

# Intuitionistic Logic

K. Kogkalidis

Logic & Language 2020

# Intuitionism



L. E. J. Brouwer

## Main Tenets

- ▶ Mathematical truth is subjective rather than fundamental
- ▶ Proof is a process of construction rather than discovery
- ▶ A mathematical object exists if it can be constructed

## Intuitionistic Logic

A different formal model to capture the notion of Intuitionistic Truth, which is stricter than Classical Truth



Arend Heyting

# Intuitionistic Logic vs Classical Logic

CL:

- ▶ Propositions are either true or false  
Law of excluded middle:  $\top \rightarrow (A \vee \neg A)$
- ▶ Negation is Falsity  
Double Negation Elimination:  $\neg(\neg A) \rightarrow A$

IL:

- ▶ Propositions are *proven* (true) or *unproven* (anything)
- ▶ Negation is refutability

Consequence

IL is a *weaker* logic, disallowing proof by contradiction

# Basic Definitions: Proposition

## Propositions

Let  $\mathcal{C}$  a set of *propositional constants*. The *propositions* (formulas)  $\mathcal{P}$  of IL are:

$$\mathcal{P} := \mathcal{C} \mid \mathcal{P}_1 \rightarrow \mathcal{P}_2 \mid \mathcal{P}_1 \times \mathcal{P}_2 \mid \mathcal{P}_1 + \mathcal{P}_2$$

where:

$\rightarrow$  is read as “implies”

$\times$  is read as “and”

$+$  is read as “or”

We will denote propositional constants by  $A, B, C, \dots$

# Basic Definitions: Assumption & Judgement

## Assumptions

An *assumption* (context)  $\mathcal{A}$  is a sequence of zero or more propositions:

$$\mathcal{A} := () \mid \mathcal{A}, \mathcal{P}$$

We will denote assumptions by  $\Gamma, \Delta, \Theta, \dots$

## Judgement

A *judgement*  $\mathcal{J}$  is a statement  $\mathcal{A} \vdash \mathcal{P}$

Read as “*from assumptions  $\mathcal{A}$ , one can conclude proposition  $\mathcal{P}$* ”

# Basic Definitions: Rule

## Rule

A *rule* is a statement consisting of zero or more **premises**  $\mathcal{J}_1, \dots, \mathcal{J}_n$  and a **conclusion**  $\mathcal{J}_c$ :

$$\frac{\mathcal{J}_1 \quad \dots \quad \mathcal{J}_n}{\mathcal{J}_c}$$

If every  $\mathcal{J}_i$  is derivable (has a proof), we can derive  $\mathcal{J}_c$

## Axiom Rule

From any formula  $A$ , one can conclude itself:

$$\overline{A \vdash A} \text{ Ax}$$

# Logical Rules

Each logical connective  $\rightarrow, \times, +$  has rules for its *introduction* and *elimination*:

# Logical Rules

Each logical connective  $\rightarrow, \times, +$  has rules for its *introduction* and *elimination*:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow I \quad \frac{\Gamma \vdash A \rightarrow B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \rightarrow E$$



# Logical Rules

Each logical connective  $\rightarrow, \times, +$  has rules for its *introduction* and *elimination*:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow I \quad \frac{\Gamma \vdash A \rightarrow B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \rightarrow E$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \times B} \times I \quad \frac{\Gamma \vdash A \times B \quad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C} \times E$$

# Logical Rules

Each logical connective  $\rightarrow, \times, +$  has rules for its *introduction* and *elimination*:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow I \quad \frac{\Gamma \vdash A \rightarrow B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \rightarrow E$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \times B} \times I \quad \frac{\Gamma \vdash A \times B \quad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C} \times E$$

$$\frac{\Gamma \vdash A + B \quad \Delta, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta \vdash C} +E$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A + B} +I_1 \quad \frac{\Gamma \vdash B}{\Gamma \vdash A + B} +I_2$$

# Structural Rules

The , of IL assumptions is *commutative*:

$$\frac{\Delta, \Gamma \vdash A}{\Gamma, \Delta \vdash A} \text{ Exchange}$$

IL propositions are free to *discard* and *replicate*:

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \text{ Contraction} \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B} \text{ Weakening}$$

# Example: Identity Function

$$\overline{\vdash A \rightarrow A}$$

## Example: Identity Function

$$\frac{\overline{A \vdash A} \quad Ax}{\vdash A \rightarrow A} \rightarrow I$$

## Example: Tuple Reversal

$$\frac{}{A \times B \vdash B \times A}$$

# Example: Tuple Reversal

$$\frac{\overline{A \times B \vdash A \times B} \quad \overline{A, B \vdash B \times A}}{A \times B \vdash B \times A} \times E$$

## Example: Tuple Reversal

$$\frac{\frac{}{A \times B \vdash A \times B} Ax \quad \frac{}{A, B \vdash B \times A}}{A \times B \vdash B \times A} \times E$$



# Example: Tuple Reversal

$$\frac{\frac{A \times B \vdash A \times B}{A \times B \vdash B \times A} Ax \quad \frac{\frac{B, A \vdash B \times A}{A, B \vdash B \times A} Ex}{A \times B \vdash B \times A} \times E$$

# Example: Tuple Reversal

$$\frac{\frac{\overline{A \times B \vdash A \times B} \quad Ax}{\overline{A \times B \vdash B \times A}} \quad \frac{\frac{\frac{\overline{B \vdash B} \quad Ax \quad \overline{A \vdash A} \quad Ax}{B, A \vdash B \times A} \quad \times I}{A, B \vdash B \times A} \quad Ex}{A \times B \vdash B \times A} \quad \times E$$

## Example: Function Composition

$$\overline{A \rightarrow B, B \rightarrow C \vdash A \rightarrow C}$$

## Example: Function Composition

$$\frac{\overline{A \rightarrow B, B \rightarrow C, A \vdash C}}{A \rightarrow B, B \rightarrow C \vdash A \rightarrow C} \rightarrow I$$

## Example: Function Composition

$$\frac{\frac{B \rightarrow C, A, A \rightarrow B \vdash C}{A \rightarrow B, B \rightarrow C, A \vdash C} \text{Ex}}{A \rightarrow B, B \rightarrow C \vdash A \rightarrow C} \rightarrow I$$

## Example: Function Composition

$$\frac{\frac{\frac{\overline{B \rightarrow C \vdash B \rightarrow C} \text{ Ax}}{B \rightarrow C, A, A \rightarrow B \vdash C} \text{ Ex}}{A \rightarrow B, B \rightarrow C, A \vdash C} \text{ Ex}}{A \rightarrow B, B \rightarrow C \vdash A \rightarrow C} \rightarrow I \quad \rightarrow E$$

# Example: Function Composition

$$\frac{\frac{\frac{\overline{B \rightarrow C \vdash B \rightarrow C} \text{ Ax}}{B \rightarrow C, A, A \rightarrow B \vdash C} \text{ Ex}}{A \rightarrow B, B \rightarrow C, A \vdash C} \text{ Ex}}{A \rightarrow B, B \rightarrow C \vdash A \rightarrow C} \rightarrow I \quad \frac{\overline{A, A \rightarrow B \vdash B}}{\rightarrow E}$$

# Example: Function Composition

$$\frac{\frac{\frac{}{B \rightarrow C \vdash B \rightarrow C} Ax}{\frac{B \rightarrow C, A, A \rightarrow B \vdash C}{A \rightarrow B, B \rightarrow C, A \vdash C} Ex} \rightarrow E}{\frac{A \rightarrow B, B \rightarrow C \vdash A \rightarrow C}{A \rightarrow B, B \rightarrow C \vdash A \rightarrow C} \rightarrow I} \rightarrow E$$



# Example: Function Composition

$$\begin{array}{c}
 \frac{\overline{B \rightarrow C \vdash B \rightarrow C} \quad Ax}{\overline{B \rightarrow C, A, A \rightarrow B \vdash C} \quad Ex} \quad \frac{\overline{A \rightarrow B \vdash A \rightarrow B} \quad Ax \quad \overline{A \vdash A} \quad Ax}{\overline{A \rightarrow B, A \vdash B} \quad Ex} \rightarrow E \\
 \frac{\overline{A \rightarrow B, A \vdash B} \quad Ex}{\overline{A, A \rightarrow B \vdash B} \quad Ex} \rightarrow E \\
 \frac{\overline{A \rightarrow B, B \rightarrow C, A \vdash C} \quad Ex}{\overline{A \rightarrow B, B \rightarrow C \vdash A \rightarrow C} \rightarrow I}
 \end{array}$$

# Example: Currying

$$\overline{(A \times B) \rightarrow C \vdash A \rightarrow B \rightarrow C}$$

## Example: Currying

$$\frac{\overline{(A \times B) \rightarrow C, A \vdash B \rightarrow C}}{(A \times B) \rightarrow C \vdash A \rightarrow B \rightarrow C} \rightarrow I$$

## Example: Currying

$$\frac{\frac{(A \times B) \rightarrow C, A, B \vdash C}{(A \times B) \rightarrow C, A \vdash B \rightarrow C} \rightarrow I}{(A \times B) \rightarrow C \vdash A \rightarrow B \rightarrow C} \rightarrow I$$

## Example: Currying

$$\frac{\frac{\frac{(A \times B) \rightarrow C \vdash (A \times B) \rightarrow C}{(A \times B) \rightarrow C, A, B \vdash C} \rightarrow I}{(A \times B) \rightarrow C, A \vdash B \rightarrow C} \rightarrow I}{(A \times B) \rightarrow C \vdash A \rightarrow B \rightarrow C} \rightarrow I \quad \frac{A \times \quad \frac{A, B \vdash A \times B}{A, B \vdash A \times B} \rightarrow E}{(A \times B) \rightarrow C \vdash (A \times B) \rightarrow C} Ax$$

## Example: Currying

$$\frac{\frac{\frac{\overline{(A \times B) \rightarrow C \vdash (A \times B) \rightarrow C} \text{Ax}}{(A \times B) \rightarrow C, A, B \vdash C} \rightarrow I}{(A \times B) \rightarrow C, A \vdash B \rightarrow C} \rightarrow I}{(A \times B) \rightarrow C \vdash A \rightarrow B \rightarrow C} \rightarrow I$$

$\frac{\overline{A \vdash A} \text{Ax} \quad \overline{B \vdash B} \text{Ax}}{A, B \vdash A \times B} \times I$   
 $\rightarrow E$

# Proof Normalization

A single judgement may have many distinct proofs:

- 😊 “Morally” distinct: different construction methods
- 😞 Redundant elongations due to chaining I/E or C/W rules

$$\frac{\frac{\dots}{A \rightarrow B, A \vdash B} \rightarrow I \quad \frac{}{A \vdash A} Ax}{\frac{}{A \rightarrow B, A \vdash B} \rightarrow E}$$

# Curry-Howard Correspondence



Haskell Curry



William Howard



Nicolas de Bruijn

## Curry-Howard Correspondence

Intuitionistic Logic describes a model of computation, known as the **simply-typed**  $\lambda$ -calculus  $\lambda \rightarrow$



Alonzo Church



# Propositions as Types

Logic	Programming
Proposition	Type
Proof	Algorithm
Provability	Type Inhabitation
Proof Normalization	$\beta$ -reduction
Propositional Constant	Primitive Type
Axiom	Variable Instantiation
Logical Connectives	Type Operators
Introduction Rules	Type Constructors
Elimination Rules	Type Destructors
Implication Introduction	Function Abstraction
Implication Elimination	Function Application

...

# Terms of the simply typed $\lambda$ -calculus

Each proposition ( $\equiv$  *type*)  $\mathcal{P}$  is assigned a unique *term* (name)  $\mathcal{T}$ :

$$\mathcal{T} : \mathcal{P}$$

Assumptions  $\mathcal{A}$  are now interpreted as a *typing environment*:

$$\mathcal{A} := () \mid \mathcal{A}, \mathcal{T} : \mathcal{P}$$

# Terms of the simply typed $\lambda$ -calculus

Let  $\mathcal{V}$  a set of *primitive terms* (variables).

We will denote primitive terms by  $x, y, z, \dots$

The terms  $\mathcal{T}$  of the simply typed  $\lambda$ -calculus then are:

$\mathcal{T} :=$

# Terms of the simply typed $\lambda$ -calculus

Let  $\mathcal{V}$  a set of *primitive terms* (variables).

We will denote primitive terms by  $x, y, z, \dots$

The terms  $\mathcal{T}$  of the simply typed  $\lambda$ -calculus then are:

$$\mathcal{T} := \mathcal{V}$$

$$\frac{}{x : A \vdash x : A} Ax$$

# Terms of the simply typed $\lambda$ -calculus

Let  $\mathcal{V}$  a set of *primitive terms* (variables).

We will denote primitive terms by  $x, y, z, \dots$

The terms  $\mathcal{T}$  of the simply typed  $\lambda$ -calculus then are:

$$\mathcal{T} := \mathcal{V} \mid \mathcal{T}(\mathcal{T})$$

$$\frac{\Gamma \vdash f : A \rightarrow B \quad \Delta \vdash x : A}{\Gamma, \Delta \vdash f(x)^1 : B} \rightarrow E$$

---

<sup>1</sup>Can also be written  $f \ x$

# Terms of the simply typed $\lambda$ -calculus

Let  $\mathcal{V}$  a set of *primitive terms* (variables).

We will denote primitive terms by  $x, y, z, \dots$

The terms  $\mathcal{T}$  of the simply typed  $\lambda$ -calculus then are:

$$\mathcal{T} := \mathcal{V} \mid \mathcal{T}(\mathcal{T}) \mid \lambda \mathcal{T}.\mathcal{T}$$

*Implication-Only*

$$\frac{\Gamma, x : A \vdash f : B}{\Gamma \vdash \lambda x.f : A \rightarrow B} \rightarrow I$$

# Terms of the simply typed $\lambda$ -calculus

Let  $\mathcal{V}$  a set of *primitive terms* (variables).

We will denote primitive terms by  $x, y, z, \dots$

The terms  $\mathcal{T}$  of the simply typed  $\lambda$ -calculus then are:

$$\begin{aligned} \mathcal{T} := \mathcal{V} \mid & \mathcal{T}(\mathcal{T}) \mid \lambda \mathcal{T}.\mathcal{T} \\ & \mid (\mathcal{T}, \mathcal{T}) \end{aligned}$$

*Implication-Only*

$$\frac{\Gamma \vdash x : A \quad \Delta \vdash y : B}{\Gamma, \Delta \vdash (x, y) : A \times B} \times I$$

# Terms of the simply typed $\lambda$ -calculus

Let  $\mathcal{V}$  a set of *primitive terms* (variables).

We will denote primitive terms by  $x, y, z, \dots$

The terms  $\mathcal{T}$  of the simply typed  $\lambda$ -calculus then are:

$$\begin{aligned} \mathcal{T} := \mathcal{V} \mid \mathcal{T}(\mathcal{T}) \mid \lambda \mathcal{T}.\mathcal{T} & \quad \text{Implication-Only} \\ \mid (\mathcal{T}, \mathcal{T}) \mid \text{case } \mathcal{T} \text{ of } (\mathcal{T}, \mathcal{T}) \rightarrow \mathcal{T} & \quad \text{/w Product} \end{aligned}$$

$$\frac{\Gamma \vdash s : A \times B \quad \Delta, x : A, y : B \vdash w : C}{\Gamma, \Delta \vdash \text{case } s \text{ of } (x, y) \rightarrow w : C} \times E$$



# Terms of the simply typed $\lambda$ -calculus

Let  $\mathcal{V}$  a set of *primitive terms* (variables).

We will denote primitive terms by  $x, y, z, \dots$

The terms  $\mathcal{T}$  of the simply typed  $\lambda$ -calculus then are:

$$\begin{aligned} \mathcal{T} := & \mathcal{V} \mid \mathcal{T}(\mathcal{T}) \mid \lambda \mathcal{T}.\mathcal{T} && \text{Implication-Only} \\ & \mid (\mathcal{T}, \mathcal{T}) \mid \text{case } \mathcal{T} \text{ of } (\mathcal{T}, \mathcal{T}) \rightarrow \mathcal{T} && \text{/w Product} \\ & \mid \text{inl}(\mathcal{T}) \mid \text{inr}(\mathcal{T}) \end{aligned}$$

$$\frac{\Gamma \vdash x : A}{\Gamma \vdash \text{inl}(x) : A + B} +l_1 \quad \frac{\Gamma \vdash x : B}{\Gamma \vdash \text{inr}(x) : A + B} +l_2$$

# Terms of the simply typed $\lambda$ -calculus

Let  $\mathcal{V}$  a set of *primitive terms* (variables).

We will denote primitive terms by  $x, y, z, \dots$

The terms  $\mathcal{T}$  of the simply typed  $\lambda$ -calculus then are:

$$\begin{aligned} \mathcal{T} := & \mathcal{V} \mid \mathcal{T}(\mathcal{T}) \mid \lambda \mathcal{T}.\mathcal{T} && \text{Implication-Only} \\ & \mid (\mathcal{T}, \mathcal{T}) \mid \text{case } \mathcal{T} \text{ of } (\mathcal{T}, \mathcal{T}) \rightarrow \mathcal{T} && \text{/w Product} \\ & \mid \text{inl}(\mathcal{T}) \mid \text{inr}(\mathcal{T}) \\ & \mid \text{case } s \text{ of } \text{inl}(x) \rightarrow w; \text{inr}(y) \rightarrow z && \text{/w Co-product} \end{aligned}$$

$$\frac{\Gamma \vdash s : A + B \quad \Delta, x : A \vdash w : C \quad \Delta, y : B \vdash z : C}{\Gamma \vdash \text{case } s \text{ of } \text{inl}(x) \rightarrow w; \text{inr}(y) \rightarrow z : C} +E$$

# Terms: Uniqueness & Structural Rules

Variable names must be **unique** for each distinct formula instantiation in a proof!

$$\frac{\Gamma, y : A, z : A \vdash u : B}{\Gamma, x : A \vdash u[x/y, x/z] : B} \text{ Contraction}$$

$$\frac{\Gamma \vdash u : B}{\Gamma, x : A \vdash u : B} \text{ Weakening}$$

## Example: Identity Function Revisited

$$\frac{\overline{A \vdash A} \quad Ax}{\vdash A \rightarrow A} \rightarrow I$$

## Example: Identity Function Revisited

$$\frac{\overline{x : A \vdash x : A}}{\vdash A \rightarrow A} \rightarrow I^{Ax}$$

## Example: Identity Function Revisited

$$\frac{\overline{x : A \vdash x : A}^{Ax}}{\vdash \lambda x. x : A \rightarrow A} \rightarrow I$$

# Example: Function Composition Revisited

$$\begin{array}{c}
 \frac{\overline{B \rightarrow C \vdash B \rightarrow C} \quad Ax}{\frac{\frac{B \rightarrow C, A, A \rightarrow B \vdash C}{A \rightarrow B, B \rightarrow C, A \vdash C} Ex}{A \rightarrow B, B \rightarrow C \vdash A \rightarrow C} \rightarrow I} \\
 \frac{\overline{A \rightarrow B \vdash A \rightarrow B} \quad Ax \quad \frac{\overline{A \vdash A} \quad Ax}{\rightarrow E} \quad \frac{\frac{A \rightarrow B, A \vdash B}{A, A \rightarrow B \vdash B} Ex}{\rightarrow E}}{\rightarrow E}
 \end{array}$$

## Example: Function Composition Revisited

$$\frac{\frac{\frac{\frac{\frac{\frac{}{g : B \rightarrow C \vdash g : B \rightarrow C}}{Ax}}{A \rightarrow B, A \vdash B}}{Ax}}{A, A \rightarrow B \vdash B}}{Ex}}{\frac{B \rightarrow C, A, A \rightarrow B \vdash C}{A \rightarrow B, B \rightarrow C, A \vdash C} Ex} \rightarrow I$$



## Example: Function Composition Revisited

$$\frac{\frac{\frac{g : B \rightarrow C \vdash g : B \rightarrow C}{\quad} Ax \quad \frac{\frac{\frac{f : A \rightarrow B \vdash f : A \rightarrow B}{\quad} Ax \quad \frac{x : A \vdash x : A}{\quad} Ax}{f : A \rightarrow B, x : A \vdash f \ x : B} \rightarrow E \quad \frac{A, A \rightarrow B \vdash B}{\quad} Ex}{\frac{B \rightarrow C, A, A \rightarrow B \vdash C}{A \rightarrow B, B \rightarrow C, A \vdash C} Ex \rightarrow E} \rightarrow I$$

# Example: Function Composition Revisited

$$\begin{array}{c}
 \frac{\frac{\frac{\overline{g : B \rightarrow C \vdash g : B \rightarrow C} \text{Ax}}{g : B \rightarrow C, x : A, f : A \rightarrow B \vdash g (f x) : C} \text{Ex}}{A \rightarrow B, B \rightarrow C, A \vdash C} \text{Ex}}{A \rightarrow B, B \rightarrow C \vdash A \rightarrow C} \rightarrow I \\
 \frac{\frac{\frac{\overline{f : A \rightarrow B \vdash f : A \rightarrow B} \text{Ax} \quad \frac{\overline{x : A \vdash x : A} \text{Ax}}{\frac{f : A \rightarrow B, x : A \vdash f x : B}{x : A, f : A \rightarrow B \vdash f x : B} \text{Ex}} \rightarrow E}{\frac{g : B \rightarrow C, x : A, f : A \rightarrow B \vdash g (f x) : C}{x : A, f : A \rightarrow B \vdash f x : B} \text{Ex}} \rightarrow E
 \end{array}$$

# Example: Function Composition Revisited

$$\begin{array}{c}
 \frac{\frac{\frac{g : B \rightarrow C \vdash g : B \rightarrow C}{Ax} \quad \frac{\frac{\frac{f : A \rightarrow B \vdash f : A \rightarrow B}{Ax} \quad \frac{x : A \vdash x : A}{\rightarrow E}}{f : A \rightarrow B, x : A \vdash f \ x : B} \quad \frac{x : A, f : A \rightarrow B \vdash f \ x : B}{Ex}}{g : B \rightarrow C, x : A, f : A \rightarrow B \vdash g \ (f \ x) : C} \quad \frac{f : A \rightarrow B, g : B \rightarrow C, x : A \vdash g \ (f \ x) : C}{\rightarrow I}}{A \rightarrow B, B \rightarrow C \vdash A \rightarrow C} \rightarrow E
 \end{array}$$

## Example: Function Composition Revisited

$$\frac{\frac{\frac{g : B \rightarrow C \vdash g : B \rightarrow C}{Ax} \quad \frac{\frac{\frac{f : A \rightarrow B \vdash f : A \rightarrow B}{Ax} \quad \frac{x : A \vdash x : A}{Ax}}{\rightarrow E} \quad \frac{f : A \rightarrow B, x : A \vdash f \ x : B}{Ex}}{\rightarrow E} \quad \frac{g : B \rightarrow C, x : A, f : A \rightarrow B \vdash g \ (f \ x) : C}{Ex}}{\rightarrow I} \quad \frac{f : A \rightarrow B, g : B \rightarrow C, x : A \vdash g \ (f \ x) : C}{\rightarrow I} \quad f : A \rightarrow B, g : B \rightarrow C \vdash \lambda x. (g \ (f \ x)) : A \rightarrow C$$

# Term Equivalence & Reduction

- ▶  $\alpha$ -conversion Changing the name of bound variables

$$\lambda x.x \stackrel{\alpha}{\equiv} \lambda y.y$$

- ▶ Substitution Changing the name of free variables

$$(f\ g)(\lambda x.(x\ g))[h/g] = (f\ h)(\lambda x.(x\ h))$$

- ▶  $\eta$ -reduction Simplifying an abstraction if the abstracted variable does not occur free in the function body

$$\lambda x.f\ x \stackrel{\eta}{\equiv} f \text{ (} x \text{ does not occur in } f \text{)}$$

- ▶  $\beta$ -reduction Removing any applicable abstraction

$$(\lambda x.g)(y) \implies g[x/y]$$

## Proof Normalization $\equiv$ Term Reduction

- ▶ Church-Rosser: order of term reduction rules is irrelevant
- ▶ Subject Reduction: term reduction rules on well-typed terms produce well-typed terms

# Proof Normalization & Term Reduction ( $\rightarrow$ )

$$\begin{array}{c}
 \overline{x : A \vdash x : A} \quad \dots \\
 \vdots \\
 \Gamma, x : A, \dots \vdash u : B \\
 \hline
 \Gamma, x : A \vdash u : B \\
 \hline
 \Gamma \vdash \lambda x. u : A \rightarrow B \quad \xrightarrow{\textcolor{red}{I}} \quad \overline{\Delta \vdash t : A} \quad \xrightarrow{\textcolor{red}{E}} \quad \frac{\Gamma, \Delta, \dots \vdash u : B}{\Gamma, \Delta \vdash u : B} \\
 \hline
 \Gamma, \Delta \vdash (\lambda x. u)(t) : B \quad \Rightarrow
 \end{array}$$

$$(\lambda x. u)(t) \Rightarrow u[t/x]$$

$\beta$ -reduction

# Proof Normalization & Term Reduction ( $\times$ )

$$\begin{array}{c}
 \frac{\frac{\frac{\vdots}{\Gamma \vdash t : A} \quad \frac{\vdots}{\Delta \vdash u : B}}{\Gamma, \Delta \vdash (t, u) : A \times B} \times I \quad \frac{\frac{\overline{x : A \vdash x : A} \quad \dots \quad \overline{y : B \vdash y : B}}{\Theta, A, \dots, B, \dots \vdash v : C} \quad \frac{\Theta, A, B \vdash C}{\Gamma, \Delta, \Theta \vdash C} \times E}{\Gamma, \Delta, \Theta \vdash C} \\
 \Downarrow \\
 \frac{\frac{\vdots}{\Gamma \vdash t : A} \quad \dots \quad \frac{\vdots}{\Delta \vdash u : B}}{\frac{\Gamma, \dots, \Delta, \dots, \Theta \vdash v : C}{\Gamma, \Delta, \Theta \vdash v : C}}
 \end{array}$$

case  $(t, u)$  of  $(x, y) \rightarrow v \implies v[t/x, u/y]$

# Proof Normalization & Term Reduction (+)

$$\begin{array}{c}
 \frac{\frac{\vdots}{\Gamma \vdash t : B}}{\Gamma \vdash \text{inr}(t) : A + B} \text{ } +I_2 \quad \frac{\frac{\overline{x : A \vdash x : A} \quad \vdots}{\Delta, x : A \vdash v : C} \quad \frac{\frac{\overline{y : B \vdash y : B} \quad \vdots}{\Delta, y : B \vdash w : C}}{\Delta, y : B \vdash w : C}}{\Gamma, \Delta \vdash \text{case inr}(t) \text{ of } \text{inl}(x) \rightarrow v; \text{inr}(y) \rightarrow w : C} \text{ } +E \\
 \Downarrow \\
 \frac{\frac{\vdots}{\Gamma \vdash t : B} \quad \dots}{\Gamma, \dots, \Delta, \vdots \vdash w : C} \\
 \hline
 \Gamma, \Delta \vdash w : C
 \end{array}$$

$\text{case inr}(t) \text{ of } \text{inl}(x) \rightarrow v; \text{inr}(y) \rightarrow w \implies w[t/y]$

Similarly for  $+I_1$



# Beyond simple types: the $\lambda$ -Cube

Three axes of extension:

$\lambda 2$  Terms can depend on types

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \lambda a. t : \Pi a. A}$$

$\lambda \Pi$  Types can depend on terms

$$\frac{\Gamma, x : A \vdash B : *}{\Gamma \vdash \Pi x. A. B : *}$$

$\lambda \omega$  Types can depend on types

$$\lambda A : *. (A \rightarrow B) \rightarrow (B \rightarrow B \rightarrow B) \rightarrow B$$

