Intuitionistic Logic

K. Kogkalidis

Logic & Language 2020

A proof

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- if c is rational, then we have $a = b = \sqrt{2}$ and we are done
- if c is irrational, let $a = c$ and $b = \sqrt{2}$,

  then $a^b = {\sqrt{2}^{\sqrt{2}}}^{\sqrt{2}} = \sqrt{2}^2 = 2$ and we are done

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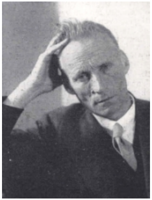
. . . what are a and b?

Intuitionism

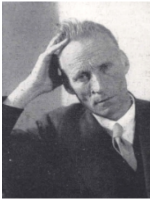


L. E. J. Brouwer

Main Tenets

- Mathematical truth is subjective rather than fundamental
- Proof is a process of construction rather than discovery
- A mathematical object exists if it can be constructed

Intuitionism



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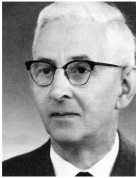
Intuitionistic Logic

A different formal model to capture the notion of Intuitionistic Truth, which is stricter than Classical Truth



Arend Heyting

Intuitionistic Logic vs Classical Logic

CL:

- Propositions are either true or false

  Law of exluded middle: $\top \to (A \vee \neg A)$

- Negation is Falsity

  Double Negation Elimination: $\neg(\neg A) \to A$

Intuitionistic Logic vs Classical Logic

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Consequence

IL is a weaker logic, disallowing proof by contradiction

Basic Definitions: Proposition

Propositions

Let $\mathcal{C}$ a set of *propositional constants*. The *propositions* (formulas) $\mathcal{P}$ of IL are:

$$\mathcal{P} := \mathcal{C} \mid \mathcal{P}_1 \rightarrow \mathcal{P}_2 \mid \mathcal{P}_1 \times \mathcal{P}_2 \mid \mathcal{P}_1 + \mathcal{P}_2$$

where:

- $\rightarrow$ is read as "implies"
- $\times$ is read as "and"
- $+$ is read as "or"

We will denote propositional constants by $A, B, C, \ldots$

Basic Definitions: Assumption & Judgement

Assumptions

An *assumption* (context) $\mathcal{A}$ is a sequence of zero or more propositions:

$$\mathcal{A} := () \mid \mathcal{A}, \mathcal{P}$$

We will denote assumptions by $\Gamma, \Delta, \Theta, \ldots$

Basic Definitions: Assumption & Judgement

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Judgement

A *judgement* $\mathcal{J}$ is a statement $\mathcal{A} \vdash \mathcal{P}$

Read as "*from assumptions $\mathcal{A}$, one can conclude proposition $\mathcal{P}$*"

Basic Definitions: Rule

Rule

A *rule* is a statement consisting of zero or more premises $\mathcal{J}_1, \ldots \mathcal{J}_n$ and a conclusion $\mathcal{J}_c$:

$$\frac{\mathcal{J}_1 \quad \ldots \quad \mathcal{J}_n}{\mathcal{J}_c}$$

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$$\frac{\mathcal{J}_1 \quad \ldots \quad \mathcal{J}_n}{\mathcal{J}_c}$$

If every $\mathcal{J}_i$ is derivable (has a proof), we can derive $\mathcal{J}_c$

Axiom Rule

From any formula A, one can conclude itself:

$$\frac{}{A \vdash A}\ Ax$$

Logical Rules

Each logical connective $\rightarrow, \times, +$ has rules for its *introduction* and *elimination*:

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$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to I \qquad \frac{\Gamma \vdash A \to B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B} \to E$$

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$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \times B} \times I \qquad \frac{\Gamma \vdash A \times B \quad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C} \times E$$

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$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \times B} \times I \qquad \frac{\Gamma \vdash A \times B \quad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C} \times E$$

$$\frac{\Gamma \vdash A + B \quad \Delta, A \vdash C \quad \Delta, B \vdash C}{\Gamma, \Delta \vdash C} + E$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A + B} + I_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A + B} + I_2$$

Structural Rules

The , of IL assumptions is *commutative*:

$$\frac{\Gamma, A, B, \Delta \vdash A}{\Gamma, B, A, \Delta \vdash A}\ \text{Exchange}$$

Structural Rules

The , of IL assumptions is *commutative*:

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IL propositions are free to *discard* and *replicate*:

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ \text{Contraction} \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ \text{Weakening}$$

Proof Example: Identity Function

$$\frac{}{\vdash A \to A}$$

Proof Example: Identity Function

$$\dfrac{\dfrac{}{A \vdash A}\ Ax}{\vdash A \to A} \to I$$

Proof Example: Function Composition

$$\frac{}{A \to B, B \to C \vdash A \to C}$$

Proof Example: Function Composition

$$\dfrac{\overline{A \to B, B \to C, A \vdash C}}{A \to B, B \to C \vdash A \to C} \to I$$

Proof Example: Function Composition

$$\dfrac{\dfrac{\dfrac{}{B \to C, A, A \to B \vdash C}}{A \to B, B \to C, A \vdash C}\ \mathsf{Ex}}{A \to B, B \to C \vdash A \to C}\ \to I$$

Proof Example: Function Composition

$$
\cfrac{\cfrac{\cfrac{\cfrac{}{B \to C \vdash B \to C}\ Ax}{B \to C, A, A \to B \vdash C}\ \to E}{A \to B, B \to C, A \vdash C}\ \mathrm{Ex}}{A \to B, B \to C \vdash A \to C}\ \to I
$$

Proof Example: Function Composition

$$
\dfrac{\dfrac{\dfrac{\overline{B \to C \vdash B \to C}\ Ax \qquad \overline{A, A \to B \vdash B}}{B \to C, A, A \to B \vdash C} \to E}{A \to B, B \to C, A \vdash C}\ \mathrm{Ex}}{A \to B, B \to C \vdash A \to C} \to I
$$

Proof Example: Function Composition

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \to C \vdash B \to C}\;Ax \qquad \dfrac{\dfrac{}{A \to B, A \vdash B}}{A, A \to B \vdash B}\;\mathrm{Ex}}{B \to C, A, A \to B \vdash C}\;\to E}{A \to B, B \to C, A \vdash C}\;\mathrm{Ex}}{A \to B, B \to C \vdash A \to C}\;\to I$$

Proof Example: Function Composition

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \to C \vdash B \to C}\,Ax \qquad \dfrac{\dfrac{\dfrac{}{A \to B \vdash A \to B}\,Ax \quad \dfrac{}{A \vdash A}\,Ax}{A \to B, A \vdash B}\,{\to}E}{A, A \to B \vdash B}\,\mathrm{Ex}}{B \to C, A, A \to B \vdash C}\,{\to}E}{A \to B, B \to C, A \vdash C}\,\mathrm{Ex}}{A \to B, B \to C \vdash A \to C}\,{\to}I$$

Proof Example: Currying

$$\frac{}{(A \times B) \to C \vdash A \to B \to C}$$

Proof Example: Currying

$$\dfrac{\overline{(A \times B) \to C, A \vdash B \to C}}{(A \times B) \to C \vdash A \to B \to C} \to I$$

Proof Example: Currying

$$
\dfrac{\dfrac{\dfrac{}{(A \times B) \to C, A, B \vdash C}}{(A \times B) \to C, A \vdash B \to C} \to I}{(A \times B) \to C \vdash A \to B \to C} \to I
$$

Proof Example: Currying

$$
\dfrac{\dfrac{\dfrac{}{(A \times B) \to C \vdash (A \times B) \to C}\ Ax \qquad \dfrac{}{A, B \vdash A \times B}}{(A \times B) \to C, A, B \vdash C}\ \to E}{\dfrac{(A \times B) \to C, A \vdash B \to C}{(A \times B) \to C \vdash A \to B \to C}\ \to I}\ \to I
$$

Proof Example: Currying

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{(A \times B) \to C \vdash (A \times B) \to C}\ Ax
      \qquad
      \dfrac{
        \dfrac{}{A \vdash A}\ Ax
        \qquad
        \dfrac{}{B \vdash B}\ Ax
      }{A, B \vdash A \times B}\ \times I
    }{(A \times B) \to C, A, B \vdash C}\ \to E
  }{(A \times B) \to C, A \vdash B \to C}\ \to I
}{(A \times B) \to C \vdash A \to B \to C}\ \to I
$$

Proof Example: Tuple Reversal

$$\dfrac{}{A \times B \vdash B \times A}$$

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$$\dfrac{\overline{A \times B \vdash A \times B} \qquad \overline{A, B \vdash B \times A}}{A \times B \vdash B \times A} \times E$$

Proof Example: Tuple Reversal

$$\dfrac{\dfrac{}{A \times B \vdash A \times B}\ Ax \qquad \dfrac{}{A, B \vdash B \times A}}{A \times B \vdash B \times A}\ \times E$$

Proof Example: Tuple Reversal

$$\dfrac{\dfrac{}{A \times B \vdash A \times B}\ Ax \qquad \dfrac{\dfrac{}{B, A \vdash B \times A}}{A, B \vdash B \times A}\ Ex}{A \times B \vdash B \times A}\ \times E$$

Proof Example: Tuple Reversal

$$\dfrac{\dfrac{}{A \times B \vdash A \times B}\,Ax \qquad \dfrac{\dfrac{\dfrac{}{B \vdash B}\,Ax \quad \dfrac{}{A \vdash A}\,Ax}{B, A \vdash B \times A}\,\times I}{A, B \vdash B \times A}\,Ex}{A \times B \vdash B \times A}\,\times E$$

Proof Normalization

A single judgement may have many distinct proofs:

- ☺ "Morally" distinct: different construction methods
- ☹ Redundant elongations due to chaining I/E or C/W rules

$$
\dfrac{\dfrac{\dfrac{\cdots}{\color{red}{A \to B, A \vdash B}}}{A \to B \vdash A \to B}\ \color{red}{\to I} \qquad \dfrac{}{A \vdash A}\ Ax}{\dfrac{A \to B, A \vdash B}{\cdots}}\ \color{red}{\to E}
$$

Curry-Howard Correspondence

Haskell Curry

William Howard

Nicolas de Bruijn

Curry-Howard Correspondence

Intuitionistic Logic describes a model of computation, known as the simply-typed λ-calculus $\lambda^{\rightarrow}$

Alonzo Church

Propositions as Types

Logic	Computer Science
Proposition	Type
Proof	Algorithm
Provability	Type Inhabitation
Proof Normalization	β-reduction
Propositional Constant	Primitive Type
Axiom	Variable Instantiation
Logical Connectives	Type Operators
Introduction Rules	Type Constructors
Elimination Rules	Type Destructors
Implication Introduction	Function Abstraction
Implication Elimination	Function Application

...

Terms of the simply typed λ-calculus

Each instance of a proposition($\equiv$ *type*) $\mathcal{P}$ is assigned a unique *term* (name) $\mathcal{T}$:

$$\mathcal{T} : \mathcal{P}$$

Assumptions $\mathcal{A}$ are now interpreted as a *typing environment*:

$$\mathcal{A} := () \mid \mathcal{A}, \mathcal{T} : \mathcal{P}$$

Terms of the simply typed λ-calculus

Let $\mathcal{V}$ a set of *primitive terms* (variables).

We will denote primitive terms by `x, y, z, ...`

The terms $\mathcal{T}$ of the simply typed λ-calculus then are:

$\mathcal{T} :=$

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$$\mathcal{T} := \mathcal{V}$$

$$\frac{}{\mathtt{x} : A \vdash \mathtt{x} : A}\ Ax$$

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The terms $\mathcal{T}$ of the simply typed λ-calculus then are:

$$\mathcal{T} := \mathcal{V} \mid \mathcal{T}_1(\mathcal{T}_2)$$

$$\frac{\Gamma \vdash \mathtt{f} : A \to B \quad \Delta \vdash \mathtt{x} : A}{\Gamma, \Delta \vdash: \mathtt{f(x)}^1 : B} \to E$$

[1] Can also be written `f x`

Terms of the simply typed λ-calculus

Let $\mathcal{V}$ a set of *primitive terms* (variables).

We will denote primitive terms by `x, y, z, ...`

The terms $\mathcal{T}$ of the simply typed λ-calculus then are:

$\mathcal{T} := \mathcal{V} \mid \mathcal{T}_1(\mathcal{T}_2) \mid \lambda\mathcal{T}_1.\mathcal{T}_2$ *Implication-Only*

$$\frac{\Gamma, \mathtt{x} : A \vdash \mathtt{f} : B}{\Gamma \vdash \lambda\mathtt{x}.\mathtt{f} : A \to B} \to I$$

Terms of the simply typed λ-calculus

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We will denote primitive terms by `x, y, z, ...`

The terms $\mathcal{T}$ of the simply typed λ-calculus then are:

$$\begin{aligned} \mathcal{T} := \mathcal{V} \mid \mathcal{T}_1(\mathcal{T}_2) \mid \lambda \mathcal{T}_1.\mathcal{T}_2 \\ \mid (\mathcal{T}_1, \mathcal{T}_2) \end{aligned}$$

Implication-Only

$$\frac{\Gamma \vdash \mathtt{x} : A \quad \Delta \vdash \mathtt{y} : B}{\Gamma, \Delta \vdash (\mathtt{x},\ \mathtt{y}) : A \times B} \times I$$

Terms of the simply typed λ-calculus

Let $\mathcal{V}$ a set of *primitive terms* (variables).

We will denote primitive terms by `x, y, z, ...`

The terms $\mathcal{T}$ of the simply typed λ-calculus then are:

$$
\begin{aligned}
\mathcal{T} := \mathcal{V} \mid \mathcal{T}_1(\mathcal{T}_2) \mid \lambda\mathcal{T}_1.\mathcal{T}_2 & \qquad \textit{Implication-Only} \\
\mid (\mathcal{T}_1, \mathcal{T}_2) \mid \texttt{case}\ \mathcal{T}_1\ \texttt{of}\ (\mathcal{T}_2, \mathcal{T}_3) \to \mathcal{T}_4 & \qquad \textit{/w Product}
\end{aligned}
$$

$$
\frac{\Gamma \vdash \texttt{s} : A \times B \quad \Delta, \texttt{x} : A, \texttt{y} : B \vdash \texttt{w} : C}{\Gamma, \Delta \vdash \texttt{case s of (x, y)} \to \texttt{w} : C} \times E
$$

Terms of the simply typed λ-calculus

Let $\mathcal{V}$ a set of *primitive terms* (variables).

We will denote primitive terms by `x, y, z, ...`

The terms $\mathcal{T}$ of the simply typed λ-calculus then are:

$$\begin{aligned}\mathcal{T} := \mathcal{V} \mid \mathcal{T}_1(\mathcal{T}_2) \mid \lambda\mathcal{T}_1.\mathcal{T}_2 & \qquad \textit{Implication-Only} \\ \mid (\mathcal{T}_1, \mathcal{T}_2) \mid \texttt{case}\ \mathcal{T}_1\ \texttt{of}\ (\mathcal{T}_2, \mathcal{T}_3) \rightarrow \mathcal{T}_4 & \qquad \textit{/w Product} \\ \mid \texttt{inl}(\mathcal{T}_1) \mid \texttt{inr}(\mathcal{T}_2) &\end{aligned}$$

$$\frac{\Gamma \vdash \texttt{x} : A}{\Gamma \vdash \texttt{inl(x)} : A + B} +I_1 \qquad \frac{\Gamma \vdash \texttt{x} : B}{\Gamma \vdash \texttt{inr(x)} : A + B} +I_2$$

Terms of the simply typed λ-calculus

Let $\mathcal{V}$ a set of *primitive terms* (variables).

We will denote primitive terms by `x, y, z, ...`

The terms $\mathcal{T}$ of the simply typed λ-calculus then are:

$$\begin{array}{llr} \mathcal{T} := & \mathcal{V} \mid \mathcal{T}_1(\mathcal{T}_2) \mid \lambda\mathcal{T}_1.\mathcal{T}_2 & \textit{Implication-Only} \\ & \mid (\mathcal{T}_1, \mathcal{T}_2) \mid \texttt{case}\ \mathcal{T}_1\ \texttt{of}\ (\mathcal{T}_2, \mathcal{T}_3) \rightarrow \mathcal{T}_4 & \textit{/w Product} \\ & \mid \texttt{inl}(\mathcal{T}_1) \mid \texttt{inr}(\mathcal{T}_2) & \\ & \mid \texttt{case}\ \mathcal{T}_1\ \texttt{of}\ \texttt{inl}(\mathcal{T}_2) \rightarrow \mathcal{T}_3;\ \texttt{inr}(\mathcal{T}_4) \rightarrow \mathcal{T}_5 & \textit{/w Co-product} \end{array}$$

$$\frac{\Gamma \vdash \texttt{s} : A + B \quad \Delta, \texttt{x} : A \vdash \texttt{w} : C \quad \Delta, \texttt{y} : B \vdash \texttt{z} : C}{\Gamma \vdash \texttt{case s of inl(x)} \rightarrow \texttt{w; inr(y)} \rightarrow \texttt{z} : C} +E$$

Terms: Uniqueness & Structural Rules

Variable names must be unique for each distinct formula instantiation in a proof!

$$\frac{\Gamma, \mathtt{y} : A, \mathtt{z} : A \vdash \mathtt{u} : B}{\Gamma, \mathtt{x} : A \vdash \mathtt{u[x/y,x/z]} : B} \text{ Contraction}$$

$$\frac{\Gamma \vdash \mathtt{u} : B}{\Gamma, \mathtt{x} : A \vdash \mathtt{u} : B} \text{ Weakening}$$

Example: Identity Function Revisited

$$
\dfrac{\dfrac{}{A \vdash A}\ Ax}{\vdash A \to A} \to I
$$

Example: Identity Function Revisited

$$\dfrac{\dfrac{}{\mathtt{x} : A \vdash \mathtt{x} : A}\ Ax}{\vdash A \to A}\ \to I$$

Example: Identity Function Revisited

$$\dfrac{\dfrac{}{\mathtt{x} : A \vdash \mathtt{x} : A}\ Ax}{\vdash \lambda\mathtt{x}.\mathtt{x} : A \to A} \to I$$

Example: Function Composition Revisited

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B \vdash A \to B}\,Ax \qquad \dfrac{}{A \vdash A}\,Ax}{A \to B, A \vdash B}\,\to E}{A, A \to B \vdash B}\,\text{Ex} \qquad \dfrac{}{B \to C \vdash B \to C}\,Ax}{\dfrac{B \to C, A, A \to B \vdash C}{\dfrac{A \to B, B \to C, A \vdash C}{A \to B, B \to C \vdash A \to C}\,\to I}\,\text{Ex}}\,\to E
$$

Example: Function Composition Revisited

$$
\dfrac{\dfrac{\dfrac{}{\mathtt{g} : B \to C \vdash \mathtt{g} : B \to C}\,Ax \quad \dfrac{\dfrac{\dfrac{}{\mathtt{f} : A \to B \vdash \mathtt{f} : A \to B}\,Ax \quad \dfrac{}{\mathtt{x} : A \vdash \mathtt{x} : A}\,Ax}{A \to B, A \vdash B}\,\to E}{A, A \to B \vdash B}\,\mathrm{Ex}}{\dfrac{B \to C, A, A \to B \vdash C}{A \to B, B \to C, A \vdash C}\,\mathrm{Ex}}\,\to E}{A \to B, B \to C \vdash A \to C}\,\to I
$$

Example: Function Composition Revisited

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\mathtt{g} : B \to C \vdash \mathtt{g} : B \to C$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\mathtt{f} : A \to B \vdash \mathtt{f} : A \to B$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\mathtt{x} : A \vdash \mathtt{x} : A$}
\RightLabel{$\to E$}
\BinaryInfC{$\mathtt{f} : A \to B, \mathtt{x} : A \vdash \mathtt{f}\ \mathtt{x} : B$}
\RightLabel{Ex}
\UnaryInfC{$A, A \to B \vdash B$}
\RightLabel{$\to E$}
\BinaryInfC{$B \to C, A, A \to B \vdash C$}
\RightLabel{Ex}
\UnaryInfC{$A \to B, B \to C, A \vdash C$}
\RightLabel{$\to I$}
\UnaryInfC{$A \to B, B \to C \vdash A \to C$}
\end{prooftree}
$$

Example: Function Composition Revisited

$$
\dfrac{
  \dfrac{
    \dfrac{}{\mathtt{g} : B \to C \vdash \mathtt{g} : B \to C}\,\mathit{Ax}
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{}{\mathtt{f} : A \to B \vdash \mathtt{f} : A \to B}\,\mathit{Ax}
        \quad
        \dfrac{}{\mathtt{x} : A \vdash \mathtt{x} : A}\,\mathit{Ax}
      }{\mathtt{f} : A \to B, \mathtt{x} : A \vdash \mathtt{f}\ \ \mathtt{x} : B}\,\to E
    }{\mathtt{x} : A, \mathtt{f} : A \to B \vdash \mathtt{f}\ \ \mathtt{x} : B}\,\mathrm{Ex}
  }{
    \dfrac{\mathtt{g} : B \to C, \mathtt{x} : A, \mathtt{f} : A \to B \vdash \mathtt{g}\ \ (\mathtt{f}\ \ \mathtt{x}) : C}{A \to B, B \to C, A \vdash C}\,\mathrm{Ex}
  }\,\to E
}{A \to B, B \to C \vdash A \to C}\,\to I
$$

Example: Function Composition Revisited

$$
\dfrac{
  \dfrac{}{\mathtt{g} : B \to C \vdash \mathtt{g} : B \to C}\;Ax
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{\mathtt{f} : A \to B \vdash \mathtt{f} : A \to B}\;Ax
      \quad
      \dfrac{}{\mathtt{x} : A \vdash \mathtt{x} : A}\;Ax
    }{\mathtt{f} : A \to B, \mathtt{x} : A \vdash \mathtt{f}\ \ \mathtt{x} : B}\;\to E
  }{\mathtt{x} : A, \mathtt{f} : A \to B \vdash \mathtt{f}\ \ \mathtt{x} : B}\;\mathrm{Ex}
}{
  \dfrac{
    \dfrac{\mathtt{g} : B \to C, \mathtt{x} : A, \mathtt{f} : A \to B \vdash \mathtt{g}\ \ (\mathtt{f}\ \ \mathtt{x}) : C}{\mathtt{f} : A \to B, \mathtt{g} : B \to C, \mathtt{x} : A \vdash \mathtt{g}\ \ (\mathtt{f}\ \ \mathtt{x}) : C}\;\mathrm{Ex}
  }{A \to B, B \to C \vdash A \to C}\;\to I
}\;\to E
$$

Example: Function Composition Revisited

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\texttt{g} : B \to C \vdash \texttt{g} : B \to C$}
\AxiomC{}
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\UnaryInfC{$\texttt{f} : A \to B \vdash \texttt{f} : A \to B$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\texttt{x} : A \vdash \texttt{x} : A$}
\RightLabel{$\to E$}
\BinaryInfC{$\texttt{f} : A \to B, \texttt{x} : A \vdash \texttt{f x} : B$}
\RightLabel{Ex}
\UnaryInfC{$\texttt{x} : A, \texttt{f} : A \to B \vdash \texttt{f x} : B$}
\RightLabel{$\to E$}
\BinaryInfC{$\texttt{g} : B \to C, \texttt{x} : A, \texttt{f} : A \to B \vdash \texttt{g (f x)} : C$}
\RightLabel{Ex}
\UnaryInfC{$\texttt{f} : A \to B, \texttt{g} : B \to C, \texttt{x} : A \vdash \texttt{g (f x)} : C$}
\RightLabel{$\to I$}
\UnaryInfC{$\texttt{f} : A \to B, \texttt{g} : B \to C \vdash \lambda\texttt{x.(g (f x))} : A \to C$}
\end{prooftree}
$$

Term Equivalence & Reduction

- α-conversion Changing the name of bound variables

$$\lambda x.x \stackrel{\alpha}{\equiv} \lambda y.y$$

- Substitution Changing the name of free variables

$$(f\ \ g)(\lambda x.(x\ \ g))[h/g] = (f\ \ h)(\lambda x.(x\ \ h))$$

- η-reduction Simplifying an abstraction if the abstracted variable does not occur free in the function body

$$\lambda x.f\ \ x \stackrel{\eta}{\equiv} f\ \ (x \text{ does not occur in } f)$$

- β-reduction Removing an applicable abstraction

$$(\lambda x.g)(y) \Longrightarrow g[y/x]$$

Term Equivalence & Reduction

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Proof Normalization $\equiv$ Term Reduction $\equiv$ Computation

- Church-Rosser: order of term reduction rules is irrevelant
- Subject Reduction: term reduction rules on well-typed terms produce well-typed terms

Proof Normalization & Term Reduction ($\rightarrow$)

$$
\dfrac{
 \dfrac{
   \dfrac{
     \begin{array}{c}\dfrac{}{\mathtt{x}:A\vdash\mathtt{x}:A}\quad\dots\\ \vdots\\ \Gamma,\mathtt{x}:A,\cdots\vdash\mathtt{u}:B\end{array}
   }{\Gamma,\mathtt{x}:A\vdash\mathtt{u}:B}
 }{\Gamma\vdash\lambda\mathtt{x}.\mathtt{u}:A\to B}\ \to I
 \qquad
 \dfrac{\vdots}{\Delta\vdash\mathtt{t}:A}
}{\Gamma,\Delta\vdash(\lambda\mathtt{x}.\mathtt{u})(\mathtt{t}):B}\ \to E
\quad\Rightarrow\quad
\dfrac{\begin{array}{c}\vdots\\ \Delta\vdash\mathtt{t}:A\quad\dots\\ \vdots\\ \Gamma,\Delta,\cdots\vdash\mathtt{u}:B\end{array}}{\Gamma,\Delta\vdash\mathtt{u}:B}
$$

$$(\lambda\mathtt{x}.\mathtt{u})(\mathtt{t}) \Longrightarrow u[t/x]$$

β-reduction

Example

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$g : B \to C \vdash g : B \to C$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$f : A \to B \vdash f : A \to B$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$x : A \vdash x : A$}
\RightLabel{$\to E$}
\BinaryInfC{$f : A \to B, x : A \vdash f\ \ x : B$}
\RightLabel{Ex}
\UnaryInfC{$x : A, f : A \to B \vdash f\ \ x : B$}
\RightLabel{$\to E$}
\BinaryInfC{$g : B \to C, x : A, f : A \to B \vdash g\ \ (f\ \ x) : C$}
\RightLabel{Ex}
\UnaryInfC{$f : A \to B, g : B \to C, x : A \vdash g\ \ (f\ \ x) : C$}
\RightLabel{$\to I$}
\UnaryInfC{$f : A \to B, g : B \to C \vdash \lambda x.g\ \ (f\ \ x) : A \to C$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$y : A \vdash y : A$}
\end{prooftree}
$$

Example

$$
\dfrac{
  \dfrac{}{\texttt{g} : B \to C \vdash \texttt{g} : B \to C}\,Ax
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{\texttt{f} : A \to B \vdash \texttt{f} : A \to B}\,Ax
      \quad
      \dfrac{}{\texttt{x} : A \vdash \texttt{x} : A}\,Ax
    }{\texttt{f} : A \to B, \texttt{x} : A \vdash \texttt{f x} : B}\,\to E
  }{\texttt{x} : A, \texttt{f} : A \to B \vdash \texttt{f x} : B}\,Ex
}{
  \dfrac{
    \dfrac{\texttt{g} : B \to C, \texttt{x} : A, \texttt{f} : A \to B \vdash \texttt{g (f x)} : C}{\texttt{f} : A \to B, \texttt{g} : B \to C, \texttt{x} : A \vdash \texttt{g (f x)} : C}\,Ex
  }{\texttt{f} : A \to B, \texttt{g} : B \to C \vdash \lambda\texttt{x.g (f x)} : A \to C}\,\to I
  \quad
  \dfrac{}{\texttt{y} : A \vdash \texttt{y} : A}\,Ax
}\,\to E
$$

$$
\text{(conclusion of the last } \to E\text{):}\quad \texttt{f} : A \to B, \texttt{g} : B \to C, \texttt{y} : A \vdash (\lambda\texttt{x.f (g x)})\ \texttt{y} : C
$$

Example

$$
\dfrac{
  \dfrac{
    \dfrac{}{\mathtt{g} : B \to C \vdash \mathtt{g} : B \to C}\;Ax
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{}{\mathtt{f} : A \to B \vdash \mathtt{f} : A \to B}\;Ax
        \quad
        \dfrac{}{\mathtt{x} : A \vdash \mathtt{x} : A}\;Ax
      }{\mathtt{f} : A \to B, \mathtt{x} : A \vdash \mathtt{f}\ \ \mathtt{x} : B}\;\to E
    }{\mathtt{x} : A, \mathtt{f} : A \to B \vdash \mathtt{f}\ \ \mathtt{x} : B}\;Ex
  }{
    \dfrac{\mathtt{g} : B \to C, \mathtt{x} : A, \mathtt{f} : A \to B \vdash \mathtt{g}\ \ (\mathtt{f}\ \ \mathtt{x}) : C}
    {
      \dfrac{\mathtt{f} : A \to B, \mathtt{g} : B \to C, \mathtt{x} : A \vdash \mathtt{g}\ \ (\mathtt{f}\ \ \mathtt{x}) : C}
      {\mathtt{f} : A \to B, \mathtt{g} : B \to C \vdash \lambda \mathtt{x}.\mathtt{g}\ \ (\mathtt{f}\ \ \mathtt{x}) : A \to C}\;\to I
    }\;Ex
  }\;\to E
  \qquad
  \dfrac{}{\mathtt{y} : A \vdash \mathtt{y} : A}\;Ax
}{\mathtt{f} : A \to B, \mathtt{g} : B \to C, \mathtt{y} : A \vdash (\lambda \mathtt{x}.\mathtt{f}\ \ (\mathtt{g}\ \ \mathtt{x}))\ \ \mathtt{y} : C}\;\to E
$$

term reduction: $(\lambda \mathtt{x}.\mathtt{g}\ \ (\mathtt{f}\ \ \mathtt{x}))\ \mathtt{y} \overset{\beta}{\leadsto}$

Example

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\mathtt{g} : B \to C \vdash \mathtt{g} : B \to C}\,Ax
        \qquad
        \dfrac{
          \dfrac{
            \dfrac{}{\mathtt{f} : A \to B \vdash \mathtt{f} : A \to B}\,Ax
            \quad
            \dfrac{}{\mathtt{x} : A \vdash \mathtt{x} : A}\,Ax
          }{\mathtt{f} : A \to B, \mathtt{x} : A \vdash \mathtt{f\ x} : B}\,{\to}E
        }{\mathtt{x} : A, \mathtt{f} : A \to B \vdash \mathtt{f\ x} : B}\,Ex
      }{\mathtt{g} : B \to C, \mathtt{x} : A, \mathtt{f} : A \to B \vdash \mathtt{g\ (f\ x)} : C}\,{\to}E
    }{\mathtt{f} : A \to B, \mathtt{g} : B \to C, \mathtt{x} : A \vdash \mathtt{g\ (f\ x)} : C}\,Ex
  }{\mathtt{f} : A \to B, \mathtt{g} : B \to C \vdash \lambda\mathtt{x.g\ (f\ x)} : A \to C}\,{\to}I
  \qquad
  \dfrac{}{\mathtt{y} : A \vdash \mathtt{y} : A}\,Ax
}{\mathtt{f} : A \to B, \mathtt{g} : B \to C, \mathtt{y} : A \vdash (\lambda\mathtt{x.f\ (g\ x)})\ \mathtt{y} : C}\,{\to}E
$$

term reduction: $(\lambda\mathtt{x.g\ (f\ x)})\ \mathtt{y} \overset{\beta}{\leadsto} \mathtt{g\ (f\ y)}$

Example

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\mathtt{g} : B \to C \vdash \mathtt{g} : B \to C}\,\mathit{Ax}
      \qquad
      \dfrac{
        \dfrac{
          \dfrac{}{\mathtt{f} : A \to B \vdash \mathtt{f} : A \to B}\,\mathit{Ax}
          \quad
          \dfrac{}{\mathtt{x} : A \vdash \mathtt{x} : A}\,\mathit{Ax}
        }{\mathtt{f} : A \to B, \mathtt{x} : A \vdash \mathtt{f}\ \mathtt{x} : B}\,\to E
      }{\mathtt{x} : A, \mathtt{f} : A \to B \vdash \mathtt{f}\ \mathtt{x} : B}\,\mathit{Ex}
    }{
      \dfrac{\mathtt{g} : B \to C, \mathtt{x} : A, \mathtt{f} : A \to B \vdash \mathtt{g}\ (\mathtt{f}\ \mathtt{x}) : C}{\mathtt{f} : A \to B, \mathtt{g} : B \to C, \mathtt{x} : A \vdash \mathtt{g}\ (\mathtt{f}\ \mathtt{x}) : C}\,\mathit{Ex}
    }\,\to E
  }{\mathtt{f} : A \to B, \mathtt{g} : B \to C \vdash \lambda\mathtt{x}.\mathtt{g}\ (\mathtt{f}\ \mathtt{x}) : A \to C}\,\color{red}{\to I}
  \qquad
  \dfrac{}{\mathtt{y} : A \vdash \mathtt{y} : A}\,\mathit{Ax}
}{\mathtt{f} : A \to B, \mathtt{g} : B \to C, \mathtt{y} : A \vdash (\lambda\mathtt{x}.\mathtt{f}\ (\mathtt{g}\ \mathtt{x}))\ \mathtt{y} : C}\,\color{red}{\to E}
$$

term reduction: $(\lambda\mathtt{x}.\mathtt{g}\ (\mathtt{f}\ \mathtt{x}))\ \mathtt{y} \overset{\beta}{\rightsquigarrow} \mathtt{g}\ (\mathtt{f}\ \mathtt{y})$
proof reduction: elimination followed by introduction

Example

normalized:

$$\dfrac{\dfrac{}{\mathtt{g} : B \to C \vdash \mathtt{g} : B \to C}\,Ax \qquad \dfrac{\dfrac{}{\mathtt{f} : A \to B \vdash \mathtt{f} : A \to B}\,Ax \quad \dfrac{}{\mathtt{y} : A \vdash \mathtt{y} : A}\,Ax}{\mathtt{f} : A \to B, \mathtt{y} : A \vdash \mathtt{f\ \ y} : B}\to E}{\mathtt{g} : B \to C, \mathtt{f} : A \to B, \mathtt{y} : A \vdash \mathtt{g\ \ (f\ \ y)} : C}\to E$$

Proof Normalization & Term Reduction ($\times$)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \mathtt{t} : A\end{array} \quad \begin{array}{c}\vdots\\ \Delta \vdash \mathtt{u} : B\end{array}}{\Gamma, \Delta \vdash (\mathtt{t},\ \mathtt{u}) : A \times B}\ {\color{red}\times I} \qquad \dfrac{\begin{array}{c}\overline{\mathtt{x} : A \vdash \mathtt{x} : A} \quad \dots \quad \overline{\mathtt{y} : B \vdash \mathtt{y} : B}\\ \vdots\\ \Theta, A, \dots, B, \dots \vdash \mathtt{v} : C\end{array}}{\Theta, \mathtt{x} : A, \mathtt{y} : B \vdash C}}{\Gamma, \Delta, \Theta \vdash C}\ {\color{red}\times E}$$

$$\Downarrow$$

$$\dfrac{\begin{array}{c}\begin{array}{c}\vdots\\ \Gamma \vdash \mathtt{t} : A\end{array} \quad \dots \quad \begin{array}{c}\vdots\\ \Delta \vdash \mathtt{u} : B\end{array}\\ \vdots\\ \Gamma, \dots, \Delta, \dots, \Theta \vdash \mathtt{v} : C\end{array}}{\Gamma, \Delta, \Theta \vdash \mathtt{v} : C}$$

$$\mathtt{case\ (t,u)\ of\ (x,y)} \to \mathtt{v} \Longrightarrow v[t/x, u/y]$$

Proof Normalization & Term Reduction (+)

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \vdash \mathtt{t} : B\end{matrix}}{\Gamma \vdash \mathtt{inr(t)} : A+B}{\color{red}+I_2} \qquad \dfrac{\begin{matrix}\overline{\mathtt{x} : A \vdash \mathtt{x} : A}\\ \vdots \\ \Delta, \mathtt{x} : A \cdots \vdash \mathtt{v} : C\end{matrix}}{\Delta, \mathtt{x} : A \vdash \mathtt{v} : C} \qquad \dfrac{\begin{matrix}\overline{\mathtt{y} : B \vdash \mathtt{y} : B}\\ \vdots \\ \Delta, \mathtt{y} : B \cdots \vdash \mathtt{w} : C\end{matrix}}{\Delta, \mathtt{y} : B \vdash \mathtt{w} : C}}{\Gamma, \Delta \vdash \mathtt{case\ inr(t)\ of\ inl(x)} \to \mathtt{v}; \mathtt{inr(y)} \to \mathtt{w} : C}{\color{red}+E}$$

$$\Downarrow$$

$$\dfrac{\begin{matrix}\vdots & \\ \Gamma \vdash \mathtt{t} : B & \ldots \\ \vdots & \\ \Gamma, \ldots, \Delta, \cdots \vdash \mathtt{w} : C & \end{matrix}}{\Gamma, \Delta \vdash \mathtt{w} : C}$$

$$\mathtt{case\ inr(t)\ of\ inl(x)} \to \mathtt{v}; \mathtt{inr(y)} \to \mathtt{w} \Longrightarrow w[t/y]$$

Similarly for $+I_1$

Beyond simple types: the λ-Cube

Three axes of extension:

$\lambda 2$ Terms can depend on types

$$\frac{\Gamma \vdash \mathtt{t} : A}{\Gamma \vdash \Lambda \mathtt{a}.\mathtt{t} : \Pi a.A}$$

$\lambda\Pi$ Types can depend on terms

$$\frac{\Gamma, \mathtt{x} : A \vdash \mathtt{B} : *}{\Gamma \vdash (\Pi \mathtt{x}\!:\!\mathtt{A}.\mathtt{B}) : *}$$

$\lambda\underline{\omega}$ Types can depend on types

$$\lambda A : *.(A \to B) \to (B \to B \to B) \to B$$

$$\begin{tikzcd}
 & \lambda\omega \arrow[rr] & & \lambda\Pi\omega \\
\lambda 2 \arrow[ur] \arrow[rr] & & \lambda\Pi 2 \arrow[ur] & \\
 & \lambda\underline{\omega} \arrow[uu] \arrow[rr] & & \lambda\Pi\underline{\omega} \arrow[uu] \\
\lambda{\to} \arrow[uu] \arrow[ur] \arrow[rr] & & \lambda\Pi \arrow[uu] \arrow[ur] &
\end{tikzcd}$$