

Linear Logic

K. Kogkalidis

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Truth vs. Resource

Linear Logic

Proposed by Girard (1987) as a resource-conscious alternative to classical & intuitionistic logic.

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“ Classical and intuitionistic logics deal with stable truths:

if A and $A \rightarrow B$, then B , but A still holds.

This is perfect in mathematics, but wrong in real life, since real implication is causal. A causal implication cannot be iterated since the conditions are modified after its use; this process of modification of the premises is known in physics as reaction. For instance, if A is to spend \$1 on a pack of a cigarettes and B is to get them, you lose \$1 in this process, and you cannot do it a second time. The reaction here was that \$1 went out of your pocket. ”

Truth vs. Resource

Linear Logic

Proposed by Girard (1987) as a resource-conscious alternative to classical & intuitionistic logic.

- ▶ Propositions now represent **resources**
 - ▶ Resources are **not** free to discard and replicate
 - ▶ \Rightarrow Contraction & Weakening are not universally applicable
- Substructural!**
- ▶ Inference rules can share contexts

Linear Logic: Syntax & Connectives

Linear propositions \mathcal{P} are defined as:

$$\mathcal{P} := \mathcal{C} \mid \mathcal{P}_1 \multimap \mathcal{P}_2 \mid \mathcal{P}_1 \otimes \mathcal{P}_2 \mid \mathcal{P}_1 \oplus \mathcal{P}_2 \mid !\mathcal{P}$$

\multimap is read as “loli”

$A \multimap B$: consume A to produce a B

\otimes is read as “tensor”

$A \otimes B$: both A and B

$\&$ is read as “with”

$A \& B$: pick from A and B

\oplus is read as “or”

$A \oplus B$: either A and B

$!$ is read as “bang”

$!A$: of course A

Universal Logic

Two kinds of resources

IL and LL can co-exist in peace: an **assumption** \mathcal{A} can be either **linear** $\langle \mathcal{A} \rangle$ or **intuitionistic** $[\mathcal{A}]$; each comes with its own identity:

$$\overline{\langle \mathcal{A} \rangle \vdash A} \quad \langle Id \rangle \qquad \overline{[\mathcal{A}] \vdash A} \quad [Id]$$

$\Gamma, \Delta, \Theta, \dots$ will now denote **sequences of assumptions**

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$\Gamma, \Delta, \Theta, \dots$ will now denote **sequences of assumptions**

Intuitionistic Resources

permit contraction & weakening:

$$\frac{\Gamma, [\mathcal{A}], [\mathcal{A}] \vdash B}{\Gamma, [\mathcal{A}] \vdash B} C \qquad \frac{\Gamma \vdash B}{\Gamma, [\mathcal{A}] \vdash B} W$$

Universal Logic

Two kinds of resources

IL and LL can co-exist in peace: an **assumption** \mathcal{A} can be either **linear** $\langle \mathcal{A} \rangle$ or **intuitionistic** $[\mathcal{A}]$; each comes with its own identity:

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$\Gamma, \Delta, \Theta, \dots$ will now denote **sequences of assumptions**

Intuitionistic Resources

permit contraction & weakening:

$$\frac{\Gamma, [A], [A] \vdash B}{\Gamma, [A] \vdash B} C \quad \frac{\Gamma \vdash B}{\Gamma, [A] \vdash B} W$$

and the introduction/elimination of !:

$$\frac{[\Gamma] \vdash A}{[\Gamma] \vdash !A} !I \quad \frac{\Gamma \vdash !A \quad \Delta, [A] \vdash B}{\Gamma, \Delta \vdash B} !E$$

Logical Rules

$$\frac{\Gamma, \langle A \rangle \vdash B}{\Gamma \vdash A \multimap B} \multimap I \quad \frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \multimap E$$

Logical Rules

$$\frac{\Gamma, \langle A \rangle \vdash B}{\Gamma \vdash A \multimap B} \multimap I \quad \frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \multimap E$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \otimes I \quad \frac{\Gamma \vdash A \otimes B \quad \Delta, \langle A \rangle, \langle B \rangle \vdash C}{\Gamma, \Delta \vdash C} \otimes E$$

Logical Rules

$$\frac{\Gamma, \langle A \rangle \vdash B}{\Gamma \vdash A \multimap B} \multimap I \quad \frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \multimap E$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \otimes I \quad \frac{\Gamma \vdash A \otimes B \quad \Delta, \langle A \rangle, \langle B \rangle \vdash C}{\Gamma, \Delta \vdash C} \otimes E$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} \& I \quad \frac{\Gamma \vdash A \& B}{\Gamma \vdash A} \& E_1 \quad \frac{\Gamma \vdash A \& B}{\Gamma \vdash B} \& E_2$$

Logical Rules

$$\frac{\Gamma, \langle A \rangle \vdash B}{\Gamma \vdash A \multimap B} \multimap I \quad \frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \multimap E$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \otimes I \quad \frac{\Gamma \vdash A \otimes B \quad \Delta, \langle A \rangle, \langle B \rangle \vdash C}{\Gamma, \Delta \vdash C} \otimes E$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} \& I \quad \frac{\Gamma \vdash A \& B}{\Gamma \vdash A} \& E_1 \quad \frac{\Gamma \vdash A \& B}{\Gamma \vdash B} \& E_2$$

$$\frac{\Gamma \vdash A \oplus B \quad \Delta, \langle A \rangle \vdash C \quad \Delta, \langle B \rangle \vdash C}{\Gamma, \Delta \vdash C} \oplus E$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \oplus I_1 \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B} \oplus I_2$$

Example

$$\langle!(A \& B)\rangle \vdash !A \otimes !B$$

Example

$$\frac{\frac{\langle!(A \& B)\rangle \vdash !(A \& B)}{\quad} \quad \frac{[A \& B] \vdash !A \otimes !B}{\quad}}{\langle!(A \& B)\rangle \vdash !A \otimes !B} !E$$

Example

$$\frac{\frac{\langle!(A \& B)\rangle \vdash!(A \& B) \quad \langle Id \rangle}{\langle!(A \& B)\rangle \vdash!(A \& B)} \quad \frac{[A \& B] \vdash!A \otimes!B}{[A \& B] \vdash!A \otimes!B}}{\langle!(A \& B)\rangle \vdash!A \otimes!B} !E$$

Example

$$\frac{\frac{\langle!(A \& B)\rangle \vdash!(A \& B)}{\langle!(A \& B)\rangle \vdash!(A \otimes B)} \langle Id \rangle \quad \frac{\frac{[A \& B], [A \& B] \vdash!A \otimes!B}{[A \& B] \vdash!A \otimes!B} C}{\langle!(A \& B)\rangle \vdash!A \otimes!B} !E$$

Example

$$\frac{\frac{\overline{\langle!(A \& B)\rangle \vdash!(A \& B)}}{\langle!(A \& B)\rangle \vdash!(A \otimes!B)} \langle Id \rangle \quad \frac{\frac{\frac{\overline{[A \& B] \vdash!A} \quad \overline{[A \& B] \vdash!B}}{[A \& B], [A \& B] \vdash!A \otimes!B} \otimes I}{[A \& B] \vdash!A \otimes!B} C}{\langle!(A \& B)\rangle \vdash!A \otimes!B} !E$$

Example

$$\frac{
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 \frac{
 \overline{\langle!(A \& B)\rangle \vdash!(A \& B)} \langle Id \rangle
 }{
 \overline{\langle!(A \& B)\rangle \vdash!(A \otimes!B)} !E
 }
 }{
 \overline{\langle!(A \& B)\rangle \vdash!(A \otimes!B)} !E
 }
 }{
 \overline{\langle!(A \& B)\rangle \vdash!(A \otimes!B)} !E
 }
 \frac{
 \frac{
 \frac{
 \overline{[A \& B] \vdash A}
 }{
 \overline{[A \& B] \vdash!A} !I
 }
 }{
 \overline{[A \& B], [A \& B] \vdash!A \otimes!B} C
 }
 \frac{
 \overline{[A \& B] \vdash!B}
 }{
 \overline{[A \& B], [A \& B] \vdash!A \otimes!B} \otimes I
 }
 }{
 \overline{\langle!(A \& B)\rangle \vdash!(A \otimes!B)} !E
 }$$

Example

$$\frac{\frac{\frac{[A \& B] \vdash A \& B}{[A \& B] \vdash A} \&E_1 \quad \frac{[A \& B] \vdash A}{[A \& B] \vdash !A} !I \quad \frac{[A \& B] \vdash !A \quad [A \& B] \vdash !B}{[A \& B], [A \& B] \vdash !A \otimes !B} \otimes I}{\frac{\langle !(A \& B) \rangle \vdash !(A \& B) \quad \langle Id \rangle \quad \frac{[A \& B], [A \& B] \vdash !A \otimes !B}{[A \& B] \vdash !A \otimes !B} C}{\langle !(A \& B) \rangle \vdash !A \otimes !B} !E}$$

Example

$$\frac{\frac{\frac{\frac{[A \& B] \vdash A \& B}{[A \& B] \vdash A} [Id]}{[A \& B] \vdash !A} \&E_1}{[A \& B] \vdash !A} !I \quad \frac{[A \& B] \vdash !B}{[A \& B], [A \& B] \vdash !A \otimes !B} \otimes I}{[A \& B] \vdash !A \otimes !B} C \quad \frac{\frac{\langle ! (A \& B) \rangle \vdash ! (A \& B)}{\langle ! (A \& B) \rangle \vdash !A \otimes !B} \langle Id \rangle}{\langle ! (A \& B) \rangle \vdash !A \otimes !B} !E$$

Example

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{[A \& B] \vdash A \& B}{[A \& B] \vdash A} \quad [Id]}{[A \& B] \vdash !A} \quad !I \quad \frac{\frac{\frac{[A \& B] \vdash B}{[A \& B] \vdash B} \quad [Id]}{[A \& B] \vdash !B} \quad !I}{[A \& B], [A \& B] \vdash !A \otimes !B} \quad \otimes I}{\frac{[A \& B], [A \& B] \vdash !A \otimes !B}{[A \& B] \vdash !A \otimes !B} \quad C} \\
 \frac{\frac{\langle ! (A \& B) \rangle \vdash ! (A \& B) \quad \langle Id \rangle}{\langle ! (A \& B) \rangle \vdash !A \otimes !B} \quad !E
 \end{array}$$

Embedding

IL in LL

Let $*$ an operator sending formulas of IL to formulas of ILL, such that:

- ▶ if $p \in \mathcal{A}$, then $p^* = p$
- ▶ otherwise:

$$(A \rightarrow B)^* = !A^* \multimap B^*$$

$$(A \times B)^* = A^* \& B^*$$

$$(A + B)^* = !A^* \oplus !B^*$$

and judgements of IL to judgements of ILL:

$$(\Gamma \vdash A)^* = [\Gamma^*] \vdash A^*$$

Embedding Example

Proving the intuitionistic judgement $A, A \rightarrow B \vdash A \times B$ in ILL:

$$(A, A \rightarrow B \vdash A \times B)^* =$$

Embedding Example

Proving the intuitionistic judgement $A, A \rightarrow B \vdash A \times B$ in ILL:

$$(A, A \rightarrow B \vdash A \times B)^* = [A^*], [(A \rightarrow B)^*] \vdash (A \times B)^*$$

Embedding Example

Proving the intuitionistic judgement $A, A \rightarrow B \vdash A \times B$ in ILL:

$$\begin{aligned}(A, A \rightarrow B \vdash A \times B)^* &= [A^*], [(A \rightarrow B)^*] \vdash (A \times B)^* \\ &= [A], [!A^* \multimap B^*] \vdash A^* \& B^*\end{aligned}$$

Embedding Example

Proving the intuitionistic judgement $A, A \rightarrow B \vdash A \times B$ in ILL:

$$\begin{aligned}(A, A \rightarrow B \vdash A \times B)^* &= [A^*], [(A \rightarrow B)^*] \vdash (A \times B)^* \\ &= [A], [!A^* \multimap B^*] \vdash A^* \& B^* \\ &= [A], [!A \multimap B] \vdash A \& B\end{aligned}$$

Embedding Example

Proving the intuitionistic judgement $A, A \rightarrow B \vdash A \times B$ in ILL:

$$\begin{aligned}(A, A \rightarrow B \vdash A \times B)^* &= [A^*], [(A \rightarrow B)^*] \vdash (A \times B)^* \\ &= [A], [!A^* \multimap B^*] \vdash A^* \& B^* \\ &= [A], [!A \multimap B] \vdash A \& B\end{aligned}$$

$$[A], [!A \multimap B] \vdash A \& B$$

Embedding Example

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$$\frac{\overline{[A], [!A \rightarrow B] \vdash A} \quad \overline{[A], [!A \multimap B] \vdash B}}{[A], [!A \multimap B] \vdash A \& B} \&I$$

Embedding Example

Proving the intuitionistic judgement $A, A \rightarrow B \vdash A \times B$ in ILL:

$$\begin{aligned}(A, A \rightarrow B \vdash A \times B)^* &= [A^*], [(A \rightarrow B)^*] \vdash (A \times B)^* \\ &= [A], [!A^* \multimap B^*] \vdash A^* \& B^* \\ &= [A], [!A \multimap B] \vdash A \& B\end{aligned}$$

$$\frac{\frac{\overline{[A] \vdash A} \quad [Id]}{[A], [!A \rightarrow B] \vdash A} \quad W \quad \frac{\overline{[A], [!A \multimap B] \vdash B}}{[A], [!A \multimap B] \vdash A \& B} \quad \&I}{[A], [!A \multimap B] \vdash A \& B}$$

Embedding Example

Proving the intuitionistic judgement $A, A \rightarrow B \vdash A \times B$ in ILL:

$$\begin{aligned}(A, A \rightarrow B \vdash A \times B)^* &= [A^*], [(A \rightarrow B)^*] \vdash (A \times B)^* \\ &= [A], [!A^* \multimap B^*] \vdash A^* \& B^* \\ &= [A], [!A \multimap B] \vdash A \& B\end{aligned}$$

$$\frac{\frac{\overline{[A] \vdash A} \quad [Id]}{[A], [!A \rightarrow B] \vdash A} \quad W \quad \frac{\frac{[!A \multimap B], [A] \vdash B}{[A], [!A \multimap B] \vdash B} \quad Ex}{[A], [!A \multimap B] \vdash A \& B} \quad \&I$$

Embedding Example

Proving the intuitionistic judgement $A, A \rightarrow B \vdash A \times B$ in ILL:

$$\begin{aligned}(A, A \rightarrow B \vdash A \times B)^* &= [A^*], [(A \rightarrow B)^*] \vdash (A \times B)^* \\ &= [A], [!A^* \multimap B^*] \vdash A^* \& B^* \\ &= [A], [!A \multimap B] \vdash A \& B\end{aligned}$$

$$\frac{\frac{\overline{[A] \vdash A} \quad [Id]}{[A], [!A \rightarrow B] \vdash A} \quad W \quad \frac{\frac{\overline{[!A \multimap B] \vdash !A \multimap B} \quad \overline{[A] \vdash !A}}{[!A \multimap B], [A] \vdash B} \multimap E}{\frac{[A], [!A \multimap B] \vdash B}{[A], [!A \multimap B] \vdash A \& B} \quad \&I} \quad Ex$$

Embedding Example

Proving the intuitionistic judgement $A, A \rightarrow B \vdash A \times B$ in ILL:

$$\begin{aligned}(A, A \rightarrow B \vdash A \times B)^* &= [A^*], [(A \rightarrow B)^*] \vdash (A \times B)^* \\ &= [A], [!A^* \multimap B^*] \vdash A^* \& B^* \\ &= [A], [!A \multimap B] \vdash A \& B\end{aligned}$$

$$\frac{\frac{\overline{[A] \vdash A} \quad [Id]}{[A], [!A \rightarrow B] \vdash A} \quad W \quad \frac{\frac{\overline{[!A \multimap B] \vdash !A \multimap B} \quad [Id] \quad \overline{[A] \vdash !A}}{[!A \multimap B], [A] \vdash B} \multimap E}{\frac{[A], [!A \multimap B] \vdash B}{[A], [!A \multimap B] \vdash A \& B} \quad \&I} \quad Ex$$

Embedding Example

Proving the intuitionistic judgement $A, A \rightarrow B \vdash A \times B$ in ILL:

$$\begin{aligned}
 (A, A \rightarrow B \vdash A \times B)^* &= [A^*], [(A \rightarrow B)^*] \vdash (A \times B)^* \\
 &= [A], [!A^* \multimap B^*] \vdash A^* \& B^* \\
 &= [A], [!A \multimap B] \vdash A \& B
 \end{aligned}$$

$$\frac{
 \frac{
 \frac{[A] \vdash A}{[A], [!A \rightarrow B] \vdash A} [Id]
 }{[A], [!A \multimap B] \vdash A \& B} W
 \quad
 \frac{
 \frac{
 \frac{[!A \multimap B] \vdash !A \multimap B}{[!A \multimap B], [A] \vdash B} [Id]
 }{[A], [!A \multimap B] \vdash B} Ex
 }{[A], [!A \multimap B] \vdash A \& B} \&I
 }{[A], [!A \multimap B] \vdash A \& B} \multimap E$$

ILL $\overset{CH}{\equiv}$ Simply typed linear λ -calculus

Linear λ -calculus

- ▶ No vacuous abstractions: abstracted variables must be used in the function body
- ▶ All variables occur exactly once

The simply typed linear λ -calculus

$\mathcal{T} :=$

The simply typed linear λ -calculus

$$\mathcal{T} := \mathcal{V}$$

$$\frac{}{\langle x : A \rangle \vdash x : A} \langle Id \rangle \quad \frac{}{[x : A] \vdash x : A} [Id]$$

The simply typed linear λ -calculus

$$\mathcal{T} := \mathcal{V} \mid !\mathcal{T} \mid \text{case } \mathcal{T} \text{ of } !\mathcal{T} \rightarrow \mathcal{T}$$

$$\frac{[\Gamma] \vdash t : A}{[\Gamma] \vdash !t : !A} !I \quad \frac{\Gamma \vdash s : !A \quad \Delta, [x : A] \vdash u : B}{\Gamma, \Delta \vdash \text{case } s \text{ of } !x \rightarrow u : B} !E$$

The simply typed linear λ -calculus

$$\mathcal{T} := \mathcal{V} \mid !\mathcal{T} \mid \text{case } \mathcal{T} \text{ of } !\mathcal{T} \rightarrow \mathcal{T} \mid \lambda\langle\mathcal{T}\rangle.\mathcal{T} \mid \mathcal{T}\langle\mathcal{T}\rangle$$

$$\frac{\Gamma, \langle x : A \rangle \vdash y : B}{\Gamma \vdash \lambda x. y : A \multimap B} \multimap I \quad \frac{\Gamma \vdash f : A \multimap B \quad \Delta \vdash x : A}{\Gamma, \Delta \vdash f \langle x \rangle^1 : B} \multimap E$$

¹Or: $f \ x$

The simply typed linear λ -calculus

$$\begin{aligned} \mathcal{T} := & \mathcal{V} \mid !\mathcal{T} \mid \text{case } \mathcal{T} \text{ of } !\mathcal{T} \rightarrow \mathcal{T} \mid \lambda\langle\mathcal{T}\rangle.\mathcal{T} \mid \mathcal{T}\langle\mathcal{T}\rangle \\ & \mid \langle\mathcal{T}, \mathcal{T}\rangle \mid \text{case } \mathcal{T} \text{ of } \langle\mathcal{T}, \mathcal{T}\rangle \rightarrow \mathcal{T} \end{aligned}$$

$$\frac{\Gamma \vdash \mathbf{t} : A \quad \Delta \vdash \mathbf{u} : B}{\Gamma \vdash \langle \mathbf{t}, \mathbf{u} \rangle : A \otimes B} \otimes I \quad \frac{\Gamma \vdash \mathbf{s} : A \otimes B \quad \Delta, \langle \mathbf{x} : A \rangle, \langle \mathbf{y} : B \rangle \vdash \mathbf{v} : C}{\Gamma, \Delta \vdash \text{case } \mathbf{s} \text{ of } \langle \mathbf{x}, \mathbf{y} \rangle \rightarrow \mathbf{v} : C} \otimes E$$

The simply typed linear λ -calculus

$$\begin{aligned}\mathcal{T} := & \mathcal{V} \mid !\mathcal{T} \mid \text{case } \mathcal{T} \text{ of } !\mathcal{T} \rightarrow \mathcal{T} \mid \lambda\langle\mathcal{T}\rangle.\mathcal{T} \mid \mathcal{T}\langle\mathcal{T}\rangle \\ & \mid \langle\mathcal{T}, \mathcal{T}\rangle \mid \text{case } \mathcal{T} \text{ of } \langle\mathcal{T}, \mathcal{T}\rangle \rightarrow \mathcal{T} \\ & \mid \langle\langle\mathcal{T}, \mathcal{T}\rangle\rangle \mid \text{fst}\langle\mathcal{T}\rangle \mid \text{snd}\langle\mathcal{T}\rangle\end{aligned}$$

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \langle\langle t, u \rangle\rangle : A \& B} \&I \quad \frac{\Gamma \vdash s : A \& B}{\Gamma \vdash \text{fst}\langle s \rangle : A} \text{fst} \quad \frac{\Gamma \vdash s : A \& B}{\Gamma \vdash \text{snd}\langle s \rangle : B} \text{snd}$$

The simply typed linear λ -calculus

$$\begin{aligned}\mathcal{T} := & \mathcal{V} \mid !\mathcal{T} \mid \text{case } \mathcal{T} \text{ of } !\mathcal{T} \rightarrow \mathcal{T} \mid \lambda\langle\mathcal{T}\rangle.\mathcal{T} \mid \mathcal{T}\langle\mathcal{T}\rangle \\ & \mid \langle\mathcal{T}, \mathcal{T}\rangle \mid \text{case } \mathcal{T} \text{ of } \langle\mathcal{T}, \mathcal{T}\rangle \rightarrow \mathcal{T} \\ & \mid \langle\langle\mathcal{T}, \mathcal{T}\rangle\rangle \mid \text{fst}\langle\mathcal{T}\rangle \mid \text{snd}\langle\mathcal{T}\rangle \\ & \mid \text{inl}\langle\mathcal{T}\rangle \mid \text{inr}\langle\mathcal{T}\rangle \\ & \mid \text{case } \mathcal{T} \text{ of } \text{inl}\langle\mathcal{T}\rangle \rightarrow \mathcal{T}; \text{inr}\langle\mathcal{T}\rangle \rightarrow \mathcal{T}\end{aligned}$$

$$\frac{\Gamma \vdash x : A}{\Gamma \vdash \text{inl}\langle x \rangle : A \oplus B} \oplus l_1 \quad \frac{\Gamma \vdash x : B}{\Gamma \vdash \text{inr}\langle x \rangle : A \oplus B} \oplus l_2$$

$$\frac{\Gamma \vdash s : A \oplus B \quad \Delta, \langle x : A \rangle \vdash u : C \quad \Delta, \langle y : B \rangle \vdash w : C}{\Gamma, \Delta \vdash \text{case } s \text{ of } \text{inl}\langle x \rangle \rightarrow u; \text{inr}\langle y \rangle \rightarrow w : C} \oplus E$$

Proof Normalization & Term Reduction (!)

$$\frac{
 \frac{
 \frac{\vdots}{\Gamma \vdash t : A}
 }{[\Gamma] \vdash !t : !A}
 \quad
 \frac{
 \frac{
 \frac{[x : A] \vdash x : A \quad \dots}{\vdots}
 }{[x : A] \vdash x : A}
 }{\Gamma, [x : A] \vdash u : B}
 }{[\Gamma], \Delta \vdash B}
 \quad
 \text{!} \quad \text{!}E
 }{[\Gamma], \Delta \vdash B}
 \Rightarrow
 \frac{
 \frac{
 \frac{
 \frac{\vdots}{[\Gamma] \vdash t : A}
 }{\vdots}
 }{[\Gamma], \dots, \Delta \vdash u : B}
 }{[\Gamma], \Delta \vdash u : B}
 }{[\Gamma], \Delta \vdash u : B}$$

case !t of !x → u ⇒ u[t/x]

Proof Normalization & Term Reduction (&)

$$\frac{\frac{\frac{\vdots}{\Gamma \vdash t : A} \quad \frac{\vdots}{\Gamma \vdash u : B}}{\Gamma \vdash \langle \langle t, u \rangle \rangle : A \& B} \&I}{\Gamma \vdash \text{fst} \langle \langle \langle t, u \rangle \rangle \rangle : A} \&E_1 \Rightarrow \frac{\vdots}{\Gamma \vdash t : A}$$

$$\text{fst} \langle \langle \langle t, u \rangle \rangle \rangle \Rightarrow t$$