1 Gravitational N-Body Problem

The N-body problem is a special case of a system of equations of the form

$$\dot{q} = M^{-1}p \tag{1}$$
$$\dot{p} = F(q) \tag{2}$$

where q and p are the vectors of position and momentum variables, M is a mass matrix which may be assumed to be a diagonal matrix with positive entries, and $F(q)$ is the force expressed as a smooth function of position. A system of this type is said to be in Hamiltonian form or we just say that it is a Hamiltonian system, named after Hamilton who studied their properties. N-body problems are special cases of (1)–(2) where the force $F(q)$ relates to the forces of interaction of a collection of bodies.

Supplemented by initial conditions, the N-body problem has a unique solution, and this can typically be assumed to exist globally, i.e. for all time. If force field F is defined to be a gravitational interaction force, then we refer to this as a gravitational N-body system. Even absent relativistic forces, the N-body problem provides a fairly accurate model for many celestial mechanics systems, such as the solar system or a star cluster. By differentiating the first equation and premultiplying by M, one obtains

$$M\ddot{q} = F$$

which may be viewed as Newton's equations of motion (mass times acceleration equals force) written in a vectorial form.

Consider a simple example of two spherical bodies moving in three dimensions. The bodies each have a centre of mass, say q_1 and q_2, each being a 3-vector. Written out in coordinates, we would have $q_1 = (x_1, y_1, z_1)$, $q_2 = (x_2, y_2, z_2)$. The combined position vector is $q = (q_1^T, q_2^T)^T \in \mathbb{R}^6$. Likewise each body has a momentum vector in $\mathbb{R}^3$, and we can set $p = (p_1^T, p_2^T)^T \in \mathbb{R}^6$. The gravitational potential energy function is $U(q) = \frac{Gm_1m_2}{\|q_1-q_2\|}$ (G the gravitational constant, and m_1 and m_2 the masses of the two bodies) and it is well known that for such a system we can obtain the force as the negative gradient of the potential energy function, thus

$$F(q) = -\nabla U(q)$$

where ∇ represents the gradient, $\nabla U(q)^T = [\partial U/\partial x_1, \partial U/\partial y_1, \partial U/\partial z_1, \partial U/\partial x_2, \partial U/\partial y_2, \partial U/\partial z_2]$. For the given potential energy function it is easy to calculate the force and thus to write the equations of motion out.

We can slightly simplify things by letting the bodies move in the plane ($\mathbb{R}^2$) instead of $\mathbb{R}^3$, and we can simplify still further by fixing the position of one of the two bodies at the origin. In this case we just have two coordinates x and y to represent the position of the moving body, and the equations of motion become

$$\begin{aligned}
\dot{x} &= m^{-1}p_x \\
\dot{y} &= m^{-1}p_y \\
\dot{p}_x &= -\gamma\frac{x}{(x^2+y^2)^{3/2}} \\
\dot{p}_y &= -\gamma\frac{y}{(x^2+y^2)^{3/2}},
\end{aligned}$$

where m is the mass of the moving body and $\gamma = GmM$ where M is the mass of the fixed body. This system is called the "Kepler Problem". Alternatively, the Kepler Problem is in the form (1)–(2) if we define

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad U(q) = U(x, y) = -\frac{\gamma}{\|q\|}.$$

2 The Verlet/Leapfrog method for solving N-body problems

If we are given a system in the form (1)–(2) we can view it as a system of ODEs in a 2d-dimensional space, where d is the total number of position coordinates needed to describe the positions of all the bodies. It is then possible to apply any ODE method, for example Euler's method, a Taylor series method, or the 4th order Runge-Kutta method, or something more exotic. However, an alternative type of scheme is more useful for this type of system.

These methods are the symplectic methods which are discussed in the lecture notes in Chapter 6 [Note this is not examinable, but you can obviously read about it id you're interested].

One particular method of this type is the Verlet or Leapfrog method which solves the equations for a timestep as follows, given (q_n, p_n) representing the positions and momenta at time step n:

$$\begin{aligned} p_{n+1/2} &= p_n + (h/2)F(q_n) \\ q_{n+1} &= q_n + hM^{-1}p_{n+1/2} \\ p_{n+1} &= p_{n+1/2} + (h/2)F(q_{n+1}) \end{aligned}$$

It is an easy-to-implement method, since for example for the first equation, we just have to evaluate the force vector, scale it and add to the momentum vector. It should be easy to see that the method is explicit. It is also symplectic. Note that only one force evaluation is needed per step, since the evaulation of F at the end of each step can be re-used at the beginning of the next one, although some care must be taken to store the F from step to step.

3 HamSolver.m

You have been provided with an m-file HamSolver.m which allows you to solve Hamiltonian systems of the form (1)–(2). You need to specify the potential energy function U, the force field $F(q) = -\nabla U(q)$, the mass matrix M, the initial conditions q_0 and p_0, the stepsize and number of steps to be performed, and the name of a numerical method that can be applied to solve the problem. The calling sequence looks like

```
>> [T,Q,P,H] = HamSolver(q0, p0, Nsteps, stepsize, @force, method_name, Pars);
```

As an illustration, to compute a solution of the Kepler problem you may use the supplied routines KeplerInit.m and KeplerForce.m and simply type

```
>> KeplerInit; %initialize q0, p0 and Pars
>> [T,Q,P,H] = HamSolver(q0, p0, 10000, 0.01, @KeplerForce, 'Verlet', Pars);
```

Then the position vectors for successive timesteps will be stored as colums of the array Q and momentum vectors will be stored similarly in P. The vector T is just a vector of time points, and H is a vector of energy values at the successive timesteps.

4 The two-fixed centres problem

In the two-fixed centres (TFC) problem we assume we have a single mass moving in the plane in the gravitational field generated by a pair of additional bodies which are assumed to be fixed in space. We simplify the problem by setting the mass of all three bodies to one.

The system is in the form (1)–(2) where the potential energy function U is

$$U(q) = -\frac{\gamma}{\|q - q_1\|} - \frac{\gamma}{\|q - q_2\|}$$

The force field for the TFC problem is given by $F(q) = -\nabla U(q)$, but now the potential is a bit more complicated. In order to work out the gradient, try treating each of the two terms in the potential separately. You will see that each one is a bit like the Kepler problem potential, but in this case there is a offset (q_1 or q_2) to the fixed body, i.e. it is not located at the origin. If you get stuck, just use the fact that $q = (x, y)$ and $q_1 = (x_1, y_1)$ then write out the norm in terms of x, y, x_1, y_1 and thus give a formula for the potential energy, then find its partial derivative with respect to x and y successively.

You will produce an m-file TFCForce.m which computes the force acting on the moving body. Use the Pars data structure to store q1 and q2, so in TFCInit.m you will have lines like "Pars.q1 = [1;0];" to define the locations of the fixed points.