Linear Logic

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Logic & Language 2020

Truth vs. Resource

Linear Logic

Proposed by Girard (1987) as a resource-conscious alternative to classical & intuitionistic logic.

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" *Classical and intuitionistic logics deal with stable truths:*

if A and $A \to B$, then B, but A still holds.

This is perfect in mathematics, but wrong in real life, since real implication is causal. A causal implication cannot be iterated since the conditions are modified after its use; this process of modification of the premises is known in physics as reaction. For instance, if A is to spend \$1 on a pack of a ~~cigarettes~~ candies and B is to get them, you lose \$1 in this process, and you cannot do it a second time. The reaction here was that \$1 went out of your pocket."

Truth vs. Resource

Linear Logic

Proposed by Girard (1987) as a resource-conscious alternative to classical & intuitionistic logic.

- Propositions now represent resources
- Resources are not free to discard and replicate
- $\Rightarrow$ Contraction & Weakening are not universally applicable

Substructural!

- Inference rules can share contexts

Linear Logic: Syntax & Connectives

Linear propositions $\mathcal{P}$ are defined as:

$$\mathcal{P} := \mathcal{C} \mid \mathcal{P}_1 \multimap \mathcal{P}_2 \mid \mathcal{P}_1 \otimes \mathcal{P}_2 \mid \mathcal{P}_1 \& \mathcal{P}_2 \mid \mathcal{P}_1 \oplus \mathcal{P}_2 \mid !\mathcal{P}$$

$\multimap$ is read as "lolli"

$A \multimap B$: consume A to produce a B

$\otimes$ is read as "tensor"

$A \otimes B$: both A and B

$\&$ is read as "with"

$A \& B$: pick from A and B

$\oplus$ is read as "or"

$A \oplus B$: either A and B

$!$ is read as "bang"

$!A$: of course A

Universal Logic

Two kinds of resources

IL and LL can co-exist in peace: an assumption $\mathcal{A}$ can be either linear $\langle\mathcal{A}\rangle$ or intuitionistic $[\mathcal{A}]$; each comes with its own identity:

$$\frac{}{\langle A\rangle \vdash A}\ \langle Id\rangle \qquad \frac{}{[A] \vdash A}\ [Id]$$

$\Gamma, \Delta, \Theta, \ldots$ will now denote sequences of assumptions

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Intuitionistic Resources

permit contraction & weakening:

$$\frac{\Gamma, [A], [A] \vdash B}{\Gamma, [A] \vdash B}\ C \qquad \frac{\Gamma \vdash B}{\Gamma, [A] \vdash B}\ W$$

Two kinds of resources

IL and LL can co-exist in peace: an assumption $\mathcal{A}$ can be either linear $\langle\mathcal{A}\rangle$ or intuitionistic $[\mathcal{A}]$; each comes with its own identity:

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$\Gamma, \Delta, \Theta, \ldots$ will now denote sequences of assumptions

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permit contraction & weakening:

$$\frac{\Gamma, [A], [A] \vdash B}{\Gamma, [A] \vdash B}\ C \qquad \frac{\Gamma \vdash B}{\Gamma, [A] \vdash B}\ W$$

and the introduction/elimination of !:

$$\frac{[\Gamma] \vdash A}{[\Gamma] \vdash !A}\ !I \qquad \frac{\Gamma \vdash !A \quad \Delta, [A] \vdash B}{\Gamma, \Delta \vdash B}\ !E$$

Logical Rules

$$\frac{\Gamma, \langle A \rangle \vdash B}{\Gamma \vdash A \multimap B} \multimap I \qquad \frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \multimap E$$

Logical Rules

$$\frac{\Gamma, \langle A \rangle \vdash B}{\Gamma \vdash A \multimap B} \multimap I \qquad \frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \multimap E$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \otimes I \qquad \frac{\Gamma \vdash A \otimes B \quad \Delta, \langle A \rangle, \langle B \rangle \vdash C}{\Gamma, \Delta \vdash C} \otimes E$$

Logical Rules

$$\frac{\Gamma, \langle A \rangle \vdash B}{\Gamma \vdash A \multimap B} \multimap I \qquad \frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \multimap E$$

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$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} \& I \qquad \frac{\Gamma \vdash A \& B}{\Gamma \vdash A} \& E_1 \qquad \frac{\Gamma \vdash A \& B}{\Gamma \vdash B} \& E_2$$

Logical Rules

$$\frac{\Gamma, \langle A \rangle \vdash B}{\Gamma \vdash A \multimap B} \multimap I \qquad \frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \multimap E$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \otimes I \qquad \frac{\Gamma \vdash A \otimes B \quad \Delta, \langle A \rangle, \langle B \rangle \vdash C}{\Gamma, \Delta \vdash C} \otimes E$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} \& I \qquad \frac{\Gamma \vdash A \& B}{\Gamma \vdash A} \& E_1 \qquad \frac{\Gamma \vdash A \& B}{\Gamma \vdash B} \& E_2$$

$$\frac{\Gamma \vdash A \oplus B \quad \Delta, \langle A \rangle \vdash C \quad \Delta, \langle B \rangle \vdash C}{\Gamma, \Delta \vdash C} \oplus E$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \oplus I_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B} \oplus I_2$$

Example

$$\frac{}{\langle !(A \& B) \rangle \vdash !A \otimes !B}$$

Example

$$\dfrac{\dfrac{}{\langle !(A\&B)\rangle \vdash\, !(A\&B)} \qquad\qquad \dfrac{}{[A\&B] \vdash\, !A \otimes !B}}{\langle !(A\&B)\rangle \vdash\, !A \otimes !B}\ !E$$

Example

$$\dfrac{\dfrac{}{\langle !(A\&B)\rangle \vdash !(A\&B)}\ \langle Id\rangle \qquad \dfrac{}{[A\&B] \vdash !A \otimes !B}}{\langle !(A\&B)\rangle \vdash !A \otimes !B}\ !E$$

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$$\dfrac{\dfrac{}{\langle !(A\&B)\rangle \vdash !(A\&B)}\ \langle Id\rangle \qquad \dfrac{\dfrac{}{[A\&B],[A\&B] \vdash !A\otimes !B}}{[A\&B] \vdash !A\otimes !B}\ C}{\langle !(A\&B)\rangle \vdash !A\otimes !B}\ !E$$

Example

$$
\dfrac{\dfrac{}{\langle !(A\&B)\rangle \vdash !(A\&B)}\ \langle Id\rangle \qquad \dfrac{\dfrac{\dfrac{}{[A\&B] \vdash !A} \qquad \dfrac{}{[A\&B] \vdash !B}}{[A\&B], [A\&B] \vdash !A \otimes !B}\ \otimes I}{[A\&B] \vdash !A \otimes !B}\ C}{\langle !(A\&B)\rangle \vdash !A \otimes !B}\ !E
$$

Example

$$
\dfrac{
  \dfrac{}{\langle !(A\&B)\rangle \vdash\, !(A\&B)}\ \langle Id\rangle
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{}{[A\&B] \vdash A}}{[A\&B] \vdash\, !A}\ !I
      \qquad
      \dfrac{}{[A\&B] \vdash\, !B}
    }{[A\&B], [A\&B] \vdash\, !A \otimes !B}\ \otimes I
  }{[A\&B] \vdash\, !A \otimes !B}\ C
}{\langle !(A\&B)\rangle \vdash\, !A \otimes !B}\ !E
$$

Example

$$
\dfrac{
\dfrac{}{\langle !(A\&B)\rangle \vdash !(A\&B)}\,\langle Id\rangle
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{[A\&B] \vdash A\&B}
}{[A\&B] \vdash A}\,\&E_1
}{[A\&B] \vdash !A}\,!I
\qquad
\dfrac{}{[A\&B] \vdash !B}
}{[A\&B], [A\&B] \vdash !A \otimes !B}\,\otimes I
}{[A\&B] \vdash !A \otimes !B}\,C
}{\langle !(A\&B)\rangle \vdash !A \otimes !B}\,!E
$$

Example

$$
\dfrac{\dfrac{}{\langle !(A\&B)\rangle \vdash !(A\&B)}\;\langle Id\rangle \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[A\&B] \vdash A\&B}\;[Id]}{[A\&B] \vdash A}\;\&E_1}{[A\&B] \vdash !A}\;!I \qquad \dfrac{}{[A\&B] \vdash !B}}{[A\&B], [A\&B] \vdash !A \otimes !B}\;\otimes I}{[A\&B] \vdash !A \otimes !B}\;C}{\langle !(A\&B)\rangle \vdash !A \otimes !B}\;!E
$$

Example

$$
\dfrac{\dfrac{}{\langle !(A\&B)\rangle \vdash !(A\&B)}\ \langle Id\rangle \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[A\&B] \vdash A\&B}\ [Id]}{[A\&B] \vdash A}\ \&E_1}{[A\&B] \vdash !A}\ !I \qquad \dfrac{\dfrac{\dfrac{}{[A\&B] \vdash B}\ [Id]}{[A\&B] \vdash B}\ \&E_2}{[A\&B] \vdash !B}\ !I}{[A\&B],[A\&B] \vdash !A \otimes !B}\ \otimes I}{[A\&B] \vdash !A \otimes !B}\ C}{\langle !(A\&B)\rangle \vdash !A \otimes !B}\ !E
$$

Embedding

IL to ILL

Let $*$ an operator sending formulas of IL to formulas of ILL, such that:

- if $p \in \mathcal{A}$, then $p^* = p$
- otherwise:
$$\begin{aligned} (A \rightarrow B)^* &= !A^* \multimap B^* \\ (A \times B)^* &= A^* \& B^* \\ (A + B)^* &= !A^* \oplus !B^* \end{aligned}$$

and judgements of IL to judgements of ILL:

$$(\Gamma \vdash A)^* = [\Gamma^*] \vdash A^*$$

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Proving the intuitionistic judgement $A, A \to B \vdash A \times B$ in ILL:

$$(A, A \to B \vdash A \times B)^* =$$

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$$(A, A \to B \vdash A \times B)^* = [A^*], [(A \to B)^*] \vdash (A \times B)^*$$

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$$\begin{aligned}(A, A \to B \vdash A \times B)^* &= [A^*], [(A \to B)^*] \vdash (A \times B)^* \\ &= [A], [!A^* \multimap B^*] \vdash A^* \& B^* \\ &= [A], [!A \multimap B] \vdash A \& B\end{aligned}$$

Embedding Example

$$\frac{}{[A], [!A \multimap B] \vdash A \& B}$$

Embedding Example

$$\dfrac{\dfrac{}{[A], [!A \to B] \vdash A} \qquad\qquad \dfrac{}{[A], [!A \multimap B] \vdash B}}{[A], [!A \multimap B] \vdash A \& B}\ \& I$$

Embedding Example

$$\dfrac{\dfrac{\overline{[A] \vdash A}}{[A], [!A \to B] \vdash A}\ W \qquad \overline{[A], [!A \multimap B] \vdash B}}{[A], [!A \multimap B] \vdash A \& B}\ \& I$$

Embedding Example

$$\dfrac{\dfrac{\dfrac{}{[A] \vdash A}\,[Id]}{[A], [!A \to B] \vdash A}\,W \qquad\qquad \dfrac{}{[A], [!A \multimap B] \vdash B}}{[A], [!A \multimap B] \vdash A \& B}\,\& I$$

Embedding Example

$$\dfrac{\dfrac{\dfrac{}{[A] \vdash A}\,[Id]}{[A], [!A \to B] \vdash A}\,W \qquad \dfrac{\dfrac{}{[!A \multimap B], [A] \vdash B}}{[A], [!A \multimap B] \vdash B}\,Ex}{[A], [!A \multimap B] \vdash A \& B}\,\&I$$

Embedding Example

$$
\dfrac{
  \dfrac{\dfrac{}{[A] \vdash A}\ [Id]}{[A], [!A \to B] \vdash A}\ W
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{[!A \multimap B] \vdash !A \multimap B}
      \qquad
      \dfrac{}{[A] \vdash !A}
    }{[!A \multimap B], [A] \vdash B}\ {\multimap} E
  }{[A], [!A \multimap B] \vdash B}\ Ex
}{[A], [!A \multimap B] \vdash A \& B}\ \& I
$$

Embedding Example

$$
\dfrac{
  \dfrac{\dfrac{}{[A] \vdash A}\,[Id]}{[A], [!A \to B] \vdash A}\,W
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{[!A \multimap B] \vdash !A \multimap B}\,[Id]
      \qquad
      \dfrac{}{[A] \vdash !A}
    }{[!A \multimap B], [A] \vdash B}\,\multimap E
  }{[A], [!A \multimap B] \vdash B}\,Ex
}{[A], [!A \multimap B] \vdash A \& B}\,\& I
$$

Embedding Example

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$[A] \vdash A$}
\RightLabel{W}
\UnaryInfC{$[A], [!A \to B] \vdash A$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$[!A \multimap B] \vdash !A \multimap B$}
\AxiomC{}
\UnaryInfC{$[A] \vdash A$}
\RightLabel{$!I$}
\UnaryInfC{$[A] \vdash !A$}
\RightLabel{$\multimap E$}
\BinaryInfC{$[!A \multimap B], [A] \vdash B$}
\RightLabel{Ex}
\UnaryInfC{$[A], [!A \multimap B] \vdash B$}
\RightLabel{$\& I$}
\BinaryInfC{$[A], [!A \multimap B] \vdash A \& B$}
\end{prooftree}
$$

Embedding Example

$$\cfrac{\cfrac{\cfrac{}{[A] \vdash A}\;[Id]}{[A], [!A \to B] \vdash A}\;W \qquad \cfrac{\cfrac{\cfrac{}{[!A \multimap B] \vdash !A \multimap B}\;[Id] \quad \cfrac{\cfrac{}{[A] \vdash A}\;[Id]}{[A] \vdash !A}\;!I}{[!A \multimap B], [A] \vdash B}\;\multimap E}{[A], [!A \multimap B] \vdash B}\;Ex}{[A], [!A \multimap B] \vdash A \& B}\;\& I$$

ILL $\overset{CH}{\equiv}$ Simply typed linear λ-calculus

Linear λ-calculus

- No vacuous abstractions: abstracted variables must be used in the function body
- All variables occur exactly once

The simply typed linear λ-calculus

$$\mathcal{V} = \{x, y, z, \dots\}$$
$$\mathcal{T} = \{s, t, u, v, w, \dots\}$$

$$\mathcal{T} :=$$

The simply typed linear λ-calculus

$$\mathcal{V} = \{x, y, z, \dots\}$$
$$\mathcal{T} = \{s, t, u, v, w, \dots\}$$

$$\mathcal{T} := \mathcal{V}$$

$$\frac{}{\langle \mathtt{x} : A \rangle \vdash \mathtt{x} : A}\ \langle Id \rangle \qquad \frac{}{[\mathtt{x} : A] \vdash \mathtt{x} : A}\ [Id]$$

The simply typed linear λ-calculus

$$\mathcal{V} = \{x, y, z, \dots\}$$
$$\mathcal{T} = \{s, t, u, v, w, \dots\}$$

$$\mathcal{T} := \mathcal{V} \mid !\mathcal{T} \mid \texttt{case}\ \ \mathcal{T}_1\ \ \texttt{of}\ \ !\mathcal{V} \to \mathcal{T}_2$$

$$\frac{[\Gamma] \vdash \texttt{t} : A}{[\Gamma] \vdash !\texttt{t} : !A}\ !I \qquad \frac{\Gamma \vdash \texttt{s} : !A \quad \Delta, [\texttt{x} : A] \vdash \texttt{t} : B}{\Gamma, \Delta \vdash \texttt{case}\ \ \texttt{s}\ \ \texttt{of}\ \ !x \to t : B}\ !E$$

The simply typed linear λ-calculus

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$$\mathcal{T} = \{s, t, u, v, w, \dots\}$$

$$\mathcal{T} := \mathcal{V} \mid !\mathcal{T} \mid \texttt{case}\ \ \mathcal{T}_1\ \ \texttt{of}\ \ !\mathcal{V} \to \mathcal{T}_2 \mid \lambda\langle\mathcal{V}\rangle.\mathcal{T} \mid \mathcal{T}_1\langle\mathcal{T}_2\rangle$$

$$\frac{\Gamma, \langle \texttt{x} : A\rangle \vdash \texttt{s} : B}{\Gamma \vdash \lambda\texttt{x}.\texttt{s} : A \multimap B} \multimap I \qquad \frac{\Gamma \vdash \texttt{s} : A \multimap B \quad \Delta \vdash \texttt{t} : A}{\Gamma, \Delta \vdash \texttt{s}\langle\texttt{t}\rangle^{1} : B} \multimap E$$

[1] Or: `s t`

The simply typed linear λ-calculus

$$\mathcal{V} = \{x, y, z, \dots\}$$
$$\mathcal{T} = \{s, t, u, v, w, \dots\}$$

$$\mathcal{T} := \mathcal{V} \mid !\mathcal{T} \mid \texttt{case}\ \ \mathcal{T}_1\ \ \texttt{of}\ \ !\mathcal{V} \to \mathcal{T}_2 \mid \lambda\langle\mathcal{V}\rangle.\mathcal{T} \mid \mathcal{T}_1\langle\mathcal{T}_2\rangle$$
$$\mid \langle\mathcal{T}_1, \mathcal{T}_2\rangle \mid \texttt{case}\ \ \mathcal{T}_1\ \ \texttt{of}\ \ \langle\mathcal{V}_1, \mathcal{V}_2\rangle \to \mathcal{T}_2$$

$$\frac{\Gamma \vdash \mathtt{s} : A \quad \Delta \vdash \mathtt{t} : B}{\Gamma \vdash \langle \mathtt{s}, \mathtt{t} \rangle : A \otimes B} \otimes I \qquad \frac{\Gamma \vdash \mathtt{s} : A \otimes B \quad \Delta, \langle \mathtt{x} : A \rangle, \langle \mathtt{y} : B \rangle \vdash \mathtt{t} : C}{\Gamma, \Delta \vdash \texttt{case}\ \ \mathtt{s}\ \ \texttt{of}\ \ \langle \mathtt{x}, \mathtt{y} \rangle \to \mathtt{t} : C} \otimes E$$

The simply typed linear λ-calculus

$$\mathcal{V} = \{x, y, z, \dots\}$$
$$\mathcal{T} = \{s, t, u, v, w, \dots\}$$

$$\begin{aligned}\mathcal{T} := \mathcal{V} \mid\ !\mathcal{T} \mid \texttt{case}\ \ \mathcal{T}_1\ \ \texttt{of}\ \ !\mathcal{V} \to \mathcal{T}_2 \mid \lambda\langle\mathcal{V}\rangle.\mathcal{T} \mid \mathcal{T}_1\langle\mathcal{T}_2\rangle \\ \mid \langle\mathcal{T}_1, \mathcal{T}_2\rangle \mid \texttt{case}\ \ \mathcal{T}_1\ \ \texttt{of}\ \ \langle\mathcal{V}_1, \mathcal{V}_2\rangle \to \mathcal{T}_2 \\ \mid \langle\langle\mathcal{T}_1, \mathcal{T}_2\rangle\rangle \mid \texttt{fst}\langle\mathcal{T}\rangle \mid \texttt{snd}\langle\mathcal{T}\rangle\end{aligned}$$

$$\frac{\Gamma \vdash \mathtt{s} : A \quad \Gamma \vdash \mathtt{t} : B}{\Gamma \vdash \langle\langle \mathtt{s}, \mathtt{t}\rangle\rangle : A\&B}\ \&I \qquad \frac{\Gamma \vdash s : A\&B}{\Gamma \vdash \texttt{fst}\langle \mathtt{s}\rangle : A}\ \&E_1 \qquad \frac{\Gamma \vdash s : A\&B}{\Gamma \vdash \texttt{snd}\langle \mathtt{s}\rangle : B}\ \&E_2$$

The simply typed linear λ-calculus

$$\mathcal{V} = \{x, y, z, \dots\}$$
$$\mathcal{T} = \{s, t, u, v, w, \dots\}$$

$$\begin{aligned}
\mathcal{T} := \mathcal{V} \mid\, !\mathcal{T} \mid \texttt{case}\ \mathcal{T}_1\ \texttt{of}\ !\mathcal{V} \to \mathcal{T}_2 \mid \lambda\langle\mathcal{V}\rangle.\mathcal{T} \mid \mathcal{T}_1\langle\mathcal{T}_2\rangle \\
\mid \langle\mathcal{T}_1, \mathcal{T}_2\rangle \mid \texttt{case}\ \mathcal{T}_1\ \texttt{of}\ \langle\mathcal{V}_1, \mathcal{V}_2\rangle \to \mathcal{T}_2 \\
\mid \langle\langle\mathcal{T}_1, \mathcal{T}_2\rangle\rangle \mid \texttt{fst}\langle\mathcal{T}\rangle \mid \texttt{snd}\langle\mathcal{T}\rangle \\
\mid \texttt{inl}\langle\mathcal{T}\rangle \mid \texttt{inr}\langle\mathcal{T}\rangle \\
\mid \texttt{case}\ \mathcal{T}_1\ \texttt{of}\ \texttt{inl}\langle\mathcal{V}_1\rangle \to \mathcal{T}_2; \texttt{inr}\langle\mathcal{V}_2\rangle \to \mathcal{T}_3
\end{aligned}$$

$$\frac{\Gamma \vdash \texttt{s} : A}{\Gamma \vdash \texttt{inl}\langle\texttt{s}\rangle : A \oplus B}\ \oplus I_1 \qquad \frac{\Gamma \vdash \texttt{s} : B}{\Gamma \vdash \texttt{inr}\langle\texttt{s}\rangle : A \oplus B}\ \oplus I_2$$

$$\frac{\Gamma \vdash \texttt{s} : A \oplus B \quad \Delta, \langle\texttt{x} : A\rangle \vdash \texttt{t} : C \quad \Delta, \langle\texttt{y} : B\rangle \vdash \texttt{u} : C}{\Gamma, \Delta \vdash \texttt{case}\ \texttt{s}\ \texttt{of}\ \texttt{inl}\langle\texttt{x}\rangle \to \texttt{t}; \texttt{inr}\langle\texttt{y}\rangle \to \texttt{u} : C}\ \oplus E$$

Proof Normalization & Term Reduction (!)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ [\Gamma] \vdash \mathtt{t} : A\end{array}}{[\Gamma] \vdash \mathtt{!t} : !A}\,!I \qquad \dfrac{\begin{array}{c}[\mathtt{x'} : A] \vdash \mathtt{x'} : A \quad \ldots\\ \vdots \\ \Delta, [\mathtt{x'} : A], \cdots \vdash \mathtt{u} : B\end{array}}{\Delta, [\mathtt{x} : A] \vdash \mathtt{u} : B}}{[\Gamma], \Delta \vdash B}\,!E \quad \Longrightarrow \quad \dfrac{\begin{array}{c}\vdots\\ [\Gamma] \vdash \mathtt{t} : A\\ \vdots\\ [\Gamma], \ldots, \Delta \vdash \mathtt{u} : B\end{array}}{[\Gamma], \Delta \vdash \mathtt{u} : B}$$

$$\mathtt{case\ \ !t\ \ of\ \ !x} \to \mathtt{u} \Longrightarrow \mathtt{u[t/x]}$$

Proof Normalization & Term Reduction (&)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \mathtt{t} : A\end{array} \quad \begin{array}{c}\vdots\\ \Gamma \vdash \mathtt{u} : B\end{array}}{\Gamma \vdash \langle\langle \mathtt{t}\,,\mathtt{u}\rangle\rangle : A\& B}\;{\color{red}\& I}}{\Gamma \vdash \mathtt{fst}\langle\ \langle\langle \mathtt{t}\,,\mathtt{u}\rangle\rangle\ \rangle : A}\;{\color{red}\& E_1} \quad \Rightarrow \quad \begin{array}{c}\vdots\\ \Gamma \vdash \mathtt{t} : A\end{array}$$

$$\mathtt{fst}\langle\ \langle\langle \mathtt{t}\,,\mathtt{u}\rangle\rangle\ \rangle \Longrightarrow \mathtt{t}$$