Numerical Ordinary Differential Equations & Applications . . . . CP4 [25 marks]

Dr Ben Goddard **due: 5pm, 4th April 2019** (turnitin)

Notes: (1) Refer to companion guidance notes for CP4 before attempting this assignment. (2) To accompany this assignment you have available the following m-files: (i) HamSolver.m, (ii) KeplerInit.m, (iii) KeplerForce.m (3) **Submit (upload) using the turnitin system by the deadline.**

This project is about solving N-body problems. N-body problems are an important special class of ordinary differential equation systems. They are characterised by the forces of interaction among the bodies and the body masses. The variables used to describe an N-body problem are the positions and momenta of the bodies. You can describe such a system by defining its energy function

$$H(q,p) = p^T M^{-1} p/2 + U(q)$$

The H stands for Hamiltonian-N-body problems are what are called Hamiltonian systems. The first term here is the kinetic energy and the second is the potential energy. The kinetic energy is a quadratic form on the inverse mass matrix (typically M is a diagonal matrix with the masses on the diagonal). U(q), the potential energy, is a function that is usually defined in terms of the distances between the bodies. The total energy is a conserved quantity which means that along solutions its value should not change.

You will study two different examples of N-body problems. The first is the Kepler problem, which describes the interaction of one moving body with a single fixed body at the origin, confined to planar motion. The second problem is the two fixed centres (TFC) problem for which two bodies are fixed and the third moves around among the other two.

You are provided with the following routines: HamSolver.m which solves Hamiltonian systems, and KeplerInit.m and KeplerForce.m which are needed to set up and solve the Kepler problem using HamSolver. Please examine the m-files to understand how they work.

P1 Kepler Problem.

- **a** Using the provided routines, set up and run a simulation of the Kepler problem with the Verlet method for the following initial conditions:

  $$q_0 = (1,0), \quad p_0 = (0.4, 0.6)$$

  You should take all masses and the coefficient γ to be one. Use a stepsize of $h = 0.001$ and solve on the time interval $[0, 10]$. Graph the solution in the xy-plane. Describe the resulting graph. It can be shown that the Kepler problem, for this initial condition, has an orbit that moves on an ellipse in the xy-plane with the fixed body at one focus. Do you observe this here? Try increasing the stepsize slowly until it changes the results. Superimpose the graphs obtained for different stepsizes on that obtained with $h = 0.001$. Describe what you observe. **[3 marks]**

- **b** Consider initial conditions of the form

  $$q_0 = (1,0), \quad p_0 = (0.6, \alpha)$$

  for various values of α, i.e. $\alpha = 0.6, 0.3, 0.1$. In each case, graph the solutions for stepsize $h = 0.001$. Discuss the orbits that result for each value of α, in particular whether or not they are closed (periodic) and whether or not they stay bounded or tend to infinity. By trying smaller stepsizes, determine if the orbits you obtain are good approximations of the exact solution or not. If you detect instability in the numerical solutions, try to explain why this is happening. You may wish to zoom in on the vicinity of the origin. **[4 marks]**

P2 Two Fixed Centres Problem.

- **a** Using KeplerInit.m and KeplerForce.m as templates, write your own routines TFCInit.m and TFCForce.m which implement a simulation of the two fixed centres problem. See the guidance note for details of the two fixed centres (TFC) problem. **[5 marks]**

- **b** Test your method first by taking both of the fixed centres at the origin. Explain why this should be identical to the Kepler Problem with a doubled value of γ. Verify that the TFC code and the Kepler code produce identical results. **[4 marks]**

- **c** Now place the two fixed points at $(0, -1)$ and $(0, 1)$. Take $\gamma = 1$. Using the initial conditions $q(0) = (0, 0)$ and $p(0) = (0.45, 0.25)$ compute a solution of the TFC problem on the interval $[0, 10]$ for $h = 0.001$

using the Verlet method; graph this solution. On the same axes, graph a solution for time interval $[0, 10]$ and stepsize $h = 0.0001$ (with a different color). Using the command axes([-1.5 1.5 -1.5 1.5]) you should be able to see the two solutions together. Discuss what causes the large stepsize solution to diverge rapidly from the small timestep one. **[4 marks]**

d Focusing on the smaller stepsize ($h = 0.0001$), increase the time interval to $[0, 20]$. Using the same initial conditions as above, simulate the system over $[0, 20]$ using the Verlet method and graph the resulting solution. Repeat the simulation using the RK4 method but using four times the stepsize (since RK4 needs 4 times the computational work as Verlet, roughly). (Replace "Verlet" in the argument list of HamSolver by "RK4", change the stepsize to $h = 0.0004$ and take 1/4 as many steps as for Verlet.)

e Next, create a series of three panels (using subplot) showing plots of "H" against "T", i.e., energy (the 4th output argument of HamSolver) against time (the first output argument) for each of Verlet and RK4 (use "hold on" and different colors). We want to understand how the two methods behave near the collision points with the fixed centres. In each of the panels you use the same commands to generate a plot showing H vs T, but in each frame you need to limit the axes differently. In the first panel of the series use `axis([0 20 -1.9 -1.5])`. This shows how the energies compare over the whole time interval. In the second panel, create the same graph, but focus on the first close approach by using `axis([1.35 1.45 -1.875 -1.865])`. You should have the two energy graphs superimposed and you should be able to easily see what happens to each graph during the time of close approach. Finally, use axis commands and the zoom feature to produce a zoomed in graph showing the energy variation for the collision nearest to $t = 7$ for each method. What do you observe? Discuss each panel separately. Which method performs best during collisions and why? **[5 marks for (d) and (e)]**

To hand in: For problem P1a, a graph showing the trajectories obtained for different stepsizes, along with a paragraph discussing it. For P1b, a graph showing the computed solutions for the three different initial conditions, along with discussion. An additional graph may be used to show how a smaller stepsize gives better accuracy. For problem P2a, the code for TFCInit.m and TFCForce.m should be included in the appendix of your hand-in. For P2b, a graph showing that the TFC and Kepler codes produce identical results. For P2c a graph showing the behavior of solutions for different stepsizes, along with a paragraph discussing it. For P2d, a graph showing the solutions for Verlet and RK4. For P2e, a series of three graphs showing the energies for Verlet and RK4 under different axes limits. For P2d and P2e a combined comparative discussion of the behavior of the two methods on this problem and the behavior of the methods near close approaches to the fixed centres. Length limit: **4** *pages including graphs but not counting the appendix containing MATLAB code.*