Spivak's Calculus: Answer Book

Konstantin Lepa

Basic Properties of Numbers

Theorem 1.1 (Triangle Inequality). *For all numbers a and b, we have*

$$|a+b| \leq |a| + |b|.$$

Axiom 1.2 (Not in the book). *For any $y \geq 0$, there is some $x \geq 0$ such that $x^2 = y$.*

Lemma 1.3 (Not in the book). *For any $y \geq 0$, there is a unique $x \geq 0$ such that $x^2 = y$. We denote $x = \sqrt{y}$.*

Proof. Let $y \geq 0$. By Axiom 1.2 we see that there is some $x \geq 0$ such that $x^2 = y$. Let $z \geq 0$, and suppose that $z^2 = y$. Then $z^2 = x^2$. If either $x = 0$ or $z = 0$, then clearly $z = x$. Now suppose that $x \neq 0$ and $z \neq 0$. Then,

$$\begin{aligned} z^2 = x^2 &\iff zz = xx \\ &\iff (zz)(x^{-1}z^{-1}) = (xx)(x^{-1}z^{-1}) \\ &\iff (zx^{-1})(zz^{-1}) = (xz^{-1})(xx^{-1}) \\ &\iff zx^{-1} = xz^{-1}. \end{aligned}$$

Because $x > 0$ and $z > 0$, we deduce that $x^{-1} > 0$ and $z^{-1} > 0$, so $x^{-1} + z^{-1} > 0$. We also know that $xx^{-1} = 1 = zz^{-1}$. Then,

$$\begin{aligned} zx^{-1} + zz^{-1} = xz^{-1} + xx^{-1} \\ &\iff z(x^{-1} + z^{-1}) = x(z^{-1} + x^{-1}) \\ &\iff z = x. \end{aligned}$$

$\square$

Lemma 1.3 (Not in the book). *For any $y \geq 0$, there is a unique $x \geq 0$ such that $x^2 = y$. We denote $x = \sqrt{y}$.*

Lemma 1.4 (Not in the book). *Let $x \geq 0$ and $y \geq 0$. Then $\sqrt{xy} = \sqrt{x}\sqrt{y}$.*

Proof. Using Lemma 1.3 we deduce that

$$\begin{aligned} xy = xy &\iff \sqrt{xy}\sqrt{xy} = \left(\sqrt{x}\sqrt{x}\right)\left(\sqrt{y}\sqrt{y}\right) \\ &\iff \sqrt{xy}\sqrt{xy} = \left(\sqrt{x}\sqrt{y}\right)\left(\sqrt{x}\sqrt{y}\right) \\ &\iff \left(\sqrt{xy}\right)^2 = \left(\sqrt{x}\sqrt{y}\right)^2 \\ &\iff \sqrt{xy} = \sqrt{x}\sqrt{y}. \end{aligned}$$

□

Problem 1-13. The maximum of two numbers x and y is denoted by $\max(x, y)$. Thus $\max(-1, 3) = \max(3, 3) = 3$ and $\max(-1, -4) = \max(-4, -1) = -1$. The minimum of x and y is denoted by $\min(x, y)$. Prove that

$$\max(x, y) = \frac{x + y + |y - x|}{2},$$
$$\min(x, y) = \frac{x + y - |y - x|}{2}.$$

Derive a formula for $\max(x, y, z)$ and $\min(x, y, z)$, using, for example

$$\max(x, y, z) = \max(x, \max(y, z)).$$

Proof. Without loss of generality, suppose that $y \geq x$. Then $y - x \geq 0$ and $|y - x| = y - x$. We also deduce that $\max(x, y) = y$, so

$$\begin{aligned} 2\max(x, y) &= 2y = y + y = (y + 0) + y \\ &= (y + [x + (-x)]) + y = ([y + x] + (-x)) + y \\ &= (y + x) + [(-x) + y] = (x + y) + (y - x) \\ &= x + y + |y - x|. \end{aligned}$$

Hence,

$$\max(x, y) = \frac{x + y + |y - x|}{2}.$$

A similar argument shows that

$$\min(x, y) = \frac{x + y - |y - x|}{2}.$$

$\square$

Using $\max(x, y, z) = \max(x, \max(y, z))$, we obtain

$$\begin{aligned} \max(x, y, z) &= \max(x, \max(y, z)) \\ &= \frac{x + \max(y, z) + |\max(y, z) - x|}{2} \\ &= \frac{x + \frac{y+z+|z-y|}{2} + \left|\frac{y+z+|z-y|}{2} - x\right|}{2}. \end{aligned}$$

Similarly, using a formula for $\min(x, y, z) = \min(x, \min(y, z))$, we obtain

$$\begin{aligned} \min(x, y, z) &= \min(x, \min(y, z)) \\ &= \frac{x + \min(y, z) - |\min(y, z) - x|}{2} \\ &= \frac{x + \frac{y+z-|z-y|}{2} - \left|\frac{y+z-|z-y|}{2} - x\right|}{2}. \end{aligned}$$

Problem 1-17.

(a) Find the smallest possible value of $2x^2 - 3x + 4$. Hint: "Complete the square," i.e., write $2x^2 - 3x + 4 = 2(x - 3/4)^2 + ?$

(b) Find the smallest possible value of $x^2 - 3x + 2y^2 + 4y + 2$.

(c) Find the smallest possible value of $x^2 + 4xy + 5y^2 - 4x - 6y + 7$.

(a) We have

$$\begin{aligned}
2x^2 - 3x + 4 &= 2\left(x^2 - \frac{3}{2}x + 2\right) \\
&= 2\left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16} + 2\right) \\
&= 2\left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) - 2\left(\frac{9}{16} - 2\right) \\
&= 2\left(x - \frac{3}{4}\right)^2 - \frac{9}{8} + 4 \\
&= 2\left(x - \frac{3}{4}\right)^2 + \frac{23}{8}.
\end{aligned}$$

Because $2(x - 3/4)^2 \geq 0$, we deduce that the smallest possible value of $2x^2 - 3x + 4$ is $23/8$.

(b) We have

$$\begin{aligned}
x^2 - 3x + 2y^2 + 4y + 2 &= (x^2 - 3x) + (2y^2 + 4y + 2) \\
&= \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right) + 2(y^2 + 2y + 1) \\
&= \left(x - \frac{3}{2}\right)^2 + 2(y + 1)^2 - \frac{9}{4}.
\end{aligned}$$

Because $(x - 3/2)^2 + 2(y + 1)^2 \geq 0$, we deduce that the smallest possible value of $x^2 - 3x + 2y^2 + 4y + 2$ is $-9/4$.

(c) We have

$$\begin{aligned}
x^2 + 4xy + 5y^2 - 4x - 6y + 7 &= (x^2 + 4xy + 4y^2) + y^2 - 4x - 6y + 7 \\
&= (x + 2y)^2 + y^2 - 4x - 8y + 2y + 7 \\
&= (x + 2y)^2 - (4x + 8y) + (y^2 + 2y + 7) \\
&= (x + 2y)^2 - 4(x + 2y) + (y^2 + 2y + 1) + 6 \\
&= (x + 2y)^2 - 4(x + 2y) + (y + 1)^2 + 6 \\
&= \left[(x + 2y)^2 - 4(x + 2y) + 4\right] + (y + 1)^2 + 2 \\
&= (x + 2y - 2)^2 + (y + 1)^2 + 2.
\end{aligned}$$

Because $(x + 2y - 2)^2 + (y + 1)^2 \geq 0$, we deduce that the smallest possible value of $x^2 + 4xy + 5y^2 - 4x - 6y + 7$ is 2.

Problem 1-18.

(a) Suppose that $b^2 - 4c \geq 0$. Show that the numbers
$$\frac{-b+\sqrt{b^2-4c}}{2}, \quad \frac{-b-\sqrt{b^2-4c}}{2}$$
both satisfy the equation $x^2 + bx + c = 0$.

(b) Suppose that $b^2 - 4c < 0$. Show that there are no numbers x satisfying $x^2 + bx + c = 0$; in fact, $x^2 + bx + c > 0$ for all x. Hint: Complete the square.

(c) Use this fact to give another proof that if x and y are not both 0, then $x^2 + xy + y^2 > 0$.

(d) For which numbers α is it true that $x^2 + \alpha xy + y^2 > 0$ whenever x and y are not both 0?

(e) Find the smallest possible value of $x^2 + bx + c$ and of $ax^2 + bx + c$, for $a > 0$.

(a) Let $d = \sqrt{b^2-4c}$. Then $d^2 = b^2 - 4c$. We deduce that
$$\begin{aligned}
&\left(\frac{-b+d}{2}\right)^2 + b\left(\frac{-b+d}{2}\right) + c \\
&\quad = \frac{1}{4}(-b+d)^2 + \frac{b}{2}(-b+d) + c \\
&\quad = \frac{1}{4}\left[(-b+d)^2 + 2b(-b+d) + 4c\right] \\
&\quad = \frac{1}{4}\left[(-b)^2 + 2(-b)d + d^2 + 2b(-b) + 2bd + 4c\right] \\
&\quad = \frac{1}{4}\left[b^2 - 2bd + d^2 - 2b^2 + 2bd + 4c\right] \\
&\quad = \frac{1}{4}\left[-b^2 + d^2 + 4c\right] \\
&\quad = \frac{1}{4}\left[-b^2 + (b^2-4c) + 4c\right] = \frac{1}{4}\cdot 0 = 0
\end{aligned}$$
and that
$$\begin{aligned}
&\left(\frac{-b-d}{2}\right)^2 + b\left(\frac{-b-d}{2}\right) + c \\
&\quad = \frac{1}{4}(-b-d)^2 + \frac{b}{2}(-b-d) + c \\
&\quad = \frac{1}{4}\left[(-b-d)^2 + 2b(-b-d) + 4c\right] \\
&\quad = \frac{1}{4}\left[(-b)^2 + 2(-b)(-d) + (-d)^2 + 2b(-b) + 2b(-d) + 4c\right] \\
&\quad = \frac{1}{4}\left[b^2 + 2bd + d^2 - 2b^2 - 2bd + 4c\right] \\
&\quad = \frac{1}{4}\left[-b^2 + d^2 + 4c\right] \\
&\quad = \frac{1}{4}\left[-b^2 + (b^2-4c) + 4c\right] = \frac{1}{4}\cdot 0 = 0.
\end{aligned}$$

(b)

Proof. Let $x \in \mathbb{R}$ be arbitrary. Then,
$$\begin{aligned}
x^2 + bx + c &= x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} + c \\
&= \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c \\
&= \left(x + \frac{b}{2}\right)^2 + \left(-\frac{1}{4}\right)(b^2 - 4c).
\end{aligned}$$
Because $(x + b/2)^2 \geq 0$, we deduce that $(-1/4)(b^2-4c)$ is the smallest possible value of x^2+bx+c. But we see that $(-1/4)(b^2-4c) > 0$ because $b^2-4c < 0$. Hence $x^2 + bx + c > 0$, and therefore there is no numbers x satisfying $x^2 + bx + c = 0$. ☐

(c)

Proof. Suppose that x and y are not both 0. We deduce that
$$x^2 + xy + y^2 = x^2 + 2xy + y^2 - xy = (x+y)^2 - xy.$$
We note that $x^2 \geq 0$ and $y^2 \geq 0$ and $(x+y)^2 \geq 0$. There are now four cases. First, suppose that $x > 0$ and $y > 0$. Then $xy > 0$, and hence $x^2 + xy + y^2 > 0$. Second, suppose that $x < 0$ and $y < 0$. Then $-x > 0$ and $-y > 0$, and then $(-x)(-y) = xy > 0$. Hence, $x^2 + xy + y^2 > 0$. Third, suppose that $x > 0$ and $y < 0$. Then $xy < 0$, and then $-xy > 0$. Hence $(x+y)^2 - xy > 0$, and therefore $x^2 + xy + y^2 > 0$. Fourth, suppose that $x < 0$ and $y > 0$. This case is just like the previous case, and we omit details. ☐

(d) Without loss of generality, suppose that $x \neq 0$. Suppose that $x^2 + \alpha xy + y^2 > 0$. We observe that
$$\begin{aligned}
x^2 + \alpha xy + y^2 &= x^2 + \alpha xy + y^2 + \frac{\alpha^2 x^2}{4} - \frac{\alpha^2 x^2}{4} \\
&= \left(y + \frac{\alpha x}{2}\right)^2 + x^2 - \frac{\alpha^2 x^2}{4} \\
&= \left(y + \frac{\alpha x}{2}\right)^2 + x^2\left(1 - \frac{\alpha^2}{4}\right).
\end{aligned}$$
Because $(y + \alpha x/2)^2 \geq 0$, we deduce that $x^2(1 - \alpha^2/4) > 0$. We already know that $x \neq 0$, so $x^2 > 0$. This implies that $1 - \alpha^2/4 > 0$, or in other words $-\alpha^2 > -4$. Then $\alpha^2 < 4$. But then $|\alpha| < 2$. Thus, $-2 < \alpha < 2$.

Now suppose that $-2 < \alpha < 2$. If $xy = 0$, then clearly $x^2 + \alpha xy + y^2 > 0$ because $x \neq 0$. If $xy > 0$, then
$$\begin{aligned}
\alpha > -2 &\iff \alpha xy > -2xy \\
&\iff x^2 + \alpha xy + y^2 > x^2 - 2xy + y^2 = (x-y)^2 \geq 0 \\
&\iff x^2 + \alpha xy + y^2 > 0.
\end{aligned}$$
Similarly, if $xy < 0$, then
$$\begin{aligned}
\alpha < 2 &\iff \alpha xy > 2xy \\
&\iff x^2 + \alpha xy + y^2 > x^2 + 2xy + y^2 = (x+y)^2 \geq 0 \\
&\iff x^2 + \alpha xy + y^2 > 0.
\end{aligned}$$

(e) We have
$$\begin{aligned}
x^2 + bx + c &= x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} + c \\
&= \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}.
\end{aligned}$$
Because $(x + b/2)^2 \geq 0$, we deduce that the smallest possible value of $x^2 + bx + c$ is $c - b^2/4$. Similarly, we have
$$\begin{aligned}
ax^2 + bx + c &= a\left[x^2 + \frac{b}{a}x\right] + c \\
&= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right] + c \\
&= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c \\
&= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.
\end{aligned}$$
Because $a(x+b/2a)^2 \geq 0$, we deduce that the smallest possible value of ax^2+bx+c is $c - b^2/4a$.

Lemma 1.4 (Not in the book). *Let $x \geq 0$ and $y \geq 0$. Then $\sqrt{xy} = \sqrt{x}\sqrt{y}$.*

Problem 1-19. The fact that $a^2 \geq 0$ for all numbers a, elementary as it may seem, is nevertheless the fundamental idea upon which most important inequalities are ultimately based. The great-granddaddy of all inequalities is the *Schwarz inequality:*

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}.$$

(A more general form occurs in Problem 2-21.) The three proofs of the Schwarz inequality outlined below have only one thing in common-their reliance on the fact that $a^2 \geq 0$ for all a.

(a) Prove that if $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for some number $\lambda \geq 0$, then equality holds in the Schwarz inequality. Prove the same thing if $y_1 = y_2 = 0$. Now suppose that y_1 and y_2 are not both 0, and that there is no number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. Then

$$\begin{aligned}0 &< (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 \\ &= \lambda^2(y_1{}^2 + y_2{}^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1{}^2 + x_2{}^2).\end{aligned}$$

Using Problem 1-18, complete the proof of the Schwarz inequality.

(b) Prove the Schwarz inequality by using $2xy \leq x^2 + y^2$ (how is this derived?) with

$$x = \frac{x_i}{\sqrt{x_1{}^2 + x_2{}^2}}, \quad y = \frac{y_i}{\sqrt{y_1{}^2 + y_2{}^2}},$$

first for $i = 1$ and then for $i = 2$.

(c) Prove the Schwarz inequality by first proving that

$$(x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.$$

(d) Deduce, from each of these three proofs, that equality holds only when $y_1 = y_2 = 0$ or when there is a number $\lambda \geq 0$ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

Proof of (a). Suppose that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for some number $\lambda \geq 0$. Using Lemma 1.4 we deduce that

$$\begin{aligned}x_1y_1 + x_2y_2 = \lambda y_1{}^2 + \lambda y_2{}^2 = \lambda(y_1{}^2 + y_2{}^2) &= \sqrt{\lambda^2}\sqrt{(y_1{}^2 + y_2{}^2)^2} \\ &= \left(\sqrt{\lambda^2}\sqrt{y_1{}^2 + y_2{}^2}\right)\sqrt{y_1{}^2 + y_2{}^2} \\ &= \sqrt{\lambda^2(y_1{}^2 + y_2{}^2)}\sqrt{y_1{}^2 + y_2{}^2} \\ &= \sqrt{(\lambda y_1)^2 + (\lambda y_2)^2}\sqrt{y_1{}^2 + y_2{}^2} \\ &= \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}.\end{aligned}$$

Now suppose that $y_1 = y_2 = 0$. We then deduce that

$$\begin{aligned}x_1y_1 + x_2y_2 = x_1 \cdot 0 + x_2 \cdot 0 = 0 + 0 = 0 &= \sqrt{x_1{}^2 + x_2{}^2} \cdot 0 \qquad (*) \\ &= \sqrt{x_1{}^2 + x_2{}^2}\sqrt{0} = \sqrt{x_1{}^2 + x_2{}^2}\sqrt{0 + 0} \\ &= \sqrt{x_1{}^2 + x_2{}^2}\sqrt{0^2 + 0^2} = \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}. \qquad (1)\end{aligned}$$

Finally, suppose that y_1 and y_2 are not both 0, and that there is no number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. Then

$$\begin{aligned}0 &< (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 \\ &= \lambda^2(y_1{}^2 + y_2{}^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1{}^2 + x_2{}^2).\end{aligned}$$

Let $a = y_1{}^2 + y_2{}^2$ and $b = -2(x_1y_1 + x_2y_2)$ and $c = x_1{}^2 + x_2{}^2$. Then $a \geq 0$. By Part (e) of Problem 1-18 we observe that

$$a\lambda^2 + b\lambda + c = a\left(\lambda + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} > 0.$$

Suppose to the contrary that $c - b^2/4a \leq 0$. If $c - b^2/4a = 0$, then $a\,(\lambda + b/2a)^2 \geq 0$ implies $a\lambda^2 + b\lambda + c \geq 0$, which is a contradiction, and hence $c - b^2/4a \neq 0$. Now suppose that $c - b^2/4a < 0$. If $a\,(\lambda + b/2a)^2 = 0$, then $a\lambda^2 + b\lambda + c < 0$, which is again a contradiction. Hence, $c - b^2/4a > 0$. Using Lemma 1.4 we then deduce that

$$\begin{aligned}& x_1{}^2 + x_2{}^2 - \frac{(-2(x_1y_1 + x_2y_2))^2}{4(y_1{}^2 + y_2{}^2)} > 0 \\ \iff\ & x_1{}^2 + x_2{}^2 - \frac{(x_1y_1 + x_2y_2)^2}{y_1{}^2 + y_2{}^2} > 0 \\ \iff\ & -\frac{(x_1y_1 + x_2y_2)^2}{y_1{}^2 + y_2{}^2} > -(x_1{}^2 + x_2{}^2) \\ \iff\ & \frac{(x_1y_1 + x_2y_2)^2}{y_1{}^2 + y_2{}^2} < x_1{}^2 + x_2{}^2 \\ \iff\ & (x_1y_1 + x_2y_2)^2 < (x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2) \\ \iff\ & |x_1y_1 + x_2y_2| < \sqrt{(x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2)} \\ \iff\ & |x_1y_1 + x_2y_2| < \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}.\end{aligned}$$

Hence, $x_1y_1 + x_2y_2 \leq \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}$. □

Proof of (b). We observe that $(x - y)^2 \geq 0$, so

$$(x - y)^2 = x^2 - 2xy + y^2 \geq 0 \iff -2xy \geq -x^2 - y^2 \iff 2xy \leq x^2 + y^2.$$

Then

$$\frac{2x_1y_1}{\sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}} \leq \frac{x_1{}^2}{x_1{}^2 + x_2{}^2} + \frac{y_1{}^2}{y_1{}^2 + y_2{}^2},$$

and then

$$\frac{2x_2y_2}{\sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}} \leq \frac{x_2{}^2}{x_1{}^2 + x_2{}^2} + \frac{y_2{}^2}{y_1{}^2 + y_2{}^2}.$$

Adding the two inequalities together we obtain

$$\begin{aligned}& \frac{2x_1y_1 + 2x_2y_2}{\sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}} \leq \frac{x_1{}^2 + x_2{}^2}{x_1{}^2 + x_2{}^2} + \frac{y_1{}^2 + y_2{}^2}{y_1{}^2 + y_2{}^2} \\ \iff\ & \frac{2(x_1y_1 + x_2y_2)}{\sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}} \leq 1 + 1 = 2 \\ \iff\ & \frac{x_1y_1 + x_2y_2}{\sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}} \leq 1 \\ \iff\ & x_1y_1 + x_2y_2 \leq \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}.\end{aligned}$$

□

Proof of (c). We deduce that

$$\begin{aligned}&(x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2) \\ &= x_1{}^2y_1{}^2 + x_1{}^2y_2{}^2 + x_2{}^2y_1{}^2 + x_2{}^2y_2{}^2 \\ &= x_1{}^2y_1{}^2 + x_1{}^2y_2{}^2 + x_2{}^2y_1{}^2 + x_2{}^2y_2{}^2 + 2x_1y_1x_2y_2 - 2x_1y_1x_2y_2 \\ &= \left[(x_1y_1)^2 + 2(x_1y_1)(x_2y_2) + (x_2y_2)^2\right] + \left[(x_1y_2)^2 - 2(x_1y_2)(x_2y_1) + (x_2y_1)^2\right] \\ &= (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.\end{aligned}$$

We observe that $(x_1y_2 - x_2y_1)^2 \geq 0$, so using Lemma 1.4 it then follows that

$$\begin{aligned}& (x_1y_1 + x_2y_2)^2 \leq (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2 \\ \iff\ & (x_1y_1 + x_2y_2)^2 \leq (x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2) \\ \iff\ & |x_1y_1 + x_2y_2| \leq \sqrt{(x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2)} \\ \iff\ & |x_1y_1 + x_2y_2| \leq \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}.\end{aligned}$$

Hence, $x_1y_1 + x_2y_2 \leq \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}$. □

Proof of (d).

(a) We have already shown that equality holds only when $y_1 = y_2 = 0$ or when there is a number $\lambda \geq 0$ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

(b) If $y_1 = y_2 = 0$, then y is undefined but we observe that equality holds in the Schwarz inequality (see the equation (1)). Now suppose that y_1 and y_2 are not both 0. Then,

$$\begin{aligned}& \frac{2x_1y_1 + 2x_2y_2}{\sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}} \leq \frac{x_1{}^2 + x_2{}^2}{x_1{}^2 + x_2{}^2} + \frac{y_1{}^2 + y_2{}^2}{y_1{}^2 + y_2{}^2} \\ \iff\ & \frac{2(x_1y_1 + x_2y_2)}{\sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}} \leq 1 + 1 = 2 \\ \iff\ & \frac{x_1y_1 + x_2y_2}{\sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}} \leq 1 \\ \iff\ & \frac{x_1y_1 + x_2y_2}{\sqrt{(x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2}} \leq 1.\end{aligned}$$

Equality holds in the last inequality when $(x_1y_2 - x_2y_1)^2 = 0$. Suppose that $(x_1y_2 - x_2y_1)^2 = 0$. Then $x_1y_2 = x_2y_1$. There is some number $\lambda \geq 0$ such that $x_1 = \lambda y_1$. Then $\lambda y_1y_2 = x_2y_1$, and then $x_2 = \lambda y_2$.

(c) In this proof we deduced that

$$(x_1y_1 + x_2y_2)^2 \leq (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.$$

Clearly, equality holds in the last inequality when $(x_1y_2 - x_2y_1)^2 = 0$. Thus, this part is just like the previous part, and hence $y_1 = y_2 = 0$ or there is some number $\lambda \geq 0$ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. □

Problem 1-20. Prove that if

$$|x - x_0| < \frac{\varepsilon}{2} \quad \text{and} \quad |y - y_0| < \frac{\varepsilon}{2},$$

then

$$\begin{aligned} |(x+y) - (x_0 + y_0)| &< \varepsilon, \\ |(x-y) - (x_0 - y_0)| &< \varepsilon. \end{aligned}$$

Proof. By the Triangle Inequality we deduce that

$$\begin{aligned} &|x - x_0| + |y - y_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \\ &\iff |(x - x_0) + (y - y_0)| \leq |x - x_0| + |y - y_0| < \varepsilon \\ &\iff |(x - x_0) + (y - y_0)| < \varepsilon \\ &\iff |x - x_0 + y - y_0| < \varepsilon \\ &\iff |(x + y) - (x_0 + y_0)| < \varepsilon. \end{aligned}$$

Because $|y - y_0| = |y_0 - y|$ we also deduce that

$$\begin{aligned} &|x - x_0| + |y_0 - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \\ &\iff |(x - x_0) + (y_0 - y)| \leq |x - x_0| + |y - y_0| < \varepsilon \\ &\iff |(x - x_0) + (y_0 - y)| < \varepsilon \\ &\iff |x - x_0 + y_0 - y| < \varepsilon \\ &\iff |(x - y) - (x_0 - y_0)| < \varepsilon. \end{aligned}$$

□

Problem 1-21. Prove that if

$$|x - x_0| < \min\left(\frac{\varepsilon}{2(|y_0|+1)},\ 1\right) \quad \text{and} \quad |y - y_0| < \frac{\varepsilon}{2(|x_0|+1)},$$

then $|xy - x_0y_0| < \varepsilon$. (The notation "min" was defined in Problem 1-13, but the formula provided by that problem is irrelevant at the moment; the first inequality in the hypothesis just means that

$$|x - x_0| < \frac{\varepsilon}{2(|y_0|+1)} \quad \text{and} \quad |x - x_0| < 1;$$

at one point in the argument you will need the first inequality, and at another point you will need the second. One more word of advice: since the hypotheses only provide information about $x - x_0$ and $y - y_0$, it is almost a foregone conclusion that the proof will depend upon writing $xy - x_0y_0$ in a way that involves $x - x_0$ and $y - y_0$.)

Proof. Suppose that

$$|x - x_0| < \min\left(\frac{\varepsilon}{2(|y_0|+1)}, 1\right) \quad \text{and} \quad |y - y_0| < \frac{\varepsilon}{2(|x_0|+1)}.$$

We note that

$$|x| = |x + x_0 - x_0| = |x_0 + (x - x_0)| \leq |x_0| + |x - x_0| < |x_0| + 1.$$

Then,

$$\begin{aligned}
|xy - x_0y_0| &= |xy - xy_0 + xy_0 - x_0y_0| \\
&= |x(y - y_0) + y_0(x - x_0)| \\
&\leq |x(y - y_0)| + |y_0(x - x_0)| \\
&= |x| \cdot |y - y_0| + |y_0| \cdot |x - x_0| \\
&< (|x_0| + 1)|y - y_0| + |y_0| \cdot |x - x_0| \\
&\leq (|x_0| + 1)|y - y_0| + |y_0| \cdot |x - x_0| + |x - x_0| \\
&= (|x_0| + 1)|y - y_0| + (|y_0| + 1)|x - x_0| \\
&< \frac{(|x_0| + 1)\varepsilon}{2(|x_0| + 1)} + \frac{(|y_0| + 1)\varepsilon}{2(|y_0| + 1)} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

$\square$

Problem 1-22. Prove that if $y_0 \neq 0$ and

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{2}\right),$$

then $y \neq 0$ and

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| < \varepsilon.$$

Proof. Suppose that $y_0 \neq 0$, and

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{2}\right).$$

Suppose to the contrary that $y = 0$. We have

$$|y_0| = |-y_0| = |y - y_0| < \frac{|y_0|}{2},$$

and as a result $1 < 1/2$, which is a contradiction. Hence, $y \neq 0$. Now note that

$$|y_0| - |y| \leq |y_0 - y| = |y - y_0| < \frac{|y_0|}{2}.$$

This implies that

$$-|y| < \frac{|y_0|}{2} - |y_0| = -\frac{|y_0|}{2},$$

or in other words $|y| > |y_0|/2$. We then deduce that

$$\begin{aligned}
\left|\frac{1}{y} - \frac{1}{y_0}\right| &= \left|\frac{1}{y_0} - \frac{1}{y}\right| = \left|\frac{y - y_0}{y_0 y}\right| = \frac{|y - y_0|}{|y_0 y|} \\
&< \frac{\varepsilon|y_0|^2}{2|y_0 y|} = \frac{\varepsilon|y_0|}{|y_0 y|} \cdot \frac{|y_0|}{2} \\
&< \frac{\varepsilon|y_0|}{|y_0 y|} \cdot |y| = \frac{\varepsilon|y_0| \cdot |y|}{|y_0 y|} = \frac{\varepsilon|y_0 y|}{|y_0 y|} = \varepsilon.
\end{aligned}$$

$\square$

Problem 1-23. Replace the question marks in the following statement by expressions involving ε, x_0, and y_0 so that the conclusion will be true:

If $y_0 \neq 0$ and

$$|y - y_0| < ? \quad \text{and} \quad |x - x_0| < ?$$

then $y \neq 0$ and

$$\left| \frac{x}{y} - \frac{x_0}{y_0} \right| < \varepsilon.$$

This problem is trivial in the sense that its solution follows from Problems 1-21 and 1-22 with almost no work at all (notice that $x/y = x \cdot 1/y$). The crucial point is not to become confused; decide which of the two problems should be used first, and don't panic if your answer looks unlikely.

Proof. Suppose that $y_0 \neq 0$ and

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon |y_0|^2}{4(|x_0| + 1)} \right)$$

and

$$|x - x_0| < \min\left(\frac{\varepsilon}{2\left(\left| \frac{1}{y_0} \right| + 1 \right)}, 1 \right).$$

Let $\delta = \varepsilon / 2(|x_0| + 1)$, so

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\delta |y_0|^2}{2} \right).$$

From Problem 1-22 we deduce that $y \neq 0$ and

$$\left| \frac{1}{y} - \frac{1}{y_0} \right| < \delta = \frac{\varepsilon}{2(|x_0| + 1)},$$

and using Problem 1-21 we then conclude that

$$\left| \frac{x}{y} - \frac{x_0}{y_0} \right| = \left| x \cdot \frac{1}{y} - x_0 \cdot \frac{1}{y_0} \right| < \varepsilon.$$

$\square$

Problem 1-24. This problem shows that the actual placement of parentheses in a sum is irrelevant. The proofs involve "mathematical induction"; if you are not familiar with such proofs, but still want to tackle this problem, it can be saved until after Chapter 2, where proofs by induction are explained.

Let us agree, for definiteness, that $a_1 + \cdots + a_n$ will denote

$$a_1 + (a_2 + (a_3 + \cdots + (a_{n-2} + (a_{n-1} + a_n))) \cdots).$$

Thus $a_1 + a_2 + a_3$ denotes $a_1 + (a_2 + a_3)$, and $a_1 + a_2 + a_3 + a_4$ denotes $a_1 + (a_2 + (a_3 + a_4))$, etc.

(a) Prove that

$$(a_1 + \cdots + a_k) + a_{k+1} = a_1 + \cdots + a_{k+1}.$$

Hint: Use induction on k.

(b) Prove that if $n \geq k$, then

$$(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) = a_1 + \cdots + a_n.$$

Hint: Use part (a) to give a proof by induction on k.

(c) Let $s(a_1, \ldots, a_k)$ be some sum formed from $a_1, \ldots, a_k$. Show that

$$s(a_1, \ldots, a_k) = a_1 + \cdots + a_k.$$

Hint: There must be two sums $s'(a_1, \ldots, a_l)$ and $s''(a_{l+1}, \ldots, a_k)$ such that

$$s(a_1, \ldots, a_k) = s'(a_1, \ldots, a_l) + s''(a_{l+1}, \ldots, a_k).$$

Proof of (a). We use mathematical induction.

Base case: Setting $k = 3$, we get

$$(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3) = a_1 + a_2 + a_3.$$

Induction step: Let $k \geq 3$, and suppose that

$$(a_i + \cdots + a_{j-1}) + a_j = a_i + \cdots + a_j,$$

where $3 \leq j - i + 1 \leq k$. Then,

$$\begin{aligned}
&(a_1 + \cdots + a_k) + a_{k+1} \\
&\quad = (a_1 + (a_2 + (a_3 + \cdots + (a_{k-2} + (a_{k-1} + a_k))) \cdots) + a_{k+1} \\
&\quad = a_1 + ((a_2 + (a_3 + \cdots + (a_{k-2} + (a_{k-1} + a_k))) \cdots) + a_{k+1}) \\
&\quad = a_1 + ((a_2 + \cdots + a_k) + a_{k+1}) \\
&\quad = a_1 + (a_2 + \cdots + a_{k+1}) \\
&\quad = a_1 + (a_2 + (a_3 + \cdots + (a_{k-1} + (a_k + a_{k+1}))) \cdots) \\
&\quad = a_1 + \cdots + a_{k+1}.
\end{aligned}$$

□

Proof of (b). Suppose that $n \geq k$. If $n = k$, we will clearly reach the desired result. Now suppose that $n > k$. We use mathematical induction.

Base case: Setting $n = k + 1$, from Part (a) of this problem we deduce that

$$(a_1 + \cdots + a_k) + a_{k+1} = a_1 + \cdots + a_{k+1}.$$

Induction step: Let $n > k + 1$, and suppose that

$$(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_{n-1}) = a_1 + \cdots + a_{n-1}.$$

Using Part (a) of this problem we obtain

$$\begin{aligned}
&(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) \\
&\quad = (a_1 + \cdots + a_k) + [(a_{k+1} + \cdots + a_{n-1}) + a_n] \\
&\quad = [(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_{n-1})] + a_n \\
&\quad = (a_1 + \cdots + a_{n-1}) + a_n \\
&\quad = a_1 + \cdots + a_n.
\end{aligned}$$

□

Proof of (c). We note that $s(a_n) = a_n$ for all $n \in \mathbb{N}$. We use mathematical induction.

Base case: Setting $k = 2$, we get $s(a_1, a_2) = a_1 + a_2$.

Induction step: Let $k > 2$, and suppose that $s(a_1, \ldots, a_k) = a_1 + \cdots + a_k$ for all $k > 2$. There must be two sums $s'(a_1, \ldots, a_l)$ and $s''(a_{l+1}, \ldots, a_k)$ such that

$$s(a_1, \ldots, a_{k+1}) = s'(a_1, \ldots, a_l) + s''(a_{l+1}, \ldots, a_{k+1}),$$

where $1 \leq l \leq k$. By an induction hypothesis, it follows that

$$s'(a_1, \ldots, a_l) = a_1 + \cdots + a_l \quad \text{and} \quad s''(a_{l+1}, \ldots, a_{k+1}) = a_{l+1} + \cdots + a_{k+1}.$$

From Part (b) of this problem we then deduce that

$$s(a_1, \ldots, a_{k+1}) = (a_1 + \cdots + a_l) + (a_{l+1} + \cdots + a_{k+1}) = a_1 + \cdots + a_{k+1}.$$

□

Problem 1-25. Suppose that we interpret "number" to mean either 0 or 1, and $+$ and $\cdot$ to be the operations defined by the following two tables.

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Check that properties P1-P9 all hold, even though $1+1=0$.

Suppose that a, b, c are "numbers".

(P1) (Associative law for addition) We have

$$\begin{aligned}
0+(0+0) &= 0+0 \\
&= (0+0)+0, \\
1+(0+0) &= 1+0 \\
&= (1+0)+0, \\
0+(1+0) &= 0+1 \\
&= (0+1)+0, \\
0+(0+1) &= 0+1 \\
&= (0+0)+1, \\
1+(1+0) &= 1+1 \\
&= (1+1)+0, \\
0+(1+1) &= 0+0 = 0 = 1+1 \\
&= (0+1)+1, \\
1+(1+1) &= 1+0 = 1 = 0+1 \\
&= (1+1)+1.
\end{aligned}$$

Hence, $a+(b+c) = (a+b)+c$.

(P2) (Existense of an additive identity) We have

$$1+0 = 0+1 = 1 \quad \text{and} \quad 0+0 = 0.$$

Hence, $a+0 = 0+a = a$.

(P3) (Existence of additive inverses) We have

$$0+0 = 0 \quad \text{and} \quad 1+1 = 0.$$

Then $0 = -0$ and $1 = -1$. Hence, $a+(-a) = (-a)+a = 0$.

(P4) (Commutative law for addition) Because $0+1 = 1+0$, we deduce that $a+b = b+a$.

(P5) (Associative law for multiplication) We have

$$\begin{aligned}
0\cdot(0\cdot 0) &= 0\cdot 0 \\
&= (0\cdot 0)\cdot 0, \\
1\cdot(0\cdot 0) &= 1\cdot 0 \\
&= (1\cdot 0)\cdot 0, \\
0\cdot(1\cdot 0) &= 0\cdot 0 \\
&= (0\cdot 1)\cdot 0, \\
0\cdot(0\cdot 1) &= 0\cdot 0 \\
&= (0\cdot 0)\cdot 1, \\
1\cdot(1\cdot 0) &= 1\cdot 0 \\
&= (1\cdot 1)\cdot 0, \\
0\cdot(1\cdot 1) &= 0\cdot 1 = 0 = 0\cdot 1 \\
&= (0\cdot 1)\cdot 1, \\
1\cdot(1\cdot 1) &= 1\cdot 1 = 1 \\
&= (1\cdot 1)\cdot 1.
\end{aligned}$$

Hence, $a\cdot(b\cdot c) = (a\cdot b)\cdot c$.

(P6) (Existence of a multiplicative identity) Because $1\cdot 1 = 1$ and $1 \neq 0$ it follows that $a\cdot 1 = 1\cdot a = a$.

(P7) (Existence of multiplicative inverses) Because $1\cdot 1 = 1$ we deduce that $1 = 1^{-1}$, and hence $a\cdot a^{-1} = a^{-1}\cdot a = 1$, for $a \neq 0$.

(P8) (Commutative law for multiplication) Because $0\cdot 1 = 1\cdot 0$, we deduce that $a\cdot b = b\cdot a$.

(P9) (Distributive law) We have

$$\begin{aligned}
0\cdot(0+0) &= 0\cdot 0 = 0 = 0+0 \\
&= 0\cdot 0 + 0\cdot 0, \\
1\cdot(0+0) &= 1\cdot 0 = 0 = 1\cdot 0 + 0 \\
&= 1\cdot 0 + 1\cdot 0, \\
0\cdot(1+0) &= 0\cdot 1 = 0\cdot 1 + 0 \\
&= 0\cdot 1 + 0\cdot 0, \\
0\cdot(0+1) &= 0\cdot 1 = 0 = 0+0 \\
&= 0\cdot 0 + 0\cdot 1, \\
1\cdot(1+0) &= 1\cdot 1 = 1\cdot 1 + 0 \\
&= 1\cdot 1 + 1\cdot 0, \\
0\cdot(1+1) &= 0\cdot 0 = 0 = 0+0 \\
&= 0\cdot 1 + 0\cdot 1, \\
1\cdot(0+1) &= 1\cdot 1 = 1 = 0+1 \\
&= 1\cdot 0 + 1\cdot 1, \\
1\cdot(1+1) &= 1\cdot 0 = 0 = 1+1 \\
&= 1\cdot 1 + 1\cdot 1.
\end{aligned}$$

Hence, $a\cdot(b+c) = a\cdot b + a\cdot c$.

Numbers of Various Sorts

Problem 2-21. TBD

Bibliography

[1] Michael Spivak, **Calculus**, 4th ed., Publish or Perish, Inc., 2008.