;
- wholly non-portable.

FORMAL SEMANTICS

Aims:

• **Rigorous** and **unambiguous** definition in terms of a well-understood formalism, e.g., logic, naive set theory, etc.

• Independence from **implementation**. Definition should describe how the language behaves as abstractly as possible.

Uses:

• Provably-correct implementations.

• Provably-correct programs.

• Basis for language comparison.

• Basis for language design.

(But usually not basis for learning a language.)

Main varieties: Operational, Denotational, Axiomatic.

Each has different purposes and strengths. In this course, we'll mostly focus on operational semantics, with brief looks at the others.

OPERATIONAL SEMANTICS

Define behavior of language on an **abstract machine**.

Abstract machine should be much **simpler** than real machines, since otherwise a compiler for a real machine would be just as good!

Typical mechanisms:

- Characterize the state of the abstract machine (typically as an **environment** mapping variables to values) and give a set of **evaluation rules** describing how each syntactic construct affects the state.
- Define a simple Von Neumann-style **stack machine** and describe how each syntactic construct can be compiled into stack machine instructions.

Some useful things to do with an operational semantics:

- Build an implementation for a real machine by interpreting or compiling the abstract machine code.
- Explicate the meaning of a language feature by proving that it has the same behavior as a combination of simpler features.
- Prove that correctly typed programs cannot “dump core” at runtime.

SEMANTICS FROM INTERPRETERS

In the homework, we're building **definitional interpreters** for small languages that display key programming language constructs.

Our goal is to study the interpreter code in order to understand **implementation** issues associated with each language.

In addition, the interpreter serves as a form of **semantic** definition for each language construct. In effect, it defines the meaning of the language in terms of the semantics of Scala.

(Of course, you'll also be learning more about the semantics of Scala as we go!)

SEMANTICS AND ERRONEOUS PROGRAMS

An important part of a language specification is distinguishing valid from invalid programs.

It is useful to define three classes of errors that make programs invalid. (Of course, even valid programs may behave differently than the programmer intended!)

Static errors are violations of the language specification that can be detected at compilation time (or, in an interpreter, before interpretation begins)

- Includes: **lexical** errors, **syntactic** errors (caught during parsing), **type** errors, etc.

- Compiler or interpreter issues an error pinpointing erroneous location in source program.

- Language **semantics** are usually defined only for programs that have no static errors.

RUNTIME ERRORS

Checked runtime errors are violations that the language implementation is required to detect and report at runtime, in a clean way.

- Examples in Scala or Python: division by zero, array bounds violations, dereferencing a null pointer.
- Depending on language, implementation may issue an error message and die, or raise an exception (which can be caught by the program).
- Language semantics must specify behavior precisely.

Unchecked runtime errors are violations that the implementation need not detect.

- Subsequent behavior of the computation is **arbitrary**. (Error is often not manifested until much later in execution.)
- Examples in C: division by zero, dereferencing a null pointer, array bounds violations.
- Language semantics probably don't specify behavior.
- **Safe** languages like Python, OCaml, and Java have **no** such errors!

FORMAL OPERATIONAL SEMANTICS

So far, we've presented operational semantics using interpreters. These have the advantage of being **precise** and **executable**. But they are not ideally **compact** or **abstract**.

Another way to present operational semantics is using **state transition judgments**, for appropriately defined machine states.

For example, consider a simple language of imperative expressions, in which variables must be defined before use, using a `let` or `fun` construct.

```
exp := var | int
     |  '(' '+' exp exp ')'
     |  '(' 'let' var exp exp ')'
     |  '(' ':=' var exp ')'
     |  '(' 'ifnz' exp exp exp ')'
     |  '(' 'fun' var exp ')'
     |  '(' '@' exp exp ')'
     |   etc.
```

For simplicity, we assume that the only values are integers and closures; `ifnz` expects an integer condition and tests whether it is non-zero.

STATE MACHINE

To evaluate this language, we choose a machine state consisting of:

• the current **environment** E, which maps each in-scope variable to a location l.

• the current **store** S, which maps each location l to an integer value v.

• the current **expression** e, to be evaluated.

We give the state transitions in the form of **judgments**:

$$\langle e, E, S \rangle \Downarrow \langle v, S' \rangle$$

Intuitively, this says that evaluating expression e in environment E and store S yields the value v and the (possibly) changed store S'.

Values v are either integers i or function closures $[x, e, E]$, where x is the formal parameter, e is the body, and E is the static environment at the point of definition.

Operational Semantics by Inference

To describe the machine's operation, we give **rules of inference** that state when a judgment can be derived from judgments about sub-expressions.

The form of a rule is

$$\frac{\textit{premises}}{\textit{conclusion}}\ \text{(Name of rule)}$$

We can view evaluation of the program as the process of building an inference tree.

This particular formulation, where each expression results in a value based on the values of the subexpessions, is called **big-step semantics**.

ENVIRONMENTS AND STORES, FORMALLY

• We write $E(x)$ means the result of looking up x in environment E. (This notation is because an environment is like a **function** taking a name as argument and returning a meaning as result.)

• We write $E + \{x \mapsto l\}$ for the environment obtained from existing environment E by **extending** it with a new binding from x to location l. If E already has a binding for x, this new binding replaces it.

The **domain** of an environment, *dom*(E), is the set of names bound in E.

Analogously with environments, we'll write

• $S(l)$ to mean the value at location l of store S

• $S + \{l \mapsto v\}$ to mean the store obtained from store S by extending (or updating) it so that location l maps to value v.

• *dom*(S) for the set of locations bound in store S.

EVALUATION RULES (1)

$$\frac{l = E(x) \quad v = S(l)}{\langle x, E, S\rangle \Downarrow \langle v, S\rangle} \text{ (Var)}$$

$$\frac{}{\langle i, E, S\rangle \Downarrow \langle i, S\rangle} \text{ (Int)}$$

$$\frac{\langle e_1, E, S\rangle \Downarrow \langle i_1, S'\rangle \quad \langle e_2, E, S'\rangle \Downarrow \langle i_2, S''\rangle}{\langle \texttt{(+ } e_1 \ e_2\texttt{)}, E, S\rangle \Downarrow \langle i_1 + i_2, S''\rangle} \text{ (Add)}$$

$$\frac{\begin{array}{c}\langle e_1, E, S\rangle \Downarrow \langle v_1, S'\rangle \qquad l \notin \mathit{dom}(S') \\ \langle e_2, E + \{x \mapsto l\}, S' + \{l \mapsto v_1\}\rangle \Downarrow \langle v_2, S''\rangle\end{array}}{\langle \texttt{(let } x \ e_1 \ e_2\texttt{)}, E, S\rangle \Downarrow \langle v_2, S''\rangle} \text{ (Let)}$$

$$\frac{\langle e, E, S\rangle \Downarrow \langle v, S'\rangle \quad l = E(x)}{\langle \texttt{(:= } x \ e\texttt{)}, E, S\rangle \Downarrow \langle v, S' + \{l \mapsto v\}\rangle} \text{ (Assgn)}$$

Evaluation Rules (2)

$$\frac{\langle e_1, E, S\rangle \Downarrow \langle i_1, S'\rangle \quad i_1 \neq 0 \quad \langle e_2, E, S'\rangle \Downarrow \langle v_2, S''\rangle}{\langle (\texttt{ifnz}\ e_1\ e_2\ e_3), E, S\rangle \Downarrow \langle v_2, S''\rangle}\ \text{(If-nzero)}$$

$$\frac{\langle e_1, E, S\rangle \Downarrow \langle 0, S'\rangle \quad \langle e_3, E, S'\rangle \Downarrow \langle v_3, S''\rangle}{\langle (\texttt{ifnz}\ e_1\ e_2\ e_3), E, S\rangle \Downarrow \langle v_3, S''\rangle}\ \text{(If-zero)}$$

$$\frac{}{\langle (\texttt{fun}\ x\ e), E, S\rangle \Downarrow \langle [x, e, E], S\rangle}\ \text{(Fun)}$$

$$\frac{\begin{array}{c}\langle e_1, E, S\rangle \Downarrow \langle [x, e', E'], S'\rangle \qquad \langle e_2, E, S'\rangle \Downarrow \langle v', S''\rangle \quad l \notin \mathit{dom}(S') \\ \langle e', E' + \{x \mapsto l\}, S'' + \{l \mapsto v'\}\rangle \Downarrow \langle v, S'''\rangle\end{array}}{\langle (\texttt{@}\ e_1\ e_2), E, S\rangle \Downarrow \langle v, S'''\rangle}\ \text{(Appl)}$$

NOTES ON THE RULES

• The structure of the rules guarantees that at most one rule is applicable at any point.

• The store relationships constrain the order of evaluation.

• If no rules are applicable, the evaluation **gets stuck**; this corresponds to a runtime error in an interpreter.

We can view the interpreter for the language as implementing a bottom-up exploration of the inference tree. A function like

```
def eval(Exp e, Env env) : Value  = { .... }
```

returns a value `v` and has side effects on a global `store` such that

$$\langle \texttt{e}, \texttt{env}, \texttt{store}_{\textit{before}} \rangle \Downarrow \langle \texttt{v}, \texttt{store}_{\textit{after}} \rangle$$

The implementation of `eval` dispatches on the syntactic form of `e`, chooses the appropriate rule, and makes recursive calls on `eval` corresponding to the premises of that rule.

Question: how deep can the derivation tree get?