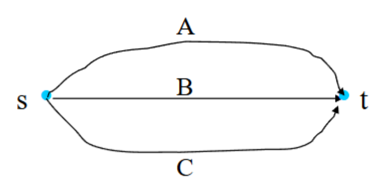
Kontovrakis for AGT Course - AGT 2026 HW2

February 2026

HW2 Notebook

Example 1: Congestion games



A simple example of a congestion game:

- A set of network users wants to move from point s to point t
- 3 possible routes, A, B, C
- Time delay in a route: depends on the number of users who have chosen this route
- $d_A(x) = 5x$, $d_B(x) = 7.5x$, $d_C(x) = 10x$,
- There is no need to examine all 243 possible profiles
- Exploiting symmetry:
  - In every route, the delay does not depend on who chose the route but only how many did so
- We can also exploit further properties
  - E.g. There can be no equilibrium where one of the routes is not used by some player

Homework: Find all the Nash equilibria of this game (if there are any)

1 Nash Equilibria Calculation

The 2 + 1 ways to calculate the Nash Equilibria in a Normal Form Game

In a finite discrete game, we learned that the Nash Equilibria in a game are strategy profiles where every player has no unilateral incentive to deviate from that profile. This means that he is "trapped" in that strategy profile , and that deviating to another strategy -given the fixed strategies of the other players- leads to a lower utility or that player.

An interesting case is when deviating from a strategy leads to same utility (we see it in this game) . This is like a Nash Equilibrium "neighborhood"

By definition

We have for each strategy profile to check that no player has an incentive to deviate.

For a game of 5 players with 3 strategies per player, we have 3^5 possible profiles. For each profile, we have to do $2 * 5$ checks, e.g. $3 - 1$ checks for each player , given a strategy profile. For n players with m strategies each, we must do $m^n * (m - 1) * n$ checks. This is the $O(n * m^{n+1})$ exhaustive search method.

By Best-Response

Fixing the strategies of other players , we can find the best response for each action. We fix the strategies of other players and find the strategy/strategies that are the best response, i.e. the strategy that yields the highest utility for the player. (In our problem the problem has a cost form). We keep the best responses for each player. The Nash Equilibria are found at strategy profiles that are made up from best-responses from each player. This reduces the computation time to $(m) * (m)^{(n-1)}$ for each player. This method has total computation time of $O(n * m^n)$

Exploit Symmetries of the Game

As we already said in the lecture, the Nash Equilibria in this game are invariant to permutations; e.g. if the strategy profile (A,A,B,C,A) which has 3 players choosing A , 1 player choosing B and one player choosing C is a Nash Equilibrium, then any profile with the same distribution is a Nash Equilibrium.

We have to prove this.

Small non-rigorous Proof

1. We notice that the delay functions only take into account the number of players that chooses a path, and not which player.

2. If we swap two players' choices, we still get the same delays.

3. Thus, since the counts are the same, they yield the same delays. This means that if no player had an incentive to move in the first arrangement, the same holds for the second arrangement.

(There is likely a way more rigorous way to prove this symmetry)

Since this symmetry holds we have shown that any profile that is not a Nash Equilibrium informs us that any other profile that has the same distribution is not a Nash Equilibrium. Equally, if a profile is a Nash Equilibrium, then any profile with the same distribution must be a Nash Equilibrium.

The Congestion Game under symmetries that lead to a game based on distributions

$d_A = 5x$
$d_B = 7.5x$
$d_C = 10x$

We need to check all the possible strategy profiles that produce delays of this form:

$$(d_A(x_1), d_B(x_2), d_C(x_3))$$

s.t.:
$\sum_{i=1}^{3} x_i = 5$ and $0 \leq x_i \leq 5$
We generate all the possible distributions, and the corresponding delays.

Table 1: Player Distributions and Corresponding Route Delays

Distribution (x_A, x_B, x_C)			**Delays** (d_A, d_B, d_C)		
x_A	x_B	x_C	d_A	d_B	d_C
0	0	5	0	0.0	50
0	1	4	0	7.5	40
0	2	3	0	15.0	30
0	3	2	0	22.5	20
0	4	1	0	30.0	10
0	5	0	0	37.5	0
1	0	4	5	0.0	40
1	1	3	5	7.5	30
1	2	2	5	15.0	20
1	3	1	5	22.5	10
1	4	0	5	30.0	0
2	0	3	10	0.0	30
2	1	2	10	7.5	20
2	2	1	10	15.0	10
2	3	0	10	22.5	0
3	0	2	15	0.0	20
3	1	1	15	7.5	10
3	2	0	15	15.0	0
4	0	1	20	0.0	10
4	1	0	20	7.5	0
5	0	0	25	0.0	0

Distribution Elimination to find NEs

To represent the player's incentive to move, we start from a distribution and if we find a distribution that is one step away, and leads to better delay for that specific player, then we eliminate the distribution. Based on this logic we compute the delay for a distribution. For each distribution , we compare the delay of each path, to the delay of other paths if they had one more player. If we don't find a path that has better delay, then the path is stable.

Results

Nash Equilibrium found at:

$$(\textbf{distribution} = (2, 2, 1)\ \textbf{delay} = (10, 15.0, 10))$$

$$(\textbf{distribution} = (3, 1, 1)\ \textbf{delay} = (15, 7.5, 10))$$