Robinson Arithmetic Notes

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1 Definition of Q

Explicit connectives	$\rightarrow$, $\forall$, $\exists$
Implicit connectives	$\neg$
Constants	0
Operations	$\mathbf{s}/1$, $+/2$, $\times/2$
Relations	$=/2$, $</2$

Q_1 $\forall x \forall y, ((\mathbf{s}x) = (\mathbf{s}y) \rightarrow x = y)$
Q_2 $\forall x, 0 \neq (\mathbf{s}x)$
Q_3 $\forall x, (x \neq 0 \rightarrow \exists y, x = (\mathbf{s}y))$
Q_4 $\forall x, (x + 0) = x$
Q_5 $\forall x \forall y, (x + (\mathbf{s}y)) = \mathbf{s}(x + y)$
Q_6 $\forall x, (x \times 0) = 0$
Q_7 $\forall x \forall y, (x \times (\mathbf{s}y)) = ((x \times y) + x)$
Q_8 $\forall x \forall y, (x < y \leftrightarrow \exists z, ((\mathbf{s}z) + x) = y)$

Q_8 seems unnecessary. It is if we just say $x < y$ by saying $\exists z, ((\mathbf{s}\ z) + x) = y$

Q shouldn't be able to define transitive closure though. Yet, **FOL**$\{\mathbb{N}, +, =\}$ can define $<$ as the transitive closure of $\mathbf{s}$ as $\exists z, ((\mathbf{s}\ z) + x) = y$? No, the other possibility is that **FOL** model theory does not constrain it to only represent the transitive closure and not something larger. We can't express that the path of ss starts from 0.

2 Redefining Q

Operations of arity n represented as relations of arity $n+1$.

Equality $x = y$ represented as $(+\ x\ 0\ y)$

Less than, $x < y$ represented as $\exists zz', (\mathbf{s}\ z\ z') \wedge (+\ x\ z'\ y)$

Explicit connectives	¬,∧,→,∀,∃
Implicit connectives	None
Constants	0
Operations	None
Relations	$\mathbf{s}/2$, $+/3$, $\times/3$

Q'_1 $\forall x \forall y, ((\exists z, (\mathbf{s}\ x\ z) \wedge (\mathbf{s}\ y\ z)) \rightarrow (+\ x\ 0\ y))$
Q'_2 $\forall x, \neg(\mathbf{s}\ x\ 0)$
Q'_3 $\forall x, (\neg(+\ x\ 0\ 0) \rightarrow \exists y, \mathbf{s}\ y\ x)$
Q'_4 $\forall x, (+\ x\ 0\ x)$
Q'_5 $\forall x \forall y, \exists y' z z', ((\mathbf{s}\ y\ y') \wedge (+\ x\ y\ z) \wedge (\mathbf{s}\ z\ z')) \rightarrow (+\ x\ y'\ z')$
Q'_6 $\forall x, (\times\ x\ 0\ 0)$
Q'_7 $\forall x \forall y, \exists y' z z', ((\mathbf{s}\ y\ y') \wedge (\times\ x\ y\ z) \wedge (+\ z\ x\ z')) \rightarrow (\times\ x\ y'\ z')$

Question: are these two systems equivalent?
What deductive system?
First-order logic: any complete deductive system.
$Q'_8, \forall x, \exists y, \mathbf{s}xy$
Remove axiom 4 due to redundancy?
$\forall x, (+\ x\ 0\ x)$

Assume an object x, then we seek to prove $+\ x\ 0\ x$ from the axioms.

By axiom 1:
$\forall x \forall y ((\exists z, (\mathbf{s}\ x\ z) \wedge (\mathbf{s}\ y\ z)) \rightarrow (+\ x\ 0\ y))$

By applying to x:
$\forall y ((\exists z, (\mathbf{s}\ x\ z) \wedge (\mathbf{s}\ y\ z)) \rightarrow (+\ x\ 0\ y))$

By applying to x
$(\exists z, (\mathbf{s}\ x\ z) \wedge (\mathbf{s}\ x\ z)) \rightarrow (+\ x\ 0\ x)$

By axiom Q'_8
$\forall x, \exists z, (\mathbf{s}\ x\ z)$

By applying to x
$\exists z, \mathbf{s}\ x\ z$

So, $\exists z, (\mathbf{s}\ x\ z) \wedge (\mathbf{s}\ x\ z)$

So, $\forall x, (+\ x\ 0\ x)$

3 Version 3

Curry the $\wedge$s into $\rightarrow$s
Remove Q'_4 and add Q'_8

Explicit connectives	$\neg,\rightarrow,\forall,\exists$
Implicit connectives	None
Constants	0
Operations	None
Relations	$\mathbf{s}/2$, $+/3$, $\times/3$

Q'_1 $\forall x, \exists y, \mathbf{s}\ x\ y$
Q'_2 $\forall x \forall y \forall z, (\mathbf{s}\ x\ z) \rightarrow (\mathbf{s}\ y\ z) \rightarrow (+\ x\ 0\ y)$
Q'_3 $\forall x, \neg(\mathbf{s}\ x\ 0)$
Q'_4 $\forall x, (\neg(+\ x\ 0\ 0) \rightarrow \exists y, \mathbf{s}\ y\ x)$
Q'_5 $\forall x \forall y \forall y' \forall z \forall z', (\mathbf{s}\ y\ y') \rightarrow (+\ x\ y\ z) \rightarrow (\mathbf{s}\ z\ z') \rightarrow (+\ x\ y'\ z')$
Q'_6 $\forall x, (\times\ x\ 0\ 0)$
Q'_7 $\forall x \forall y \forall y' \forall z \forall z', (\mathbf{s}\ y\ y') \rightarrow (\times\ x\ y\ z) \rightarrow (+\ z\ x\ z') \rightarrow (\times\ x\ y'\ z')$

Zero function $zero(x) = 0$
$zero\ x\ y = \times\ x\ 0\ y$
* By Q'_6 and the assumption that $\times$ is a function.

Successor function $suc(x) = x + 1$
$suc\ x\ y = \mathbf{s}\ x\ y$
* By the assumption that $\mathbf{s}$ is a function
* Other assumptions?

Projection function $P_i^n(x_0, ..., x_{n-1}) = x_i$ is represented by:
$P_i^n\ x_0\ ...\ x_{n-1}\ y = +\ x_i\ 0\ y$

Characteristic function of $=$,
$X(x_0, x_1) = 1$ if $x_0 = x_1$, 0 otherwise
$((+\ x_0\ 0\ x_1) \wedge (+\ y\ 0\ 1)) \vee (\neg(+\ x_0\ 0\ x_1) \wedge (+\ y\ 0\ 0))$

Composition

Regular minimization

4 Version 4

$\neg(+\ x\ 0\ 0)$ to be represented as $\exists y y', (\mathbf{s}\ y\ y') \wedge (+\ y'\ 0\ x)$. Only one negation operation used in the axioms now.

Explicit connectives	$\neg$,$\rightarrow$,$\forall$,$\exists$
Implicit connectives	None
Constants	0
Operations	None
Relations	$\mathbf{s}/2$, $+/3$, $\times/3$

Q'_1	$\forall x, \exists y, \mathbf{s}\ x\ y$
Q'_2	$\forall x \forall y \forall z, (\mathbf{s}\ x\ z) \rightarrow (\mathbf{s}\ y\ z) \rightarrow (+\ x\ 0\ y)$
Q'_3	$\forall x, \neg(\mathbf{s}\ x\ 0)$
Q'_4	$\forall x, ((\exists y y', (\mathbf{s}\ y\ y') \wedge (+\ 0\ y'\ x)) \rightarrow \exists z, \mathbf{s}\ z\ x)$
Q'_5	$\forall x \forall y \forall y' \forall z \forall z', (\mathbf{s}\ y\ y') \rightarrow (+\ x\ y\ z) \rightarrow (\mathbf{s}\ z\ z') \rightarrow (+\ x\ y'\ z')$
Q'_6	$\forall x, (\times\ x\ 0\ 0)$
Q'_7	$\forall x \forall y \forall y' \forall z \forall z', (\mathbf{s}\ y\ y') \rightarrow (\times\ x\ y\ z) \rightarrow (+\ z\ x\ z') \rightarrow (\times\ x\ y'\ z')$

Note that we can't do the same with $Q'_3 = \forall x, \neg(\mathbf{s}\ x\ 0)$ as this is what's keeping $\exists y y', (\mathbf{s}\ y\ y') \wedge (+\ y'\ 0\ x)$ sound as the property of $x \neq 0$.

Can we show that Q'_4 is redundant?
$\forall x, ((\exists y y', (\mathbf{s}\ y\ y') \wedge (+\ 0\ y'\ x)) \rightarrow \exists z, \mathbf{s}\ z\ x)$

Assume an object x
$((\exists y y', (\mathbf{s}\ y\ y') \wedge (+\ 0\ y'\ x)) \rightarrow \exists z, \mathbf{s}\ z\ x)$

Assume $((\exists y y', (\mathbf{s}\ y\ y') \wedge (+\ 0\ y'\ x))$

Prove $\exists z, \mathbf{s}\ z\ x)$

$(+\ y'\ 0\ x)$
$\mathbf{s}..x$

5 Version 5

Explicit connectives	$\neg,\wedge,\rightarrow,\forall,\exists$
Implicit connectives	None
Constants	0
Operations	None
Relations	$\mathbf{s}/2,\ +/3,\ \times/3$

Q_1' $\forall x, \exists y, \mathbf{s}\ x\ y$
Q_2' $\forall x \forall y \forall z, (\mathbf{s}\ x\ z) \rightarrow (\mathbf{s}\ y\ z) \rightarrow (+\ x\ 0\ y)$
Q_3' $\forall x, \neg(\mathbf{s}\ x\ 0)$
Q_4' $\forall x, ((\exists y y', (\mathbf{s}\ y\ y') \wedge (+\ 0\ y'\ x)) \rightarrow \exists z, \mathbf{s}\ z\ x)$
Q_5' $\forall x \forall y \forall y' \forall z, (\mathbf{s}\ y\ y') \rightarrow (+\ x\ y\ z) \rightarrow \exists z', (+\ x\ y'\ z') \wedge (\mathbf{s}\ z\ z')$
Q_6' $\forall x, (\times\ x\ 0\ 0)$
Q_7' $\forall x \forall y \forall y' \forall z, (\mathbf{s}\ y\ y') \rightarrow (\times\ x\ y\ z) \rightarrow \exists z', (\times\ x\ y'\ z') \wedge (+\ z\ x\ z')$

6 Simulating primitive recursion with composition and regular minimization

7 A function is representable in Q if and only if it is computable

8 Decidability

9 Deductive systems for classical propositional logic

9.1 Frege

Rules:
$A, A \to B \vdash B$

Axioms:
$A \to (B \to A)$
$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
$(A \to B) \to (\neg B \to \neg A)$
$\neg\neg A \to A$
$A \to \neg\neg A$
Proof of completeness?

9.2 Hilbert's axiom system

Rules:
$A, A \to B \vdash B$

Axioms:
$A \to (B \to A)$
$(A \to (B \to C)) \to (B \to (A \to C))$
$(B \to C) \to ((A \to B) \to (A \to C))$
$A \to (\neg A \to B)$
$(A \to B) \to ((\neg A \to B) \to B)$

10 References

[1] http://openlogicproject.org/files/2017/02/phil479-screen.pdf
[2] https://math.stackexchange.com/questions/873489/the-undecidability-of-robinson-arithmetic-without-multiplication