Lambek Calculus

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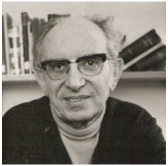
Logic & Language 2020

Categorial Grammars: History

Kazimierz Ajdukiewicz



Yehoshua Bar-Hillel

AB Grammars

An AB Grammar is a tuple $(\Sigma, \mathcal{A}, S, L)$

- Σ a finite set of symbols
- $\mathcal{A}$ a finite set of primitives, deriving:

$$\mathcal{T}_\mathcal{A} := \mathcal{A} \mid \mathcal{T}_\mathcal{A}/\mathcal{T}_\mathcal{A} \mid \mathcal{T}_\mathcal{A}\backslash\mathcal{T}_\mathcal{A}$$

- S a distinguished type, $S \in \mathcal{T}_\mathcal{A}$
- L a mapping $\Sigma \to \mathcal{T}_\mathcal{A}$

Inference Rules

$$X \longleftarrow X/Y, Y$$

$$X \longleftarrow Y, Y\backslash X$$

AB Grammars & Constituency Parsing

Consider a grammar where

$\mathcal{A}$ $:= \{s, n, np\}$

Σ a (simple) lexicon of english

L a mapping from:

common nouns to n

proper nouns to np

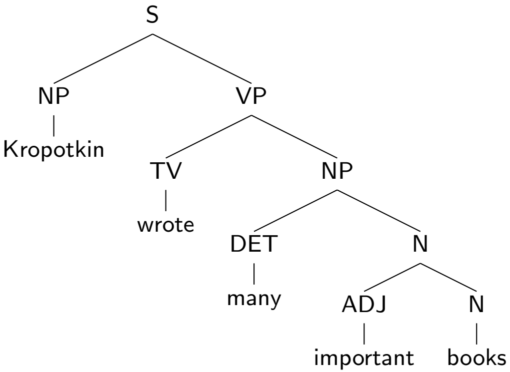
determiners to np/n

adjectives to n/n

intransitive verbs to $np \backslash s$

transitive verbs to $(np \backslash s)/np$

...



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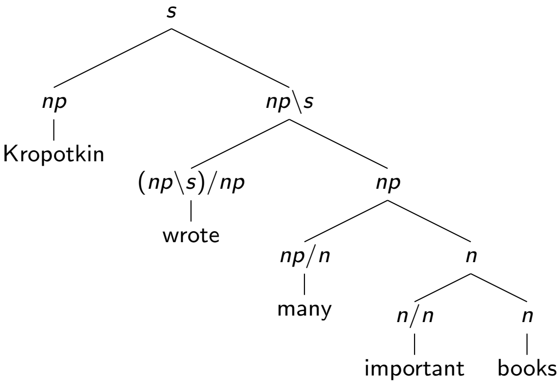
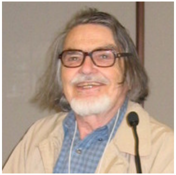
  - ...



Refinement: Lambek Calculus L



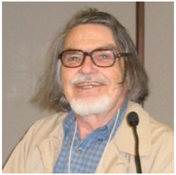
Joachim Lambek

The Mathematics of Sentence Structure (1958):

$$\mathcal{T} := \mathcal{A} \mid \mathcal{T}_1/\mathcal{T}_2 \mid \mathcal{T}_1\backslash\mathcal{T}_2 \mid \mathcal{T}_1 \otimes \mathcal{T}_2$$

- $/$ 'right' division (*over*)
- $\backslash$ 'left' division (*under*)
- $\otimes$ concatenation (*and*)

Refinement: Lambek Calculus L



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$\rightsquigarrow$ $\mathrm{ILL}_{\multimap,\otimes}$ without Exchange:

$$\frac{\Gamma \vdash B/A \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}\ /E \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash B/A}\ /I$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash A\backslash B}{\Gamma, \Delta \vdash B}\ \backslash E \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A\backslash B}\ \backslash I$$

$$\frac{\Gamma \vdash A \otimes B \quad \Delta, A, B, \Theta \vdash C}{\Delta, \Gamma, \Theta \vdash C}\ \otimes E \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes I$$

Lambek Calculus L

The Lambek Calculus L

- is the grammar of strings, being order-sensitive
- is a substructural logic coinciding with the non-commutative fragment of multiplicative intuitionistic linear logic $\mathrm{ILL}_{\otimes,/,\backslash}$

  $\Longrightarrow$ assumptions of L are no longer multisets, but sequences
- has equal generative capacity to AB- and CF-grammars

Further Refinement: NL

The , of L assumptions still hides an implicit structural rule:

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On the Calculus of Syntactic Types (1961):
Assumptions are now bracketed structures $\mathcal{S} := \mathcal{T}_{\mathcal{A}} \mid (\mathcal{S}, \mathcal{S})$

Further Refinement: NL

The , of L assumptions still hides an implicit structural rule: associativity

On the Calculus of Syntactic Types (1961):
Assumptions are now bracketed structures $\mathcal{S} := \mathcal{T}_{\mathcal{A}} \mid (\mathcal{S}, \mathcal{S})$

$$\frac{\Gamma \vdash B/A \quad \Delta \vdash A}{(\Gamma, \Delta) \vdash B} /E \qquad \frac{(\Gamma, A) \vdash B}{\Gamma \vdash B/A} /I$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash A\backslash B}{(\Gamma, \Delta) \vdash B} \backslash E \qquad \frac{(A, \Gamma) \vdash B}{\Gamma \vdash A\backslash B} \backslash I$$

$$\frac{\Gamma \vdash A \otimes B \quad \Delta[(A, B)] \vdash C}{\Delta[\Gamma] \vdash C} \otimes E \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{(\Gamma, \Delta) \vdash A \otimes B} \otimes I$$

where $\Gamma[\Delta]$: Δ a sub-structure of Γ

from NL one can recover L via explicit associativity:

$$\frac{\Gamma[(\Delta, (\Theta, \Phi))] \vdash C}{\Gamma[((\Delta, \Theta), \Phi)] \vdash C} A$$

Non-Associative Lambek Calculus NL

The N/A Lambek Calculus NL

- is the grammar of trees, being order- and constituency-sensistive
- is a substructural logic coinciding with the non-commutative non-associative fragment of $\mathrm{ILL}_{\otimes,/,\backslash}$

$\Longrightarrow$ assumptions of NL are no longer sequences, but binary trees

Example: L vs NL

$$B/C \vdash (A/B)\backslash(A/C)$$

Derivation in L

$$\frac{}{B/C \vdash (A/B)\backslash(A/C)}$$

Example: L vs NL

$$B/C \vdash (A/B)\backslash(A/C)$$

Derivation in L

$$\dfrac{\overline{A/B, B/C \vdash A/C}}{B/C \vdash (A/B)\backslash(A/C)}\ \backslash I$$

Example: L vs NL

$$B/C \vdash (A/B)\backslash(A/C)$$

Derivation in L

$$\dfrac{\dfrac{\overline{A/B, B/C, C \vdash A}}{A/B, B/C \vdash A/C}\,/I}{B/C \vdash (A/B)\backslash(A/C)}\,\backslash I$$

Example: L vs NL

$$B/C \vdash (A/B)\backslash(A/C)$$

Derivation in L

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A/B \vdash A/B}\,Ax \qquad \dfrac{}{B/C, C \vdash B}}{A/B, B/C, C \vdash A}\,/E}{A/B, B/C \vdash A/C}\,/I}{B/C \vdash (A/B)\backslash(A/C)}\,\backslash I$$

Example: L vs NL

$$B/C \vdash (A/B)\backslash(A/C)$$

Derivation in L

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A/B \vdash A/B}\,Ax \qquad \dfrac{\dfrac{}{B/C \vdash B/C}\,Ax \quad \dfrac{}{C \vdash C}\,Ax}{B/C, C \vdash B}\,/E}{A/B, B/C, C \vdash A}\,/E}{A/B, B/C \vdash A/C}\,/I}{B/C \vdash (A/B)\backslash(A/C)}\,\backslash I$$

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$$B/C \vdash (A/B)\backslash(A/C)$$

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Example: L vs NL

$$B/C \vdash (A/B)\backslash(A/C)$$

Derivation in NL

$$\dfrac{\overline{(A/B, B/C) \vdash A/C}}{B/C \vdash (A/B)\backslash(A/C)}\ \backslash I$$

Example: L vs NL

$$B/C \vdash (A/B)\backslash(A/C)$$

Derivation in NL

$$\dfrac{\dfrac{\overline{((A/B, B/C), C) \vdash A}}{(A/B, B/C) \vdash A/C}\ /I}{B/C \vdash (A/B)\backslash(A/C)}\ \backslash I$$

Example: L vs NL

$$B/C \vdash (A/B)\backslash(A/C)$$

Derivation in NL

$$\cfrac{\cfrac{\cfrac{\cfrac{\dots}{(A/B,(B/C,C)) \vdash C}}{((A/B,B/C),C) \vdash A}\,{\color{red}A}}{(A/B,B/C) \vdash A/C}\,/I}{B/C \vdash (A/B)\backslash(A/C)}\,\backslash I$$

Directional linear λ-calculus

$$\mathcal{T} = \mathcal{V} \mid \lambda^l \mathcal{V}.\mathcal{T} \mid \lambda^r \mathcal{V}.\mathcal{T} \mid \mathcal{T}_1 \triangleright \mathcal{T}_2 \mid \mathcal{T}_1 \triangleleft \mathcal{T}_2 \mid \ldots$$

$$\frac{\Gamma \vdash \mathtt{s} : B/A \quad \Delta \vdash \mathtt{t} : A}{(\Gamma, \Delta) \vdash \mathtt{s} \triangleleft \mathtt{t} : B} \; /E \qquad \frac{(\Gamma, \mathtt{x} : A) \vdash \mathtt{s} : B}{\Gamma \vdash \lambda \mathtt{x}^r.\mathtt{s} : B/A} \; /I$$

$$\frac{\Gamma \vdash \mathtt{s} : A \quad \Delta \vdash \mathtt{t} : A \backslash B}{(\Gamma, \Delta) \vdash \mathtt{s} \triangleright \mathtt{t} : B} \; \backslash E \qquad \frac{(\mathtt{x} : A, \Gamma) \vdash \mathtt{s} : B}{\Gamma \vdash \lambda \mathtt{x}^l.\mathtt{s} : A \backslash B} \; \backslash I$$

Parsing $\equiv$ Deduction

Parsing as Deduction

For categorial grammars, syntactic parsing becomes equated with a logical deduction process, proving the well-formedness of a sentence and finding its structure

$$\dfrac{\dfrac{\text{Kropotkin}}{\mathrm{np}} \qquad \dfrac{\dfrac{\text{wrote}}{(np\backslash s)/s} \qquad \dfrac{\dfrac{\text{many}}{np/n} \qquad \dfrac{\dfrac{\text{important}}{\mathrm{n/n}} \qquad \dfrac{\text{books}}{\mathrm{n}}}{(\text{important} \cdot \text{books}) \vdash n}\,/E}{(\text{many} \cdot (\text{important} \cdot \text{books})) \vdash np}\,/E}{(\text{wrote} \cdot (\text{many} \cdot (\text{important} \cdot \text{books}))) \vdash np\backslash s}\,/E}{(\text{Kropotkin} \cdot (\text{wrote} \cdot (\text{many} \cdot (\text{important} \cdot \text{books})))) \vdash s}\,\backslash E$$

Lexicon & Ambiguity

$$\frac{?}{(\mathrm{I} \cdot (\mathrm{saw} \cdot (\mathrm{the} \cdot (\mathrm{man} \cdot (\mathrm{with} \cdot (\mathrm{the} \cdot \mathrm{binoculars})))))) \vdash s}$$

vs.

$$\frac{?}{(\mathrm{I} \cdot ((\mathrm{saw} \cdot (\mathrm{the} \cdot \mathrm{man})) \cdot (\mathrm{with} \cdot (\mathrm{the} \cdot \mathrm{binoculars}))) \vdash s}$$

Lexicon & Ambiguity: Reading 1

$$\frac{}{(\mathsf{I} \cdot (\mathsf{saw} \cdot (\mathsf{the} \cdot (\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})))))) \vdash s}$$

Lexicon & Ambiguity: Reading 1

$$\dfrac{\dfrac{\mathsf{I}}{\mathsf{np}} \qquad \dfrac{}{\mathsf{saw} \cdot (\mathsf{the} \cdot (\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})))) \vdash np\backslash s}}{(\mathsf{I} \cdot (\mathsf{saw} \cdot (\mathsf{the} \cdot (\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars}))))) \vdash s}$$

Lexicon & Ambiguity: Reading 1

$$\dfrac{\dfrac{\mathsf{I}}{\mathsf{np}} \qquad \dfrac{\dfrac{\mathsf{saw}}{(np\backslash s)/np} \qquad \dfrac{}{(\mathsf{the} \cdot (\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})))) \vdash np}}{\mathsf{saw} \cdot (\mathsf{the} \cdot (\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})))) \vdash np\backslash s}}{(\mathsf{I} \cdot (\mathsf{saw} \cdot (\mathsf{the} \cdot (\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})))))) \vdash s}$$

Lexicon & Ambiguity: Reading 1

$$\dfrac{\dfrac{\text{I}}{\text{np}} \qquad \dfrac{\dfrac{\text{saw}}{(np\backslash s)/np} \qquad \dfrac{\dfrac{\text{the}}{\text{np}/\text{n}} \qquad \dfrac{}{(\text{man} \cdot (\text{with} \cdot (\text{the} \cdot \text{binoculars}))) \vdash n}}{(\text{the} \cdot (\text{man} \cdot (\text{with} \cdot (\text{the} \cdot \text{binoculars})))) \vdash np}}{\text{saw} \cdot (\text{the} \cdot (\text{man} \cdot (\text{with} \cdot (\text{the} \cdot \text{binoculars})))) \vdash np\backslash s}}{(\text{I} \cdot (\text{saw} \cdot (\text{the} \cdot (\text{man} \cdot (\text{with} \cdot (\text{the} \cdot \text{binoculars}))))) \vdash s}$$

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\frac{\dfrac{\mathsf{I}}{\mathsf{np}} \qquad \dfrac{\dfrac{\mathsf{saw}}{(np\backslash s)/np} \qquad \dfrac{\dfrac{\mathsf{the}}{\mathsf{np}/\mathsf{n}} \qquad \dfrac{\dfrac{\mathsf{man}}{n} \qquad \dfrac{}{(\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})) \vdash n\backslash n}}{(\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars}))) \vdash n}}{(\mathsf{the} \cdot (\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})))) \vdash np}}{\mathsf{saw} \cdot (\mathsf{the} \cdot (\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})))) \vdash np\backslash s}}{(\mathsf{I} \cdot (\mathsf{saw} \cdot (\mathsf{the} \cdot (\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})))))) \vdash s}
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$$
\dfrac{\dfrac{\mathsf{I}}{np} \qquad \dfrac{\dfrac{\mathsf{saw}}{(np\backslash s)/np} \qquad \dfrac{\dfrac{\mathsf{the}}{np/n} \qquad \dfrac{\dfrac{\mathsf{man}}{n} \qquad \dfrac{\dfrac{\mathsf{with}}{(n\backslash n)/np} \qquad (\mathsf{the} \cdot \mathsf{binoculars}) \vdash np}{(\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})) \vdash n\backslash n}}{(\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars}))) \vdash n}}{(\mathsf{the} \cdot (\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})))) \vdash np}}{\mathsf{saw} \cdot (\mathsf{the} \cdot (\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})))) \vdash np\backslash s}}{(\mathsf{I} \cdot (\mathsf{saw} \cdot (\mathsf{the} \cdot (\mathsf{man} \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})))))) \vdash s}
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$$

Lexicon & Ambiguity: Reading 2

Need an alternative type for "with"

- with (producing noun modifier): $(n\backslash n)/np$
- with (producing verb-phrase modifier): $((np\backslash s)\backslash(np\backslash s))/np$

Lexicon & Ambiguity: Reading 2

$$\dfrac{\dfrac{\mathsf{I}}{\mathsf{np}} \qquad \dfrac{}{(\mathsf{saw} \cdot (\mathsf{the} \cdot \mathsf{man})) \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})) \vdash np\backslash s}}{(\mathsf{I} \cdot ((\mathsf{saw} \cdot (\mathsf{the} \cdot \mathsf{man})) \cdot (\mathsf{with} \cdot (\mathsf{the} \cdot \mathsf{binoculars})))) \vdash s} \ \backslash E$$

Lexicon & Ambiguity: Reading 2

$$\dfrac{\dfrac{\text{I}}{\text{np}} \qquad \dfrac{\dfrac{\dfrac{\text{saw}}{(np\backslash s)/np} \qquad \begin{array}{c}\vdots\\ \text{the}\cdot\text{man}\vdash np\end{array}}{\text{saw}\cdot(\text{the}\cdot\text{man})\vdash np\backslash s}\,/E}{(\text{saw}\cdot(\text{the}\cdot\text{man}))\cdot(\text{with}\cdot(\text{the}\cdot\text{binoculars}))\vdash np\backslash s}}{(\text{I}\cdot((\text{saw}\cdot(\text{the}\cdot\text{man}))\cdot(\text{with}\cdot(\text{the}\cdot\text{binoculars})))\vdash s}\,\backslash E$$

Lexicon & Ambiguity: Reading 2

$$
\dfrac{\dfrac{\text{I}}{np} \qquad \dfrac{\dfrac{\dfrac{\text{saw}}{(np\backslash s)/np} \qquad \begin{array}{c}\vdots\\ \text{the} \cdot \text{man} \vdash np\end{array}}{\text{saw} \cdot (\text{the} \cdot \text{man}) \vdash np\backslash s}\,/E \qquad \dfrac{}{\text{with} \cdot (\text{the} \cdot \text{binoculars}) \vdash (np\backslash s)\backslash(np\backslash s)}}{(\text{saw} \cdot (\text{the} \cdot \text{man})) \cdot (\text{with} \cdot (\text{the} \cdot \text{binoculars})) \vdash np\backslash s}\,/E}{(\text{I} \cdot ((\text{saw} \cdot (\text{the} \cdot \text{man})) \cdot (\text{with} \cdot (\text{the} \cdot \text{binoculars}))) \vdash s}\,\backslash E
$$

Lexicon & Ambiguity: Reading 2

$$
\dfrac{\dfrac{\mathrm{I}}{\mathrm{np}} \qquad \dfrac{\dfrac{\dfrac{\text{saw}}{(np\backslash s)/np} \qquad \begin{matrix}\vdots \\ \text{the} \cdot \text{man} \vdash np\end{matrix}}{\text{saw} \cdot (\text{the} \cdot \text{man}) \vdash np\backslash s}\,/E \qquad \dfrac{\dfrac{\text{with}}{((np\backslash s)\backslash(np\backslash s))\,/\,np} \qquad \begin{matrix}\vdots \\ \text{the} \cdot \text{binoculars}\end{matrix}}{\text{with} \cdot (\text{the} \cdot \text{binoculars}) \vdash (np\backslash s)\backslash(np\backslash s)}\,/E}{(\text{saw} \cdot (\text{the} \cdot \text{man})) \cdot (\text{with} \cdot (\text{the} \cdot \text{binoculars})) \vdash np\backslash s}}{(\mathrm{I} \cdot ((\text{saw} \cdot (\text{the} \cdot \text{man})) \cdot (\text{with} \cdot (\text{the} \cdot \text{binoculars})))) \vdash s}\,\backslash E
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Lexicon & Ambiguity: Reading 2

Syntactic/structural ambiguity becomes lexical ambiguity

contrapose: $VP \rightarrow VP\ PP$ vs. $N \rightarrow N\ PP$

lexicaly ambiguous types can be treated with the & connective

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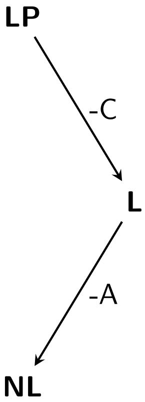
"consistently" ambiguous words can be assigned polymorphic types, i.e. types of the form

$$\Pi\sigma.s$$

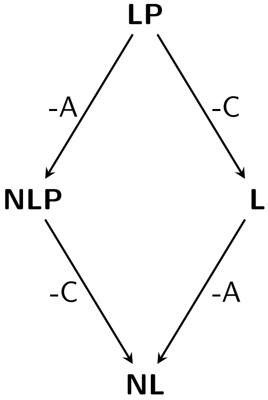
remember polymorphic identity type: $\Pi\iota.(\iota \rightarrow \iota)$

The Full Substructural Picture



logic	struct	assoc	commut
LP	multiset	✓	✓
L	string	✓	-
NL	tree	-	-

The Full Substructural Picture



logic	struct	assoc	commut
LP	multiset	✓	✓
L	string	✓	-
NL	tree	-	-
NLP	mobile	-	✓

Comparison with CFGs

- More "formal"
  The Lambek Calculus defines a substructural logic and an algebra.
- More general
  Rule size constant with vocabulary size. Lexicalization happens on the lexicon, assigning a type to each "type" of word.
- Natural syntax-semantics interface
  Connection to I(L)L allow easy translation from syntactic to semantic calculus (tbd ...)