Intelligent and Adaptive Control Systems Project

Direct Model Reference Adaptive Control

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January 2023

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1 Introduction - Linearization

** Please note that in the context of this report we will be reffering to Direct Model Reference Adaptive Control as DMRAC **

Let the system be:

$$M\ddot{q} + Gsin(q) + C\dot{q} = u \tag{1}$$

, which describes the operation of an inverted pendulum. In (1) by $q \in \mathbf{R}$ we denote the yaw angle in rad, by $\dot{q}$ the angular velocity in rad/s and by $\ddot{q}$ the angular acceleration in rad/s^2. Additionally $u \in \mathbf{R}$ is the control input expressed as torque $(N \cdot m)$, exerted on the base of the inverted pendulum, while the output of the system is the yaw angle q. The parameters $M\ [N \cdot m \cdot s^2/rad]$, $G\ [N \cdot m]$, $C[N \cdot m \cdot s/rad]$, appearing in (1) are assumed to be constant, but unknown. For simulation purposes, the values $M = 1/2, G = 10, C = 1$ will be used. The goal is to design a controller, which controls the operation of the system in the neighborhood of zero.

So we start with the 1st requirement, the linearization of (1) around 0. First, we bring the system into the form of state equations:

$$\begin{cases} x_1 = q \\ x_2 = \dot{q} \end{cases} \Rightarrow \begin{cases} \dot{x_1} = \dot{q} \\ \dot{x_2} = \ddot{q} \end{cases} \Rightarrow$$

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\dfrac{G}{M}sin(x_1) - \dfrac{C}{M}x_2 + \dfrac{1}{M}u \end{cases} \tag{2}$$

The output of the system is described by the relation:

$$y = x_1 \text{ or } y = [1\ 0]\mathbf{x} \tag{3}$$

Then we calculate the equilibrium points of the system:

$$\begin{cases} x_2 = 0 \\ -\dfrac{G}{M}sin(x_1) - \dfrac{C}{M}x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = 0 \\ sin(x_1) = 0 \end{cases} \Rightarrow$$

$$\begin{cases} x_2 = 0 \\ x_1 = k\pi,\ k \in \mathbf{Z} \end{cases}$$

Therefore the point $\mathbf{x}^* = (0, 0)$ is an equilibrium point and we can proceed to the linearization in the neighborhood of zero with the well-known procedure:

$$A = \left.\frac{\partial F}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -\dfrac{G}{M}cos(x_1) & -\dfrac{C}{M} \end{bmatrix}_{(0,0)} \Rightarrow$$

$$A = \begin{bmatrix} 0 & 1 \\ -\dfrac{G}{M} & -\dfrac{C}{M} \end{bmatrix}$$

$$B = \left.\frac{\partial F}{\partial u}\right|_{(0,0)} = \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix}$$

So, the system linearized about $\mathbf{x}^* = (0, 0)$ is:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\dfrac{G}{M} & -\dfrac{C}{M} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix}u,\ y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} \tag{4}$$

2 Reference Model Controller

2.1 Design

In the specific section, the constants of the system are considered known, $M = \frac{1}{2}, G = 10, C = 1$. We substitute into (3) and then calculate the transfer function of the system:

$$(3) \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ -20 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u,\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \tag{5}$$

Therefore:

$$\begin{aligned} \ddot{y} = \ddot{x}_1 = \dot{x}_2 = -20y - 2\dot{y} + 2u \Rightarrow \\ \ddot{y} + 20y + 2\dot{y} = 2u \xRightarrow{\mathcal{L}(\cdot)} \\ s^2 Y + 2sY + 20Y = 2U \Rightarrow \\ Y = \frac{2}{s^2 + 2s + 20} U = G_p(s) U \end{aligned} \tag{6}$$

, where $G_p(s) = \frac{2}{s^2 + 2s + 20}$, the transfer function of the linearized system.

Now, let the reference model with transfer function:

$$G_m(s) = \frac{1}{s^2 + \alpha s + \beta} \tag{7}$$

, with $a, b > 0$ such that the system meets the requested specifications of zero overshoot and less than $10s$ recovery time (more on the choices of a, b below).

According to the utterance only the output y and the input u are available for measurement, so the controller is designed according to the course notes, pages 11-15 (Chapter 2). Thus, before proceeding to the formulation of the control law, we check that the necessary assumptions of page 12 hold true:

For the controlled system:

- $Z_p(s) = 1$, is regular and stable (Hurwitz) polynomial.
- The degree of $R_p(s) = s^2 + 2s + 20$, is known, $n_p = 2$, so also the upper bound $n = n_p = 2$.
- The relative degree of $G_p(s)$ is known, $n^* = 2$.
- The sign of the high-frequency gain $k_p = 2$ is obviously known.

For the reference model:

- $Z_m(s), R_m(s)$, are regular and stable (Hurwitz) polynomials and obviously $p_m = 2 = n$.
- The relative degree of the reference model is ${n_m}^* = n^* = 2$.

Then we choose a stable filter $\Lambda_0(s)$ such that $\Lambda(s) = \Lambda_0(s) Z_m(s)$ is a regular, stable filter $n - 1 = 2 - 1 = 1^u$ degree. But $Z_m(s) = 1$, so:

$$\Lambda(s) = \Lambda_0(s) = s + \lambda,\ \lambda > 0 \tag{8}$$

λ will then be chosen in a suitable way to help ensure the constraints and generally good operation of the closed-loop system.

In addition, we have the filter vector $\alpha(s)$ which is defined as follows:

$$\alpha(s) = \alpha_{n-2}(s) = \begin{bmatrix} s^{n-2} & s^{n-3} & ... & s & 1 \end{bmatrix}^T = [1]^T = 1 \tag{9}$$

, which, as we see in our case is scalar, and equal to 1.

We are now able to define the control input:

$$\begin{aligned} U &= {\theta_1}^* \frac{\alpha(s)}{\Lambda(s)} U + {\theta_2}^* \frac{\alpha(s)}{\Lambda(s)} Y + {\theta_3}^* Y + {c_0}^* R \Rightarrow \\ U &= {\theta_1}^* \frac{1}{s+\lambda} U + {\theta_2}^* \frac{1}{s+\lambda} Y + {\theta_3}^* Y + {c_0}^* R \end{aligned} \tag{10}$$

To calculate the controller parameters ${\theta_1}^*, {\theta_2}^*, {\theta_3}^*, {c_0}^*$:

$${c_0}^* = \frac{k_m}{k_p} = 1/2 \tag{11}$$

$${\theta_1}^* \alpha(s) R_p(s) + k_p({\theta_2}^* \alpha(s) + {\theta_3}^* \Lambda(s)) Z_p(s) = \Lambda(s) R_p(s) - Z_p(s) \Lambda_0(s) R_m(s) \Rightarrow$$

$${\theta_1}^*(s^2 + 2s + 20) + 2({\theta_2}^* + {\theta_3}^*(s+\lambda)) = (s+\lambda)(s^2 + 2s + 20) - (s+\lambda)(s^2 + \alpha s + \beta) \Rightarrow$$

$${\theta_1}^* s^2 + (2{\theta_1}^* + 2{\theta_3}^*)s + (20{\theta_1}^* + 2{\theta_2}^* + 2\lambda{\theta_3}^*) = (2-a)s^2 + (20 + 2\lambda - b - la)s + (20\lambda - \lambda b)$$

From the above equation we get the system:

$$\begin{cases} {\theta_1}^* = (2-a) \\ 2{\theta_1}^* + 2{\theta_3}^* = 20 + 2\lambda - \beta - \lambda\alpha \\ 20{\theta_1}^* + 2{\theta_2}^* + 2\lambda{\theta_3}^* = 20\lambda - \lambda b \end{cases} \Rightarrow$$

$$\begin{cases} {\theta_1}^* = 2a \\ {\theta_2}^* = 10a + 2\lambda - al + \dfrac{al^2}{2} - \lambda^2 - 20 \\ {\theta_3}^* = \alpha - \dfrac{\beta}{2} + \lambda - \dfrac{\alpha}{2} + 8 \end{cases} \tag{12}$$

Therefore for the closed loop system we have:

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + Bu \\ u = {\theta_1}^* \omega_1 + {\theta_2}^* \omega_2 + {\theta_3}^* y + {c_0}^* r \\ \dot{\omega_1} = -\lambda\omega_1 + u, \ \omega_1(0) = 0 \\ \dot{\omega_2} = -\lambda\omega_2 + y, \ \omega_2(0) = 0 \end{cases} \tag{13}$$

2.2 Scenario a

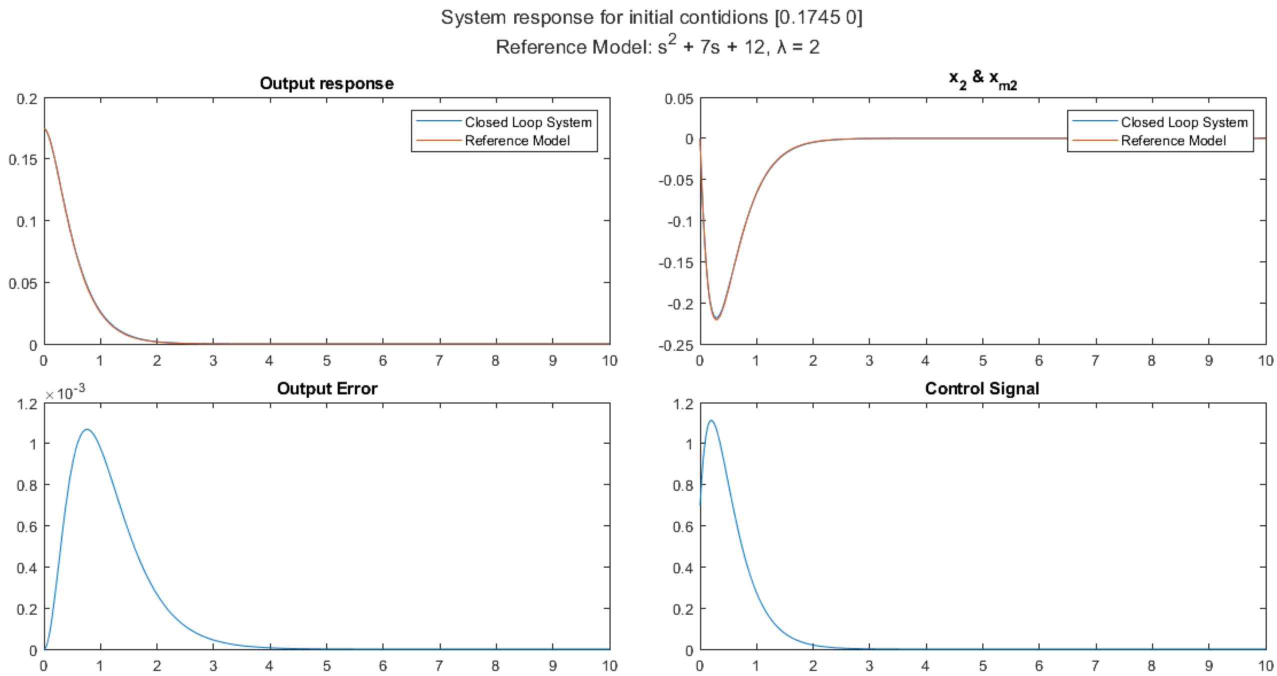
In this request, we need to use the reference model controller we designed above, to stop the pendulum at the position $q = 0$, when it starts from some angles $(0.1745\, rad, 0.8727\, rad)$, with zero elevation and recovery time not exceeding $10s$. The above will happen in the simulation frameworks in matlab.

Starting with the specifications, to be met by the closed loop system we must first ensure that the reference model meets them. Thus, to have zero elevation it is enough that the poles of the reference model are real (and obviously stable). Choosing poles at $-3, -4$ we get the values $\alpha = 7, \beta = 12$. Moreover, we know that the recovery time $t_s \approx 4/Re\{\text{latest pole}\}$, so in our case $t_s \approx 4/3 = 1.33 < 10$. So with this option the specifications from the reference model are met which has now become:

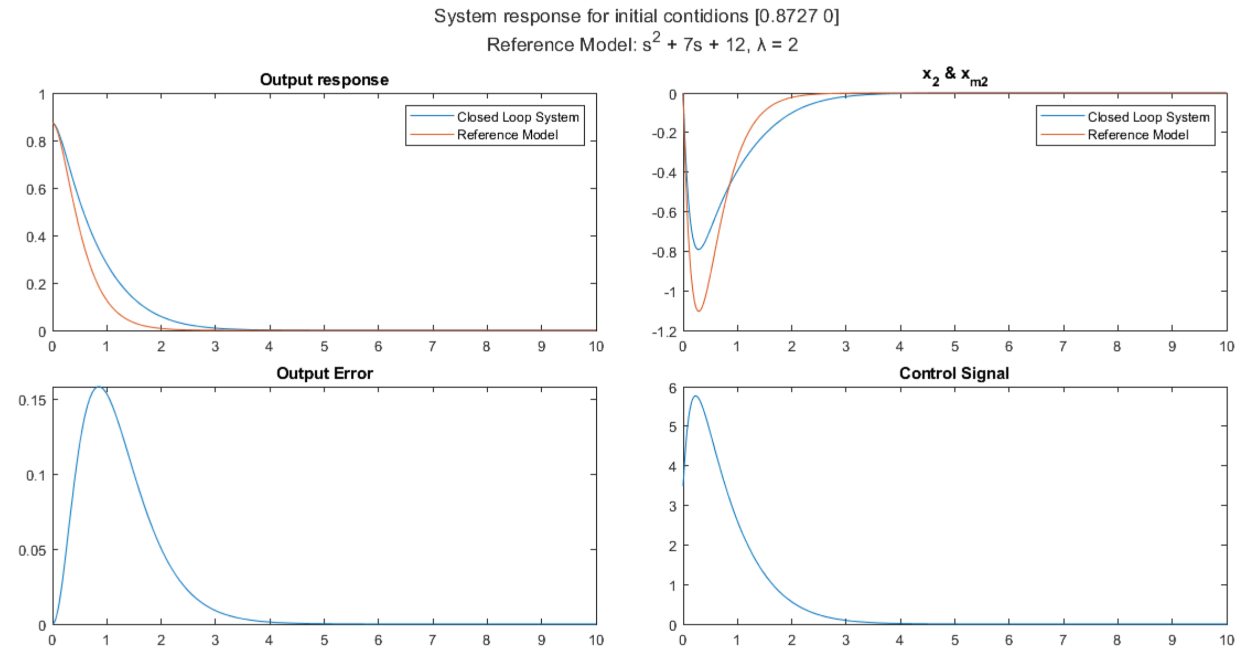
$$G_m(s) = \frac{1}{s^2 + 7s + 12}$$

For $\Lambda(s)$ we randomly choose $\lambda = 2$ and we are ready to simulate the system.

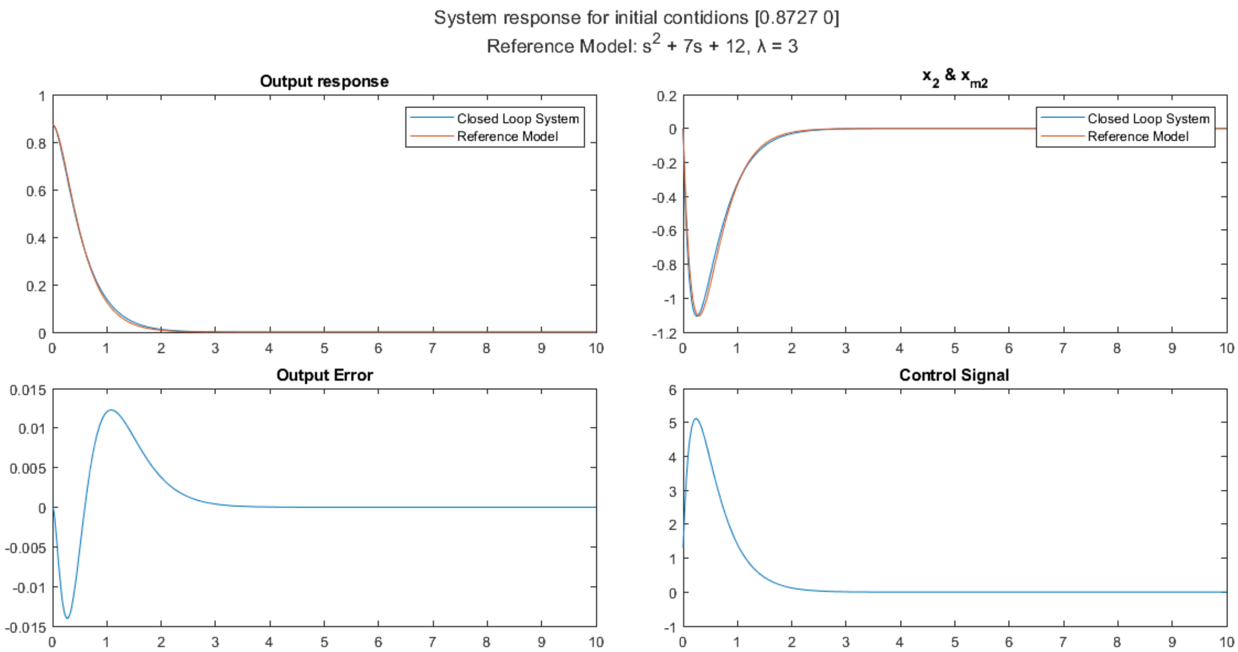
First we simulate the case where $q(0) = 0.1745rad$. Both the output of the system $(y = x_1 = q)$ and its 2nd state $(x_2 = \dot{q})$ closely track the states of the reference model $(x_{m1} = q_m, x_{m2} = \dot{q}_m)$. More specifically, the maximum error observed in the system output $(y - y_m)$ is equal to $\varepsilon_{max} = 1.1 * 10^{-3}$. It is evident that both the zero lift specification and the recovery time are met with great comfort. Moreover, observing the control input (the torque we introduce into the system) we see that it has a similar behavior to the error, it takes the maximum value at the beginning of the simulation and goes to zero when the pendulum stabilizes at the position $q = 0$. Finally, all closed-loop signals are blocked. The corresponding graphs are shown below: a) the output of the system and the reference model, b) the second state of the system and the reference model, c) the error of the output, d) the control input u.



Now, we simulate the second case, where $q(0) = 0.8727\, rad$. Again we have similar behavior to the first case, only this time x_1, x_2 follows x_{m1}, x_{m2} with a delay and generally less accuracy. Nevertheless, again the system specifications are satisfied and all signals in the closed loop are blocked.



The above restoration that we observe is due to the fact that we have a greater deflection angle than the first case. Therefore, the position change $q_m = x_{m1}$ is also larger and the closed-loop system has difficulty tracking it. To improve the behavior of the system and speed up the convergence, we can increase the value of λ. As we will deal with the change of the controller's free parameters in detail in Topic 5, the response of the system for $\lambda = 3$ is given indicatively



2.3 Scenario b

In the specific request we want the deflection angle (output of the system) to track some sinusoidal signals. In the first phase we have to calculate the reference signal r which the reference model needs to track the specific signal.

Before proceeding to the calculation of r it is obvious that the reference model $y_m = \frac{1}{s^2 + \alpha s + \beta} u$, can be easily expressed with equations of state as:

$$\mathbf{\dot{x}_m} = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix} \mathbf{x_m} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x_m} \tag{14}$$

Now let $y_d = Asin(\omega t)$ be the desired output, then:

$$y_m = y_d \Rightarrow x_{m1} = y_d = Asin(\omega t) \Rightarrow$$
$$\dot{x}_{m1} = A\omega cos(\omega t)$$

But, $\dot{x}_{m1} = x_{m2}$, so $x_{m2} = A\omega cos(\omega t)$ and $\dot{x}_{m2} = -A\omega^2 sin(\omega t)$. Therefore it is:

$$\dot{x}_{m2} = -\beta x_{m1} - \alpha x_{m2} + r \Rightarrow$$
$$-A\omega^2 sin(\omega t) = -\beta A sin(\omega t) - \alpha A\omega cos(\omega t) + r \Rightarrow$$
$$r = A(\beta - \omega^2) sin(\omega t) + A\alpha\omega cos(\omega t) \tag{15}$$

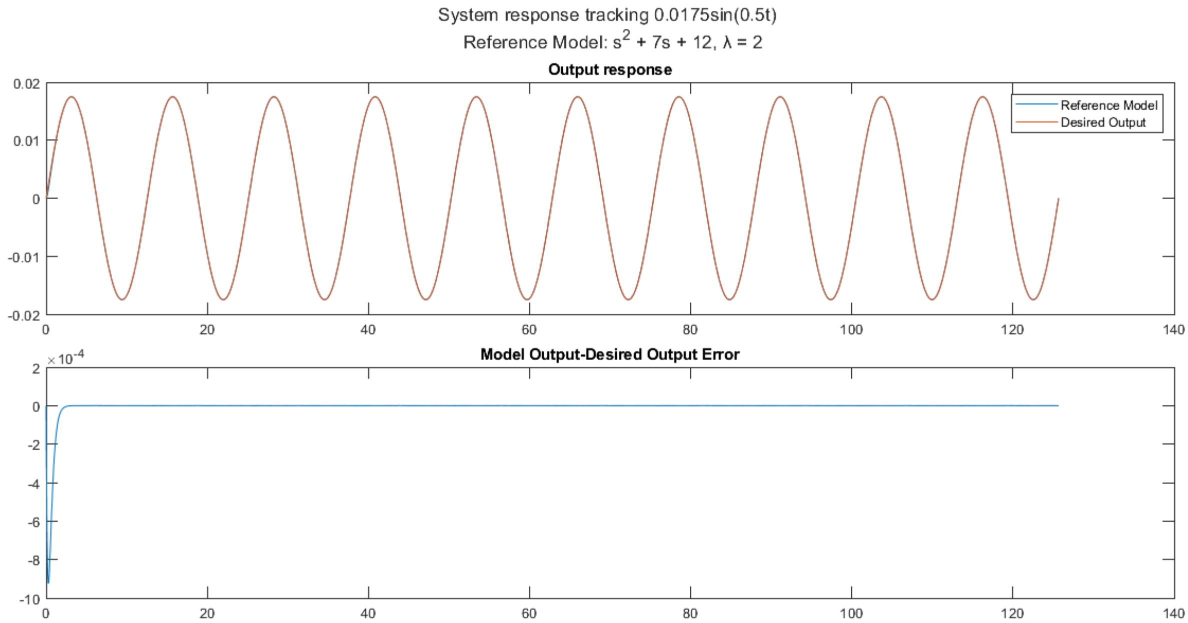
While for the initial conditions of the reference model we have:

$$\begin{cases} x_{m1}(0) = Asin(\omega \cdot 0) = 0 \\ x_{m2}(0) = A\omega cos(\omega \cdot 0) = A\omega \end{cases} \tag{16}$$

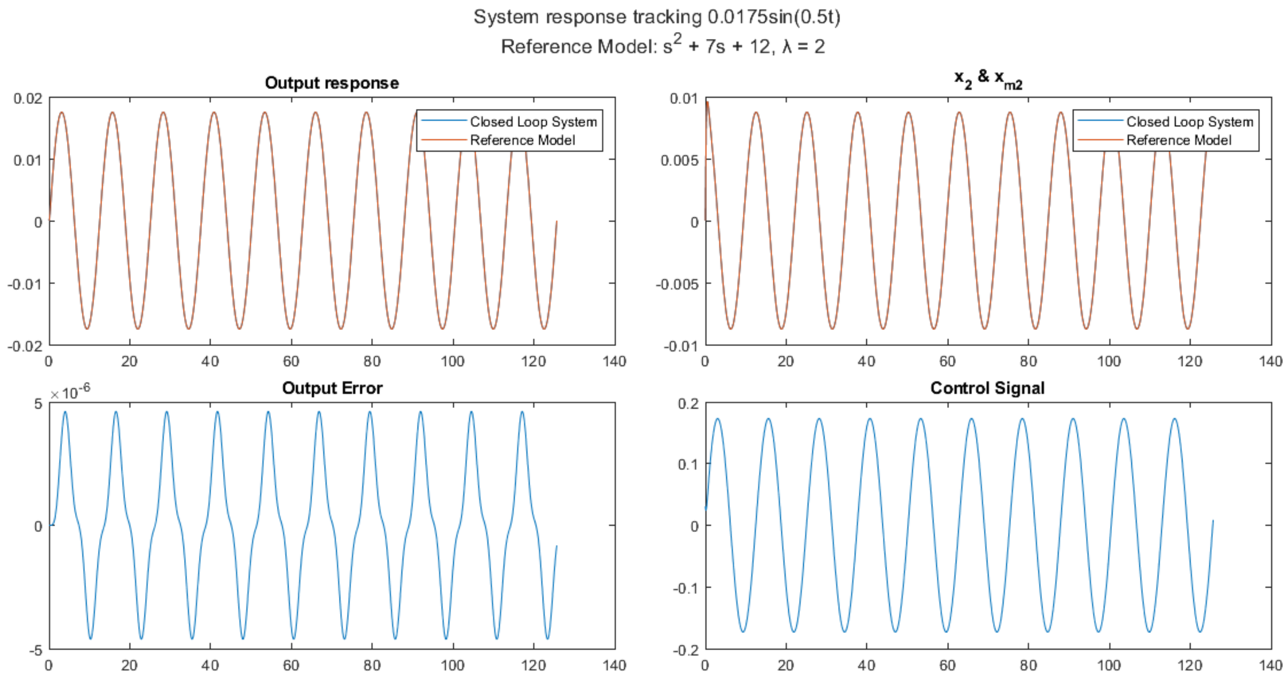
These should be the initial conditions of the reference model to have perfect tracking, however we consider that the pendulum is initially stationary, i.e. $x_m(0) = [0\,0]$. We expect this variation in initial conditions to lead to a transient before the reference model and closed-loop system follow the desired output.

Let's see what happens in practice starting with the first simulation. The desired output is $y_d = 0.0175 sin(0.5t)$, while the reference model is the same as before, $G_m(s) = \frac{1}{s^2 + 7s + 12}$, as is λ =2.

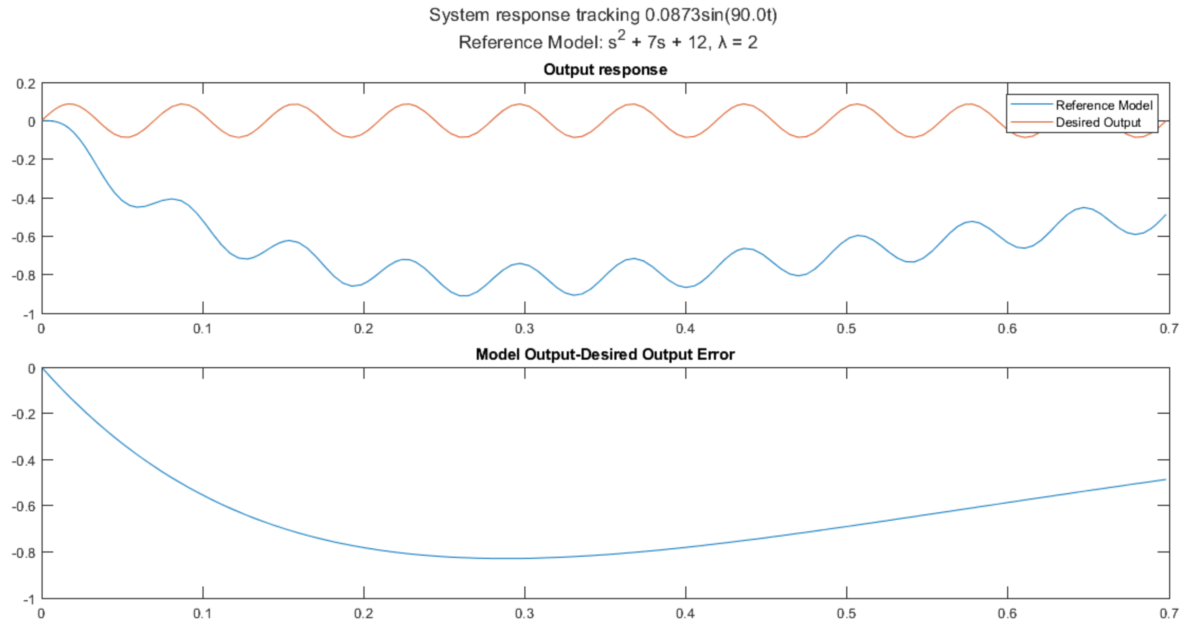
In the first phase, we simulate y_d, y_m and compare them in a time range of 10 periods. Indeed, as we expected, there is a very short transient at the beginning of the simulation, and then the reference model accurately follows y_d . Moreover, the error never exceeds the value 10^{-3}, even in the transient phenomenon. Therefore, we consider the behavior of the reference model to be sufficiently good and continue simulating the closed-loop system.



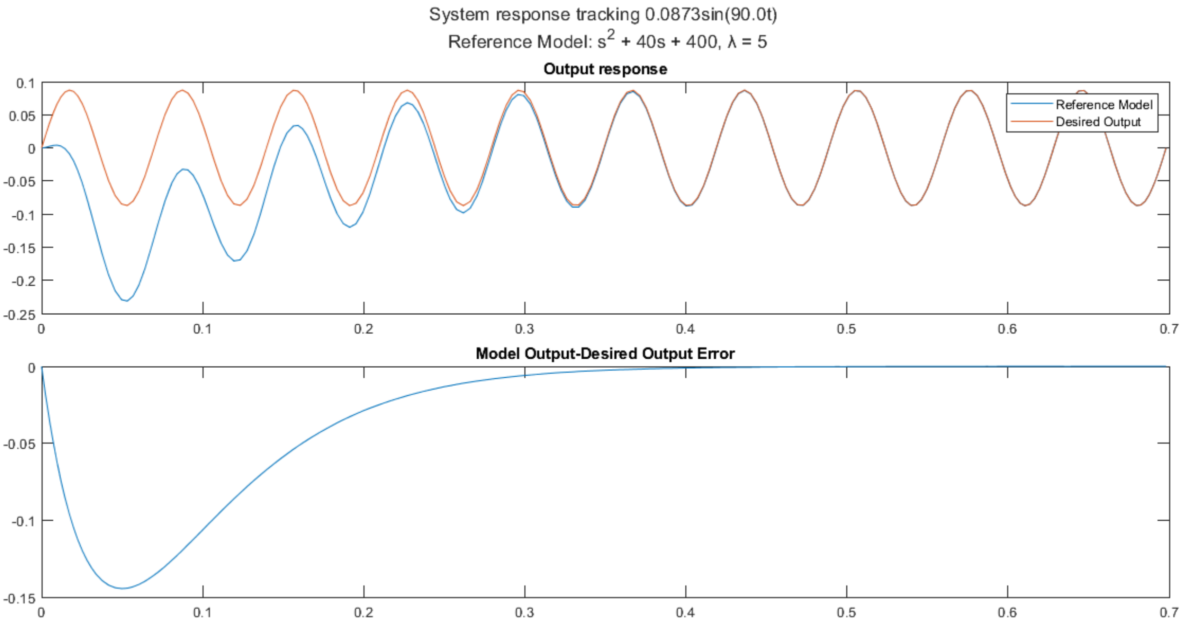
Simulating the closed-loop system for 10 periods as well, we observe that $q, \dot{q}$ coincides with $q_m, \dot{q}_m$ throughout the simulation. More specifically, we notice that there is a periodic error in the output, which is of the order of 10^{-6} and therefore we can consider it negligible. Therefore, since the output of the system accurately follows that of the reference model, which as we have seen is identical to the desired output, we have achieved the goal. Finally, we can see that the control signal is also sinusoidal, something expected since the system is linear (even in the operating range around 0).



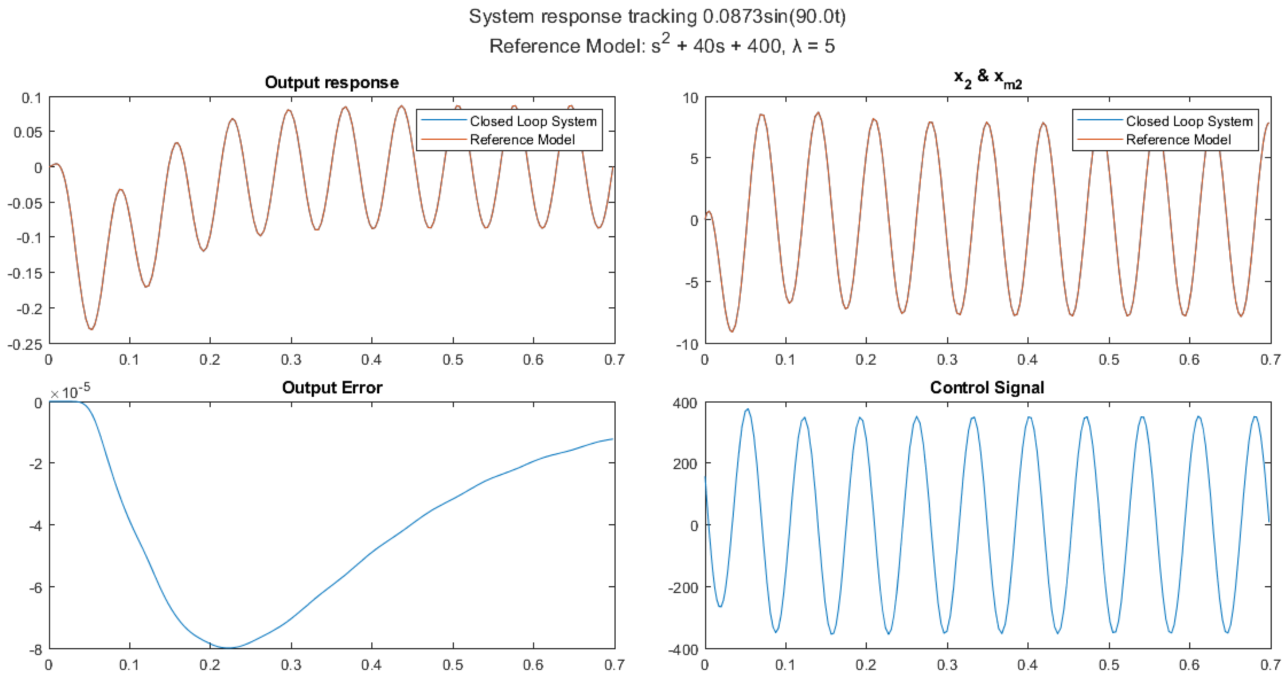
Next, we need to track $y_d = 0.0873sin(90t)$. Notice that the new sine wave has a significantly larger amplitude and frequency. To see how these changes affect, we simulate again y_d, y_m for 10 periods.



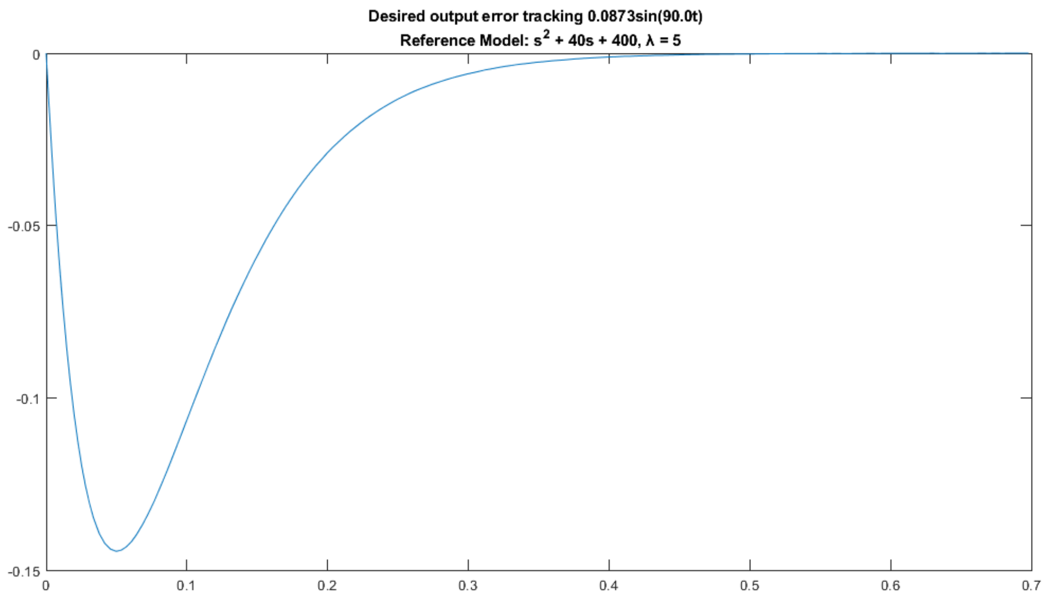
This time, due to the increased amplitude and frequency, the transient effect is larger, and the reference model does not converge to the desired output quickly enough. So, to deal with this problem, we choose a faster reference model. After a few tests we end up with a reference model with a double pole at 20 (ie $G_m(s) = 1/(s^2 + 40 + 400)$).



As can be seen in the figure above, again the transient phenomenon is more pronounced than in the first case, but now the output of the model converges to the desired output in a satisfactory time. Furthermore, before we proceed to simulate the closed-loop system, we change the value of λ to $\lambda = 5$ to help the system track the largest and fastest changes in the output.

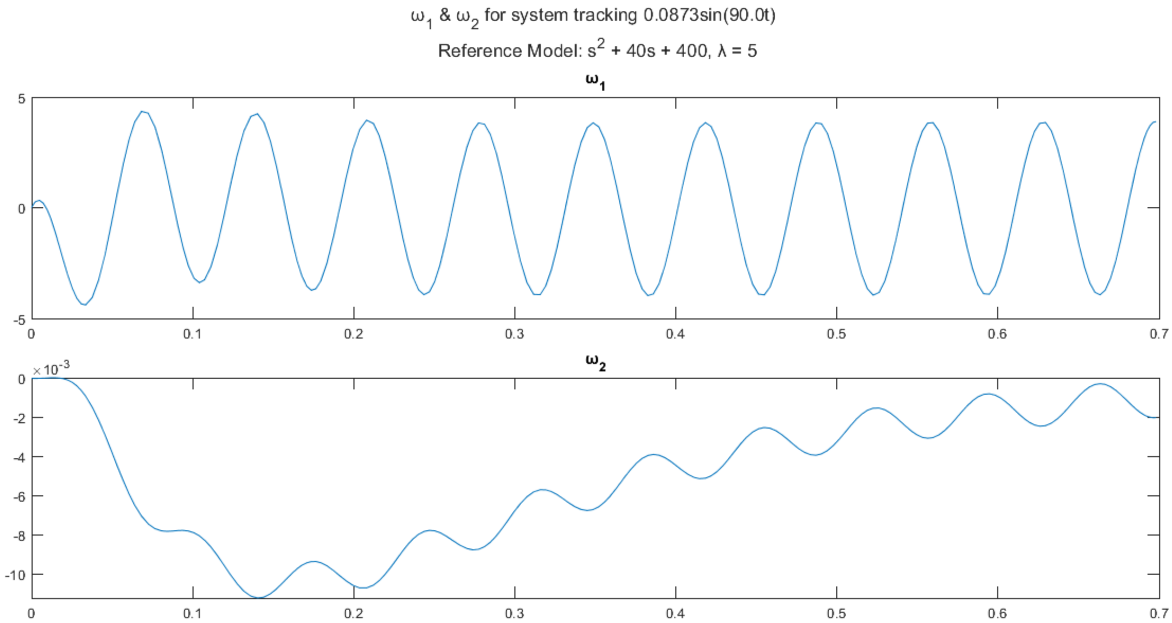


Again we see that we have an identification of $q, \dot{q}$ with $q_m, \dot{q}_m$ with the error taking values of the order of 10^{-5}. The control signal in the steady state is sinusoidal, of course having a much larger amplitude than before. In this particular case it is interesting to see the error of the output y with respect to the desired output y_d.



As we explained earlier, since the error $\varepsilon = y - y_m$ is practically negligible, the output shows an error at the beginning of the simulation, during the transient phenomenon, and after a few periods it coincides with y_d. Practically, it is $y - y_d = y_m - y_d$ which can be easily seen from the corresponding figures.

Finally, as in simulation a), all signals in the closed loop were blocked. In the graphs above, the signals x_1, x_2, u are shown, while the signals ω_1, ω_2 for the last simulation are also shown below, where $G_m(s) = 1/(s^2 + 40 + 200)$ and $\lambda = 5$.



3 State Feedback DMRAC

3.1 Design

As we have shown earlier the system close to zero has the form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\dfrac{G}{M} & -\dfrac{C}{M} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix} u,\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

, while the reference model is expressed by the equations of state:

$$\dot{\mathbf{x}}_{\mathbf{m}} = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix} \mathbf{x}_{\mathbf{m}} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_{\mathbf{m}}$$

According to the theoretical analysis of notes pp. 9-11 (Chapter 2), we must first show that there exist matrices K^*, L^*, which satisfy the conditions:

$$A - BK^* = A_m,\ BL^* = B_m$$

For the first condition we have:

$$\begin{aligned} A - BK^* &= A_m \Rightarrow \\ BK^* &= A - A_m \Rightarrow \\ \frac{B^T B}{|B|^2} K^* &= \frac{B^T}{|B|^2}(A - A_m) \Rightarrow \\ K^* &= \frac{1}{|B|^2} \begin{bmatrix} 0 & 1/M \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -G/M + \beta & -C/M + \alpha \end{bmatrix} \Rightarrow \\ K^* &= \frac{1}{|B|^2} \begin{bmatrix} \beta/M - G/M^2 & \alpha/M - C/M^2 \end{bmatrix} \end{aligned} \tag{17}$$

For the second condition:

$$\begin{aligned} BL^* &= B_m \Rightarrow \\ L^* &= \frac{B^T B_m}{|B|^2} \Rightarrow \\ L^* &= \frac{1}{|B|^2} \begin{bmatrix} 0 & 1/M \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow nonumber \qquad (18) \\ L^* &= \frac{1}{|B|^2} \frac{1}{M} \qquad (19) \end{aligned}$$

So, the arrays K^*, L^* exist and we can proceed to design the controller, which will have the form:

$$\begin{cases} u = -K(t)\mathbf{x} + L(t)r \\ \dot{K} = B_m^T P \mathbf{e} \mathbf{x}^T sgn(L^*) \\ \dot{L} = -B_m^T P \mathbf{e} r sgn(L^*) \end{cases} \tag{20}$$

, where:

- $\mathbf{e} = \mathbf{x} - \mathbf{x}_{\mathbf{m}}$, the tracking error
- $P = P^T > 0$, the solution of the equation Lyapunov $A_m^T P + P A_m = -Q$, $Q = Q^T > 0$
- $sgn(L^*) = 1$, if $L^* > 0$, while $sgn(L) = -1$, if $L^* < 0$

But, from (18) obviously $L^* > 0$, so substituting $sgn(L^*) = 1$ in (19) we get the control law:

$$\begin{cases} u = -K(t)\mathbf{x} + L(t)r \\ \dot{K} = B_m^T P\mathbf{e}\mathbf{x}^T \\ \dot{L} = -B_m^T P\mathbf{e}r \end{cases} \tag{21}$$

Before proceeding to the simulations, we analyze the equations of (20), in order to obtain the scalar quantities k_1, k_2. After some obvious operations, we arrive at the following adjustment laws of the controller's gains:

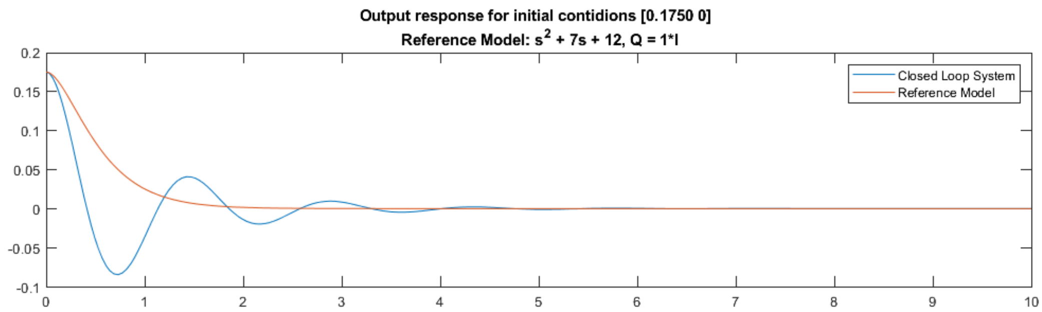
$$\begin{cases} \dot{k}_1 = P_{12}e_1x_1 + P_{22}e_2x_1 \\ \dot{k}_2 = P_{12}e_1x_2 + P_{22}e_2x_2 \\ \dot{L} = -(P_{12}e_1 + P_{22}e_2)r \end{cases} \tag{22}$$

, where $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ and $\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$.

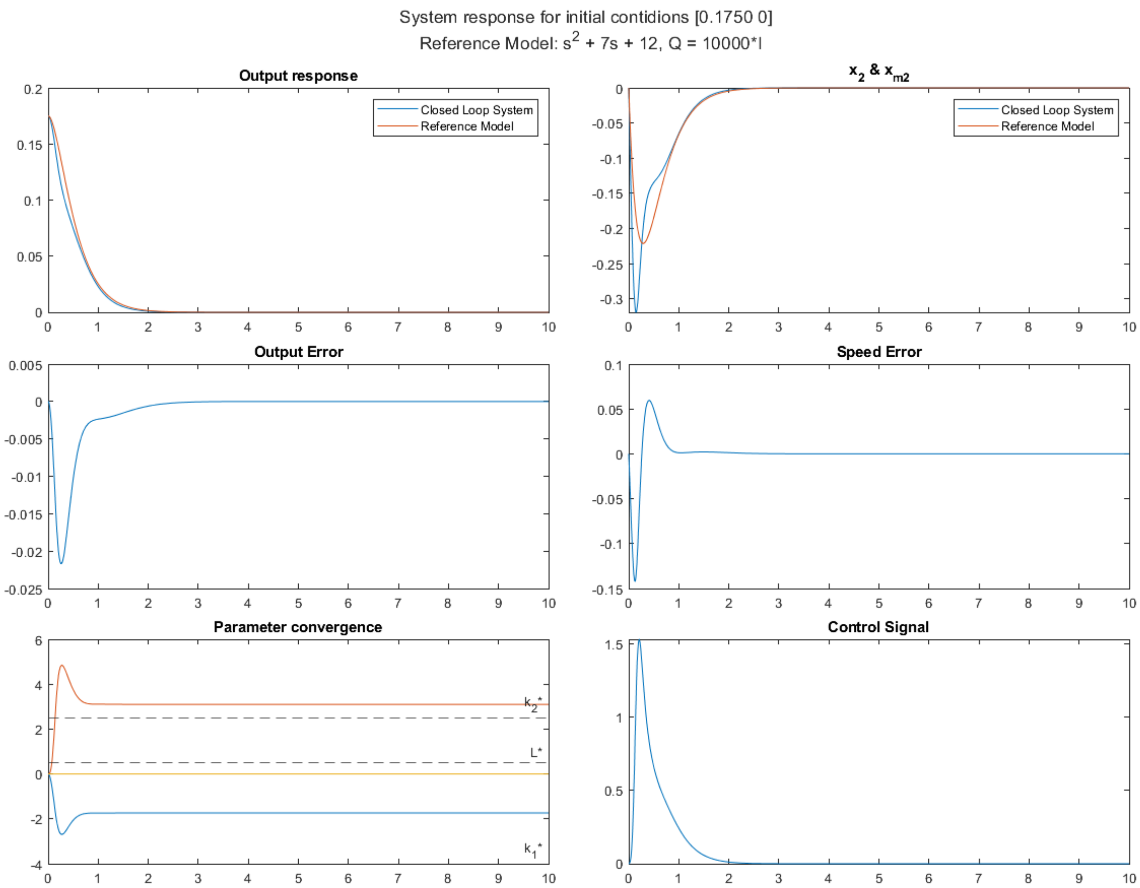
Then, in the context of the simulations, we will consider initial conditions $k_1(0) = k_2(0) = L(0) = 0$, assuming that we have no information about the values K^*, L^*.

3.2 Scenario a

In this particular question we need to simulate the same scenario as Topic 2.a simply using the adaptive state feedback controller we designed above. We start our tests with the first case, where $q(0) = 0.1745\,rad$. To determine the matrix P, we use $Q = I_2$. Below is the output of the system relative to the output of the reference model.

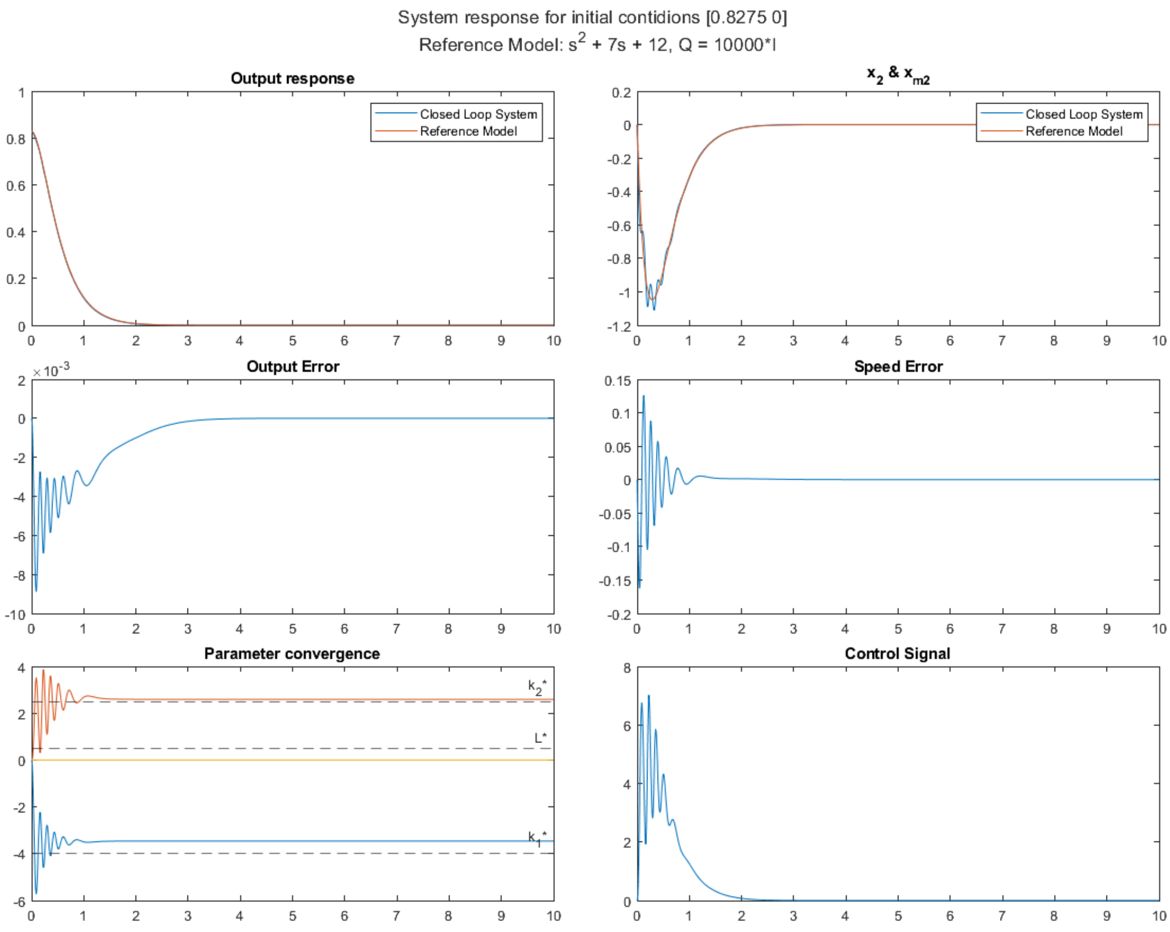


We notice that the system in this particular case exhibits overshoot, and in general has difficulty following the reference model. This is probably due to the value that the pincer P takes. Indeed, printing the matrix P obtained by solving the equation Lyapunov $P = \begin{bmatrix} 1.22 & 0.042 \\ 0.042 & 0.078 \end{bmatrix}$, we see that the elements P_{12}, P_{22}, which determine the rate of change of the controller's profits, as shown in relation (21), take very small values. So we increase the value of Q and after a few trials we end up with a value of $10^4 \cdot I$ for which we have good behavior of the closed-loop system.



It is clear that with the new Q (4 orders of magnitude larger) we achieve very accurate tracking of the reference model, satisfying the zero overshoot and recovery time specifications. The error of the output takes its maximum value shortly after the start of the simulation, in the transient phenomenon, and then goes to zero after the pendulum has balanced. A similar behavior is also displayed by the pendulum speed error, which of course receives higher values than those of the output error. The control signal shows a similar behavior to the reference model controller of Subject 2, initially having an increasing trend, while decreasing as we approach the equilibrium point. Regarding the controller gains, we see that k_1, k_2 move towards the values ${k_1}^*, {k_2}^*$, but end up in wrong values, while L remains constant and equal to $L(0) = 0$, since we have $r = 0 \Rightarrow \dot{L} = 0$. Finally, apparently all signals circulating in the closed loop are blocked.

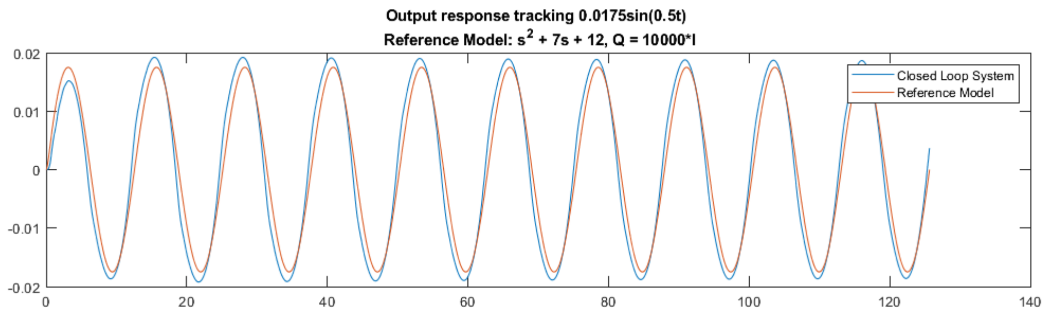
We continue with the simulation for $q(0) = 0.8275\,rad$. We again use $Q = 10^4 \cdot I$.



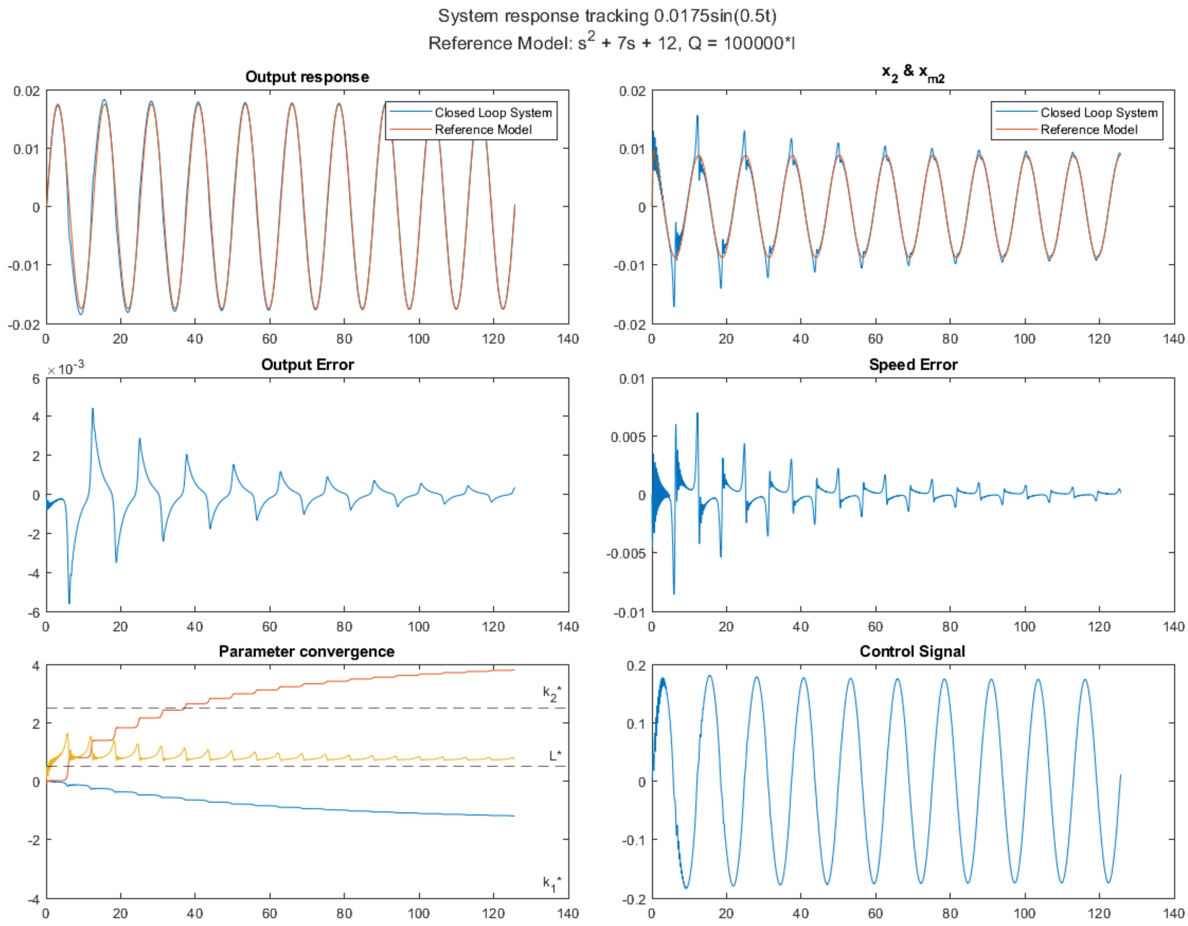
First, we observe that the given specifications are again satisfied, while in this particular case we also have better tracking with smaller errors, in both the 2 states of the system. Moreover, k_1, k_2 result in values quite close to ${k_1}^*, {k_2}^*$, while L, obviously, remains constant and equal to 0. The above leads us to the conclusion that we have better behavior of the system in this case, but looking more closely at all the graphs, we notice that the closed loop signals show fluctuations/oscillations that in the long run can cause damage to the system.

3.3 Scenario b

In this case we want the output of the system to track some sine, just like in scenario b of topic 2. So we use the same input that we proved above to be necessary, $r = A(\beta - \omega^2)sin(\omega t) + Aa\omega cos(\omega t)$. We start our tests for $y_d = 0.0175sin(0.5t)$. Based on the results of scenario a, we first simulate the system for $10^4 \cdot I$.

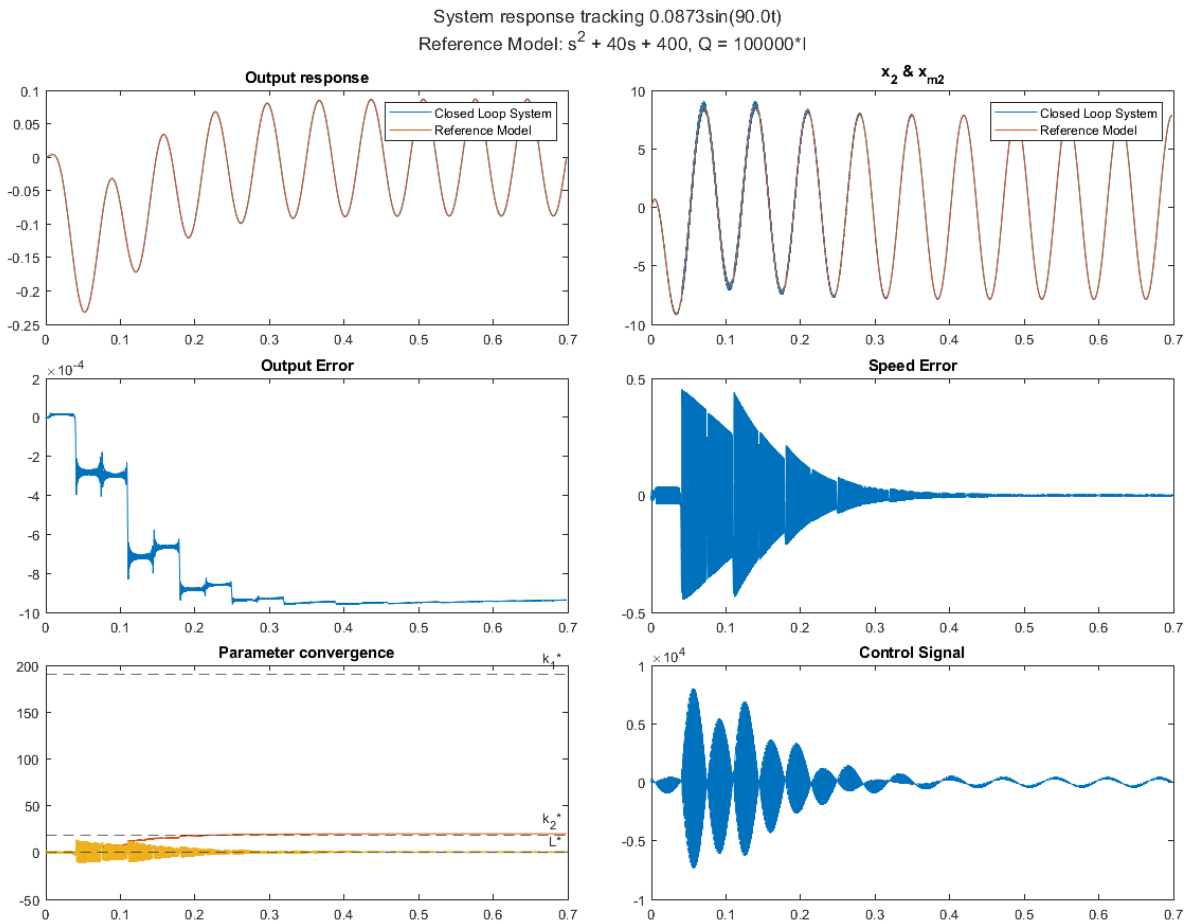


As shown above, the system output has a sinusoidal shape, but it does not accurately track the desired signal, so we increase Q to 10^5 and repeat the simulation.



Now the system output tracks the reference model very accurately. More specifically, the output error takes a maximum value of $\approx 5 * 10^{-3}$ and over time tends to zero. A similar behavior is displayed by the pendulum speed error. The controller's profits seem to converge to some values which, of course, do not coincide with the ideal values. Finally, the control signal has a sinusoidal form with some small disturbances during the first period.

Continuing, the output should now track $y_d = 0.0873sin(90t)$. Taking advantage of the tests done in Topic 2, we will again use the reference model $G_m(s) = 1/(s^2 + 40 + 400)$ in order to speed up the transient effect.



Again, we have good tracking of y_d with sufficiently small error. In this case, the transient effect is more intense in all signals, something to be expected. In terms of controller gains, 2 out of 3 are in a region close to optimal values, while the 3rd is far away. The control input takes large values, for the transient up to 8000, which could be problematic in the real world, where we could not apply torque indefinitely. Finally, observing the figures we see that the closed loop signals perform very high frequency oscillations, which as commented above is a concern.

4 Output Feedback DMRAC

4.1 Design

Although in this particular Topic the parameters of the system are considered unknown and again the necessary assumptions of page 12 of the notes apply, as in Topic 2.

We again choose a stable filter $n-1=2-1=1^{th}$ of order $\Lambda(s)=s+\lambda,\ \lambda>0$, but also a stable filter 1^{th} of order $P(s)=s+p_0,\ p_0>0$.

Before proceeding to the formulation of the control law, we must show that the transfer function $G_\varphi(s)=(s+p_0)*G_m(s)=\dfrac{s+p_0}{s^2+\alpha s+\beta}$ is textbfStrictly Positive Real.

a) $G_\varphi(s)$ is analytic with respect to $\sigma \geq 0$ as an explicit function.

b) $Re\{G_\varphi(j\omega)\}>0, \forall\omega \in \mathbf{R}$.

First we calculate $G_\varphi(j\omega)$:

$$
\begin{aligned}
G_\varphi(j\omega) &= \frac{j\omega+p_0}{(j\omega)^2+\alpha j\omega+\beta} = \frac{j\omega+p_0}{\beta-\omega^2+j\alpha\omega} \\
&= \frac{(j\omega+p_0)(\beta-\omega^2-j\alpha\omega)}{(\beta-\omega^2+j\alpha\omega)(\beta-\omega^2-j\alpha\omega)} \\
&= \frac{j(b\omega-\omega^3)+a\omega^2+p_0\beta-p_0\omega^2-jp_0\alpha\omega}{(\beta-\omega^2)^2-(ja\omega)^2} \\
&= \frac{(\alpha-p_0)\omega^2+p_0\beta+j(\beta\omega-\omega^3-p_0\alpha\omega)}{(\beta-\omega^2)^2+\alpha^2\omega^2}
\end{aligned}
$$

So:

$$
\begin{gathered}
Re\{G_\varphi(j\omega)\}>0 \Rightarrow \\
\frac{(\alpha-p_0)\omega^2+p_0\beta}{(\beta-\omega^2)^2+\alpha^2\omega^2}>0 \Rightarrow \\
(\alpha-p_0)\omega^2+p_0\beta>0 \Rightarrow \\
\begin{cases} \alpha-p_0>0 \\ \Delta=-4(\alpha-p_0)p_0\beta<0 \end{cases} \quad \Rightarrow \\
p_0<a
\end{gathered}
$$

c)

$$
\begin{aligned}
\lim_{|\omega|\to\infty} \omega^2 Re\{G_\varphi(j\omega)\} &= \lim_{|\omega|\to\infty} \omega^2 \frac{(\alpha-p_0)\omega^2+p_0\beta}{(\beta-\omega^2)^2+\alpha^2\omega^2} \\
&= \lim_{|\omega|\to\infty} \frac{(\alpha-p_0)\omega^4}{\omega^4} \\
&= \alpha-p_0>0, \text{ from b).}
\end{aligned}
$$

Therefore, $G_\varphi(s)$ is Strictly Positive Real since $p_0<\alpha$, which we will consider later when we choose the free parameters of the controller.

Having done the above analysis we can substitute in the control law on page 22 (Chapter 2):

$$\begin{cases} u = \theta^T \omega - \varphi^T \Gamma e \varphi \\ \dot{\omega_1} = -\lambda\omega_1 + u, \omega_1(0) = 0 \\ \dot{\omega_2} = -\lambda\omega_2 + y, \omega_2(0) = 0 \\ \dot{\varphi} = -p_0\varphi + \omega, \varphi(0) = \mathbf{0} \in \mathbf{R^{4x1}} \\ \dot{\theta} = -\Gamma\varepsilon\varphi sgn(k_p/k_m), \theta(0) = \mathbf{0} \in \mathbf{R^{4x1}} \end{cases}$$

, where:

- $\varepsilon = y - y_m$, the tracking error of the output
- $\omega = \begin{bmatrix} \omega_1 & \omega_2 & y & r \end{bmatrix}^T$
- $\theta = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & c_0 \end{bmatrix}^T$
- $\Gamma = \Gamma^T > 0$
- $sgn(k_p/k_m) = sgn(1/M) = 1$

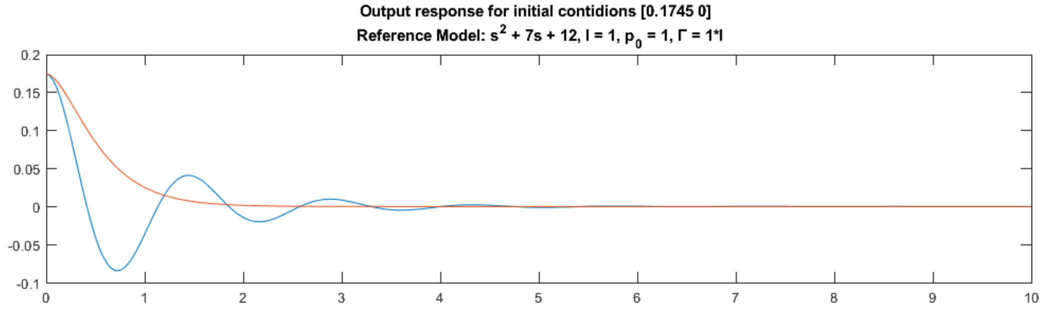
So, finally we have:

$$\begin{cases} u = \theta^T \omega - \varphi^T \Gamma e \varphi \\ \dot{\omega_1} = -\lambda\omega_1 + u, \omega_1(0) = 0 \\ \dot{\omega_2} = -\lambda\omega_2 + y, \omega_2(0) = 0 \\ \dot{\varphi} = -p_0\varphi + \omega, \varphi(0) = \mathbf{0} \in \mathbf{R^{4x1}} \\ \dot{\theta} = -\Gamma\varepsilon\varphi, \theta(0) = \mathbf{0} \in \mathbf{R^{4x1}} \end{cases} \tag{23}$$

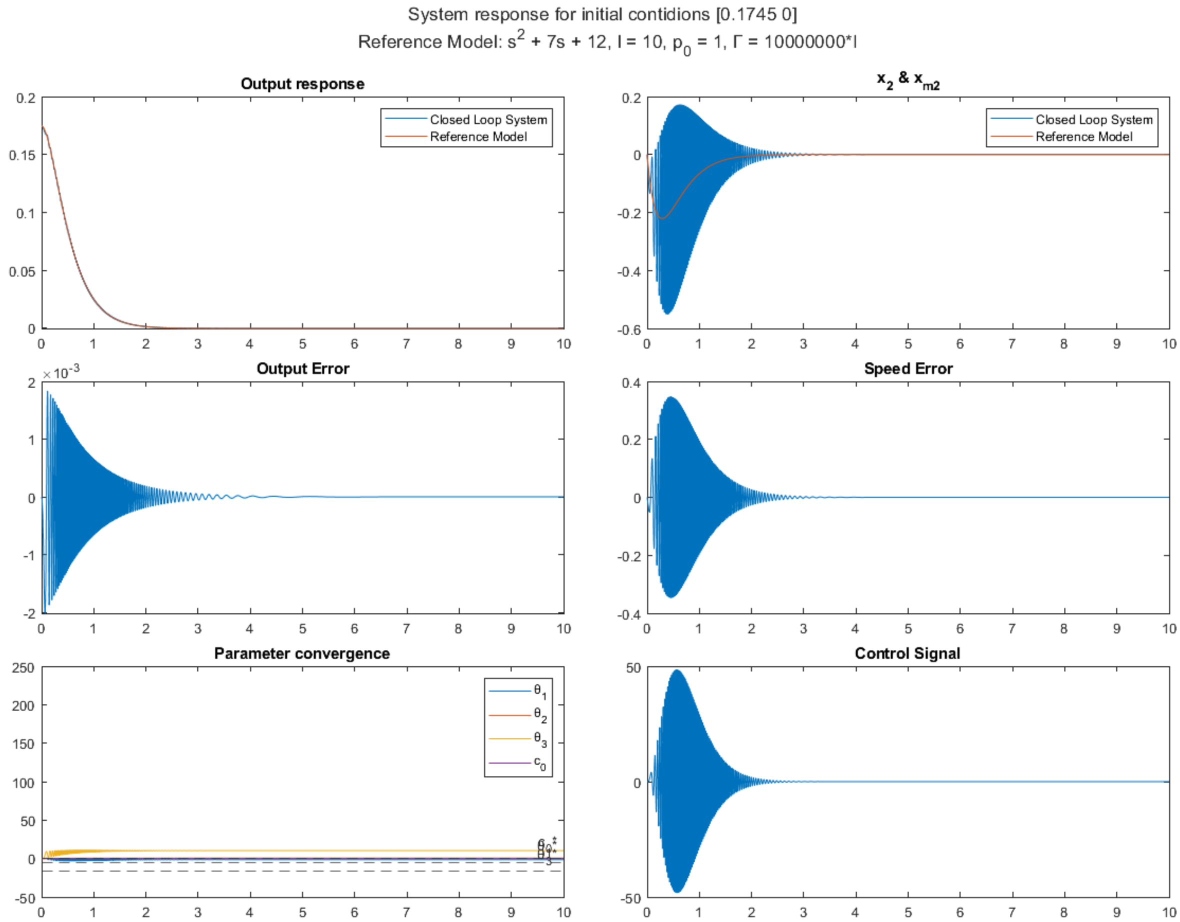
Finally, we have chosen for the initial conditions of φ, θ to be zero, which is not necessary. More specifically if we had some knowledge about the region to which θ^* belongs we could adjust the initial conditions appropriately (without of course ensuring that θ will interfere with the starting region).

4.2 Scenario a

We simulate again the case where the pendulum starts from a non-zero position, with zero velocity and without a reference input r. We start with the case where $q(0) = 0.1745\, rad$, using the reference model $G_m(s) = 1/(s^2 + 7s + 12)$ while testing the values $\lambda = 1, p_0 = 1, C = I_4$. According to the above analysis, the specific value of p_0 is permissible, since it satisfies the condition $p_0 < \alpha$.

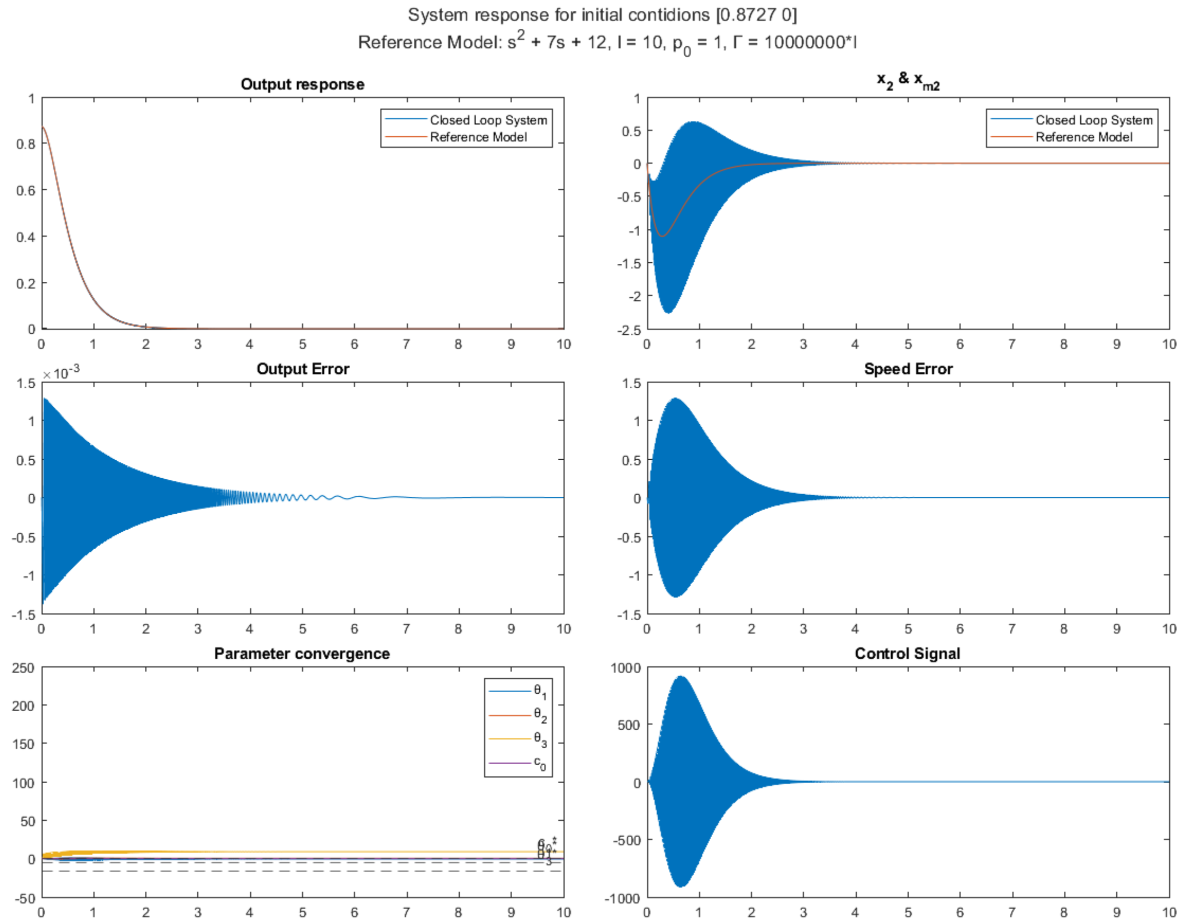


As shown above, for the values of the free parameters used, the controller does not have much effect on the system and fails to track the reference model. This is mainly due to the small values of the matrix Γ, but by doing tests we also change the values of λ, p_0 and we end up with the following values for which the response of the system is shown below: $\lambda = 10, p_0 = 1, \Gamma = 10^7 \cdot I$.



With the new controller parameters, tracking of the reference model is achieved while also satisfying the specifications for the closed loop system. More specifically, the error of the output never exceeds the value $2 * 10^{-3}$, while in the steady state it is practically zero. With greater difficulty, the system follows the speed (x_2), while the values of θ do not approach those of θ^*. Finally, it is clear that all signals in the closed loop oscillate rapidly during the transient, behavior that can cause damage in the real world.

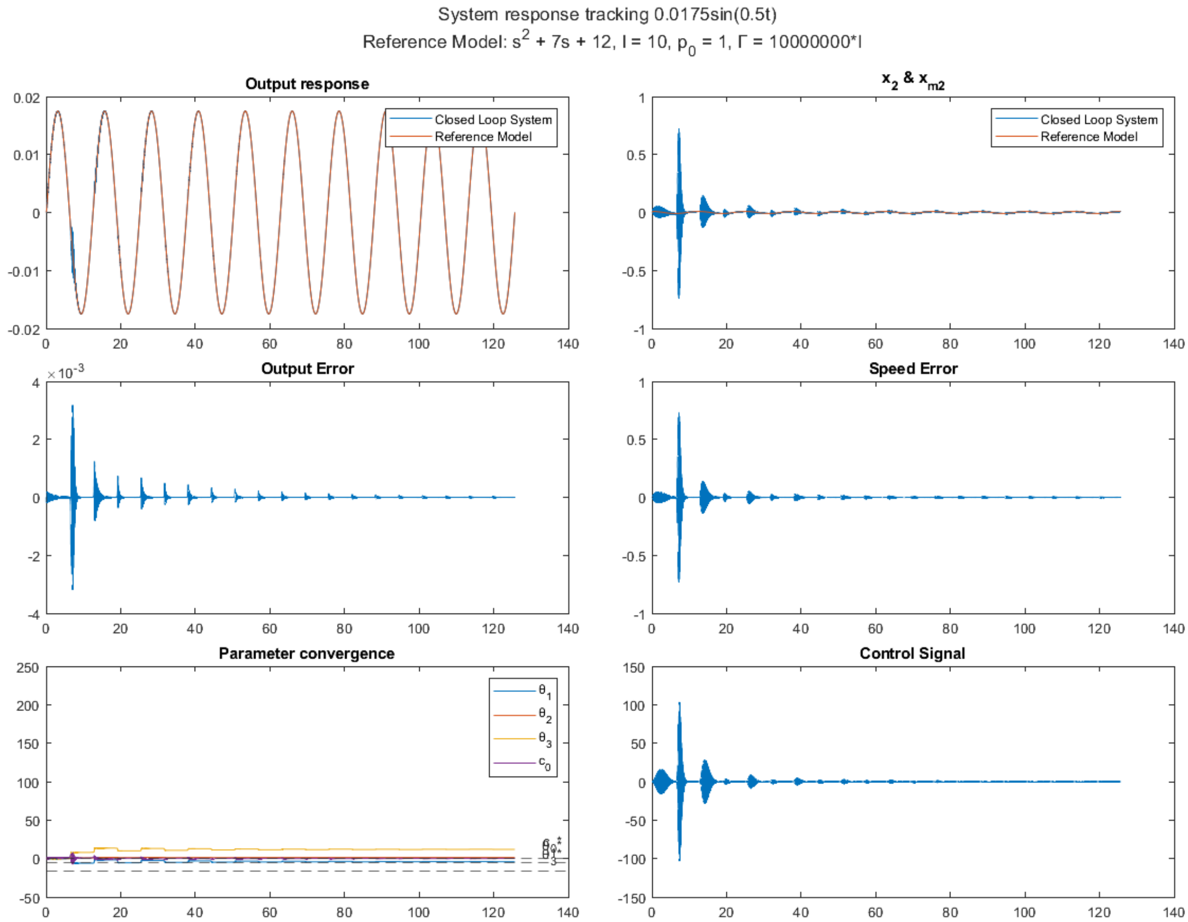
We continue with the second case, where $q(0) = 0.8727\, rad$. For the reference model and the free parameters of the controller we use the same parameters as above.



The behavior of the system is identical to the previous case. We have good tracking and specification satisfaction, while the θ parameters end up at "random" values and there is the disturbing phenomenon of strong oscillations.

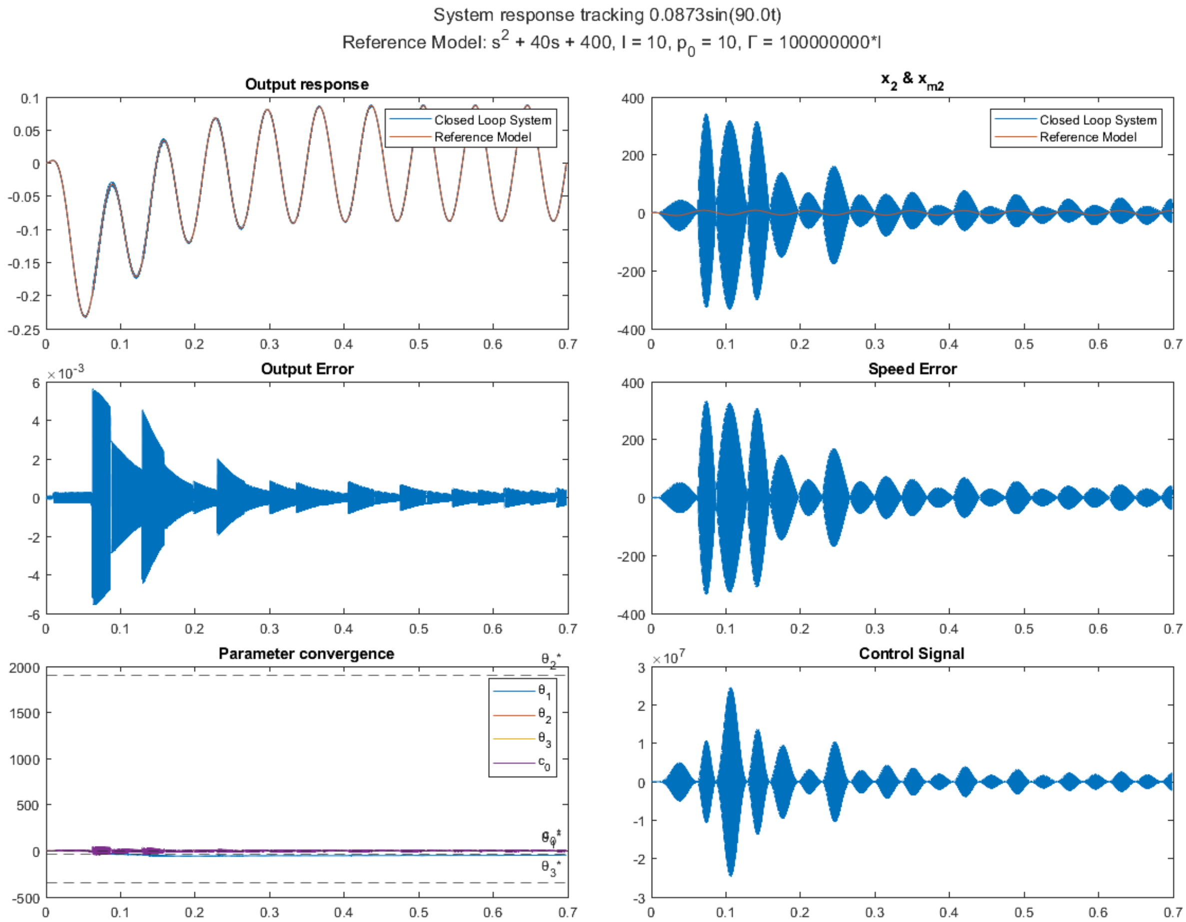
4.3 Scenario b

In this case we want the output of the system to track some sine, just like in scenario b of subjects 2,3. Thus we use the same input that we proved above to be necessary, $r = A(\beta - \omega^2)sin(\omega t) + A\alpha\omega cos(\omega t)$. We start our tests for $y_d = 0.0175sin(0.5t)$. We use reference model $G_m(s) = 1/(s^2 + 7s + 12)$, while based on the results of scenario a, we simulate the system for $\lambda = 10, p_0 = 1, \Gamma = 10^7 \cdot I$.



We observe that the output of the system follows the reference model with satisfactory accuracy. Both output and velocity error are practically zero after the second period of the sine. Once again the adaptive controller gains fail to approach θ^* while the closed-loop signals exhibit oscillatory behavior (in addition to the necessary sinusoidal excitation and response).

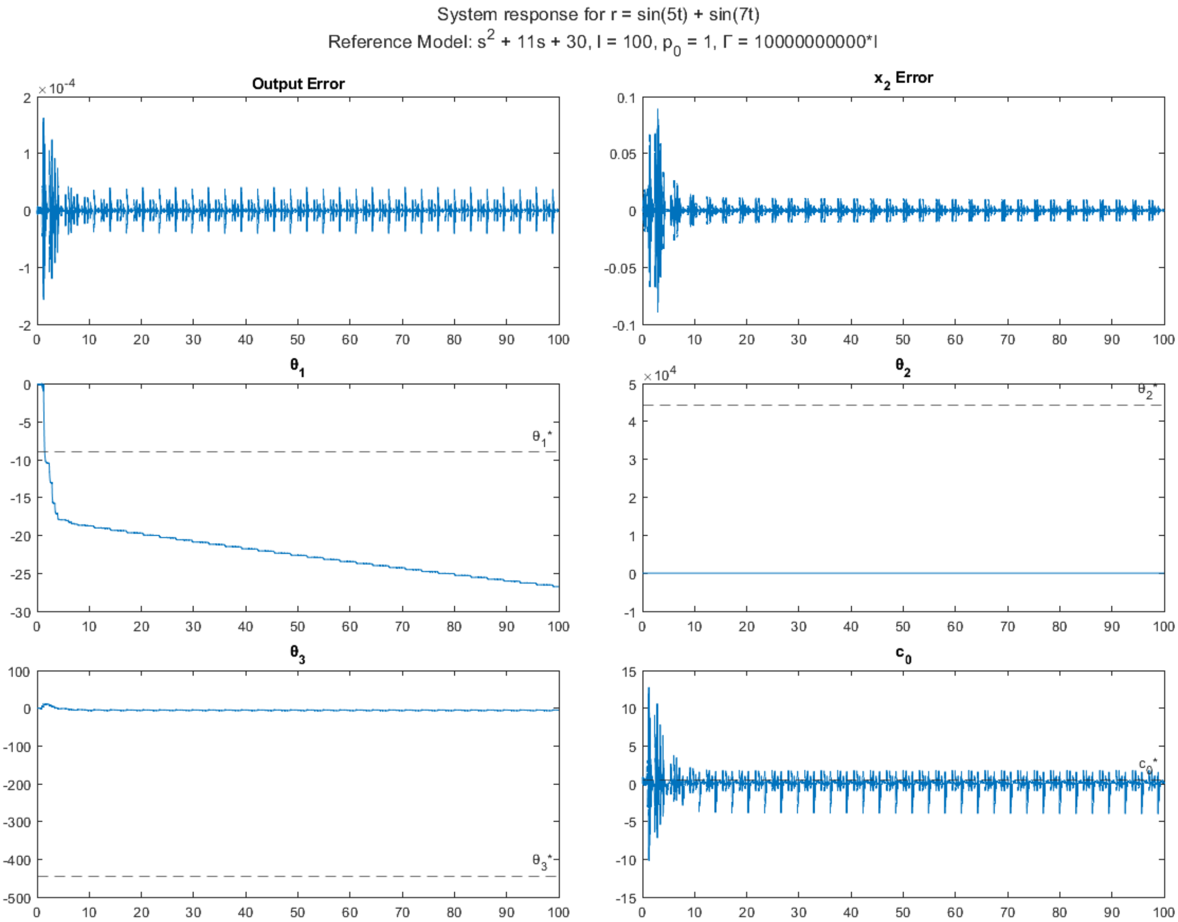
Continuing, the output should now track $y_d = 0.0873sin(90t)$. Taking advantage of the tests done in Topic 2, we will again use the reference model $G_m(s) = 1/(s^2 + 40 + 400)$ in order to speed up the transient effect.



Again, we have good tracking of y_d with sufficiently small error. In this case, the transient effect is more intense in all signals, something to be expected. Of course, in this particular case the speed error never goes to zero, but on the contrary takes large values. The parameters θ move again in a region far from θ^*, while in addition to the phenomenon of strong oscillations, the problem of the huge values taken by the control signal reappears ($\approx 3 * 10^7$). As we explained before, this is a problem because under normal conditions for the input u we would use a rotor, which would have limited torque capabilities.

4.4 Persistent Excitation Parametric Convergence

Although in the above simulations we managed to minimize the system error, in no case did we achieve the convergence of the parameters in θ^*. According to the theory, a signal r sufficiently rich $2n$ is able to ensure the convergence of $\tilde{\theta} \to 0$. We choose the signal $r = 0.05sin(5t) + 0.05sin(7t)$, which is obviously rich enough $2n$. The ω of the sinuses have been chosen 5.7 in order to speed up the parametric convergence a bit, while their amplitudes have been adjusted so that we do not stray far from the neighborhood of zero, where the linearization approximation is good. Below are the simulation results:



Although, we expected to have exponential convergence $\tilde{\theta} \to 0$ something like this does not happen. In fact, some of the parameters have not even managed to stabilize, but by increasing the simulation time we do not see a significant difference. A second consideration is that despite the careful choice of r, the error introduced by the linearized model does not allow us to approximate θ^*. However, by simulating the linearized system, instead of the real one, we get identical behavior. Thus we conclude that the complexity of the controller, in combination with the discrete-time nature of the simulation, do not allow us to guarantee parametric convergence.

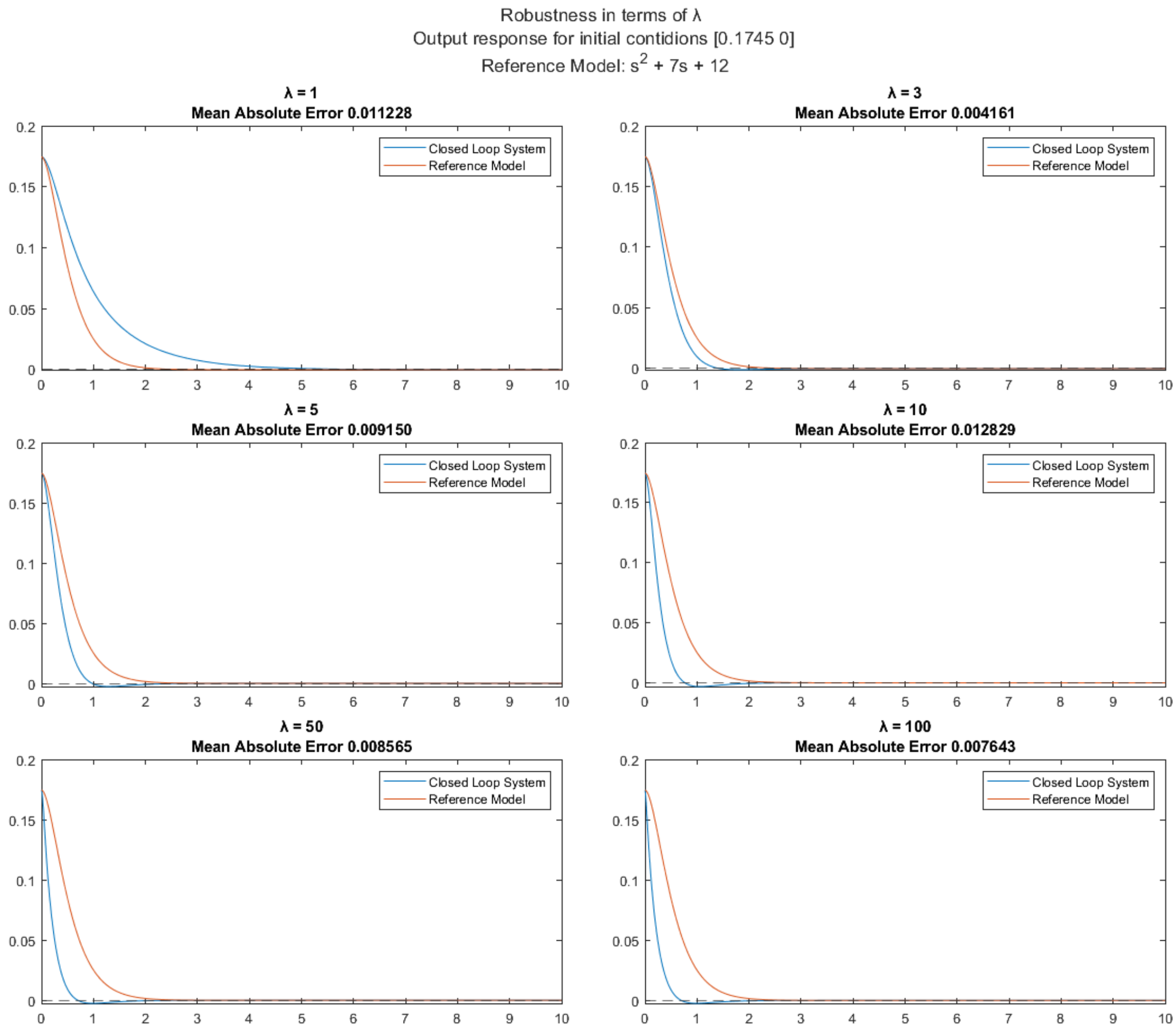
5 Robustness

In this specific topic, we are asked to perform a robustness study (through simulations) for the 3 controllers studied, in terms of changes in the free parameters of each controller. For each of the controllers, the 4 simulations made in subjects 2,3,4 (scenario a, scenario b) will be used. For the best presentation of the results and drawing conclusions, for scenario a we use a graph, where the system and model output responses are presented, while for scenario b a graph with the error between the 2 outputs ($\varepsilon = y - y_m$) . In addition, the mean absolute error is used as an aid in evaluating/comparing responses.

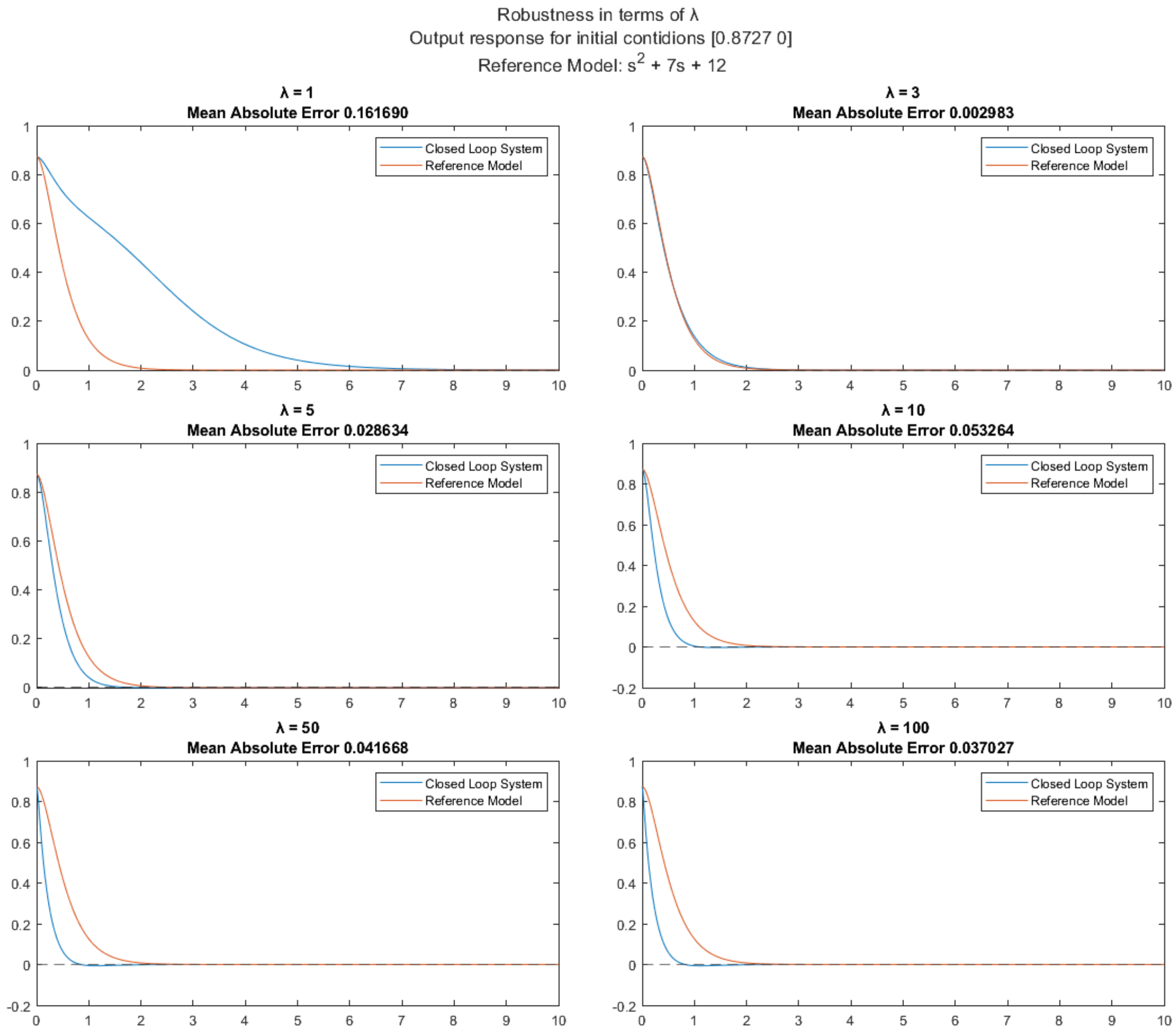
5.1 Reference Model Controller

5.1.1 Scenario a

Robustness to λ



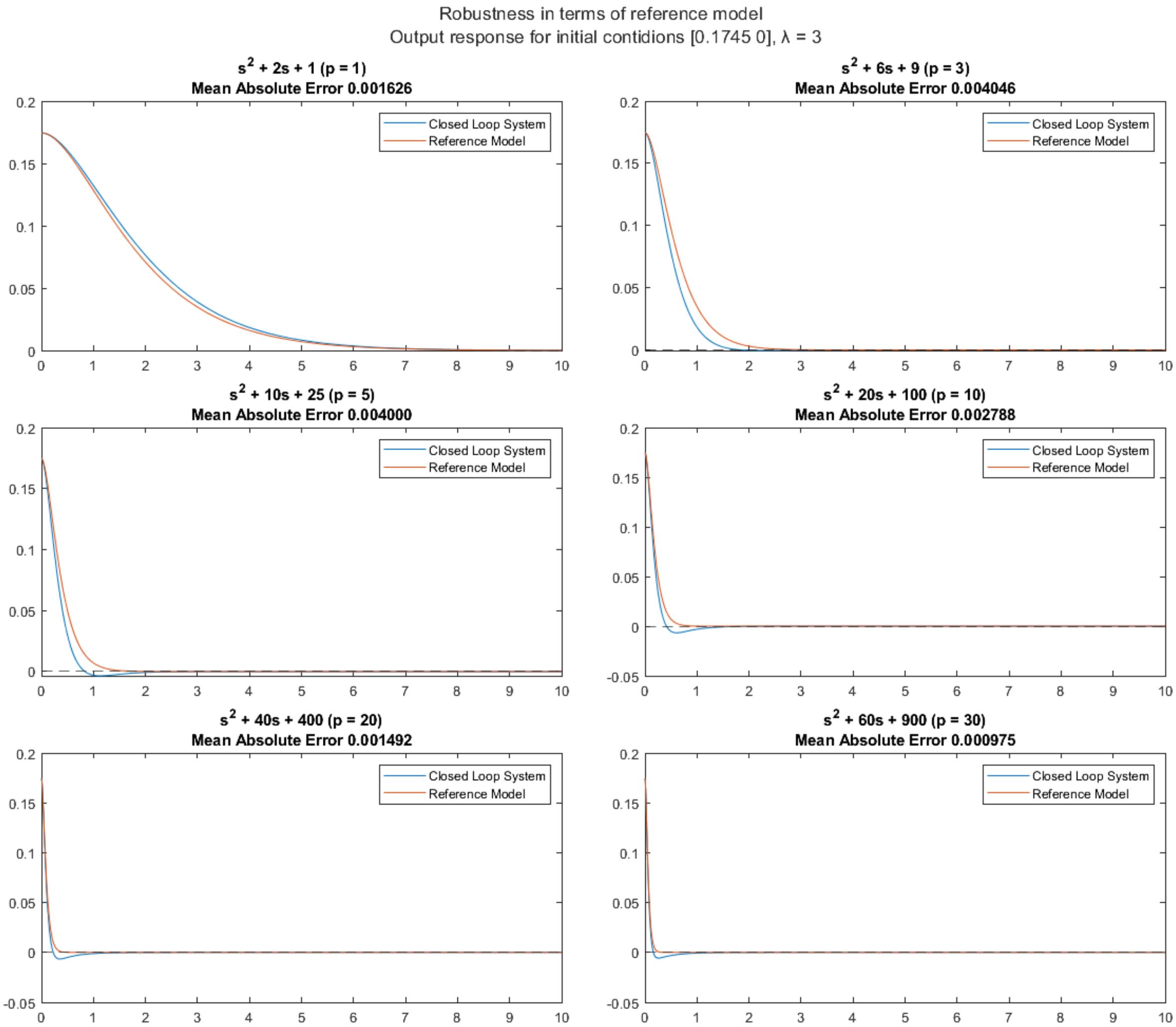
We start the robustness study with respect to λ and specifically with the case where $q(0) = 0.1745rad$. For the value $\lambda = 1$ we observe that the system satisfies our specifications, following the model with a small delay. Increasing λ the system becomes faster and faster, converging to 0 faster than the reference model. It is also certain that for $\lambda \geq 5$ an elevation occurs, while based on the average absolute error the best case is for $\lambda = 3$.



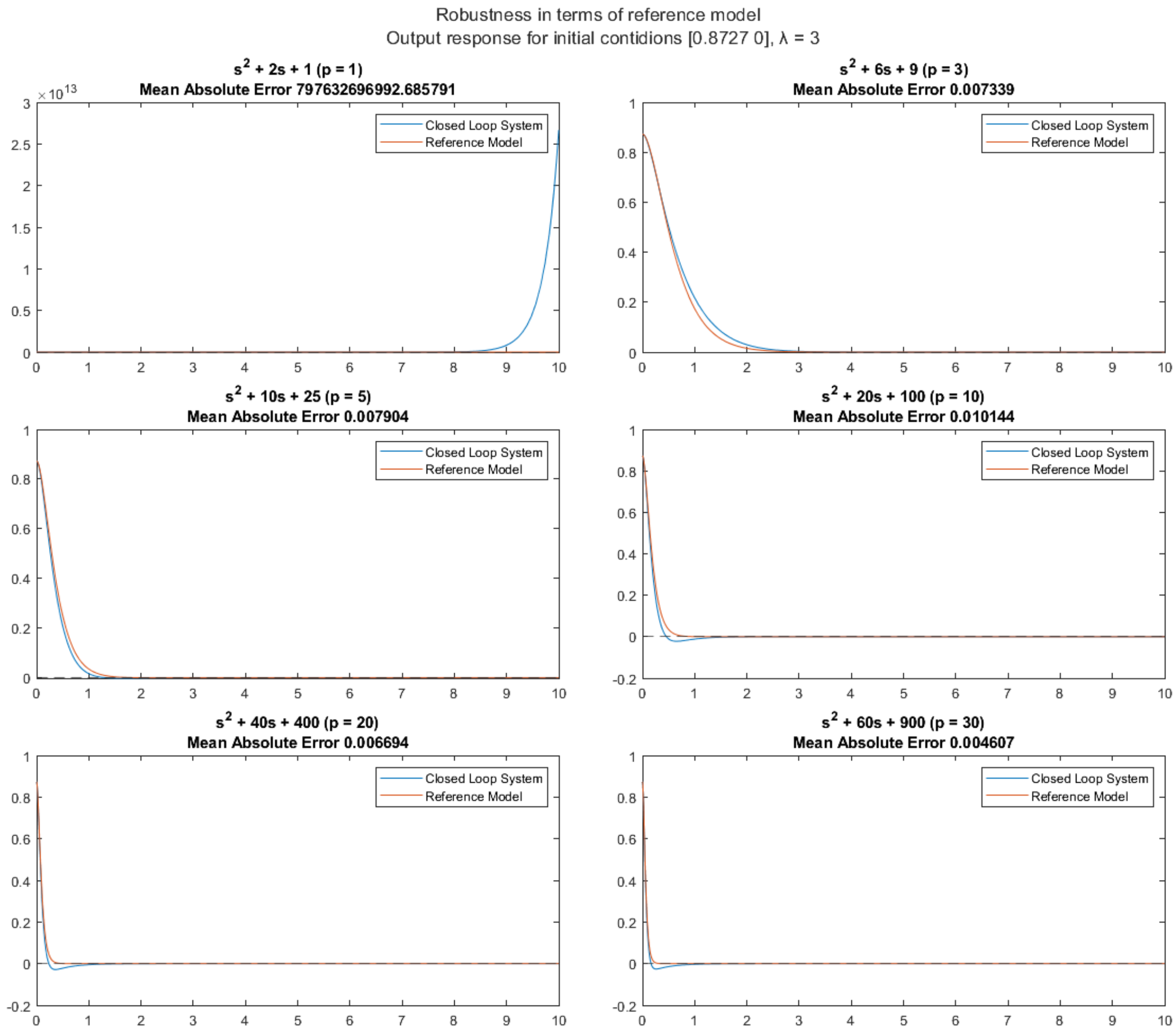
And in the case where $q(0) = 0.8727rad$ we observe a similar behavior. The main differences are that, for $\lambda = 1$ we have a much slower output response (of course it continues to meet the specifications), while overshoot occurs for $\lambda > 10$. Again the minimum mean error occurs in the case where $\lambda = 3$.

Robustness to reference model

At this point we study the robustness of the system with respect to the reference model. More specifically, we assume that we have a transfer function with a double pole, without loss of generality (since in any case the dominant pole prevails), and we observe the response of the output to changes in the value of the double pole.



As we expected, moving the poles away from 0 leads to an acceleration of the response of both the model and the system, while for all cases ($p \in [1, 30]$) the recovery time specification is satisfied, for values of $p \geq 5$ superelevation occurs.

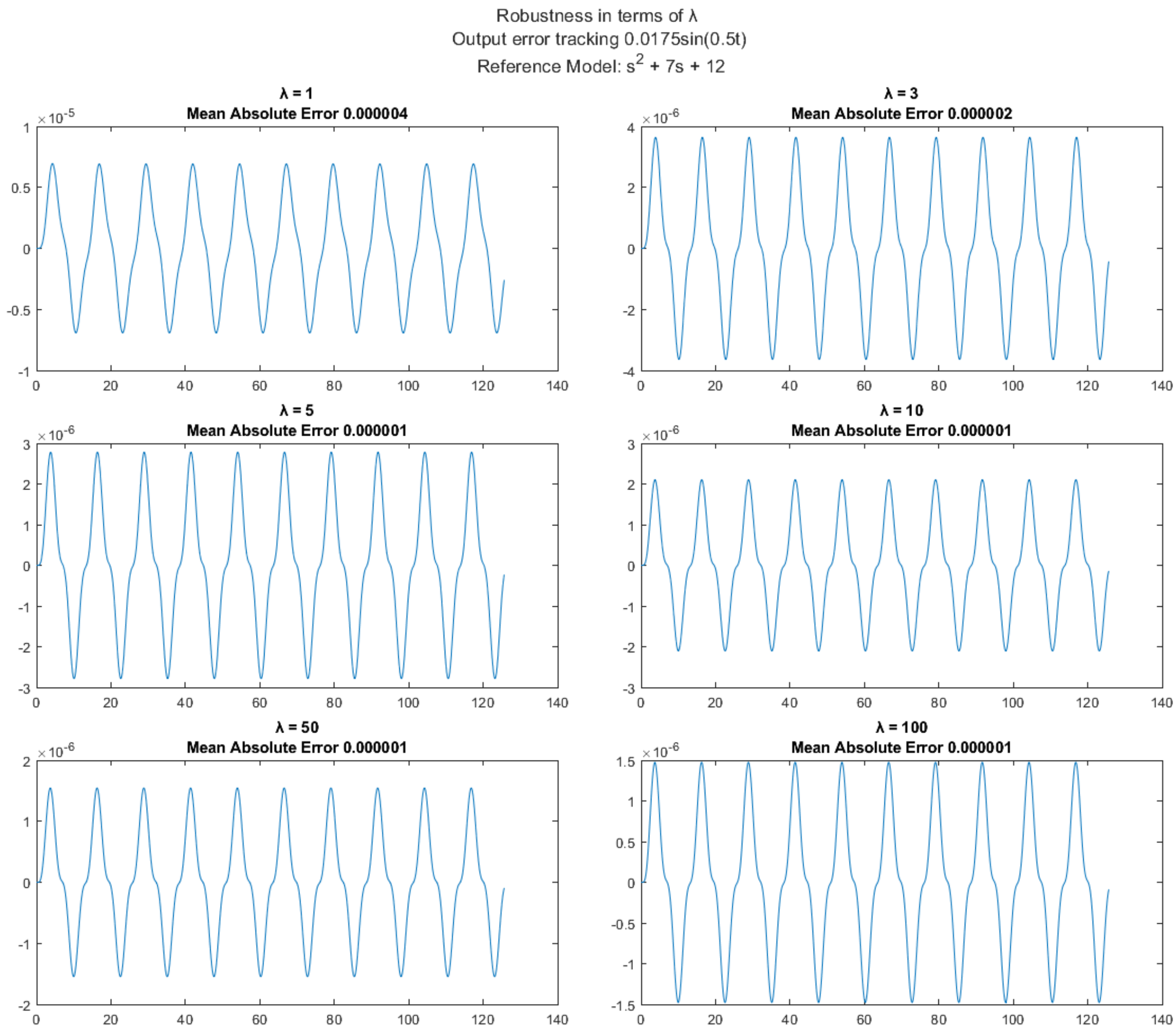


The results for $q(0) = 0.8727$ are more interesting. More specifically for $p = 1$, we observe that the closed loop system is unstable, with the output going to infinity. This happens because the slow response of the model, combined with the relatively large initial displacement, keep the system for a long enough time in a region relatively far from 0, in the vicinity of which the error of the linearization is small. This time the overshoot is shown for $p \geq 10$.

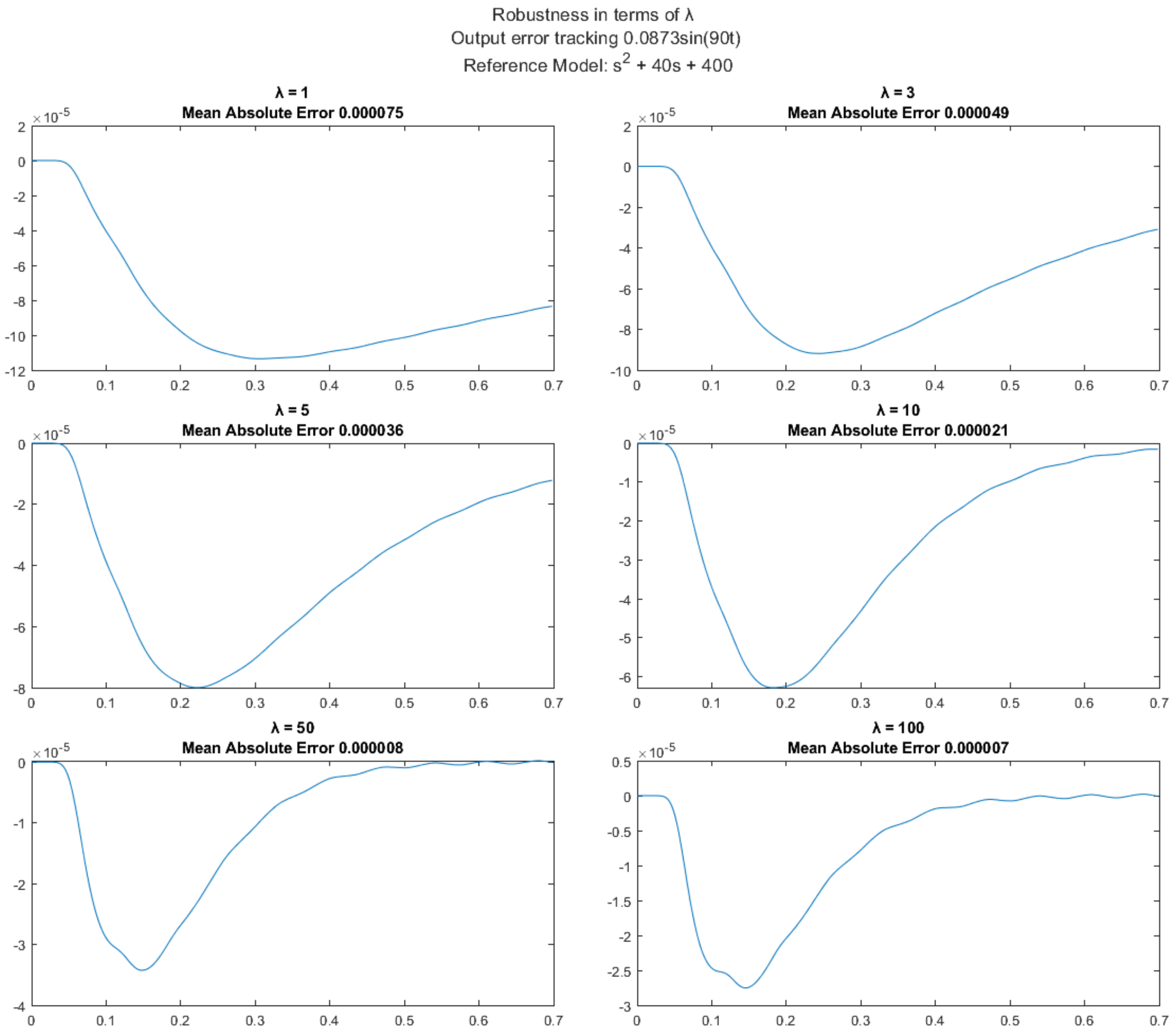
5.1.2 Scenario b

We again perform the same robustness study tests, for scenario b, with the only difference being the change of the reference model to the 2 desired output circumscriptions, based on the logic presented in the previous topics.

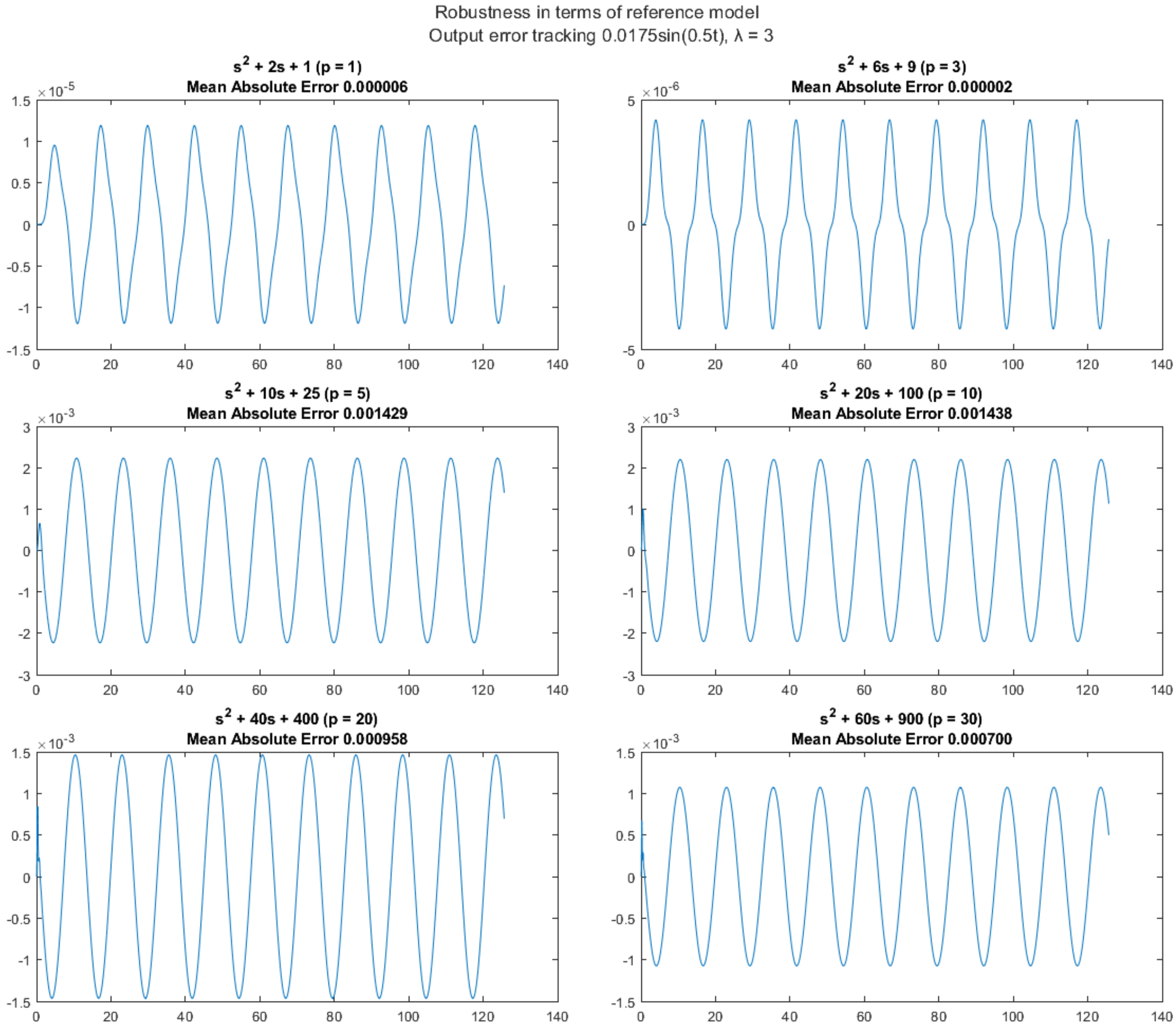
Robustness to λ



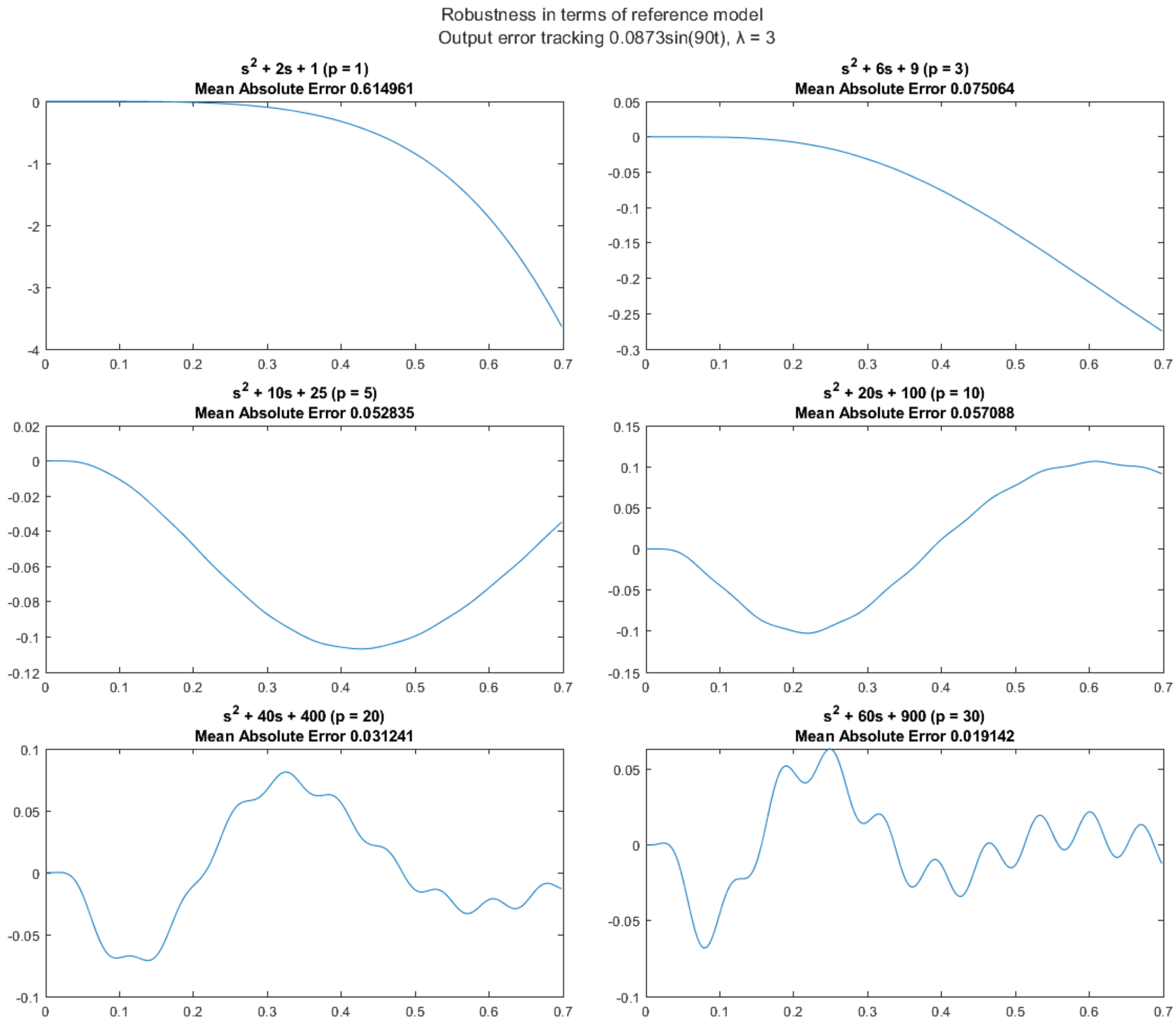
Studying the error of the output response in terms of changes in λ we do not observe any significant difference. The error, which already takes very small values (practically zero), is slightly reduced by gradually increasing λ. We draw the same conclusions by simulating the second case where $y_d = 0.0873sin(90t)$. The results of the second simulation are shown below.


Robustness in terms of λ
Output error tracking 0.0873sin(90t)
Reference Model: s2 + 40s + 400
λ = 1
Mean Absolute Error 0.000075
λ = 3
Mean Absolute Error 0.000049
λ = 5
Mean Absolute Error 0.000036
λ = 10
Mean Absolute Error 0.000021
λ = 50
Mean Absolute Error 0.000008
λ = 100
Mean Absolute Error 0.000007

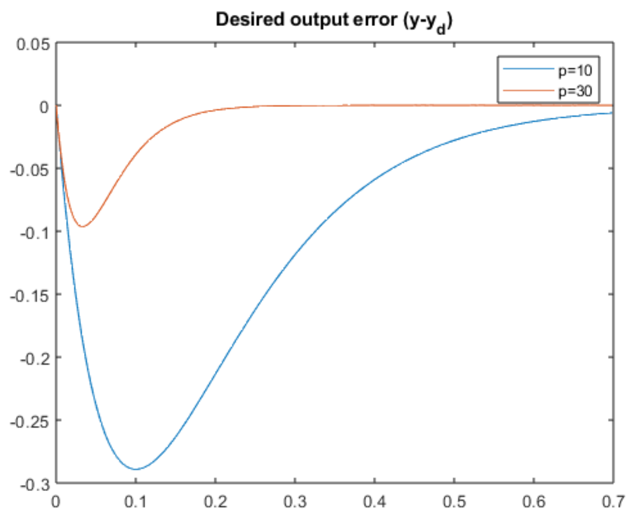
Robustness to reference model



In this particular case, moving the pole away from zero usually leads to a gradual increase in the output error. The best case is observed for $p = 3$, however depending on the monitoring stringency of the respective application all values could be accepted.



This time the second case, where $y_d = 0.0873sin(90t)$ is more interesting. For the case $p = 1$ corresponding to scenario a, the system appears to be unstable with the error increasing exponentially. Contrary to the first case, here we observe that for larger pole values the error of the output becomes smaller. In addition, in the specific case, where a strong and relatively long-lasting transient phenomenon occurs in practice, the model with large poles has a better behavior compared to the rest, as shown in the specific graphs. Thus, the adjacent graph is illustratively presented where the error $\varepsilon_d = y - y_d$ is compared for the cases $p = 10$ $p = 30$. Indeed for $p = 30$ the error converges to 0 noticeably faster.

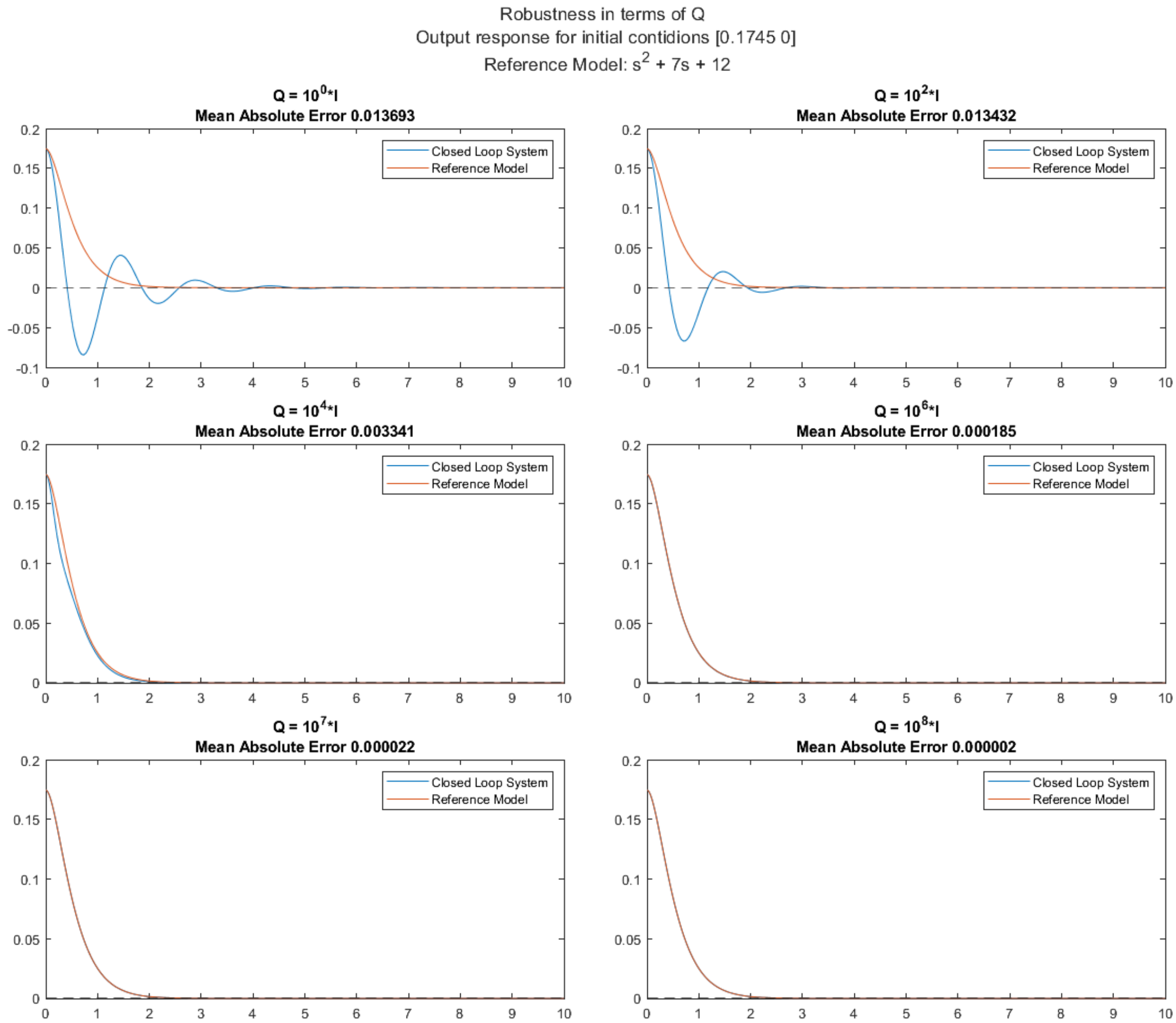


5.2 State Feedback DMRAC

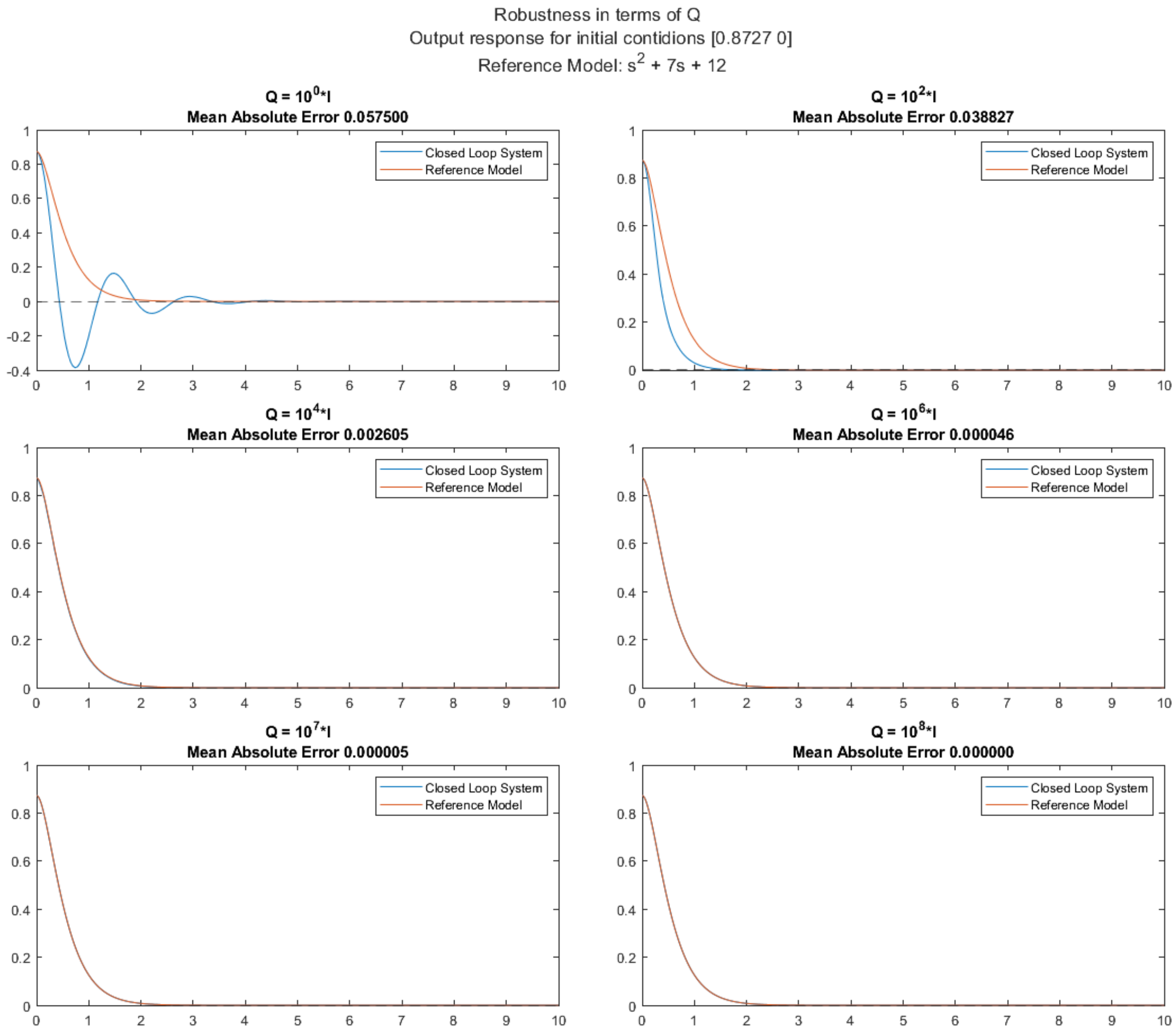
We continue with the State Feedback DMRAC. In this case the free variables are the matrix Q and the poles of the reference model. We start again with scenario a.

5.2.1 Scenario a

Robustness to Q

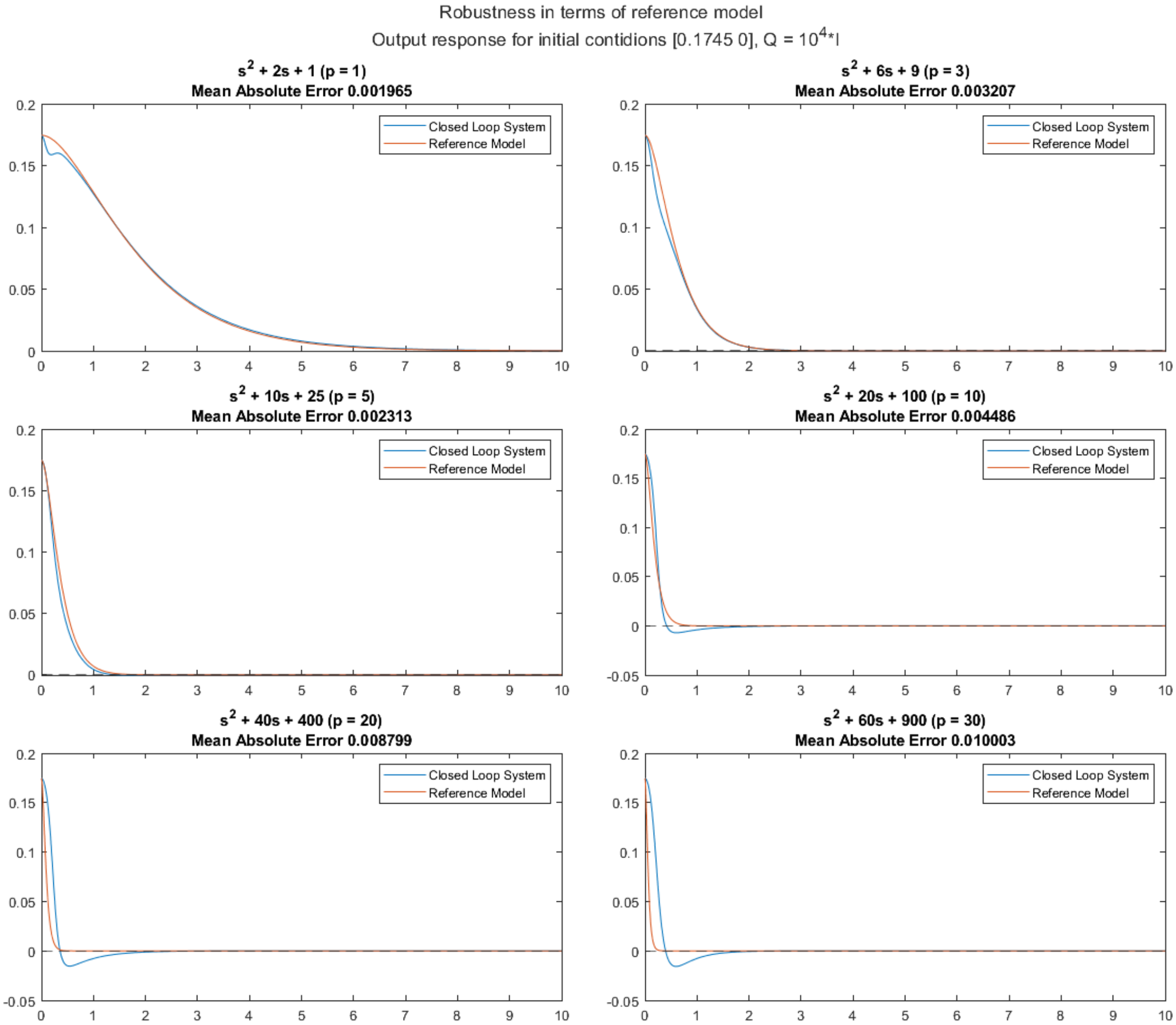


For small values of Q ($I, 10^2 \cdot I$) we observe that the closed-loop system fails to track the output of the reference model, with the output response more closely resembling it in the absence of a controller, appearing thus elevation in both cases. For the larger values, the design specifications are satisfied in all cases, while observing the average absolute error we see that an increase (of the scaling factor) of Q leads to a reduction of the already very small error.

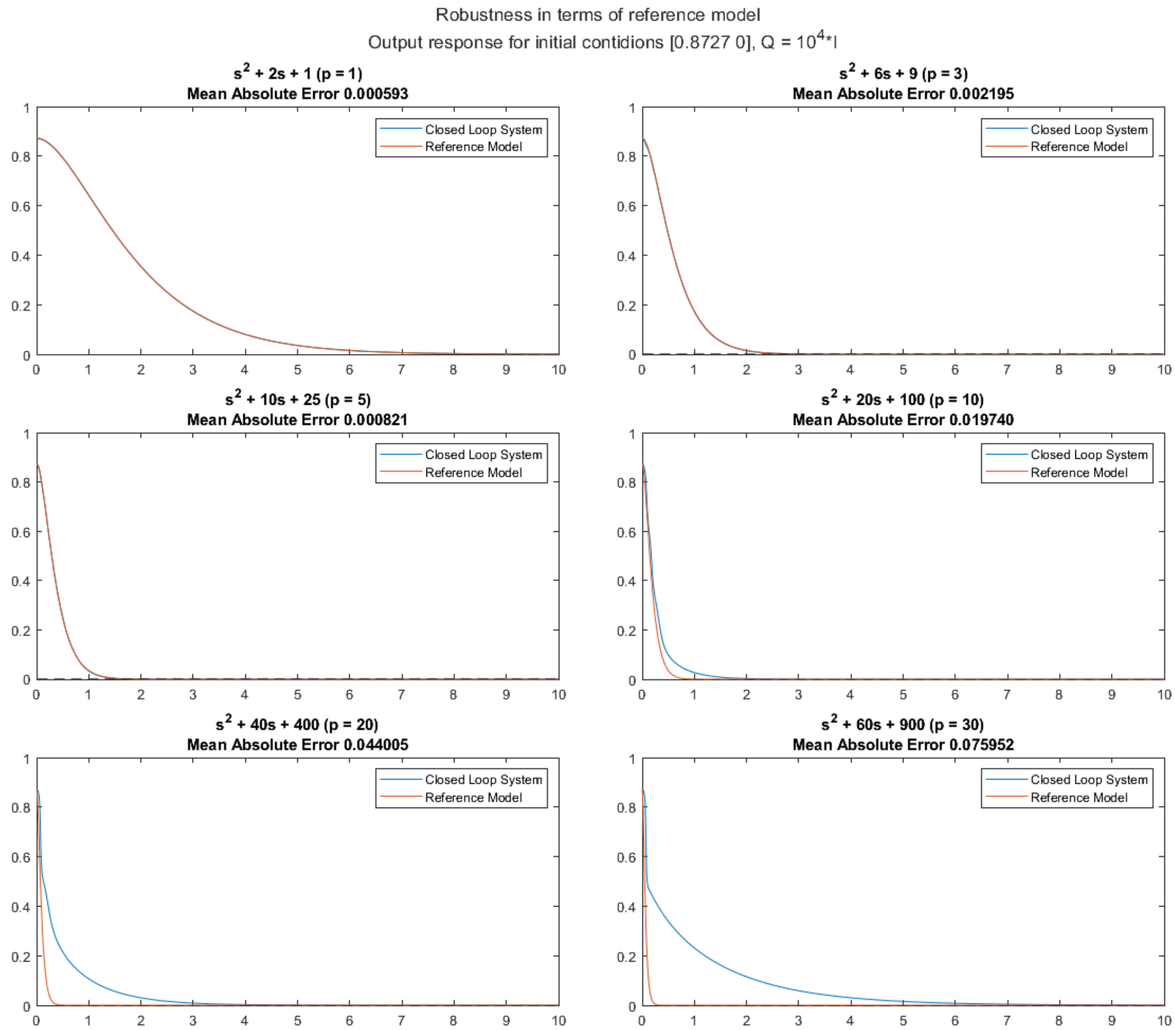


And in the case where $q(0) = 0.8727$ we observe similar results. For $Q = I$ the controller has no substantial effect and the system executes damped oscillation. For the remaining values the closed loop system monitors the output of the reference model satisfying the requested specifications.

Robustness to reference model



We now examine the robustness of the system with respect to the poles of the reference model. Starting with the case where $q(0) = 0.1745$ we observe that in all cases the output converges sufficiently quickly to zero $(1 \geq p \geq 30)$, but for values of $p \geq 10$ the output of system shows an elevation and thus does not meet the specifications.

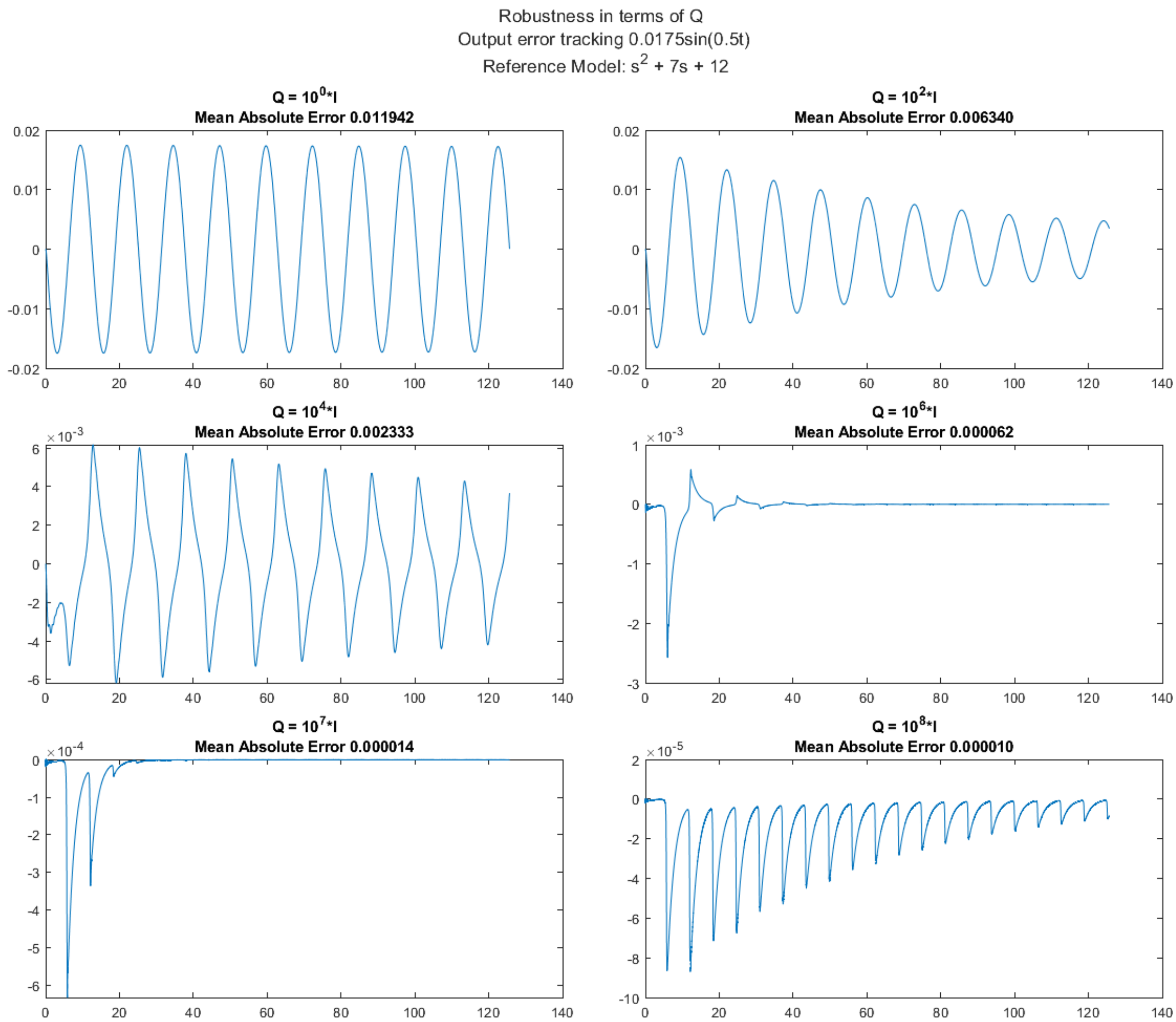


Continuing with the case $q(0) = 0.8727$, it is first worth noting that unlike the reference model controller we studied above, regardless of the value of p the system is stable. More specifically, in all cases the system meets both design specifications, of course we notice that by increasing the value of p, thus making the response of the reference model faster, the system has difficulty tracking the desired output and the error $gradually increasese$.

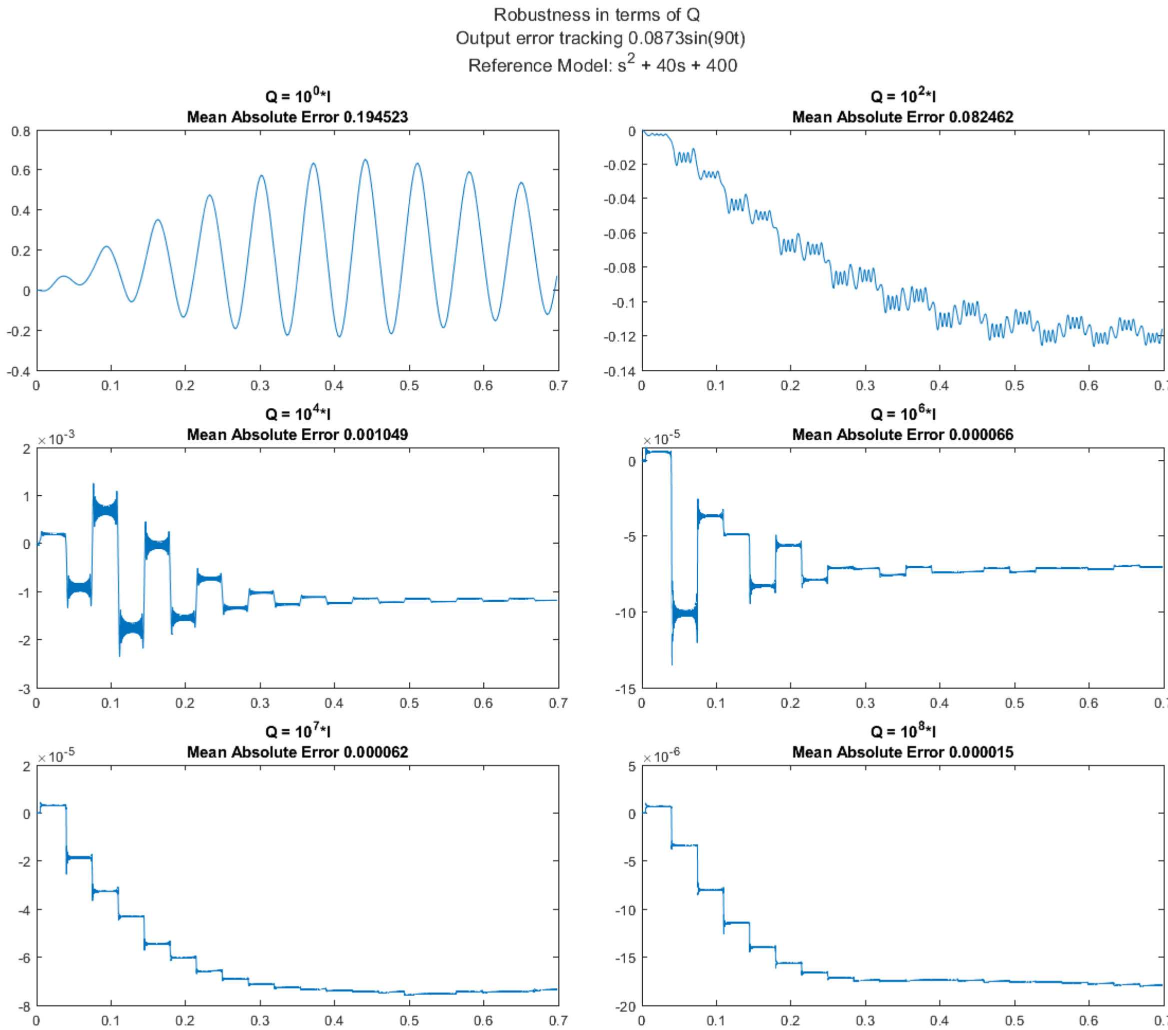
5.2.2 Scenario b

We proceed to scenario b, again changing the reference model to study different desired outputs.

Robustness to Q

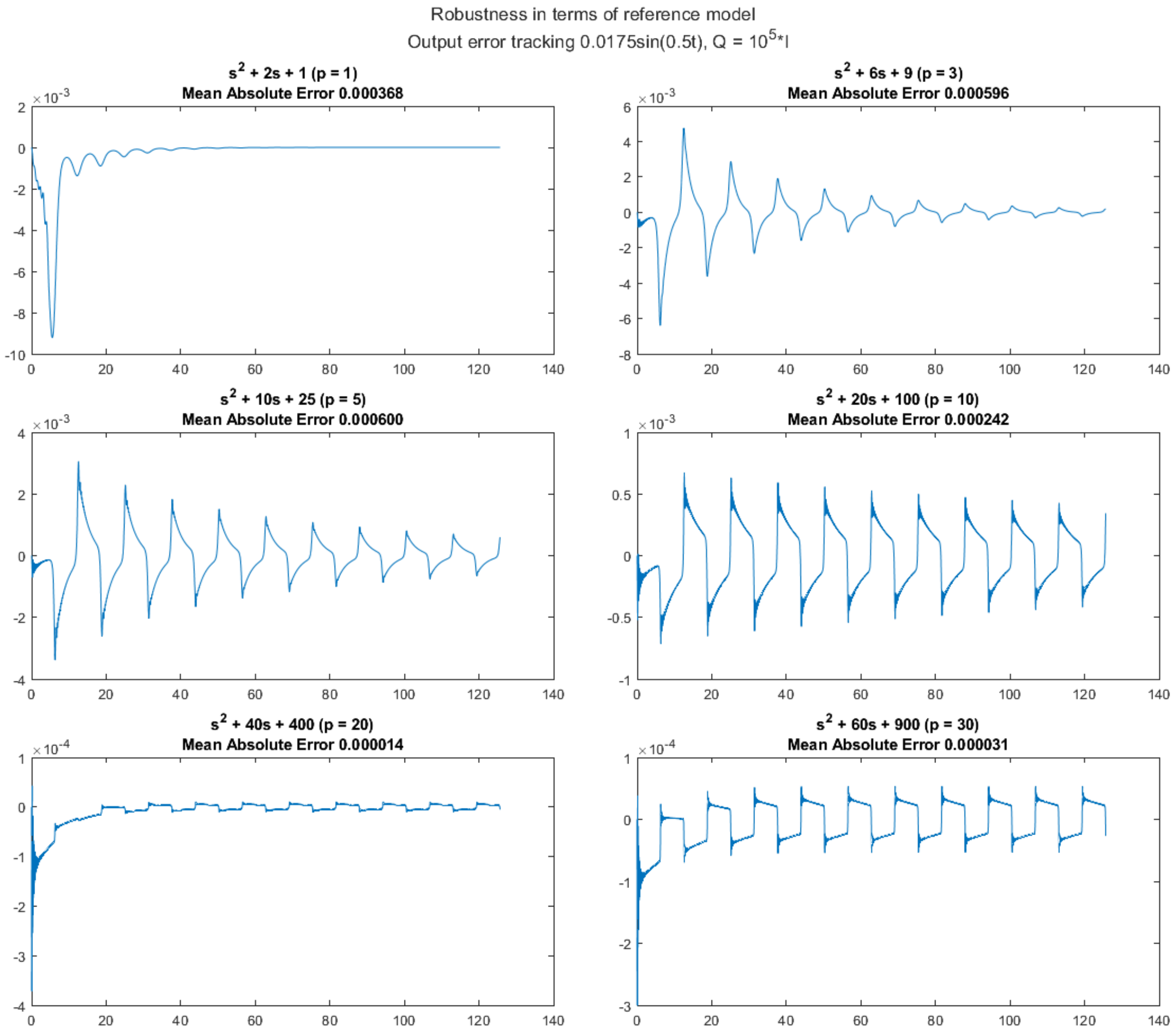


The results in this particular case are similar to those of scenario a. For small values of Q the system does not succeed in tracking y_m, while more specifically for $Q = I$ the output of the system is practically 0. By increasing Q the system successfully tracks the output of the reference model , enabling us with a suitable choice of Q (large enough) to achieve any small error.

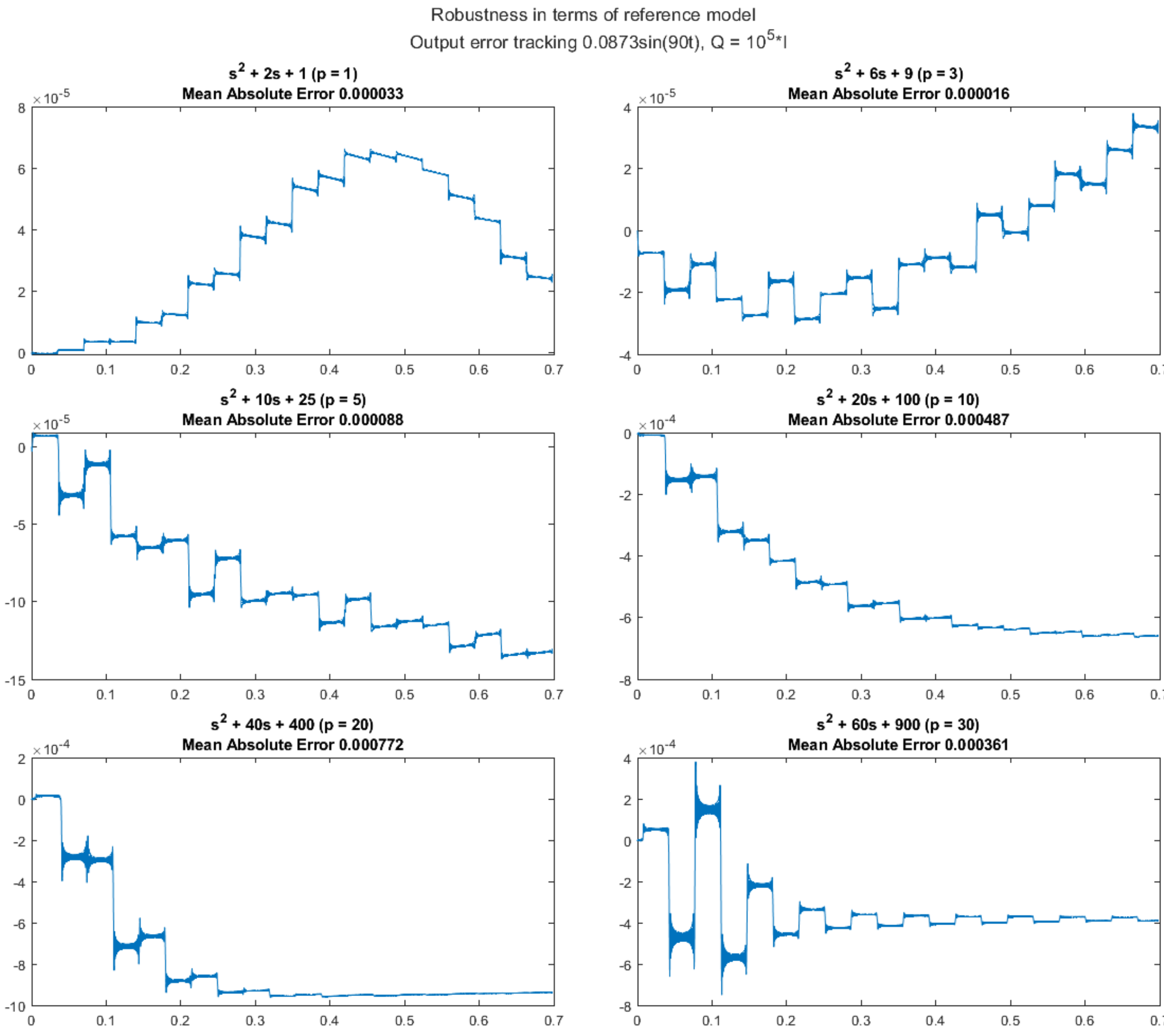


Although for this particular test we have used a different reference model than the previous one, the results remain practically the same. For small values of Q the system fails showing a very large error, while increasing Q the error decreases continuously.

Robustness to reference model



Studying this particular case, the poles of the reference model seem to have little effect on the output tracking error, as long as they are stable. In general, for larger pole values we have a smaller error, with all values of course showing a sufficiently small error.



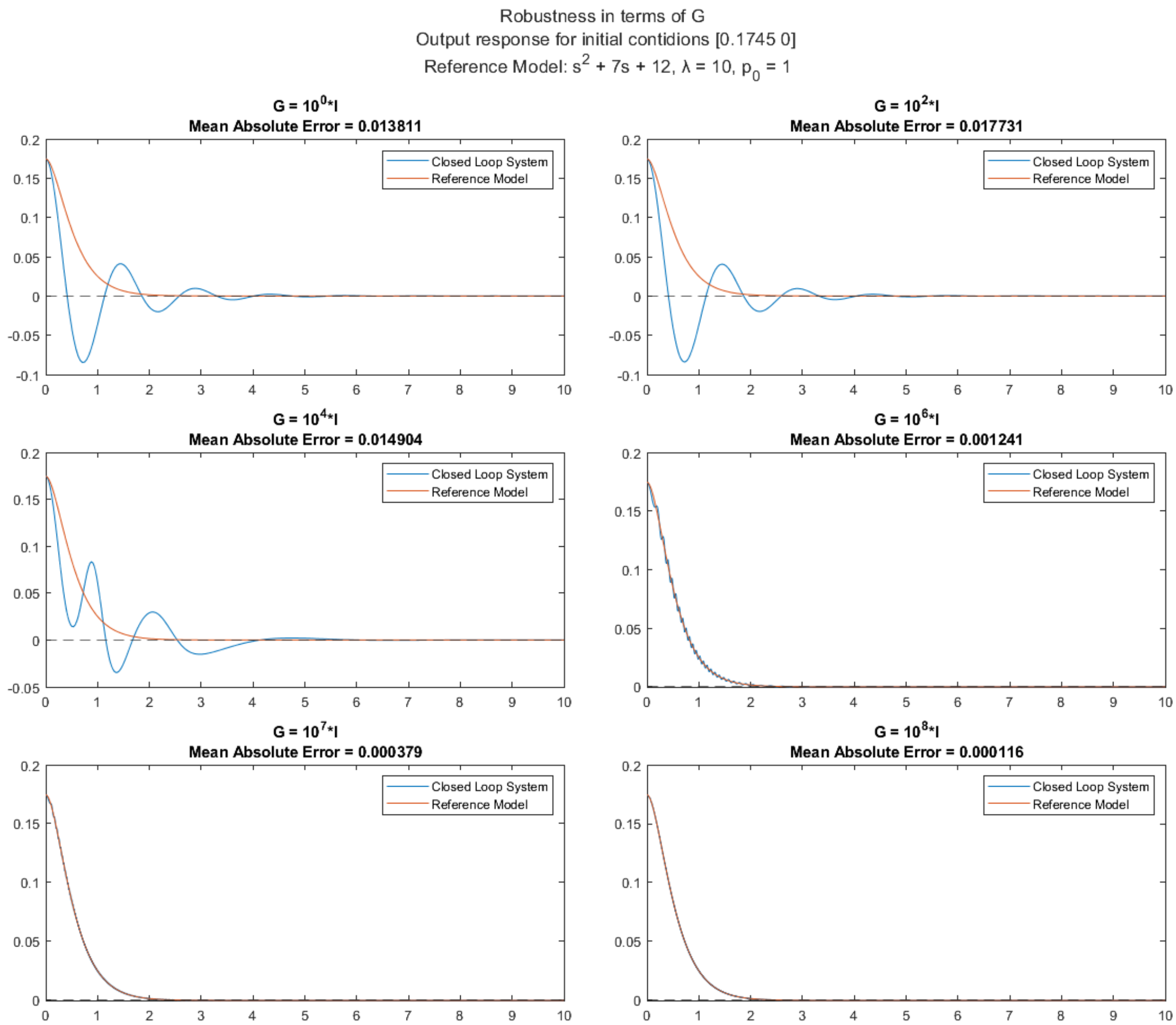
Unlike the previous case, here we get the minimum error for $p = 3$. Of course, as previously mentioned the $y - y_m$ error is not so indicative of the optimal behavior in this particular experiment with the largest pole value having the smallest desired output error in reality.

5.3 Output Feedback DMRAC

We continue with the Output Feedback DMRAC. In this case the free variables are the matrix Γ, λ, p_0 and the poles of the reference model. We start again with scenario a.

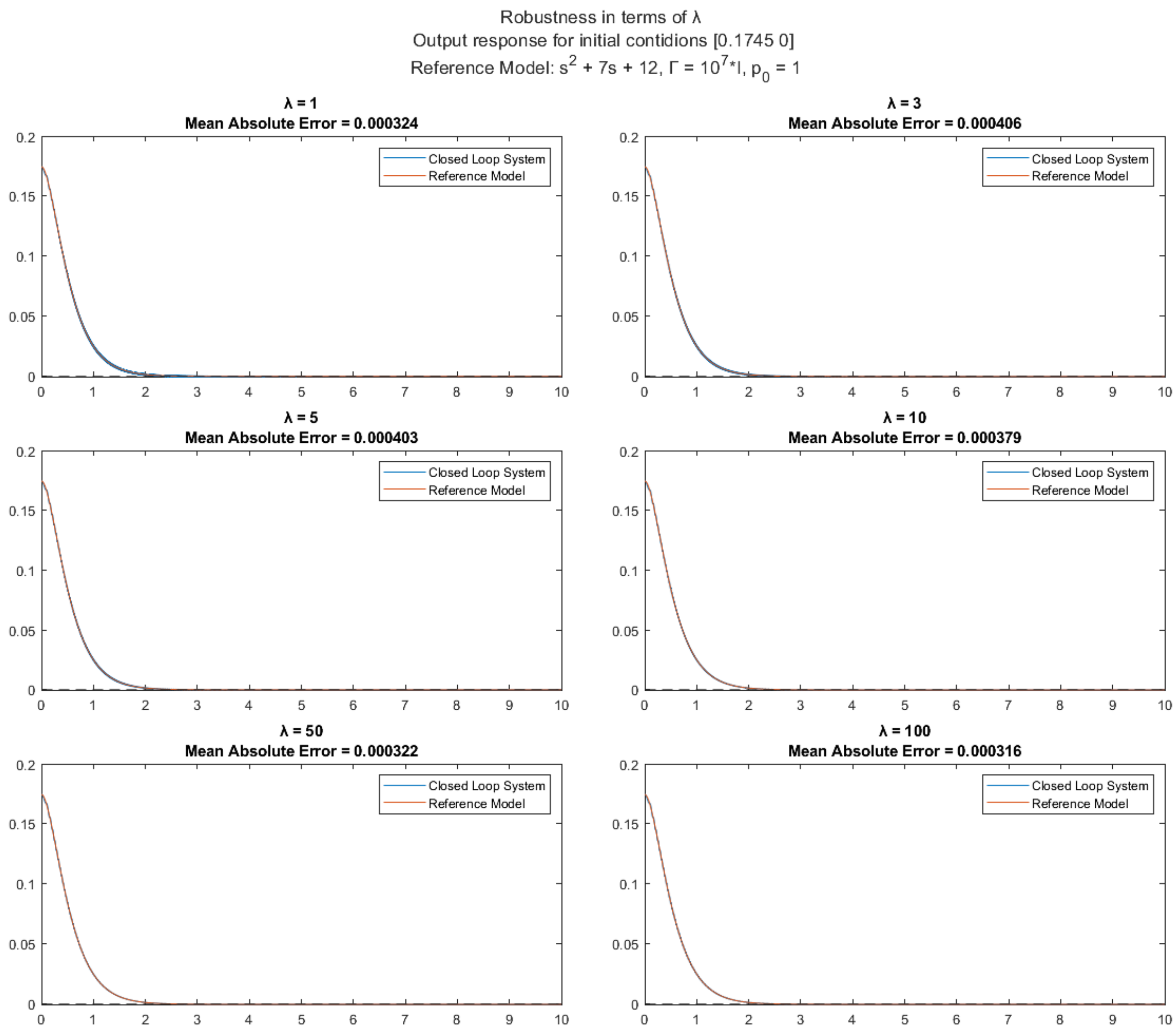
5.3.1 Scenario a

Robustness to C



Similar to State Feedback DMRAC and robustness to Q, we observe that for small Γ the controller does not have much effect on the closed-loop system whose output performs damped oscillation. By gradually increasing Γ we reach a value of $10^6 \cdot I$, where we now have acceptably good tracking with the overshoot gone. By further increasing Γ we can further reduce the error of the output response. The results are identical for the 2nd case with $q(0) = 0.8727$ and thus the corresponding graphs are omitted. The main difference is that in the second case we can achieve the correct operation of the controller with smaller Γ.

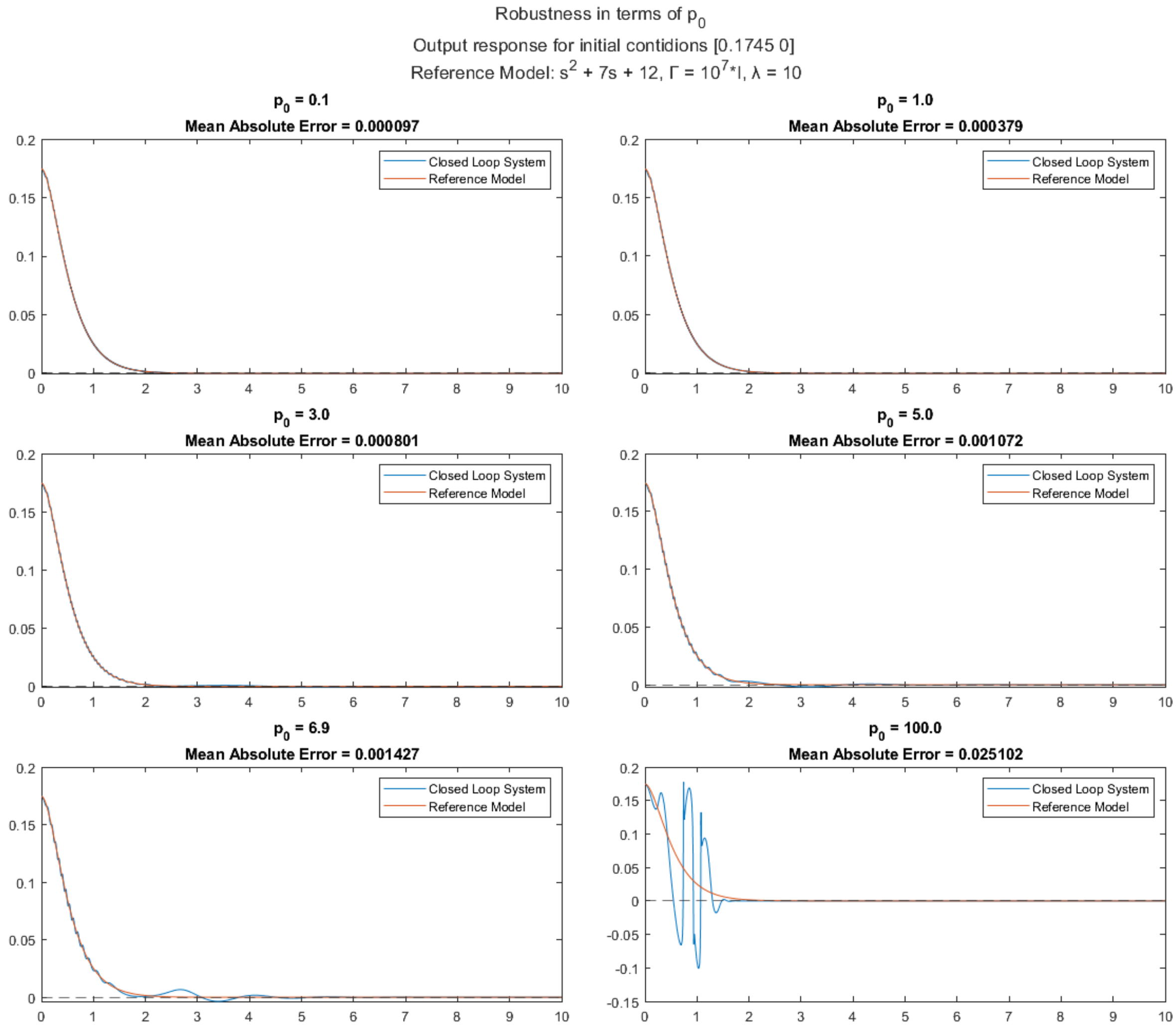
Robustness to λ



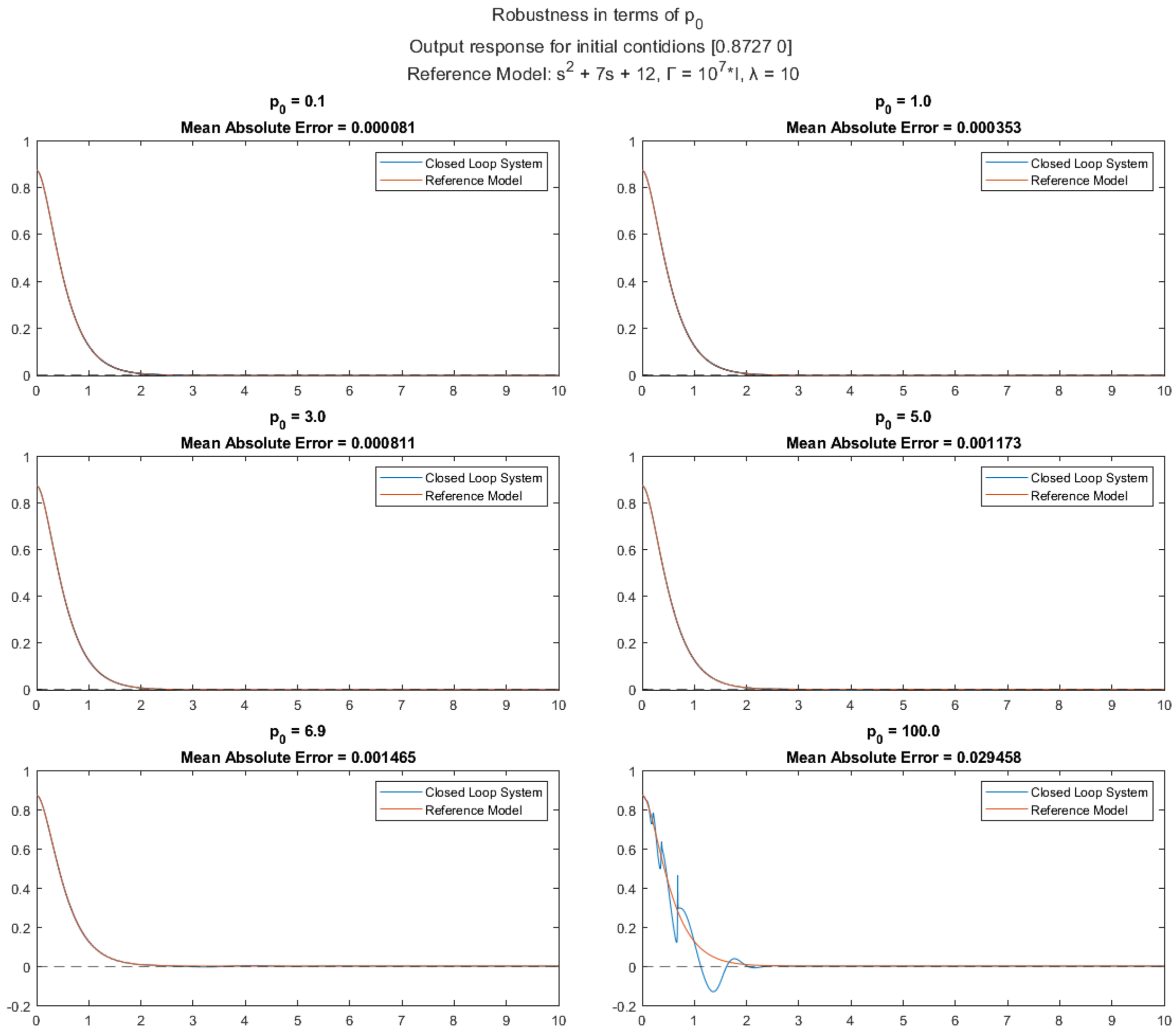
Changing the value of λ we observe that it has a negligible effect on the response of the output. In all cases the system meets the specifications, while the average absolute error remains practically the same. The behavior of the system is absolutely similar for $q(0) = 0.8727$ and thus the corresponding graphs are omitted, which of course can be produced by the corresponding matlab script.

Robustness to p_0

At this point we study the robustness of the output response with respect to p_0. As we have shown in the design of the controller for this particular parameter, our options are limited to the interval $(0, a)$, that is, in the specific case $(0, 7)$. However, we give p_0 the extreme value of 100 to observe the behavior of the system for a value outside the allowed interval.

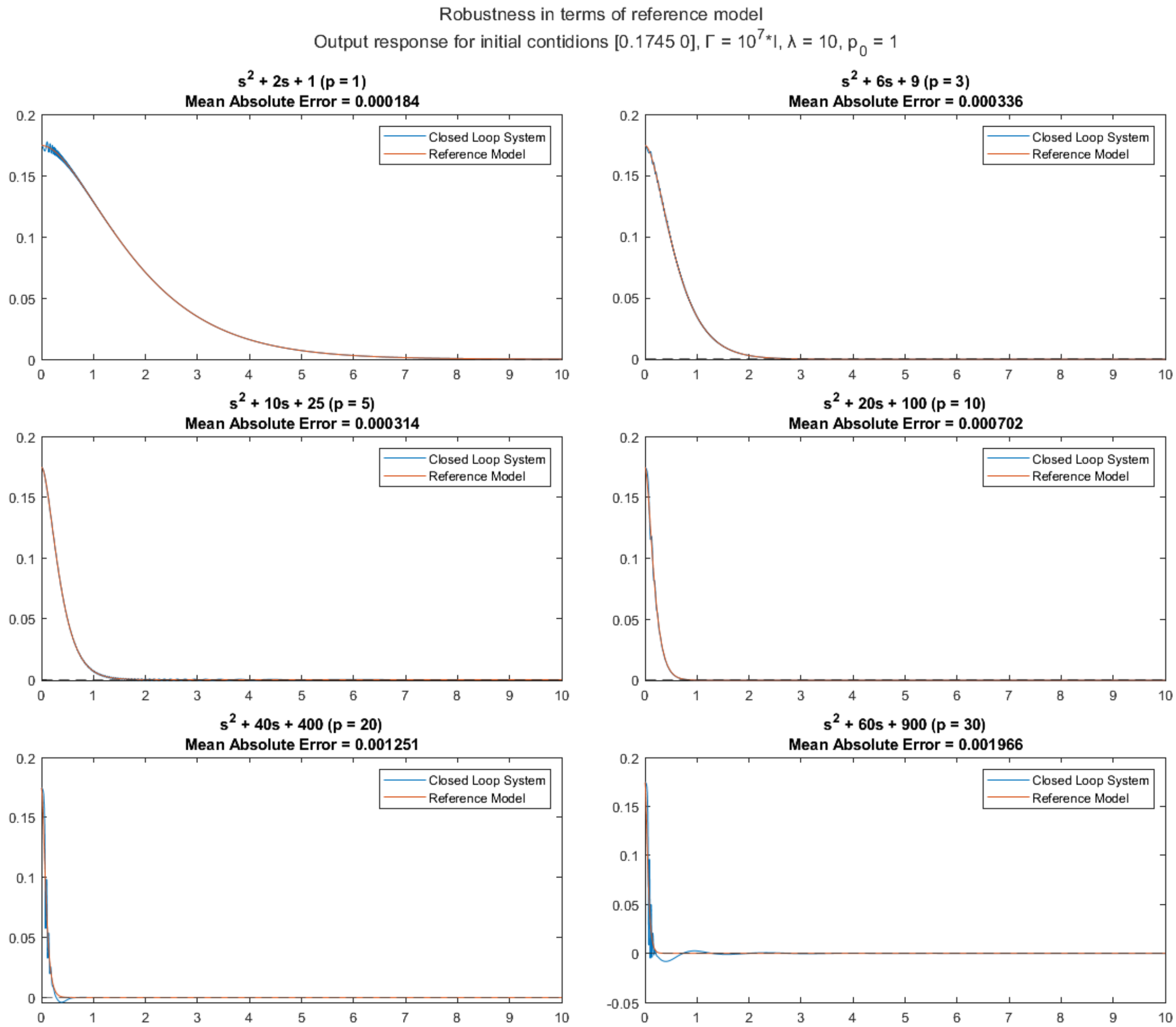


For the first case, where $q(0) = 0.1745$ we notice that approaching the value $p_0 = 7$ the error of the system output becomes a little bigger. A bigger problem, however, is the fact that for $p_0 \geq 5$ the output starts to show elevation. Regarding the case for $p_0 = 100$ it is clear that the controller is not working properly and the output does not follow the reference model.

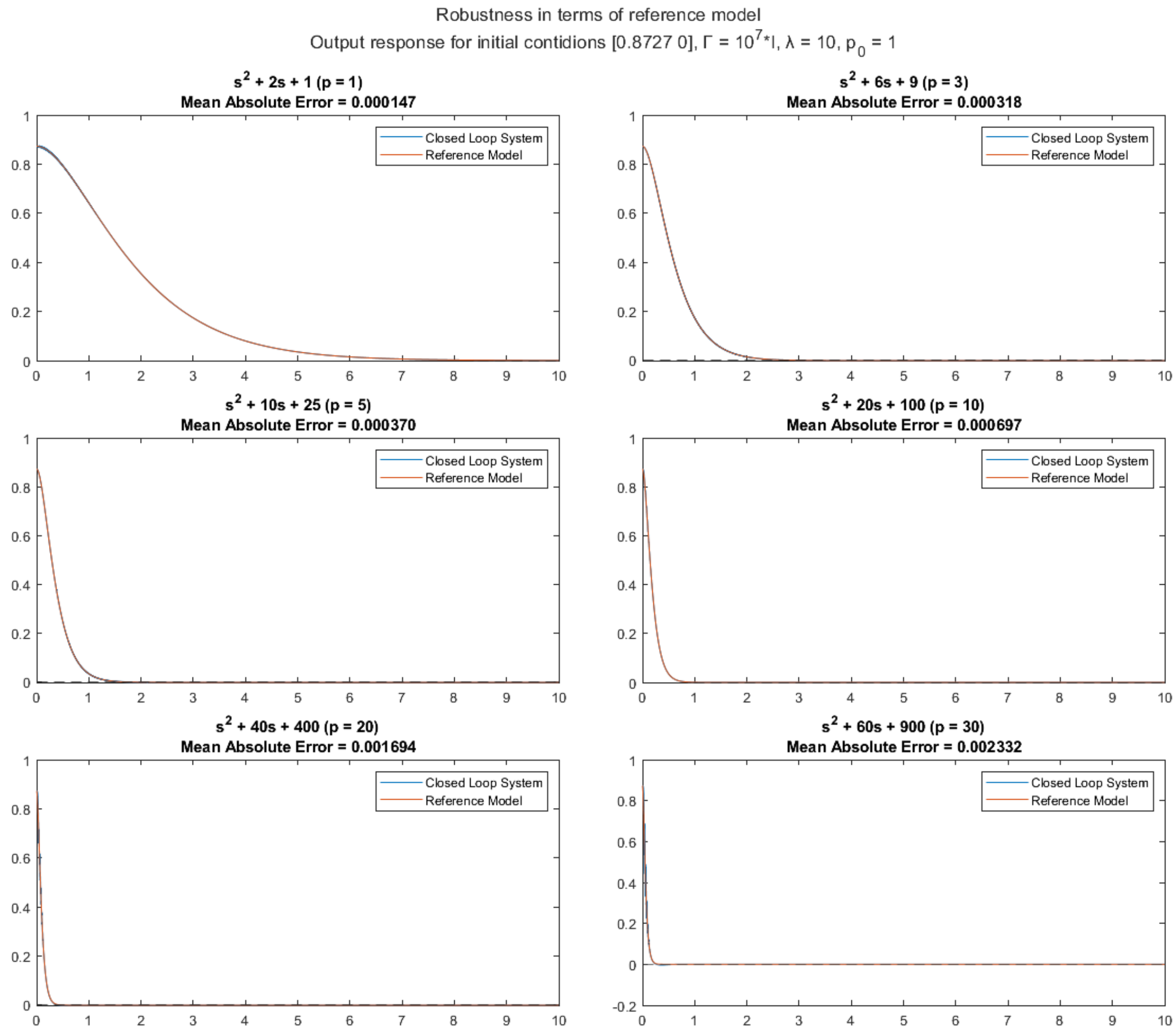


Now simulating the 2nd case for $q(0) = 0.8727$, we observe correspondingly a very small increase in the tracking error for larger values of p_0, without of course overshooting appearing this time. Again for $p_0 = 100$ the response of the output shows relatively large and unpredictable fluctuations and obviously does not satisfy the specifications, since it shows an overshoot.

Robustness to reference model



By changing the poles of the reference model, for $q(0) = 0.1745$, we observe that the system tracks with a smaller error the later models, i.e. those with smaller poles. Furthermore for $p \geq 20$ the output of the closed-loop system exhibits overshoot and thus does not meet our design specifications.

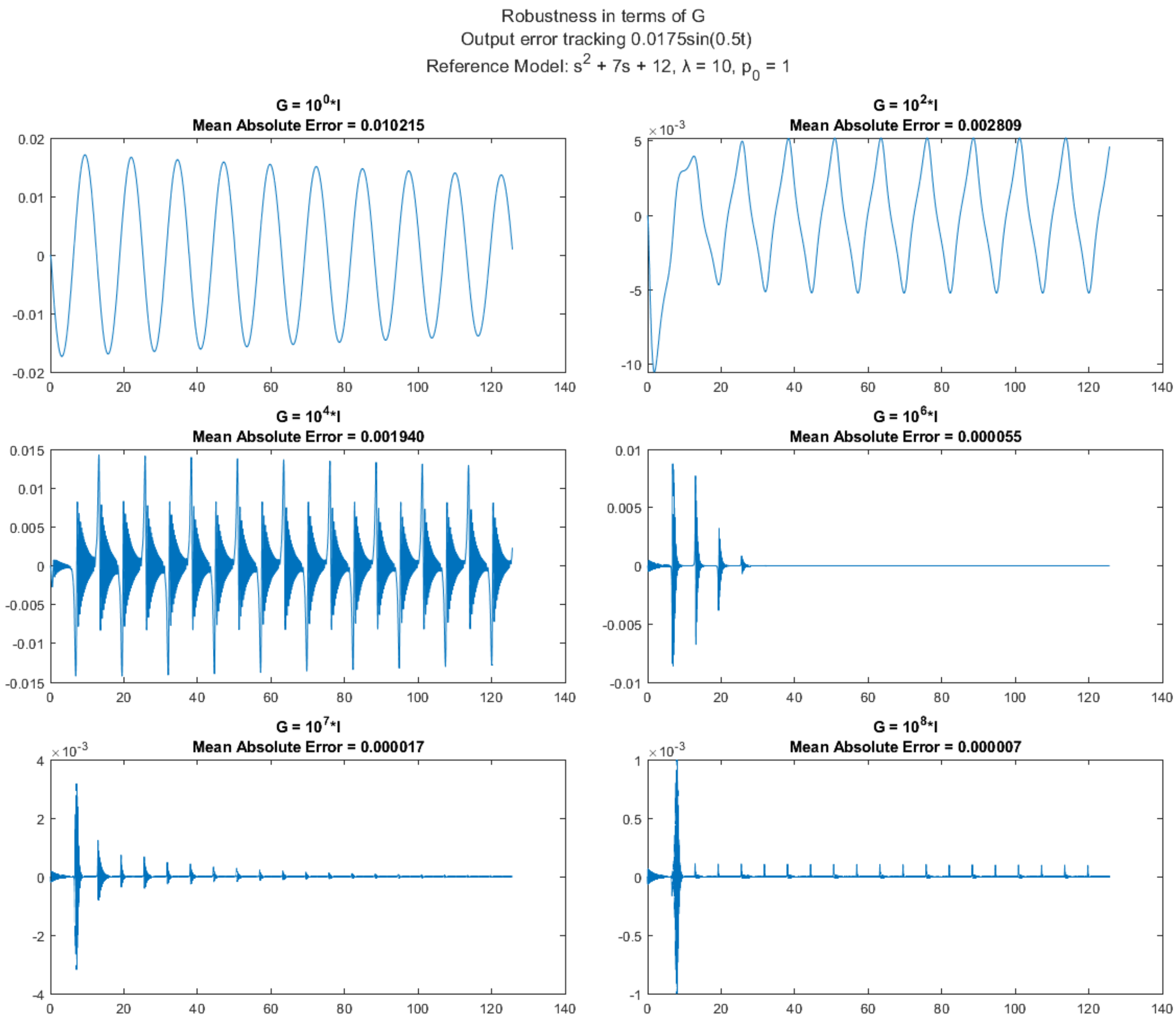


Changing now the poles of the reference model, for $q(0) = 0.8727$, we again observe that the system tracks with a smaller error the later models, i.e. those with smaller poles. However, this time the system meets the specifications of acquisition time and zero overshoot in all cases, regardless of the reference model (which of course also does not show overshoot).

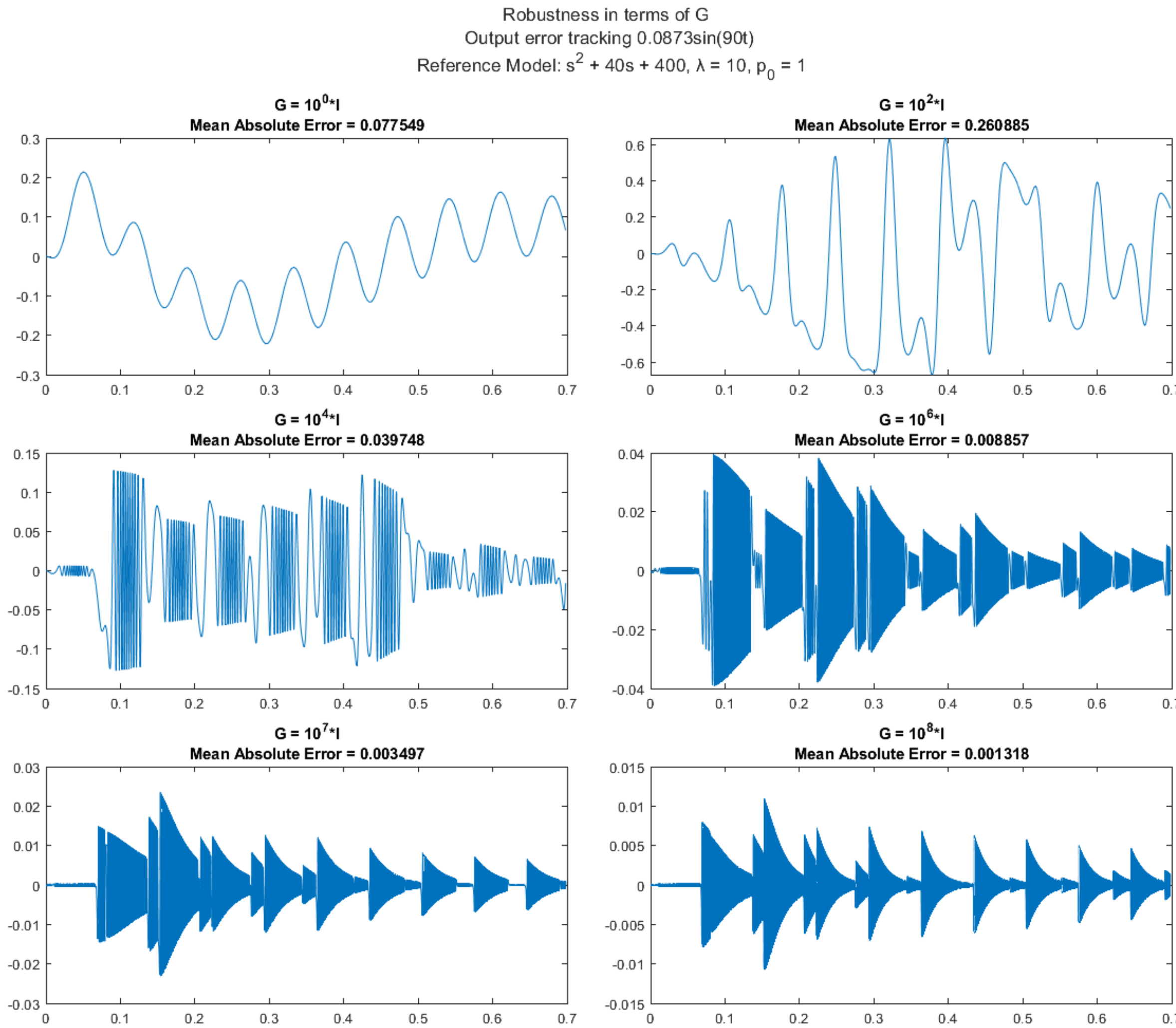
5.3.2 Scenario b

Moving on to scenario b, once again, we will use different reference models for the 2 instances of y_d.

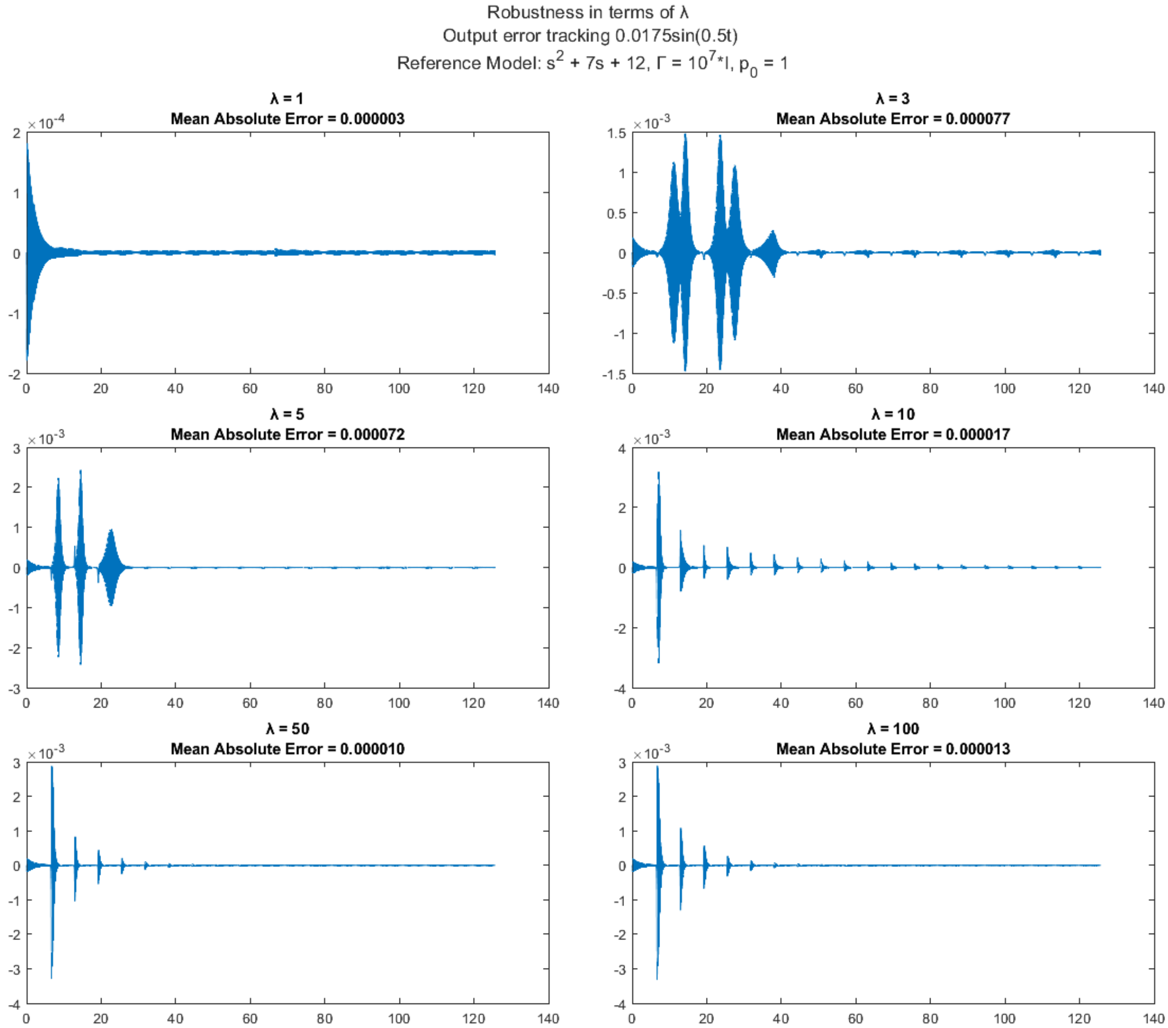
Robustness to C



In full correspondence with scenario a, here too we observe that for small Γ the controller essentially does not work, while increasing Γ leads to an increasingly smaller tracking error. Similar are the results for the 2nd case, the results of which are shown below:

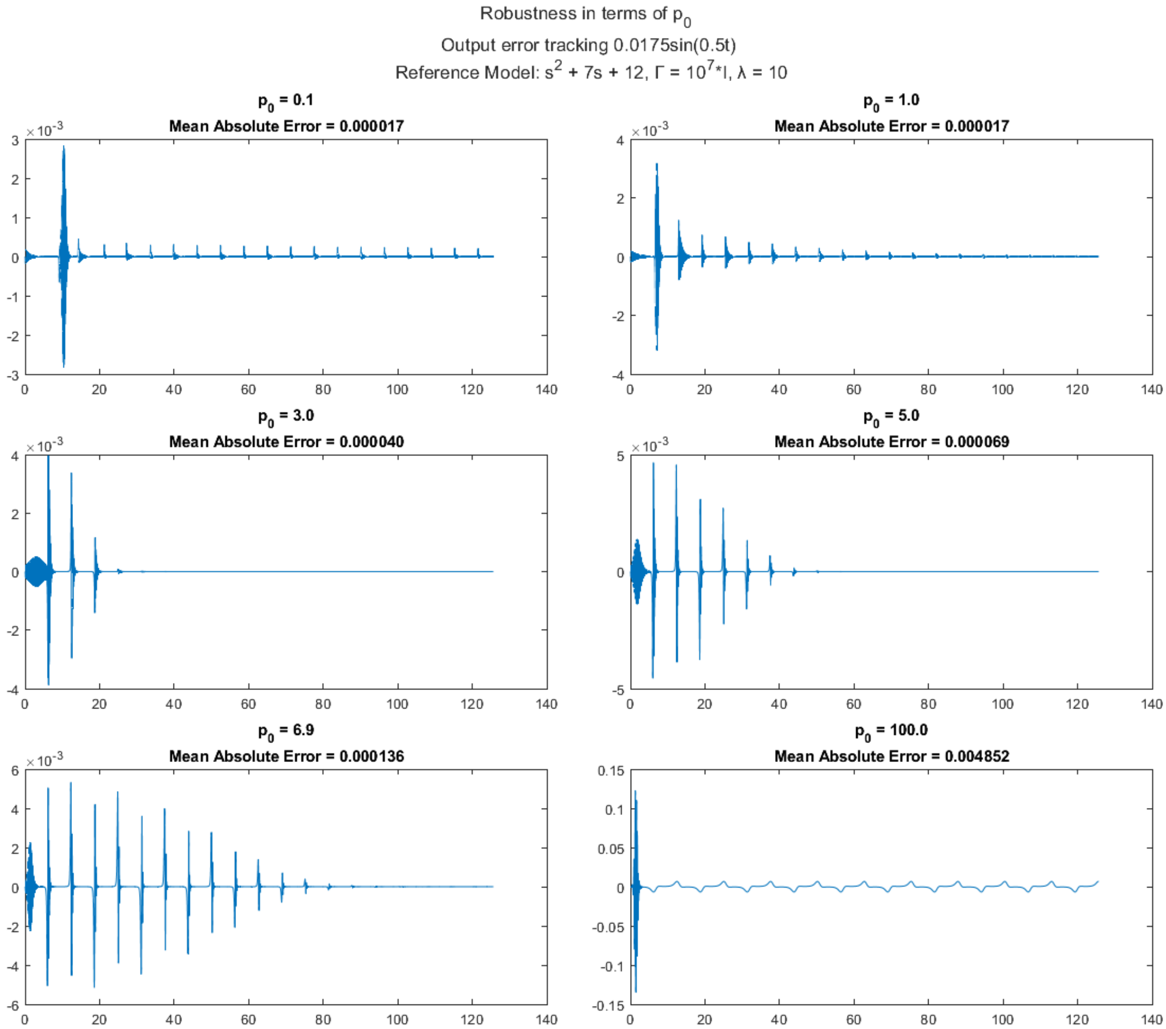

Robustness in terms of G
Output error tracking 0.0873sin(90t)
Reference Model: s2 + 40s + 400, λ = 10, p0 = 1
G = 10^0*I
Mean Absolute Error = 0.077549
G = 10^2*I
Mean Absolute Error = 0.260885
G = 10^4*I
Mean Absolute Error = 0.039748
G = 10^6*I
Mean Absolute Error = 0.008857
G = 10^7*I
Mean Absolute Error = 0.003497
G = 10^8*I
Mean Absolute Error = 0.001318

Robustness to λ

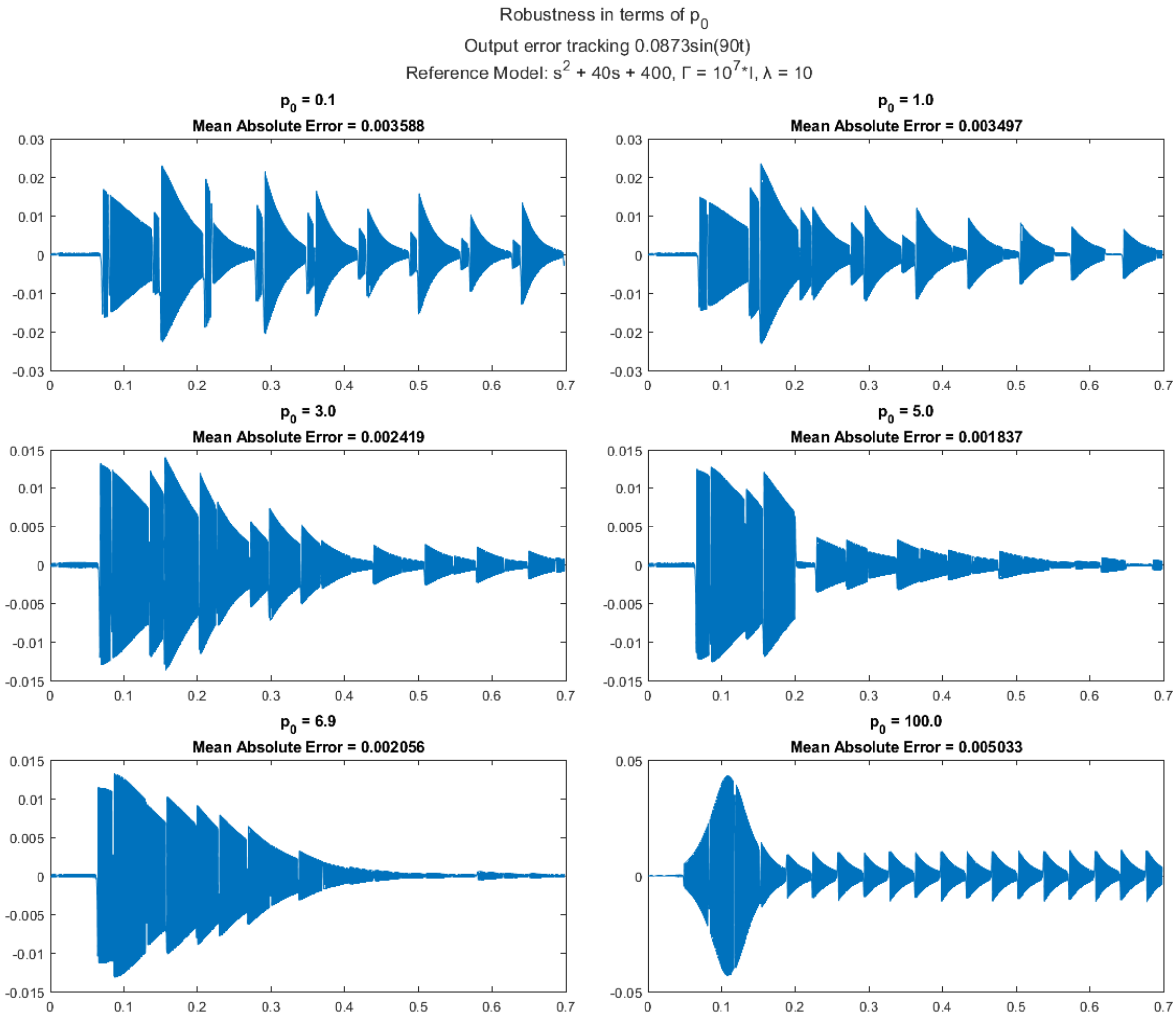


Again our conclusions are the same as those of scenario a. Changes in the value of λ have a negligible effect on the output response of the system, with the tracking error taking consistently low values with very small changes of λ regardless. Although the form of the error differs, exactly the same conclusions are reached for the 2nd case, where $y_d = 0.0873sin(90t)$ and thus the corresponding graphs are omitted.

Robustness to p_0

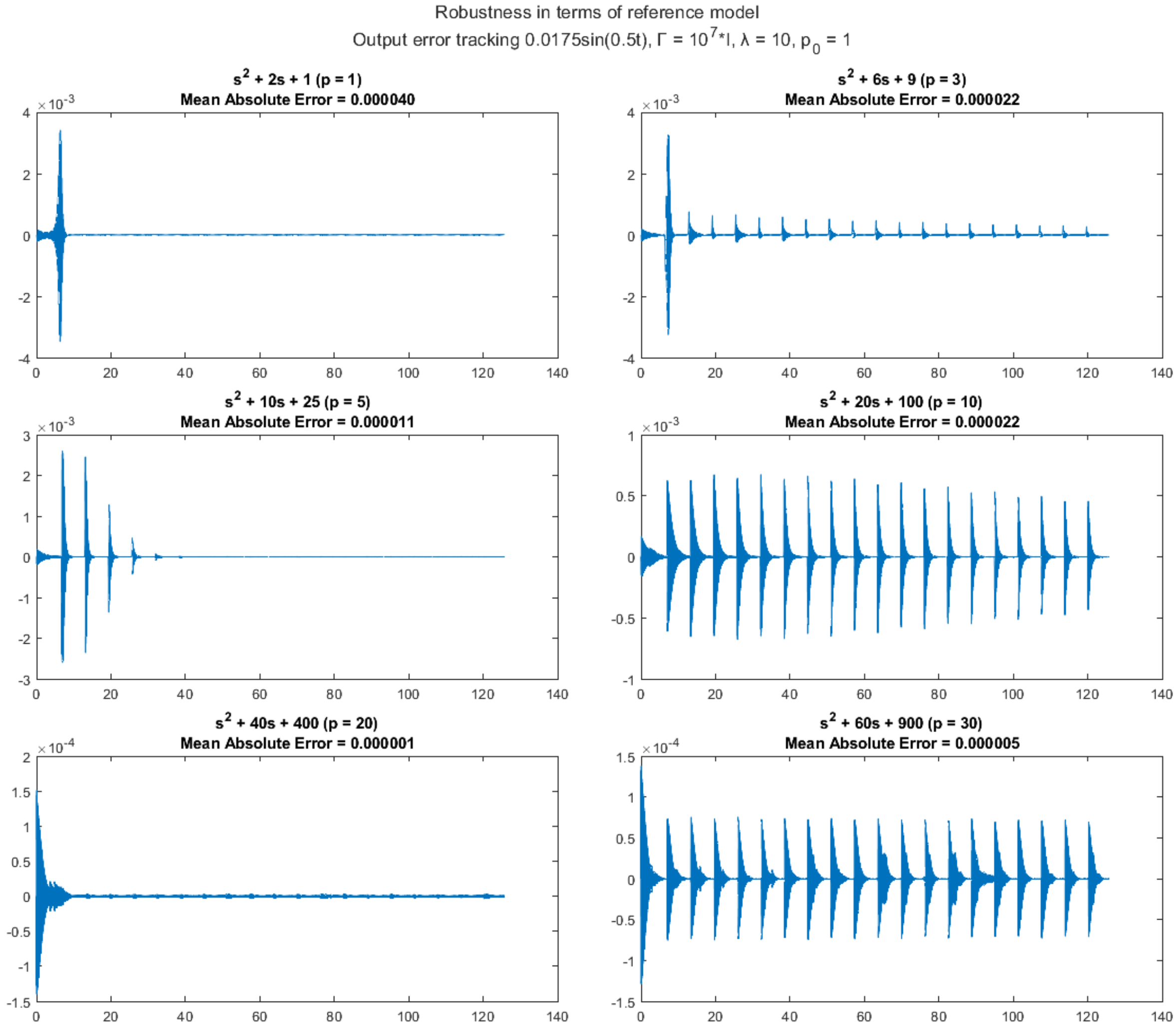


Observing the above graphs, we come to the conclusion that the changes of p_0 do not greatly affect the response of the output. As it increases, a small increase in the average error is observed, while for $p_0 = 100$ (outside the allowed interval) we see a large error in the transient phenomenon and a smaller error in the steady state.

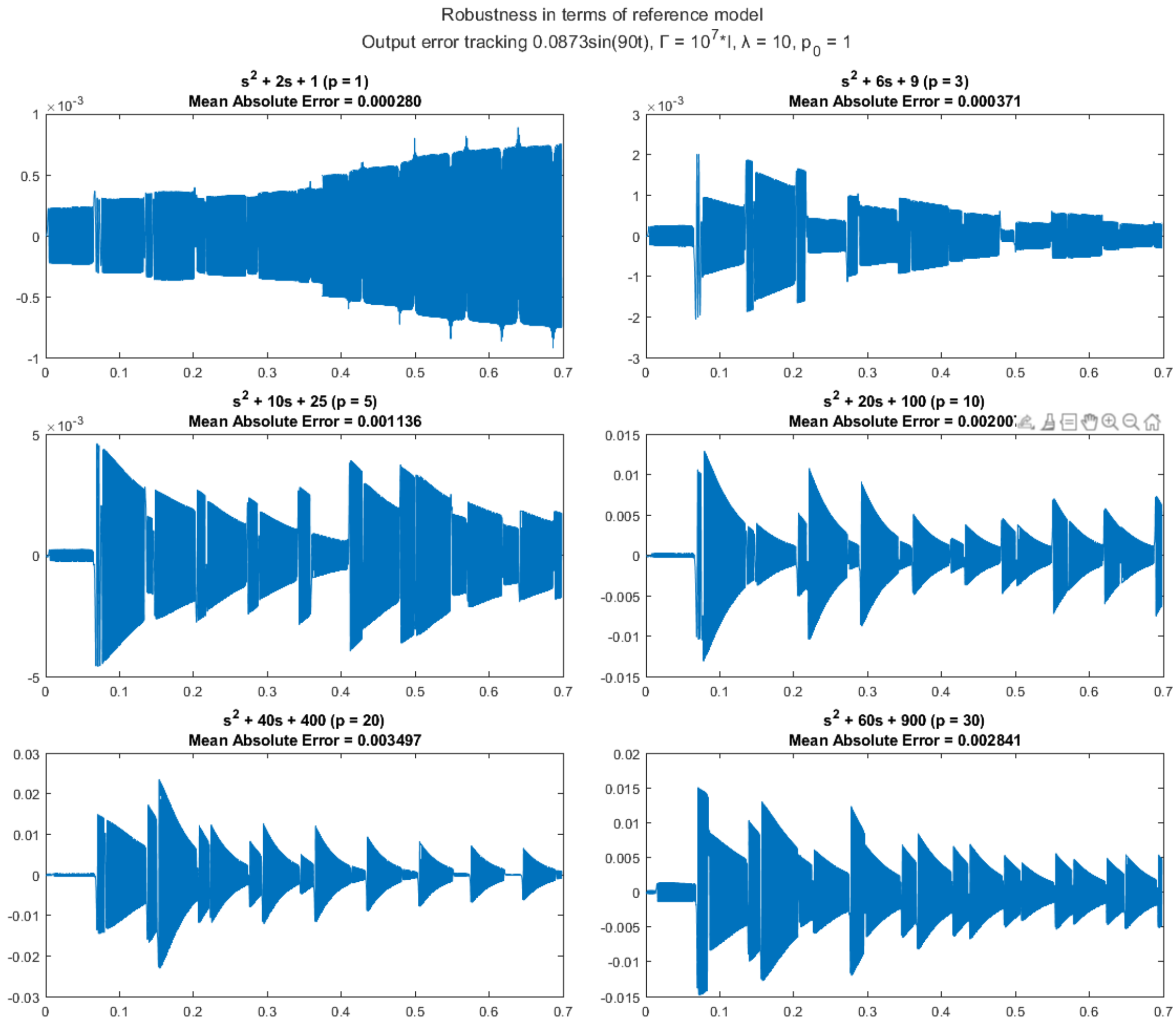


Simulating the system for $y_d = 0.0873sin(90t)$ and the fastest reference model, we observe that again the value of p_0 does not greatly affect it, but unlike before now the increase in the value of p_0 is accompanied from a small reduction in output error. Again the increase of p_0 has some limits and testing the extreme value $p_0 = 100$ we see that the error in both the transient phenomenon and the steady state is increased.

Robustness to reference model



In the case where $y_d = 0.01745 sin(0.5t)$ we observe that the poles of the model do not significantly affect the output error, the latter being very low in all cases.



On the contrary, when $y_d = 0.0873sin(90t)$ we see that for larger reference model poles we have a larger output error ($\varepsilon = y - y_m$), which is of course misleading, as in reality by moving the poles away from the 0, we get smaller and smaller desired output error ($\varepsilon_d = y - y_d$), as we showed earlier.