Computing Assignment C2

Instructions

This assessment is due by 1700 on Friday week 11. Submission will open at 1200 on Monday week 9.

You should submit

- The requested report, summarising your experiments, results, and conclusions. This should be a single PDF file submitted via Turnitin.
- Script and/or function .m files you use to conduct your experiments, perform calculations, or to generate figures. These should be submitted via Learn.

Please make sure you have read and understand the School of Mathematics information on academic misconduct at https://teaching.maths.ed.ac.uk/main/undergraduate/studies/assessment/academic-misconduct.

Please also make sure you have read and understand the information on Learn under "Assessments" regarding Turnitin.

If you have any questions regarding academic misconduct policies or Turnitin please contact Dr. James R. Maddison, j.r.maddison@ed.ac.uk.

1 Problem outline

Consider the two-dimensional hyperbolic equation

$$\partial_t \Psi + \partial_x \chi \partial_y \Psi - \partial_y \chi \partial_x \Psi = 0 \quad \text{for } (x, y) \in (0,1)^2, t \in (0, T), \tag{1}$$

where $\chi(x, y)$ is a given function to be specified, defined such that it is zero on all boundaries

$$\chi(x = 0, y) = \chi(x = 1, y) = \chi(x, y = 0) = \chi(x, y = 1) = 0. \tag{2}$$

An initial condition is given

$$\Psi(x, y, t = 0) = P(x, y), \tag{3}$$

defined such that $P(x, y)$ is also zero on all boundaries. It follows that

$$\Psi(x = 0, y, t) = \Psi(x = 1, y, t) = \Psi(x, y = 0, t) = \Psi(x, y = 1, t) = 0. \tag{4}$$

Consider discretisations of the form

$$\frac{\psi^{1/2}_{m,p} - \psi^0_{m,p}}{\frac{1}{2}\Delta t} + J^0_{m,p} = 0 \quad \text{for } m, p \in \{1, \ldots, M-1\}, \tag{5}$$

$$\frac{\psi^{1}_{m,p} - \psi^0_{m,p}}{\Delta t} + J^{1/2}_{m,p} = 0 \quad \text{for } m, p \in \{1, \ldots, M-1\}, \tag{6}$$

$$\frac{\psi^{n+1}_{m,p} - \psi^{n-1}_{m,p}}{2\Delta t} + J^n_{m,p} = 0 \quad \text{for } m, p \in \{1, \ldots, M-1\}, n \in \{1, \ldots, N-1\}, \tag{7}$$

with

$$\psi^n_{m,0} = \psi^n_{m,M} = \psi^n_{0,p} = \psi^n_{M,p} \quad \text{for } m, p \in \{0, \ldots, M\}, n \in \{0, 1/2, 1, \ldots, N\}, \tag{8a}$$

$$\psi^0_{m,p} = P\left(x = m\Delta x, y = p\Delta x\right) \qquad \text{for } m, p \in \{1, \ldots, M-1\}\,. \tag{8b}$$

$\psi^n_{m,p}$ is the discrete solution at $x = m\Delta x$, $y = p\Delta x$, and $t = n\Delta t$, where $M\Delta x = 1$ and $N\Delta t = T$ with M and N positive integers. Note that here n is allowed to take the value 1/2, but is otherwise an integer, $n \in \{0, 1/2, 1, \ldots, N\}$.

Consider three different definitions for $J^n_{m,p}$, each defined for $m, p \in \{1, \ldots, M-1\}$ and $n \in \{0, 1/2, 1, \ldots, N-1\}$ (from Arakawa, 1966, equations (36)–(38)),

$$J^n_{m,p} \to J^{++,n}_{m,p} = \frac{1}{4\Delta x^2}\left[\left(\chi_{m+1,p} - \chi_{m-1,p}\right)\left(\psi^n_{m,p+1} - \psi^n_{m,p-1}\right) - \left(\chi_{m,p+1} - \chi_{m,p-1}\right)\left(\psi^n_{m+1,p} - \psi^n_{m-1,p}\right)\right], \tag{9a}$$

$$\begin{aligned} J^n_{m,p} \to J^{+\times,n}_{m,p} = \frac{1}{4\Delta x^2}\Big[&\chi_{m+1,p}\left(\psi^n_{m+1,p+1} - \psi^n_{m+1,p-1}\right) - \chi_{m-1,p}\left(\psi^n_{m-1,p+1} - \psi^n_{m-1,p-1}\right) \\ &-\chi_{m,p+1}\left(\psi^n_{m+1,p+1} - \psi^n_{m-1,p+1}\right) + \chi_{m,p-1}\left(\psi^n_{m+1,p-1} - \psi^n_{m-1,p-1}\right)\Big], \end{aligned} \tag{9b}$$

$$\begin{aligned} J^n_{m,p} \to J^{\times+,n}_{m,p} = \frac{1}{4\Delta x^2}\Big[&\chi_{m+1,p+1}\left(\psi^n_{m,p+1} - \psi^n_{m+1,p}\right) - \chi_{m-1,p-1}\left(\psi^n_{m-1,p} - \psi^n_{m,p-1}\right) \\ &-\chi_{m-1,p+1}\left(\psi^n_{m,p+1} - \psi^n_{m-1,p}\right) + \chi_{m+1,p-1}\left(\psi^n_{m+1,p} - \psi^n_{m,p-1}\right)\Big], \end{aligned} \tag{9c}$$

where $\chi_{m,p} = \chi\left(x = m\Delta x, y = p\Delta x\right)$.

The provided script `hyperbolic_2d.m` provides a starting point implementation for such a numerical discretisation of the two-dimensional hyperbolic equation.

Complete and modify this script, and use this to investigate the properties of the discretisation. You are strongly recommended to make appropriate use of functions.

You should at least

- Generate one or more figures showing the "time" evolution (how the numerical solution develops with increasing t) for the case where

$$P\left(x, y\right) = \exp\left(-\frac{\left(x - 0.25\right)^2 + \left(y - 0.6\right)^2}{0.08}\right)\sin\left(\pi x\right)\sin\left(\pi y\right),$$
$$\chi\left(x, y\right) = \sin\left(\pi x\right)\sin\left(\pi y\right),$$

  for *any one* of the given forms for $J^n_{m,p}$. You will need to choose appropriate values for Δt, Δx, and T.

- For this $P\left(x, y\right)$ and $\chi\left(x, y\right)$, plot

$$\Delta x^2 \sum_{m=1}^{M-1}\sum_{p=1}^{M-1} \psi^n_{m,p},$$
$$\Delta x^2 \sum_{m=1}^{M-1}\sum_{p=1}^{M-1} \psi^n_{m,p}\psi^n_{m,p},$$

  against $n\Delta t$, for *all three* of the given forms for $J^n_{m,p}$. You will again need to choose appropriate values for Δt, Δx, and T.

Extend your investigation, considering for example combinations of the three given forms for $J^n_{m,p}$, other properties of the discretisation, other discretisations, or related partial differential equations or boundary conditions.

Summarise your experiments, results, and conclusions in a report. You should prepare your report using an appropriate typesetting or word processing package. The use of LaTeX is recommended. You should include all relevant figures and references in a single PDF document of not more than 6 pages in length, with font size of not less than 11pt.

Marking

This assignment is marked out of 40. Marks will be awarded for

- Structure and presentation (10 marks)
  - The report should be well organised and well laid out.
  - Numerical experiments should be clearly summarised.
  - Results should be clearly presented.
  - There should be appropriate use of references.
- Analysis and interpretation (15 marks)
  - The numerical experiments should be clearly motivated.
  - The interpretation of results should be supported with appropriate mathematical analysis.
  - Conclusions should be appropriately justified and/or qualified.
- Correctness and quality of code (5 marks)
  - Codes should be error free.
  - Codes should be logically laid out and easily understood.
  - The intent of the codes should be clear.
- Depth of investigation (10 marks)
  - The investigation should be extended with appropriate additional numerical experiments.
  - The new experiments should be clearly motivated, with explanation and interpretation of results, and appropriate conclusions.

References

Arakawa, A. (1966). Computational design for long-term numerical integration of the equations of fluid motion: Two-dimensional incompressible flow. Part I. *Journal of Computational Physics*, 1(1):119–143.