Consider finite element spatial discretization of the position and density

$$\boldsymbol{x}(\boldsymbol{\xi}) = \sum_{a=1}^{n_{\mathrm{en}}} N_a(\boldsymbol{\xi}) \boldsymbol{x}_a \tag{1}$$

$$\rho(\boldsymbol{\xi}) = \sum_{a=1}^{n_{\mathrm{en}}} N_a(\boldsymbol{\xi}) \rho_a \tag{2}$$

Further, let us introduce a squared distance function as

$$d(\boldsymbol{\xi}) = \frac{1}{2} \left\| \boldsymbol{x}(\boldsymbol{\xi}) - \boldsymbol{x}_g \right\|^2 \tag{3}$$

The problem at hand can be formulated as

$$\begin{cases} \text{find} & \boldsymbol{\xi} = \arg\min d(\boldsymbol{\xi}) \\ \text{subjected to} & \rho(\boldsymbol{\xi}) = \rho_{\mathrm{t}} \\ & \xi_i = \bar{\xi} \end{cases} \tag{4}$$

This constrained problem can be refolmulated as an unconstrained one using the method of Lagrange multipliers. To this end, let us introduce the Lagrangian as

$$\mathcal{L}(\boldsymbol{\xi}, \lambda) = d(\boldsymbol{\xi}) + \lambda_1 \left(\rho(\boldsymbol{\xi}) - \rho_{\mathrm{t}}\right) + \lambda_2 \left(\xi_i - \bar{\xi}\right) \tag{5}$$

Its extremal point $(\boldsymbol{\xi}, \boldsymbol{\lambda})$ have to satisfy

$$\frac{\partial \mathcal{L}(\boldsymbol{\xi}, \lambda)}{\partial \boldsymbol{\xi}} = \frac{\partial d(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} + \frac{\partial \rho(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \lambda_1 + \frac{\partial \xi_i}{\partial \boldsymbol{\xi}} \lambda_2 = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}(\boldsymbol{\xi}, \lambda)}{\partial \lambda_1} = \rho(\boldsymbol{\xi}) - \rho_{\mathrm{t}} = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}(\boldsymbol{\xi}, \lambda)}{\partial \lambda_2} = \xi_i - \bar{\xi} = 0 \tag{8}$$

This is a system of non-linear algebraic equation which can be resolved for instance by the Newton-Raphson method. To this end one must perform the Taylor expansion around point $(\boldsymbol{\xi}_k, \lambda_k)$ up to first order

$$\frac{\partial \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \boldsymbol{\xi}} + \frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \boldsymbol{\xi}^2} \Delta\boldsymbol{\xi} + \frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \boldsymbol{\xi} \partial \lambda_1} \Delta\lambda_1 + \frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \boldsymbol{\xi} \partial \lambda_2} \Delta\lambda_2 = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \lambda_1} + \frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \lambda_1 \partial \boldsymbol{\xi}} \Delta\boldsymbol{\xi} + \frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \lambda_1^2} \Delta\lambda_1 + \frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \lambda_1 \partial \lambda_2} \Delta\lambda_2 = 0 \tag{10}$$

$$\frac{\partial \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \lambda_2} + \frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \lambda_2 \partial \boldsymbol{\xi}} \Delta\boldsymbol{\xi} + \frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \lambda_2 \partial \lambda_1} \Delta\lambda_1 + \frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \lambda_2^2} \Delta\lambda_2 = 0 \tag{11}$$

which can be written in matrix notation as

$$
\begin{bmatrix} \frac{\partial^2 \mathcal{L}_k}{\partial \boldsymbol{\xi}^2} & \frac{\partial^2 \mathcal{L}_k}{\partial \boldsymbol{\xi} \partial \lambda_1} & \frac{\partial^2 \mathcal{L}_k}{\partial \boldsymbol{\xi} \partial \lambda_2} \\ \frac{\partial^2 \mathcal{L}_k}{\partial \lambda_1 \partial \boldsymbol{\xi}} & \mathbf{0} & \mathbf{0} \\ \frac{\partial^2 \mathcal{L}_k}{\partial \lambda_2 \partial \boldsymbol{\xi}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta \boldsymbol{\xi} \\ \Delta \lambda_1 \\ \Delta \lambda_2 \end{Bmatrix} = - \begin{Bmatrix} \frac{\partial \mathcal{L}_k}{\partial \boldsymbol{\xi}} \\ \frac{\partial \mathcal{L}_k}{\partial \lambda_1} \\ \frac{\partial \mathcal{L}_k}{\partial \lambda_2} \end{Bmatrix} \tag{12}
$$

In the sequal, all required derivatives will be derived

$$
\frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \lambda_1^2} = \frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \lambda_2^2} = \frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}_k, \boldsymbol{\lambda}_k)}{\partial \lambda_1 \partial \lambda_2} = 0 \tag{13}
$$

$$
\frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}, \boldsymbol{\lambda})}{\partial \lambda_1 \partial \boldsymbol{\xi}} = \frac{\partial \rho(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \tag{14}
$$

$$
\frac{\partial^2 \mathcal{L}(\boldsymbol{\xi}, \boldsymbol{\lambda})}{\partial \lambda_2 \partial \boldsymbol{\xi}} = \frac{\partial \xi_i}{\partial \boldsymbol{\xi}} \tag{15}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathcal{L}_k}{\partial \boldsymbol{\xi}^2} &= \frac{\partial}{\partial \boldsymbol{\xi}} \left(\frac{\partial d(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} + \frac{\partial \rho(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \lambda_1 + \frac{\partial \rho_i}{\partial \boldsymbol{\xi}} \lambda_2 \right) && (16) \\
&= \frac{\partial^2 d(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^2} + \frac{\partial^2 \rho(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^2} \lambda_1 && (17)
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial d(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} &= \frac{\partial}{\partial \boldsymbol{\xi}} \left(\frac{1}{2} \left\| \boldsymbol{x}(\boldsymbol{\xi}) - \boldsymbol{x}_g \right\|^2 \right) && (18) \\
&= \frac{\partial}{\partial \boldsymbol{\xi}} \left(\frac{1}{2} \left\{ \boldsymbol{x}(\boldsymbol{\xi}) - \boldsymbol{x}_g \right\} \cdot \left\{ \boldsymbol{x}(\boldsymbol{\xi}) - \boldsymbol{x}_g \right\} \right) && (19) \\
&= \frac{\partial \boldsymbol{x}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \cdot \left\{ \boldsymbol{x}(\boldsymbol{\xi}) - \boldsymbol{x}_g \right\} && (20)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 d(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^2} &= \frac{\partial}{\partial \boldsymbol{\xi}} \left(\frac{\partial \boldsymbol{x}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \cdot \left\{ \boldsymbol{x}(\boldsymbol{\xi}) - \boldsymbol{x}_g \right\} \right) && (21) \\
&= \frac{\partial^2 \boldsymbol{x}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^2} \cdot \left\{ \boldsymbol{x}(\boldsymbol{\xi}) - \boldsymbol{x}_g \right\} + \frac{\partial \boldsymbol{x}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \cdot \frac{\partial \boldsymbol{x}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} && (22)
\end{aligned}
$$