

# Custom Representations of Inductive Families

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**Abstract.** Inductive families provide a convenient way of programming with dependent types. Yet, when it comes to compilation, their default linked-tree runtime representations, as well as the need to convert between different indexed views of the same data when programming with dependent types, can lead to unsatisfactory runtime performance. In this paper, we aim to introduce a language with dependent types, and inductive families with custom representations. Representations are a version of Wadler’s views [15], refined to inductive families like in Epi-gram [13]. However, representations come with compilation guarantees: a represented inductive family will not leave any runtime traces behind, without having to rely on automated optimisations such as deforestation [16]. This way, we can build a library of convenient inductive families based on a minimal set of primitives, whose re-indexing and conversion functions are erased at compile-time. In addition, we show how we can express inductive data optimisation techniques, such as representing **Nat**-like types as GMP-style big integers, without special casing in the compiler. With dependent types, reasoning about data representations is also possible; for example, we get computationally irrelevant isomorphisms between the original and represented data.

**Keywords:** Dependent types · Memory representation · Inductive families

## 1 Introduction

Inductive families are a generalisation of inductive data types found in some programming languages with dependent types. Every inductive definition is equipped with an eliminator that captures the notion of mathematical induction over the data, and in particular, enables structural recursion over the data. This is a powerful tool for programming as well as theorem proving. However, this abstraction has a cost when it comes to compilation: the runtime representation of inductive types is a linked tree structure. This representation is not always the most efficient for all operations, and often forces users to rely on more efficient machine primitives to achieve desirable performance, at the cost of structural recursion and dependent pattern matching. This is the first problem we aim to address in this paper.

Despite advances in the erasure of irrelevant indices in inductive families [5] and the use of theories with irrelevant fragments [3,14], there is still a need

to convert between different indexed views of the same data. For example, the function to convert from `List T` to `Vec T n` by forgetting the length index  $n$  is *not* erased by any current language with dependent types, unless vectors are defined as a refinement of lists with an erased length field (which hinders dependent pattern matching due to the presence of non-structural witnesses), or a Church encoding is used in a Curry-style context [9] (which restricts the flexibility of data representation). This is the second problem we aim to address in this paper.

Wadler’s views [15] provide a way to abstract over inductive interfaces, so that different views of the same data can be defined and converted between seamlessly. In the context of inductive families, views have been used in Epigram [13] that utilise the index refinement machinery of dependent pattern matching to avoid certain proof obligations with eliminator-like constructs. While views exhibit a nice way to transport across a bijection between the original data and the viewed data, they do not utilise this bijection to erase the view from the program. Despite deforestation handling this erasure to some extent, it is not guaranteed to erase all traces of the view from the program, and the optimisation might be difficult to predict.

In this paper, we propose an extension  $\lambda_{\text{REP}}$  to a core language with dependent types and inductive families  $\lambda_{\text{IND}}$ , which allows programmers to define custom, correct-by-construction data representations. This is done through user-defined translations of the constructors and eliminators of an inductive type to a concrete implementation, which form a bijective view of the original data called a ‘representation’. Representations are defined internally to the language, and require coherence properties that ensure a representation is faithful to its the original inductive family. In the final version of the paper, we plan to contribute the following:

- A dependent type system with inductive families  $\lambda_{\text{IND}}$ , and its extension by representations  $\lambda_{\text{REP}}$ .
- A formulation of common optimisations such as the ‘Nat-hack’, and similarly for other inductive types, as representations.
- A demonstration of zero-cost data reuse when reindexing by using representations.
- A translation from  $\lambda_{\text{REP}}$  to  $\lambda_{\text{IND}}$  that erases all inductive types with representations from the program.
- An implementation of this system and accompanying examples in SUPERFLUID, a programming language with inductive types and dependent pattern matching.

## 2 A tour of data representations

A common optimisation done by programming languages with dependent types such as Idris 2 and Lean is to represent natural numbers as GMP-style big integers. The definition of natural numbers looks like

$$\text{data Nat} \left\{ \begin{array}{l} 0 : \text{Nat} \\ 1+ : \text{Nat} \rightarrow \text{Nat} \end{array} \right\} \quad (1)$$

and generates a Peano-style induction principle  $\text{elim}_{\text{Nat}}$  of type<sup>1</sup>

$$(P : \text{Nat} \rightarrow \mathcal{U}) \rightarrow P \ 0 \rightarrow ((n : \text{Nat}) \rightarrow \overline{P \ n} \rightarrow P \ (1 + n)) \rightarrow (s : \text{Nat}) \rightarrow P \ s.$$

Without further intervention, the  $\text{Nat}$  type is represented in unary form, where each digit becomes an empty heap cell at runtime. This is inefficient for a lot of the basic operations on natural numbers, especially since computers are particularly well-equipped to deal with numbers natively, so many real-world implementations will treat  $\text{Nat}$  specially, swapping the default inductive type representation with one based on GMP integers. This is done by performing the replacements

$$|0| = 0 \tag{2}$$

$$|1 +| = 1 + \tag{3}$$

$$|\text{elim}_{\text{Nat}} P \ m_0 \ m_{1+} \ s| = \text{ubig-elim} \ |s| \ |m_0| \ |m_{1+}| \tag{4}$$

where  $|\cdot|$  denotes a source translation into a compilation target language with primitives  $\text{ubig-}$ .<sup>2</sup>

In addition to the constructors and eliminators, the compiler might also define translations for commonly used definitions which have a more efficient counterpart in the target, such as recursively-defined addition, multiplication, etc. The recursively-defined functions are well-suited to proofs and reasoning, while the GMP primitives are more efficient for computation.

The issue with this approach is that it only works for the data types which the compiler can recognise as special. Particularly in the presence of dependent types, other data types might end up being equivalent to  $\text{Nat}$  or another ‘nicely-representable’ type, but in a non-trivial way that the compiler cannot recognise. Hence, one of our goals is to extend this optimisation to work for any data type. To achieve this this, our framework requires that representations are fully typed in a way that ensures the behaviour of the representation of a data type matches the behaviour of the data type itself.

## 2.1 The well-typed Nat-hack

A representation definition looks like

$$\text{repr Nat as UBig} \left\{ \begin{array}{l} 0 \text{ as } 0 \\ 1 + n \text{ as } 1 + n \\ \text{elim}_{\text{Nat}} \text{ as } \text{ubig-elim} \\ \text{by } \text{ubig-elim-zero-id}, \\ \quad \text{ubig-elim-add-one-id} \end{array} \right\}$$

<sup>1</sup> Recursive parameters like  $\overline{P \ n}$  are lazy, which makes the eliminator more efficient when they are not used.

<sup>2</sup> Idris 2 will in fact look for any ‘Nat-like’ types and apply this optimisation. A Nat-like type is any type with two constructors, one with arity zero and the other with arity one. A similar optimisation is also done with list-like and boolean-like types because they have a canonical representation in the target runtime, Chez Scheme.

**Nat** is represented as the type **UBig** of GMP-style unlimited-size unsigned integers, with translations for the constructors **0** and **1+**, and the eliminator **elim<sub>Nat</sub>**. Additionally, the eliminator satisfies the expected computation rules of the **Nat** eliminator, which are postulated as propositional equalities. This representation is valid in a signature containing the primitives

$$\begin{aligned} &0, 1 : \mathbf{UBig} \quad +, \times : \mathbf{UBig} \rightarrow \mathbf{UBig} \rightarrow \mathbf{UBig} \\ &\mathbf{ubig-elim} : (P : \mathbf{UBig} \rightarrow \mathcal{U}) \rightarrow P \ 0 \rightarrow ((n : \mathbf{UBig}) \rightarrow \overline{P \ n} \rightarrow P \ (1 + n)) \\ &\quad \rightarrow (s : \mathbf{UBig}) \rightarrow P \ s \end{aligned}$$

and propositional equalities

$$\begin{aligned} &\mathbf{ubig-elim-zero-id} :_{\forall P b r} \mathbf{ubig-elim} \ P \ b \ r \ 0 = b \\ &\mathbf{ubig-elim-add-one-id} :_{\forall P b r n} \mathbf{ubig-elim} \ P \ b \ r \ (1 + n) = r \ n \ (\lambda \_ . \mathbf{ubig-elim} \ P \ b \ r \ n). \end{aligned}$$

Representations can also be defined for functions on **Nat**, such as addition, multiplication, and other numeric operations, in terms of **UBig** primitives.

$$\mathbf{repr} \ \mathbf{add} \ \mathbf{as} \ + \ \mathbf{by} \ +\text{-id} \quad \mathbf{repr} \ \mathbf{mul} \ \mathbf{as} \ \times \ \mathbf{by} \ \times\text{-id}$$

These will be replaced during a translation process back to  $\lambda_{\text{IND}}$ , like rewriting rules [7], given that we have the appropriate lemmas to justify them in the signature.

This will effectively erase the **Nat** type from the compiled program, replacing all occurrences with the **UBig** type and its primitives. In a way, the hard work is done by the postulates above; we expect that the underlying implementation of **UBig** indeed satisfies them, which is a separate concern from the correctness of the representation itself. However, postulates are only needed when the representation target is a primitive; the next examples use defined types as targets, so that the coherence of the target eliminator follows from the coherence of other eliminators used in its implementation.

## 2.2 Vectors are just certain lists

In addition to representing inductive types as primitives, we can use representations to share the underlying data when converting between indexed views of the same data. For example, we can define a representation of **Vec** in terms of **List**, so that the conversion from one to the other is ‘compiled away’. We can do this by representing the indexed type as a refinement of the unindexed type by an appropriate relation. For the case of **Vec**, we know intuitively that

$$\mathbf{Vec} \ T \ n \simeq \{l : \mathbf{List} \ T \mid \mathbf{length} \ l = n\} := \mathbf{List}' \ T \ n$$

so we can start by choosing  $\mathbf{List}' \ T \ n$  as the representation of  $\mathbf{Vec} \ T \ n$ .<sup>3</sup> We are then tasked with providing terms that correspond to the constructors of **Vec** but

<sup>3</sup> We will take the subset  $\{x : A \mid P \ x\}$  to mean a  $\Sigma$ -type  $(x : A) \times P \ x$  where the right component is irrelevant and erased at runtime.

for  $\text{List}'$ . These can be defined as

$$\begin{aligned} \text{nil} &: \text{List}' T 0 & \text{cons} &: T \rightarrow \text{List}' T n \rightarrow \text{List}' T (1+ n) \\ \text{nil} &= (\text{nil}, \text{refl}) & \text{cons } x \ (xs, p) &= (\text{cons } x \ xs, \text{cong } (1+) \ p) \end{aligned}$$

Next we need to define the eliminator for  $\text{List}'$ , which should have the form

$$\begin{aligned} \text{elim-List}' &: (E : (n : \text{Nat}) \rightarrow \text{List}' T n \rightarrow \text{Type}) \\ &\rightarrow E 0 \ \text{nil} \\ &\rightarrow ((x : T) \rightarrow (n : \text{Nat}) \rightarrow (xs : \text{List}' T n) \rightarrow \overline{E \ n \ xs} \rightarrow E \ (1+ \ n) \ (\text{cons } x \ xs)) \\ &\rightarrow (n : \text{Nat}) \rightarrow (v : \text{List}' T n) \rightarrow E \ n \ v \end{aligned}$$

Dependent pattern matching does a lot of the heavy lifting by refining the length index and equality proof by matching on the underlying list. However we still need to substitute the lemma  $\text{cong } (1+) \ (1+-\text{inj } p) = p$  in the recursive case.

$$\begin{aligned} \text{elim-List}' \ P \ b \ r \ 0 \ (\text{nil}, \text{refl}) &= b \\ \text{elim-List}' \ P \ b \ r \ (1+ \ m) \ (\text{cons } x \ xs, e) &= \text{subst } (\lambda p. \ P \ (1+ \ m) \ (\text{cons } x \ xs, p)) \\ &\quad (1+-\text{cong-id } e) \ (r \ x \ (xs, 1+-\text{inj } e)) \\ &\quad (\lambda \_ . \ \text{elim-List}' \ P \ b \ r \ m \ (xs, 1+-\text{inj } e)) \end{aligned}$$

Finally, we need to prove that the eliminator satisfies the expected computation rules propositionally. These are

$$\begin{aligned} \text{elim-List}'\text{-nil-id} &: \text{elim-List}' \ P \ b \ r \ 0 \ (\text{nil}, \text{refl}) = b \\ \text{elim-List}'\text{-cons-id} &: \text{elim-List}' \ P \ b \ r \ (1+ \ m) \ (\text{cons } x \ xs, \text{cong } (1+) \ p) \\ &= r \ x \ (xs, p) \ (\lambda \_ . \ \text{elim-List}' \ P \ b \ r \ m \ (xs, p)) \end{aligned}$$

which we leave as an exercise, though they are evident from the definition of  $\text{elim-List}'$ . This completes the definition of the representation of  $\text{Vec}$  as  $\text{List}'$ , which would be written as

$$\text{repr } \text{Vec } T \ n \ \text{as } \text{List}' T \ n \ \left\{ \begin{array}{l} \text{nil as nil} \\ \text{cons as cons} \\ \text{elim}_{\text{Vec}} \text{ as elim-List}' \\ \text{by elim-List}'\text{-nil-id,} \\ \text{elim-List}'\text{-cons-id} \end{array} \right\}$$

Now the hard work is done; Every time we are working with a  $v : \text{Vec } T \ n$ , its form will be  $(l, p)$  at runtime, where  $l$  is the underlying list and  $p$  is the proof that the length of  $l$  is  $n$ . Under the assumption that the  $\Sigma$ -type's right component is irrelevant and erased at runtime, every vector is simply a list at runtime, where the length proof has been erased. In the full paper we will show how this erasure is achieved in practice in SUPERFLUID using Quantitative Type Theory [3].

We can utilise this representation to convert between `Vec` and `List` at zero runtime cost, by using the `repr` and `unrepr` operators of the language (defined in section 3). Specifically, we can define the functions

```
forget-length : Vec T n → List T
forget-length v = let (l, _) = repr v in l

recall-length : (l : List T) → Vec T (length l)
recall-length l = unrepr (l, refl)
```

and it holds by reflexivity that `forget-length` is a left inverse of `recall-length`.

### 2.3 General reindexing

The idea from the previous example can be generalised to any data type. In general, suppose that we have two inductive families

$$F : P \rightarrow \mathcal{U} \quad G : P \rightarrow X \rightarrow \mathcal{U}$$

for some index family  $X : P \rightarrow \mathcal{U}$ . If we hope to represent  $G$  as some refinement of  $F$  then we must be able to provide a way to compute  $G$ 's extra indices  $X$  from  $F$ , like we computed `Vec`'s extra `Nat` index from `List` with `length` in the previous example. This means that we need to provide a function

$$\text{comp} : \forall p. F\ p \rightarrow X\ p$$

which can then be used to form the family

$$F^{\text{comp}}\ p\ x := \{f : F\ p \mid \text{comp}\ f = x\}.$$

If  $G$  is ‘equivalent’ to the algebraic ornament of  $F$  by the algebra defining `comp` (given by an isomorphism between the underlying polynomial functors), then it is also equivalent to the  $\Sigma$ -type above. The ‘recomputation lemma’ of algebraic ornaments [8] then arises from its projections. Our system allows us to *set* the representation of  $G$  as  $F^{\text{comp}}$ , so that the forgetful map from  $G$  to  $F$  is the identity at runtime.

### 2.4 Zero-copy deserialisation

The machinery of representations can be used to implement zero-copy deserialisation of data formats into inductive types. For example, consider the following record for a player in a game:

$$\text{data Player} \left\{ \begin{array}{l} \text{player} : (\text{position} : \text{Position}) \\ \quad \rightarrow (\text{direction} : \text{Direction}) \\ \quad \rightarrow (\text{items} : \text{Fin MAX\_INVENTORY}) \\ \quad \rightarrow (\text{inventory} : \text{Inventory items}) \rightarrow \text{Player} \end{array} \right\}$$

We can use the `Fin` type to maintain the invariant that the inventory has a maximum size. Additionally, we can index the `Inventory` type by the number of items it contains, which might be defined similarly to `Vec`:

$$\text{data } \text{Inventory} \ (n : \text{Nat}) \left\{ \begin{array}{l} \text{empty} : \text{Inventory } 0 \\ \text{add} : \text{Item} \rightarrow \text{Inventory } n \rightarrow \text{Inventory } (1 + n) \end{array} \right\}$$

We can use the full power of inductive families to model the domain of our problem in the way that is most convenient for us. If we were writing this in a lower-level language, we might choose to use the serialised format directly when manipulating the data, relying on the appropriate pointer arithmetic to access the fields of the serialised data, to avoid copying overhead. Representations allow us to do this while still being able to work with the high-level inductive type.

We can define a representation for `Player` as a pair of a byte buffer and a proof that the byte buffer contents correspond to a player record. Similarly, we can define a representation for `Inventory` as a pair of a byte buffer and a proof that the byte buffer contents correspond to an inventory record of a certain size. The projection `inventory : (p : Player) → Inventory p.items` is compiled into some code to slice into the inventory part of the player's byte buffer. We assume that the standard library already represents `Fin` in the same way as `Nat`, so that reading the `items` field is a constant-time operation (we do not need to build a unary numeral). We can thus define the representation of `Player` as

$$\text{repr } \text{Player} \text{ as } \{ \text{Buf} \mid \text{IsPlayer} \} \left\{ \begin{array}{l} \text{player as buf-is-player} \\ \text{elim}_{\text{Player}} \text{ as elim-buf-is-player} \\ \text{by elim-buf-is-player-id} \end{array} \right\}$$

with an appropriate definition of `IsPlayer` which refines a byte buffer. We will provide the full details of this construction in the final paper.

## 2.5 Transitivity

Representations are transitive, so in the previous example, the ‘terminal’ representation of `Vec` also depends on the representation of `List`. It is possible to define a custom representation for `List` itself, for example a heap-backed array or a finger tree, and `Vec` would inherit this representation. However it will still be the case that `Repr (Vec T n) ≡ List T`, which means the `repr/Repr` operators only look at the immediate representation of a term, not its terminal representation. Regardless, we can construct predicates that find types which satisfy a certain ‘eventual’ representation. For example, given a `Buf` type of byte buffers, we can consider the set of all types which are eventually represented as a `Buf`:

$$\text{data } \text{ReprBuf} \ (T : \mathcal{U}) \left\{ \begin{array}{l} \text{buf} : \text{ReprBuf } \text{Buf} \\ \text{from} : \text{ReprBuf } (\text{Repr } T) \rightarrow \text{ReprBuf } T \\ \text{refined} : \text{ReprBuf } T \rightarrow \text{ReprBuf } \{ t : T \mid P \ t \} \end{array} \right\}$$

Every such type comes with a projection function to the `Buf` type

```

as-buf :  $\forall T. \text{ReprBuf } T \rightarrow T \rightarrow \text{Buf}$ 
as-buf buf x = x
as-buf (from t) x = as-buf t (repr x)
as-buf (refined t) (x, _) = as-buf t x

```

which eventually computes to the identity function after applying `repr` the appropriate amount of times. Upon compilation, every type is converted to its terminal representation, and all `repr` calls are erased, so the `as-buf` function is effectively the identity function at runtime.<sup>4</sup>

### 3 A type system for data representations

In this section, we will develop an extension of dependent type theory with inductive families and custom data representations. We start in section 3.2 by exploring the semantics of data representations in terms of algebras for signatures. In section 3.4 we define a core language with inductive families  $\lambda_{\text{IND}}$ . In section 3.6, we extend this language with data representations to form  $\lambda_{\text{REP}}$ . All of the examples in the paper are written in a surface language that elaborates to  $\lambda_{\text{REP}}$ . The dependent pattern matching can be elaborated to internal eliminators by standard techniques [10,6]. The  $\lambda_{\text{IND}}/\lambda_{\text{REP}}$  languages use eliminators instead of pattern matching.

We work in an extensional metatheory with a small universe **Set**,  $(a : A) \times B$  for dependent pairs,  $(a : A) \rightarrow B$  for dependent products, and  $=$  for equality. The metatheory also supports quotient-inductive-inductive definitions, which are used to define the syntaxes of the languages presented in this paper in the style of Kaposi and Altenkirch [2]. Weakening of terms in defined type theories is generally also left implicit to reduce syntactic noise, and often higher-order abstract syntax notation is used when de-Brujin indices are implied.

#### 3.1 Algebraic signatures

A representation of a data type must be able to emulate the behaviour of the original data type. In turn, the behaviour of the original data type is determined by its elimination, or induction principle. This means that a representation of a data type should provide an implementation of induction of the same ‘shape’ as the original. Induction can be characterised in terms of algebras and displayed algebras of algebraic signatures.

Algebraic signatures consist of a list of operations, each with a specified arity. There are many flavours of algebraic signatures with varying degrees of

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<sup>4</sup> We do not guarantee that an invocation of `as-buf` will be entirely erased, but rather that any invocation will eventually produce the identity function without having to perform a case analysis on its  $T$  subject.



expressiveness. For this paper, we are interested in algebraic signatures which can be used as a syntax for defining inductive families in a type theory. Thus, we define

$$\begin{aligned}
 \text{Theory} &: (\Gamma : \text{Con}) \rightarrow \text{Tel } \Gamma \rightarrow \mathbf{Set} \\
 \bullet &: \text{Theory } \Gamma \ P \\
 \triangleright &: (T : \text{Theory } P) \rightarrow \text{Op } P \rightarrow \text{Theory } P \\
 \text{Op} &: (\Gamma : \text{Con}) \rightarrow \text{Tel } \Gamma \rightarrow \mathbf{Set} \\
 \Pi &: (A : \text{Ty } \Gamma) \rightarrow \text{Op } (\Gamma \triangleright A) \ P \rightarrow \text{Op } \Gamma \ P \\
 \Pi\iota &: (p : \text{Tms } \Gamma \ P) \rightarrow \text{Op } \Gamma \ P \rightarrow \text{Op } \Gamma \ P \\
 \iota &: (p : \text{Tms } \Gamma \ P) \rightarrow \text{Op } \Gamma \ P
 \end{aligned}$$

The `Theory` sort represents a simple class of algebraic signatures with values in some type theory  $(\text{Con}, \text{Ty}, \text{Tm}, \dots)$ . We do not call it `Signature` to avoid a name clash with another kind of ‘signatures’ that we will define later. Indeed, algebraic theories [1] are a generalization of algebraic signatures. Each theory is described by an associated telescope of indices  $P$  as a telescope of indices, a *finite list* of operations:

- $\Pi A B$ , a (dependent) abstraction over some type  $A$  from the external type theory, of another operation  $B$ .
- $\Pi\iota p B$ , an abstraction over a recursive occurrence of the object being defined, with indices  $p$ , of another operation  $B$ .
- $\iota p$ , a constructor of the object being defined, with indices  $p$ .

For example, the theory of natural numbers lives in the empty context has an empty telescope of indices. It is defined by

$$\text{NatTh} : \text{Theory } \bullet \bullet = \bullet \triangleright \iota \ () \triangleright \Pi\iota \ () \ (\iota \ ())$$

We can add labels to aid readability, omitting parameters if they are empty, and use  $\Rightarrow$  for simple arrows:

$$\text{NatTh} := \bullet \triangleright \text{zero} : \iota \triangleright \text{succ} : \iota \Rightarrow \iota$$

Notice that this syntax only allows occurrences of  $\iota$  in positive positions, which is a requirement for inductive types. Different classes of theories and quantification are explored in detail by Kovács [12].

### 3.2 Algebras, displayed algebras and inductive algebras

In order to make use of our definition for theories, we would like to be able to interpret the structure into a semantic universe. An algebra  $(X, a) : \text{Alg } T$  for a carrier  $X$  and theory  $T$  defines a way to interpret the structure of  $T$  in terms of a type in a type theory  $X : \text{Ty } \Gamma$ . This produces a type which matches the

structure of  $T$  but replaces each occurrence of  $\iota$  with  $X$ . The function arrows in  $T$  are interpreted as function arrows in the target universe. Algebras for the theory of natural numbers might look like

$$\text{Alg NatTh} \simeq (X : \text{Ty } \Gamma) \times (\text{zero} : \text{Tm } \Gamma \ X) \times (\text{succ} : \text{Tm } \Gamma \ (\Pi \ X \ X))$$

We have a choice in terms of how much we want to interpret  $T$  in the external type theory, and how much we want to interpret it in the *metatheory*. Here we have chosen to interpret a theory as a metatheoretical iterated pair type, but an operation as a term in the type theory.

Very special classes of algebras support *induction*. To formulate induction, we first need to define displayed algebras. A displayed algebra  $(M, m)$  over an algebra  $(X, a)$  for a theory  $T$  with carrier  $M$  mirrors the shape of  $T$  like an algebra does, but each recursive occurrence  $\iota$  is now replaced by  $M$  applied to the corresponding value of the algebra. The displayed algebras for natural numbers are

$$\begin{aligned} \text{DispAlg } (X, \text{zero}, \text{succ}) &\simeq (M : \text{Ty } (\Gamma \triangleright X)) \\ &\quad \times (\text{zero}' : \text{Tm } \Gamma \ M[\text{zero}]) \\ &\quad \times (\text{succ}' : \text{Tm } \Gamma \ (\Pi (x : X) (\Pi M[x] M[\text{app succ } x]))) \end{aligned}$$

The type  $M$  is often called the *motive*, and  $m$  the *methods*.

**Definition 1.** *An algebra is inductive if every displayed algebra over it has a section.*

A section is a dependent function from  $X$  to  $M$  which takes its values from the displayed algebra. For natural numbers,

$$\begin{aligned} \text{Section } \{(X, \text{zero}, \text{succ})\} (M, \text{zero}', \text{succ}') & \\ \simeq (f : \text{Tm } \Gamma \ (\Pi (x : X) M[x])) & \\ \times (\text{app } f \ \text{zero} = \text{zero}') & \\ \times ((x : \text{Tm } \Gamma \ X) \rightarrow \text{app } f \ (\text{app succ } x) = \text{app } (\text{app succ}' x) \ (\text{app } f \ x)) & \end{aligned}$$

A section is the output of induction: a proof of  $M$  for all  $X$ .

### 3.3 Internal and external constructions

For the remainder of the paper we choose a fixed representation for algebras, displayed algebras and sections. We will omit the full definitions here for space, but where missing they can be found in section 8.1 of the appendix.

We define these constructions in two ways: one that is fully internal to the type theory and the other that is partially external (using the metatheory). In particular, all the external constructions are *positive* in the syntax of the type theory ( $\text{Ty}$ ,  $\text{Tm}$ ,  $\text{Tel}$ , etc) so that they can be added into the syntax retroactively.

First, we define an ‘external’ version of algebras

$$\begin{aligned} \text{Algebra} &: (T : \text{Theory } \Gamma \ P) \rightarrow \text{Ty } (\Gamma \triangleright P) \\ \text{Algebra } T \ X &:= (a : \text{In } T \ X) \rightarrow \text{Tm } \Gamma \ X[\langle \text{out } a \rangle] \\ \text{Alg } T &:= (X : \text{Ty } (\Gamma \triangleright P)) \times \text{Algebra } T \ X, \end{aligned}$$

which take some arguments `In` and produce the output `X` evaluated at the appropriate index `out a` based on the arguments. This is the uncurried version of the presentation with  $\Pi$  types. The type `In` is defined as  $(v : \text{Var } T) \times \text{Tms } \Gamma \ (\text{in } (T \ v) \ X)$  for where `Var` `T` indexes operations in theories. The function

$$\begin{aligned} \text{in} &: \text{Op } \Gamma \ P \rightarrow (\text{Ty } \Gamma \triangleright P) \rightarrow \text{Tel } \Gamma \\ \text{in } (\Pi \ A \ B) \ X &:= \bullet \triangleright (a : A) \triangleright \text{in } B[\langle a \rangle] \ X \\ \text{in } (\Pi \iota \ p \ B) \ X &:= \bullet \triangleright X[\langle p \rangle] \triangleright \text{in } B \ X \\ \text{in } (\iota \ p) \ X &:= \bullet, \end{aligned}$$

computes a telescope in the type theory, of the input arguments to the algebra element of the operation the induction is on. For the output, there is a function to compute the indices. It is defined as `out {T v} a X`, where

$$\begin{aligned} \text{out} &: \{O : \text{Op } \Gamma \ P\} \rightarrow \text{Tms } \Gamma \ (\text{in } O \ X) \rightarrow \text{Ty } \Gamma \rightarrow \text{Tms } \Gamma \ P \\ \text{out } \{\Pi \ A \ B\} \ (a, t) \ X &:= \text{out } \{B[\langle a \rangle]\} \ t \ X \\ \text{out } \{\Pi \iota \ p \ B\} \ (r, t) \ X &:= \text{out } \{B\} \ t \ X \\ \text{out } \{\iota \ p\} \ () \ X &:= p. \end{aligned}$$

We can also define an internal version of algebras as telescopes  $\text{alg } T := \bullet \triangleright (X : \Pi \ P \ U) \triangleright \text{algebra } T \ X$  where

$$\begin{aligned} \text{algebra} &: \text{Theory } \Gamma \ P \rightarrow \text{Ty } (\Gamma \triangleright P) \rightarrow \text{Tel } \Gamma \\ \text{algebra } \bullet \ X &= \bullet \\ \text{algebra } (T \triangleright O) \ X &:= \text{algebra } T \ X \triangleright \Pi \ (a : \text{in } O \ X) \ X[\text{out } O \ a \ X]. \end{aligned}$$

All internal constructions have ‘realisation’ functions into the metatheory

$$\begin{aligned} \ulcorner \_ \urcorner &: \text{Tms } \Gamma \ (\text{alg } T) \rightarrow \text{Alg } T \\ \ulcorner \_ \urcorner &: \text{Tms } \Gamma \ (\text{algebra } T \ X) \rightarrow \text{Algebra } T \ X. \end{aligned}$$

A similar construction can be performed for displayed algebras over external algebras

$$\begin{aligned} \text{dispAlgebra } (X, x) \ M &: \text{Tel } \Gamma \\ \text{DispAlg } (X, x) &:= (M : \text{Ty } (\Gamma \triangleright P \triangleright X)) \times \text{Tms } \Gamma \ (\text{dispAlgebra } (X, x) \ M) \end{aligned}$$

and we can use the realisation function for algebras to get internal displayed algebras over internal algebras

$$\begin{aligned} \text{dispAlg} &: \text{Tel } (\Gamma \triangleright \text{alg } T) \\ \text{dispAlg} &:= a. \bullet \triangleright (M : \Pi \ P \ (\Pi \ X \ U)) \triangleright \text{dispAlgebra } \ulcorner a \urcorner \ (\text{El } M @ @). \end{aligned}$$

with realisation functions  $\ulcorner\_ \urcorner : \mathsf{Tms} \Gamma (\mathsf{dispAlg}[\langle t \rangle]) \rightarrow \mathsf{DispAlg} \ulcorner t \urcorner$  and  $\ulcorner\_ \urcorner : \mathsf{Tms} \Gamma (\mathsf{dispAlgebra} t M) \rightarrow \mathsf{DispAlgebra} t M$ .

Finally, we construct external sections over displayed algebras

$$\mathsf{Section} (M, m) := (f : \mathsf{Sec} M) \times \mathsf{Coh} f \quad \mathsf{Sec} M := \mathsf{Tm} (\Gamma \triangleright P \triangleright X) M$$

$$\mathsf{Coh} f := \forall a. f[\langle \mathsf{out} a; x a \rangle] = \mathsf{apply} m f a$$

$$\mathsf{IntCoh} f := \forall a. \mathsf{Tm} \Gamma (\mathsf{Id} f[\langle \mathsf{out} a; x a \rangle] (\mathsf{apply} m f a))$$

which have coherence rules using either the equality of the metatheory ( $\mathsf{Coh}$ ) or the propositional equality of the type theory ( $\mathsf{IntCoh}$ ). The  $\mathsf{apply}$  function takes a displayed algebra, a section, and some arguments  $\mathsf{In}$ , and evaluates the section at those arguments. Sections also have internal analogues

$$\begin{aligned} \mathsf{section} &: \mathsf{Tel} (\Gamma \triangleright \mathsf{alg} T \triangleright \mathsf{dispAlg}) & \mathsf{sec} &: \mathsf{Tel} (\Gamma \triangleright \mathsf{alg} T \triangleright \mathsf{dispAlg}) \\ \mathsf{section} &:= X x M m. \mathsf{sec} \triangleright \mathsf{coh} & \mathsf{sec} &:= X x M m. \Pi P (\Pi X M) \end{aligned}$$

$$\mathsf{coh} : \mathsf{Tel} (\Gamma \triangleright (X, x) : \mathsf{alg} T \triangleright (M, m) : \mathsf{dispAlg} \triangleright \Pi P (\Pi X M))$$

which only use propositional equality.

We can define realisations  $\ulcorner\_ \urcorner_0 : \mathsf{Tms} \Gamma (\mathsf{section}[\langle t \rangle, \langle m \rangle]) \rightarrow \mathsf{Sec} \ulcorner m \urcorner$  and  $\ulcorner\_ \urcorner_1 : (t : \mathsf{Tms} \Gamma (\mathsf{section}[\langle t \rangle, \langle m \rangle])) \rightarrow \mathsf{IntCoh} \ulcorner t \urcorner_0$  which produce only internal coherence proofs. Now we can define a synonym for internal inductive algebras as a telescope

$$\mathsf{indAlg} T := \bullet \triangleright \mathsf{alg} T \triangleright \Pi \mathsf{dispAlg} \mathsf{section}$$

which has a realisation

$$\begin{aligned} \ulcorner\_ \urcorner_0 &: \mathsf{Tms} \Gamma (\Pi \mathsf{dispAlg}[\langle X \rangle] \mathsf{section}[\langle X \rangle]) \rightarrow \mathsf{DispAlg} X \rightarrow \mathsf{Sec} X \\ \ulcorner\_ \urcorner_1 &: (i : \mathsf{Tms} \Gamma (\Pi \mathsf{dispAlg}[\langle X \rangle] \mathsf{section}[\langle X \rangle])) \rightarrow \mathsf{DispAlg} \ulcorner X \urcorner \rightarrow \mathsf{IntCoh} \ulcorner i \urcorner_0. \end{aligned}$$

Next we will make use of the external versions of algebras, displayed algebras, and sections in order to add inductive algebras as part of the well-typed syntactical definition of a type theory in a strictly-positive but fully-applied manner. Later, we will make use of the internal versions in order to be able to package inductive algebras as a single syntactic entity that corresponds to data representations.

### 3.4 A type theory with inductive families, $\lambda_{\mathsf{IND}}$

The language  $\lambda_{\mathsf{IND}}$ , is a dependent type theory with  $\Pi$ ,  $\mathsf{Id}$ , and a universe  $\mathcal{U} : \mathcal{U}$ . We will not concern ourselves with a universe hierarchy but our results should be readily extensible to such a type theory. This language also has inductive families and global definitions. We follow a similar approach to Cockx and Abel [6] by packaging named inductive constructions and function definitions into a signature  $\Sigma : \mathsf{Sig}$ , and indexing contexts by signatures. The contexts  $\mathsf{Con}$  in the resulting theory are pairs  $(\Sigma : \mathsf{Sig}) \times \mathsf{Loc} \Sigma$  where  $\mathsf{Loc} \Sigma$  are local contexts given by a closed telescope of types as usual. Substitutions only occur between contexts of the same signature. Items in a signature  $\Sigma$  can be either

- function definitions **def**  $P \ A \ t$  for some parameters  $P : \text{Loc } \Sigma$ , return type  $A : \text{Ty } (\Sigma, P)$  and implementation  $t : \text{Tm } A \ (\Sigma, P)$ ,
- postulates **post**  $P \ A$  for some parameters  $P : \text{Loc } \Sigma$  and return type  $A : \text{Ty } (\Sigma, P)$ , or
- inductive type definitions **data**  $P \ T$  for some indices  $P : \text{Loc } \Sigma$  and theory  $T : \text{Theory } (\Sigma, \bullet) \ P$ .

We reuse the **Theory** type defined in the previous section. This allows us to define data types such as vectors

```

Vect : data
  (•▷  $\mathcal{U} \triangleright \text{Nat}$ )
  (•▷  $\text{nil} : \Pi (T : \mathcal{U}) (\iota (T, \text{zero}))$ )
  ▷  $\text{cons} : \Pi (T : \mathcal{U}) \Pi (n : \text{Nat}) \Pi \iota (T, n) \Pi T (\iota (T, \text{succ } n))$ .
    
```

We do not make a distinction between parameters and indices for data types, though this might be desirable in an implementation because it generates elimination rules which are structurally uniform in the parameters. Nevertheless, our system is extensible to one with uniform parameters and we leave its formulation as an implementation detail.

In order to actually construct inductive types in  $\lambda_{\text{IND}}$ , we need to extend the syntax with some term and type formers. First, we add a type former

$$\mathbf{D} : \mathbf{data} \ P \ T \in \Sigma \rightarrow \text{Ty } (\Sigma, \Delta \triangleright P)$$

which, given a data definition  $i$  in  $\Sigma$ , and terms for its indices  $p$ , constructs the data type  $(\mathbf{D} \ i)[p]$ . The relation  $I \in \Sigma$  finds items in a signature, to be thought of in a similar way to how  $\mathbf{Var} \ \Gamma \ A$  defines variables for type  $A$  in a local context  $\Gamma$ . It is evidently a decidable relation, and is a proposition for a fixed  $I$  and  $\Sigma$ .

Additionally, we add a constructor term

$$\mathbf{C} : \forall i. \text{Algebra } T \ (\mathbf{D} \ i)$$

which fully applied, defines the data constructor  $\mathbf{C} \ a$  of type  $(\mathbf{D} \ i)[\langle p \rangle]$  for arguments  $a$ , by ‘freely’ extending the syntax with an algebra for the type family  $\mathbf{D} \ i$ . The (strictly positive) occurrences of **Tm** in **Algebra** are part of the inductive syntax of the type theory. We can now construct, for example, natural numbers as

$$\mathbf{C}_{\text{Nat}} (\text{succ}, (\mathbf{C}_{\text{Nat}} (\text{zero}, ())) : \text{Tm } \Gamma \ (\mathbf{D} \ \text{Nat}).$$

Next, we add an eliminator term

$$\mathbf{E} : \forall i. (m : \text{DispAlg } \mathbf{C}_i) \rightarrow \text{Sec } m$$

which given a data definition  $i$  in  $\Sigma$ , a motive and methods for  $i$ , eliminates each  $d : (\mathbf{D} \ i)[\langle p \rangle]$  into  $M[\langle p; d \rangle]$ . This captures the induction principle of the data type. The coherence part of the section is captured by an equality constructor in the syntax

$$\mathbf{E}\text{-id} : \forall i. (m : \text{DispAlg } \mathbf{C}_i) \rightarrow \text{Coh } (\mathbf{E}_i \ m).$$

**Lemma 1.** *The constructor algebra  $C_i$  of a data type  $i$  is inductive.*

*Proof.* For every displayed algebra  $m$  over  $C$  we get a section  $(E\ m, E\text{-id}\ m)$ .

Finally, we add terms to the language for global function definitions and postulates

$$\begin{aligned} F &: \text{def } P\ A\ t \in \Sigma \rightarrow \text{Tm } (\Sigma, \Delta \triangleright P)\ A \\ P &: \text{post } P\ A \in \Sigma \rightarrow \text{Tm } (\Sigma, \Delta \triangleright P)\ A \end{aligned}$$

along with an equality constructor for function definitions

$$F\text{-id} : \forall i. F\ i = t.$$

The language we have defined thus far is comparable to the core language of a proof assistant. Next, we will explore how to extend this language to support data representations.

### 3.5 Extending $\lambda_{\text{IND}}$ with representations

We extend the language  $\lambda_{\text{IND}}$  to form  $\lambda_{\text{REP}}$ , which allows users to define custom representations for inductive types and global functions. The machinery of algebras that we have developed in section 3.2 allows for a very direct definition of representations of data types.

**Definition 2.** *A representation of a data type  $\text{data } P\ T$  is an inductive algebra for  $T$ .*

Representations live alongside items in signatures, and each item only corresponds to at most one representation. We achieve this by adding a constructor to signatures

$$\triangleright : (\Sigma : \text{Sig}) \rightarrow (I : \text{Item } \Sigma) \rightarrow \text{Rep } \Sigma\ I \rightarrow \text{Sig}.$$

Representations are defined inductively as

$$\begin{aligned} \text{Rep} &: (\Sigma : \text{Sig}) \rightarrow \text{Item } \Sigma \rightarrow \text{Set} \\ \text{datarep} &: \text{Tms } (\Sigma, \epsilon)\ (\text{indAlg } T) \rightarrow \text{Rep } \Sigma\ (\text{data } P\ T) \\ \text{defrep} &: (x : \text{Tm } (\Sigma, P)\ A) \rightarrow \text{Tm } (\Sigma, P)\ (\text{Id } x\ t) \rightarrow \text{Rep } \Sigma\ (\text{def } P\ A\ t) \end{aligned}$$

A representation of an item is some data which can be used to *replace* an item in a program, in a semantically equivalent way. We will write  $\text{datarep } (R, r, Q)$  to unpack the telescope of an inductive algebra for a data representation with carrier  $R$ , algebra  $r$  and induction  $Q$ . Representations for definitions are also included, where a definition can be represented by a term propositionally equal to original definition, but perhaps with better computational properties.

### 3.6 Reasoning about representations

To allow reasoning about representations internally to  $\lambda_{\text{REP}}$  we add a type former

$$\text{Repr} : \text{Ty } \Gamma \rightarrow \text{Ty } \Gamma \quad (5)$$

along with two new terms in the syntax, forming an isomorphism

$$\text{repr} : \text{Tm } \Gamma \ T \simeq \text{Tm } \Gamma \ (\text{Repr } T) : \text{unrepr} \quad (6)$$

which preserves  $\Pi/\text{Id}/\mathcal{U}$ . The type  $\text{Repr } T$  is the defined representation of the type  $T$ . The term  $\text{repr}$  takes a term of type  $T$  to its representation of type  $\text{Repr } T$ , and  $\text{unrepr}$  undoes the effect of  $\text{repr}$ , treating a represented term as an inhabitant of its original type. These new constructs come with equality constructors in the syntax shown in fig. 1.

$$\begin{array}{ll}
\text{repr} : \text{unrepr } (\text{repr } t) \equiv t & \text{Repr-}\mathcal{U} : \text{Repr } \mathcal{U} \equiv \mathcal{U} \\
\text{reprl} : \text{repr } (\text{unrepr } t) \equiv t & \text{repr-code} : \text{repr } (\text{code } T) \equiv \text{code } T \\
& \text{unrepr-code} : \text{unrepr } (\text{code } T) \equiv \text{code } T \\
\text{Repr-}\Pi : \text{Repr } (\Pi \ T \ U) \equiv \Pi \ T \ (\text{Repr } U) & \\
\text{repr-}\lambda : \text{repr } (\lambda \ u) \equiv \lambda \ (\text{repr } u) & \text{Repr-Id} : \text{Repr } (\text{Id } a \ b) \equiv \text{Id } (\text{repr } a) \ (\text{repr } b) \\
\text{unrepr-}\lambda : \text{unrepr } (\lambda \ u) \equiv \lambda \ (\text{unrepr } u) & \text{repr-refl} : \text{repr } (\text{refl } u) \equiv \text{refl } (\text{repr } u) \\
\text{repr-}@ : \text{repr } (f \ @) \equiv (\text{repr } f) \ @ & \text{unrepr-refl} : \text{unrepr } (\text{refl } u) \equiv \text{refl } (\text{unrepr } u) \\
\text{unrepr-}@ : \text{unrepr } (f \ @) \equiv (\text{unrepr } f) \ @ & \text{repr-J} : \text{repr } (\text{J } C \ w \ e) \\
& \equiv \text{J } (\text{Repr } C) \ (\text{repr } w) \ e \\
\text{repr}[] : \text{repr } (t[\sigma]) \equiv (\text{repr } t)[\sigma] & \text{unrepr-J} : \text{unrepr } (\text{J } (\text{Repr } C) \ w \ e) \\
\text{unrepr}[] : \text{unrepr } (t[\sigma]) \equiv (\text{unrepr } t)[\sigma] & \equiv \text{J } C \ (\text{unrepr } w) \ e \\
\text{Repr}[] : \text{Repr } (T[\sigma]) \equiv (\text{Repr } T)[\sigma] &
\end{array}$$

**Fig. 1.** Coherence of the representation operators with substitutions,  $\Pi$ ,  $\text{Id}$ , and universes. The terms  $\text{Repr } (\text{El } t)$ ,  $\text{repr } (\pi_2 \sigma)$  and  $\text{unrepr } (\pi_2 \sigma)$  do not reduce.

So far the representation operators do not really do much other than commute with almost everything in the syntax. In order to make them useful, we need to define how they compute when they encounter data types which are represented in the signature. We use a decidable relation  $R \in_i \Sigma'$  to mean that  $R : \text{Rep } \Sigma \ I$  is the representation of an item  $I : \text{Item } \Sigma$  where  $i : I \in \Sigma'$ . This relation is a proposition, so it is proof-irrelevant. Furthermore, it is stable under weakening of contexts and signatures, because each item can only be represented once in a signature. In the following rules,  $r : \text{datarep } (R, r, Q) \in_i \Sigma$ .

Firstly, we define the reduction that occurs when a type  $\text{D } i$  is represented,

$$\text{Repr-D}_i : \forall r. \text{Repr } (\text{D } i) = \text{El } R@, \quad (7)$$

yielding the carrier  $R$  of the inductive algebra that represents it (after converting it from a function into the universe to a type family).

Additionally, we can add a rule for representing constructors, albeit in propositional form, where

$$\text{repr-}C_i : \forall r. \text{Tm } (\Sigma, \Delta) \text{ (Id (repr (C } a)) (\ulcorner r \urcorner a^{\text{Repr}}))$$

Here, the operation  $\_^{\text{Repr}}$  is used to apply the term former `repr` to the recursive part of the arguments  $a$ . The full construction can be found in section 8.2 of the appendix

One might be tempted to make this equality definitional too. Unfortunately, this would render conversion checking undecidable, because if one applies `unrepr` to a term `repr (C a)` which has already been reduced to its representation, `unrepr (\ulcorner r \urcorner a^{\text{Repr}})`, there is no clear way to decide that this is convertible to `C a` even though the definitional equality rules would imply that it is (due to the annihilation of `repr` and `unrepr`). There is no equivalent of `unrepr` for types, so (7) preserves the decidability of conversion checking.

We can also add a propositional equality rules for representing eliminators. First, representing an eliminator just applies `repr` to the motive and methods:

$$\begin{aligned} \text{repr-}E_i &: \forall r. \text{Tm } (\Sigma, \Delta) \text{ (Id (repr (E } m)) (E \text{ } m^{\text{Repr}})) \\ \text{unrepr-}E_i &: \forall r. \text{Tm } (\Sigma, \Delta) \text{ (Id (unrepr (E } m)) (E \text{ } m^{\text{Unrepr}})) \end{aligned}$$

Additionally, eliminating something using `E` should be the same as eliminating the representation of that thing using the represented eliminator  $Q$ :

$$\text{repr-equiv-}E_i : \forall r. \text{Tm } (\Sigma, \Delta) \text{ (Id (E } m) (s. (\ulcorner Q \urcorner_0 m^{\text{Repr}*})[\langle \text{repr } s \rangle]))$$

Above we use more auxilliary definitions which represent the carriers of algebras, as well as displayed algebras (appendix section 8.2):

$$\begin{aligned} \_^{\text{Repr}} &: \text{Algebra } T \text{ } X \rightarrow \text{Algebra } T \text{ (Repr } X) \\ \_^{\text{Repr}} &: \text{DispAlgebra } a \text{ } M \rightarrow \text{DispAlgebra } a \text{ (Repr } M) \\ \_^{\text{Repr}*} &: \text{DispAlgebra } a \text{ } M \rightarrow \text{DispAlgebra } a^{\text{Repr}} \text{ (} p \text{ } x. M[\langle p; \text{unrepr } x \rangle]) \end{aligned}$$

We do not need an additional equality rule for representing function definitions as this is given by the equality proof  $p$  in the definition of a representation `defrepr t p`, when accounting for the definitional equality between a definition and its implementation.

### 3.7 Translating representations away

We now define a translation step  $\mathcal{R}$  from  $\lambda_{\text{REP}}$  to  $\lambda_{\text{IND}}^{\text{EXT}}$ , meant to be applied during the compilation process. More specifically, the translation target is the extensional flavour of  $\lambda_{\text{IND}}$  by adding the equality reflection rule. General undecidability of conversion is not a problem because type checking is decidable



for  $\lambda_{\text{REP}}$ <sup>5</sup> and we only need to apply this transformation after type checking, on fully-typed terms. The translation is defined over the syntax of  $\lambda_{\text{REP}}$  [4] such that definitional equality is preserved. Overall,  $\mathcal{R}$  preserves the structure of  $\lambda_{\text{REP}}$ , but maps constructs to their ‘terminal’ representations. First, we define a translation of signatures  $\mathcal{R} : \text{Sig}_{\text{REP}} \rightarrow \text{Sig}_{\text{IND}}^{\text{EXT}}$  as

$$\mathcal{R} \bullet := \bullet \quad \mathcal{R} (\Sigma \triangleright I) := \mathcal{R}\Sigma \triangleright \mathcal{R}I \quad \mathcal{R} (\Sigma \sqsupseteq I R) := \mathcal{R}\Sigma$$

which erases all items with defined representations. This utilises a translation of items  $\mathcal{R} : \text{Item}_{\text{REP}} \Sigma \rightarrow \text{Item}_{\text{IND}}^{\text{EXT}} \mathcal{R}\Sigma$  which simply recurses on all subterms with  $\mathcal{R}$ . Types are translated as

$$\begin{aligned} \mathcal{R} : \text{Ty}_{\text{REP}} (\Sigma, \Delta) &\rightarrow \text{Ty}_{\text{IND}}^{\text{EXT}} (\mathcal{R}\Sigma, \mathcal{R}\Delta) \\ \mathcal{R} (\text{D } i) &:= \begin{cases} \mathcal{R} (\text{El } R @) & \text{if } \text{datarep } (R, r, Q) \in_i \Sigma \\ \text{D } \mathcal{R}i & \text{otherwise} \end{cases} \quad \mathcal{R} (\text{Repr } T) := \mathcal{R}T \end{aligned}$$

(otherwise recurse on all subterms with  $\mathcal{R}$ ).

The definitional equality rules of  $\text{Repr}$  and  $\text{D}$  are mirrored, but  $\mathcal{R}$  is now applied to all subterms. Similarly, terms are translated as

$$\begin{aligned} \mathcal{R} : \text{Tm}_{\text{REP}} (\Sigma, \Delta) T &\rightarrow \text{Tm}_{\text{IND}}^{\text{EXT}} (\mathcal{R}\Sigma, \mathcal{R}\Delta) \mathcal{R}T \\ \mathcal{R} (\text{C}_i a) &= \begin{cases} \ulcorner \mathcal{R}r \urcorner \mathcal{R}a & \text{if } \text{datarep } (R, r, Q) \in_i \Sigma \\ \text{C}_{\mathcal{R}i} \mathcal{R}a & \text{otherwise} \end{cases} \\ \mathcal{R} (\text{E}_i m) &= \begin{cases} \ulcorner \mathcal{R}Q \urcorner_0 \mathcal{R}m & \text{if } \text{datarep } (R, r, Q) \in_i \Sigma \\ \text{E}_{\mathcal{R}i} \mathcal{R}m & \text{otherwise} \end{cases} \\ \mathcal{R} (\text{F}_i) &= \begin{cases} \mathcal{R}t & \text{if } \text{defrep } t p \in_i \Sigma \\ \text{F}_{\mathcal{R}i} & \text{otherwise} \end{cases} \quad \mathcal{R} (\text{repr } t), \mathcal{R} (\text{unrepr } t) := \mathcal{R}t \\ \mathcal{R} (\text{repr-C}_i a), \mathcal{R} (\text{repr-E}_i m), \mathcal{R} (\text{unrepr-E}_i m), \mathcal{R} (\text{repr-equiv-E}_i m) &:= \text{refl} \end{aligned}$$

(otherwise recurse on all subterms with  $\mathcal{R}$ )

Constructor, eliminator and definition translations mirror the equality rules in section 3.6, but apply  $\mathcal{R}$  to all subterms rather than only the recursive occurrences of the data type being represented. As a result, all of the propositional equality constructors are translated to reflexivity, since after applying  $\mathcal{R}$  both sides are identical.

The equality constructors of the syntax of  $\lambda_{\text{REP}}$  must also be translated. The base equalities of the type theory are preserved by their counterparts in  $\lambda_{\text{IND}}^{\text{EXT}}$ . The coherence rules for representation operators (fig. 1) are preserved by metatheoretic reflexivity on the other side, since all representation operators

<sup>5</sup> Not formalised in this paper.

are erased. Finally, coherence rules for definitions  $F$  and eliminators  $E$  are preserved by reflecting the propositional coherence rules provided by their defined representations:

$$\begin{aligned} \text{ap}_{\mathcal{R}} (E\text{-id}_i \ m) &:= \begin{cases} \text{reflect } \ulcorner \mathcal{R}Q \urcorner_1 \ \mathcal{R}m & \text{if } \text{datarep } (R, r) \ Q \in_i \Sigma \\ E\text{-id}_{\mathcal{R}i} \ \mathcal{R}m & \text{otherwise} \end{cases} \\ \text{ap}_{\mathcal{R}} (F\text{-id}_i) &:= \begin{cases} \text{reflect } \mathcal{R}p & \text{if } \text{defrep } t \ p \in_i \Sigma \\ F\text{-id}_{\mathcal{R}i} & \text{otherwise} \end{cases} \\ & \text{(otherwise recurse on all equality constructors with } \text{ap}_{\mathcal{R}}) \end{aligned}$$

**Theorem 1.**  $\mathcal{R}$  preserves typing and definitional equality:

$$(t_1, t_2 : \text{Tm}_{\text{IND}} \ \Gamma \ A) \rightarrow t_1 = t_2 \rightarrow \mathcal{R}t_1 = \mathcal{R}t_2$$

*Proof.* By  $\text{ap}_{\mathcal{R}}$ .

$\mathcal{R}$  is not injective in general, because two distinct (by their location in the signature) data types might be defined to have the same representation.

**Theorem 2.**  $\mathcal{R}$  is a left-inverse of the evident inclusion  $i : \lambda_{\text{IND}} \hookrightarrow \lambda_{\text{REP}}$ :

$$(t : \text{Tm}_{\text{IND}} \ \Gamma \ A) \rightarrow \mathcal{R}(it) = t.$$

*Proof.* The inclusion produces signatures in  $\lambda_{\text{REP}}$  without the  $\geq$  constructor. Thus no items have defined representations. Also, the action of  $\mathcal{R}$  on the image of  $i$  does not invoke the equality reflection rule. With that constraint, and by induction on the syntax,  $\mathcal{R} \circ i$  is the identity function on  $\lambda_{\text{IND}}$ .

### 3.8 Computational irrelevance

In order to reason about computational irrelevance, we assume that there is an additional program extraction step  $\mathcal{E}$  from  $\lambda_{\text{IND}}$  into some simply-typed calculus, denoted by vertical bars  $|x|$ . As opposed to  $\mathcal{R}$ ,  $\mathcal{E}$  operates on the unquotiented syntax of  $\lambda_{\text{IND}}$ . This can be justified by interpreting the quotient-inductive constructions from before into setoids [11]. This kind of transformation is used because we might want to compile two definitionally equal terms differently. For example, we might not always want to reduce function application redexes. We will use the `monospace` font for terms in  $\lambda$ .

**Definition 3.** A function  $f : \text{Tm} \ \Gamma \ (\Pi \ A \ B)$ , is computationally irrelevant if  $|\mathcal{R}A| = |\mathcal{R}B|$  and  $|\mathcal{R}f| = \text{id}$ .

**Theorem 3.** The type former *Repr* is injective up to internal isomorphism, i.e.

$$\text{Repr } T = \text{Repr } T' \rightarrow \text{Tm} \ \Gamma \ (\text{Iso } T \ T') \quad (8)$$

Moreover, this isomorphism is computationally irrelevant.

*Proof.* The forward direction is given by first applying `repr` to  $t$ , transporting over the given equality and then applying `unrepr`. The backward direction is given by applying `repr` to  $t'$ , transporting over the equality and then applying `unrepr`. The coherence holds by the rules `repr` and `reprl`. After applying  $\mathcal{R}$ , all representation operators are erased and the isomorphism is the identity on both sides (even before extraction).

Consider extending our languages with usage-aware subset  $\Sigma$ -types

$$\{ \_ \mid \_ \} : (A : \text{Ty } \Gamma) \rightarrow \text{Ty } (\Gamma \triangleright A) \rightarrow \text{Ty } \Gamma$$

in such a way that `Repr` and  $\mathcal{R}$  preserve them, but such that the extraction step erases the right component, i.e.  $|\{A \mid B\}| = |A|$ ,  $|(x, y)| = |x|$  and  $|\pi_1 x| = |x|$ .<sup>6</sup> Suppose we have an inductive family  $G : \text{data } (\bullet \triangleright I) T_G \in \Sigma$  over some index type  $I$ , and an inductive type  $F : \text{data } \bullet T_F \in \Sigma$  such that  $G$  is represented by a refinement  $f : \text{Tm } (\Sigma, \bullet) (\Pi (D F) I)$  of  $F$ ,

$$\text{datarep } (i. \{x : D F \mid \text{Id } (\text{app } f \ x) \ i\}, r, Q) \in_G \Sigma.$$

Then, we can construct computationally irrelevant functions

$$\begin{aligned} \text{forget}_i &: \text{Tm } \Gamma (\Pi (D G)[i] (D F)) & \text{remember} &: \text{Tm } \Gamma (\Pi (x : D F) (D G)[f \ x]) \\ \text{forget}_i &= \lambda g. \pi_1 (\text{repr } g) & \text{remember} &= \lambda x. \text{unrepr } (x, \text{refl}). \end{aligned}$$

Clearly  $|\mathcal{R} \text{ forget}_i| = |\mathcal{R} \text{ remember}| = \text{id}$ .

## 4 Implementation

SUPERFLUID is a programming language with dependent types,  $\mathcal{U} : \mathcal{U}$ , quantities, inductive families and dependent pattern matching. Its compiler is written in Haskell and the compilation target is JavaScript. Dependent pattern matching in SUPERFLUID is elaborated to a core language with internal eliminators. The  $\mathcal{R}$  transformation is then applied to the core program, which erases all inductive constructs with defined representations. This is finally translated to JavaScript, erasing all irrelevant data. As a result, we are able to represent `Nat` as JavaScript's `BigInt`, and `List T/SnocList T/Vec T n` as JavaScript's arrays with the appropriate index refinement, such that we can convert between them without any runtime overhead.

## 5 Related work

- Views by wadler - Views by McBride/McKinna - Cedile - Guillaume stuff - Refinements and induction by Atkey - Bit stealing - Inductive types deconstructed

<sup>6</sup> This can be implemented using quantitative type theory for example.

## 6 Future work

- Formalisation of NbE and decidability of equality - Show that  $\lambda$  can be translated into  $w$  types
  - Quotient types - Induction-induction and induction-recursion
  - Metaprogramming and deriving representations, ornaments, requesting that things are the identity
  - Exploration of CWF semantics

## 7 Conclusion

## References

1. Adamek, J., Rosicky, J., Vitale, E.M.: Cambridge tracts in mathematics: Algebraic theories: A categorical introduction to general algebra series number 184. Cambridge University Press, Cambridge, England (18 Nov 2010), <https://www.cambridge.org/academic/subjects/mathematics/logic-categories-and-sets/algebraic-theories-categorical-introduction-general-algebra?format=HB&isbn=9780521119221>
2. Altenkirch, T., Kaposi, A.: Type theory in type theory using quotient inductive types. In: Proceedings of the 43rd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages. pp. 18–29. POPL '16, Association for Computing Machinery, New York, NY, USA (11 Jan 2016), <https://doi.org/10.1145/2837614.2837638>
3. Atkey, R.: Syntax and semantics of quantitative type theory. In: Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science. pp. 56–65. LICS '18, Association for Computing Machinery, New York, NY, USA (9 Jul 2018), <https://doi.org/10.1145/3209108.3209189>
4. Boulier, S., Pédro, P.M., Tabareau, N.: The next 700 syntactical models of type theory. In: Proceedings of the 6th ACM SIGPLAN Conference on Certified Programs and Proofs. pp. 182–194. CPP 2017, Association for Computing Machinery, New York, NY, USA (16 Jan 2017), <https://doi.org/10.1145/3018610.3018620>
5. Brady, E., McBride, C., McKinna, J.: Inductive families need not store their indices. In: Types for Proofs and Programs. pp. 115–129. Springer Berlin Heidelberg (2004), [http://dx.doi.org/10.1007/978-3-540-24849-1\\_8](http://dx.doi.org/10.1007/978-3-540-24849-1_8)
6. Cockx, J., Abel, A.: Elaborating dependent (co)pattern matching. Proc. ACM Program. Lang. **2**(ICFP), 1–30 (30 Jul 2018), <https://doi.org/10.1145/3236770>
7. Cockx, J., Tabareau, N., Winterhalter, T.: The taming of the rew: a type theory with computational assumptions. Proc. ACM Program. Lang. **5**(POPL), 1–29 (4 Jan 2021), <https://doi.org/10.1145/3434341>
8. Dagand, P.E., McBride, C.: A categorical treatment of ornaments. arXiv [cs.PL] (16 Dec 2012), <http://arxiv.org/abs/1212.3806>
9. Diehl, L., Firsov, D., Stump, A.: Generic zero-cost reuse for dependent types. Proc. ACM Program. Lang. **2**(ICFP), 1–30 (30 Jul 2018), <https://doi.org/10.1145/3236799>

10. Goguen, H., McBride, C., McKinna, J.: Eliminating dependent pattern matching. In: Algebra, Meaning, and Computation, pp. 521–540. Lecture notes in computer science, Springer Berlin Heidelberg, Berlin, Heidelberg (2006), <https://research.google.com/pubs/archive/99.pdf>
11. Kovács, A.: Staged compilation with two-level type theory. arXiv [cs.PL] (20 Sep 2022), <http://arxiv.org/abs/2209.09729>
12. Kovács, A.: Type-theoretic signatures for algebraic theories and inductive types. Ph.D. thesis (2023), [https://andraskovacs.github.io/pdfs/phdthesis\\_compact.pdf](https://andraskovacs.github.io/pdfs/phdthesis_compact.pdf)
13. McBride, C., McKinna, J.: The view from the left. *J. Funct. Programming* **14**(1), 69–111 (Jan 2004), <https://www.cambridge.org/core/services/aop-cambridge-core/content/view/F8A44CAC27CCA178AF69DD84BC585A2D/S0956796803004829a.pdf/div-class-title-the-view-from-the-left-div.pdf>
14. Moon, B., Eades, III, H., Orchard, D.: Graded modal dependent type theory. In: Programming Languages and Systems. pp. 462–490. Springer International Publishing (2021), [http://dx.doi.org/10.1007/978-3-030-72019-3\\_17](http://dx.doi.org/10.1007/978-3-030-72019-3_17)
15. Wadler, P.: Views: a way for pattern matching to cohabit with data abstraction. In: Proceedings of the 14th ACM SIGACT-SIGPLAN symposium on Principles of programming languages. pp. 307–313. POPL '87, Association for Computing Machinery, New York, NY, USA (1 Oct 1987), <https://doi.org/10.1145/41625.41653>
16. Wadler, P.: Deforestation: transforming programs to eliminate trees. *Theor. Comput. Sci.* **73**(2), 231–248 (1 Jun 1990), <https://www.sciencedirect.com/science/article/pii/030439759090147A>

## 8 Appendix

### 8.1 Utilities when working with algebras

### 8.2 Utilities when working with representations