1 Signal Model

The signal model is as follows:

$$x[t] = A\cos(\omega_0(t-\nu)+\phi)+\epsilon[t] \tag{1}$$

for $t = 0,\ldots,T-1$ where A is the signal amplitude, $\omega_0 \in (0,\pi)$ is the true signal frequency, $\nu = \frac{T-1}{2}$, ϕ is the initial signal phase and is uniformly distributed $[0,2\pi)$, and ϵ is an independent, identically normally distributed, zero mean process with variance σ_ϵ^2.

The signal-to-noise ratio is given by

$$\texttt{SNR} = \frac{A^2}{2\sigma_\epsilon^2} \tag{2}$$

2 Maximum Likelihood Estimation

Much of this follows Kay's **Example 7.16** in [1].

Assuming that the noise ϵ is normally distributed, then the probability density function of

$$\mathbf{x} = \Big[x[0], x[1], \ldots, x[T-1]\Big]^T \tag{3}$$

is

$$p(\mathbf{x};\theta) = \frac{1}{(2\pi\sigma_\epsilon^2)^{T/2}}\exp\Big(-\frac{1}{2\sigma_\epsilon^2}\sum_{t=0}^{T-1}\big(x[t]-\tilde{A}\cos(\tilde{\omega}_0(t-\nu)+\tilde{\phi})\big)^2\Big) \tag{4}$$

where our parameters of interest are $\theta = \Big[\tilde{A}, \tilde{\omega}_0, \tilde{\phi}\Big]^T$[1].

Then we can take the negative logarithm to get

$$L(\theta) = \frac{T}{2}\log_e(2\pi\sigma_\epsilon^2) + \frac{1}{2\sigma_\epsilon^2}\sum_{t=0}^{T-1}\big(x[t]-\tilde{A}\cos(\tilde{\omega}_0(t-\nu)+\tilde{\phi})\big)^2 \tag{5}$$

and the maximum likelihood problem because one of *minimizing* L:

$$\hat{\theta} = \Big[\hat{A}, \hat{\omega}_0, \hat{\phi}\Big]^T = \arg\min_\theta L(\theta) \tag{6}$$

2.1 Linearizing

First, Kay [1] uses the trigonometric identity

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \tag{7}$$

to rewrite

$$L(\theta) = \frac{T}{2}\log_e(2\pi\sigma_\epsilon^2) + \frac{1}{2\sigma_\epsilon^2}\sum_{t=0}^{T-1}\big(x[t]-\tilde{A}\cos(\tilde{\omega}_0(t-\nu))\cos(\phi)+\tilde{A}\sin(\tilde{\omega}_0(t-\nu))\sin(\phi)\big)^2 \tag{8}$$

$$L(\theta') = \frac{T}{2}\log_e(2\pi\sigma_\epsilon^2) + \frac{1}{2\sigma_\epsilon^2}\sum_{t=0}^{T-1}\big(x[t]-\tilde{\alpha}_c\cos(\tilde{\omega}_0(t-\nu))-\tilde{\alpha}_s\sin(\tilde{\omega}_0(t-\nu))\big)^2 \tag{9}$$

[1] Do we want to add σ_ϵ^2?

where

$$\tilde{\alpha}_c = \tilde{A}\cos\phi \tag{10}$$

$$\tilde{\alpha}_s = -\tilde{A}\sin\phi \tag{11}$$

So now we've reparamaterized the problem to use

$$\theta' = [\tilde{\alpha}_c, \tilde{\alpha}_s, \tilde{\omega}_0]^T \tag{12}$$

2.2 Vectorizing

Kay [1] takes (9) and forms the vector version:

$$\begin{aligned} L(\theta') &= \frac{T}{2}\log_e(2\pi\sigma_\epsilon^2) + (\mathbf{x} - \tilde{\alpha}_c\mathbf{c} - \tilde{\alpha}_s\mathbf{s})^T(\mathbf{x} - \tilde{\alpha}_c\mathbf{c} - \tilde{\alpha}_s\mathbf{s}) && (13)\\ &= \frac{T}{2}\log_e(2\pi\sigma_\epsilon^2) + (\mathbf{x} - \mathbf{H}\underline{\alpha})^T(\mathbf{x} - \mathbf{H}\underline{\alpha}) && (14)\end{aligned}$$

where

$$\begin{aligned} \mathbf{x} &= [x[0], x[1], \ldots, x[T-1]]^T && (15)\\ \mathbf{c} &= \left[\cos\left(-\tilde{\omega}_0\frac{T-1}{2}\right), \ldots, \cos(-\frac{\tilde{\omega}_0}{2}), \cos(\frac{\tilde{\omega}_0}{2}), \ldots \cos\left(\tilde{\omega}_0\frac{T-1}{2}\right)\right]^T && (16)\\ \mathbf{s} &= \left[\sin\left(-\tilde{\omega}_0\frac{T-1}{2}\right), \ldots, \sin(-\frac{\tilde{\omega}_0}{2}), \sin(\frac{\tilde{\omega}_0}{2}), \ldots \sin\left(\tilde{\omega}_0\frac{T-1}{2}\right)\right]^T && (17)\\ \underline{\alpha} &= [\tilde{\alpha}_c\ \tilde{\alpha}_s]^T && (18)\\ \mathbf{H} &= [\mathbf{c}\ \mathbf{s}] && (19)\end{aligned}$$

assuming T is even.

Kay [1] then notes that this is a least squares problem which, for a given $\mathbf{H}$ has a solution for $\underline{\hat{\alpha}}$ of

$$\underline{\hat{\alpha}} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{x} \tag{20}$$

This means

$$\begin{aligned} L(\theta'|\underline{\hat{\alpha}}) &= \frac{T}{2}\log_e(2\pi\sigma_\epsilon^2) + (\mathbf{x} - \mathbf{H}\underline{\hat{\alpha}})^T(\mathbf{x} - \mathbf{H}\underline{\hat{\alpha}}) && (21)\\ &= \frac{T}{2}\log_e(2\pi\sigma_\epsilon^2) + (\mathbf{x} - \mathbf{H}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{x})^T(\mathbf{x} - \mathbf{H}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{x}) && (22)\\ &= \mathbf{x}^T\left(\mathbf{I} - \mathbf{H}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\right)\mathbf{x} && (23)\end{aligned}$$

Which means that the maximum likelihood estimator of frequency can be found by maximizing:

$$\mathbf{x}^T\mathbf{H}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{x} = \begin{bmatrix}\mathbf{c}^T\mathbf{x}\\ \mathbf{s}^T\mathbf{x}\end{bmatrix}^T\begin{bmatrix}\mathbf{c}^T\mathbf{c} & \mathbf{c}^T\mathbf{s}\\ \mathbf{s}^T\mathbf{c} & \mathbf{s}^T\mathbf{s}\end{bmatrix}\begin{bmatrix}\mathbf{c}^T\mathbf{x}\\ \mathbf{s}^T\mathbf{x}\end{bmatrix} \tag{24}$$

3 What does this look like?

Kay [1] states near equation (7.66) that when ω_0 is not close to 0 or π and when $\mathbf{c}^T\mathbf{c}/N \approx \frac{1}{2}$ and $\mathbf{s}^T\mathbf{s}/N \approx \frac{1}{2}$ then the MLE of frequency is approximately the same as the periodogram maximizer:

$$\hat{\omega}_0 = \arg\max_{\tilde{\omega}_0} \left| \sum_{t=0}^{T-1} x[t] \exp(-\jmath \tilde{\omega}_0 t) \right|^2 \tag{25}$$

So it might be interesting to plot (24) and (25) on the same scale. Figure 1 shows this for the signal

$$x[t] = \cos(2\pi 0.0123456789 t + \pi/4) + \cos(2\pi(0.0123456789 + 1/200)t + \pi/4) + \epsilon[t]; \tag{26}$$

where $T = 32$, and $\epsilon[t]$ is independent and identically normally distributed with zero mean and standard deviation 0.5.

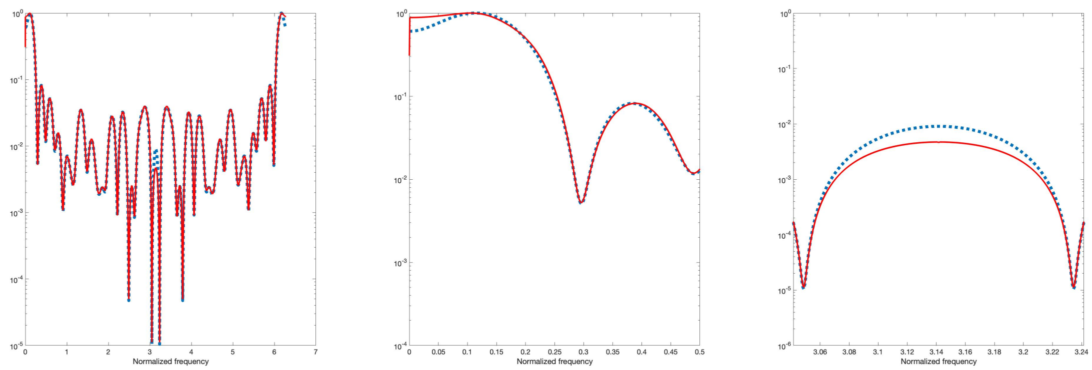

Figure 1: Full frequency range comparison, zoom in around 0, zoom in around π.

References

[1] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory.* Prentice Hall, 1997.