Elektrodynamik
Uebung 02

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Reo ②

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(a) $\operatorname{grad}\phi = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$; $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$. Vgl.

mit $d\phi = m_x dx + m_y dy + m_z dz \Rightarrow m_x = \frac{\partial\phi}{\partial x}, m_y = \frac{\partial\phi}{\partial y}, m_z = \frac{\partial\phi}{\partial z}$

$\Rightarrow \operatorname{grad} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$.

Für bel. d analog $\operatorname{grad}_d = \begin{pmatrix} \partial/\partial x^1 \\ \partial/\partial x^2 \\ \vdots \\ \partial/\partial x^d \end{pmatrix}$.

- $\underline{r} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$, $r = \sqrt{x^2+y^2}$, $\varphi = \operatorname{atan} y/x$, $z = z$.

$d\phi = \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{\partial\varphi}d\varphi + \frac{\partial\phi}{\partial z}dz = \frac{\partial\phi}{\partial r}\left(\frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy\right) + \frac{\partial\phi}{\partial\varphi}\left(\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy\right) + \frac{\partial\phi}{\partial z}dz = \left(\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\phi}{\partial\varphi}\frac{\partial\varphi}{\partial x}\right)dx + \left(\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\phi}{\partial\varphi}\frac{\partial\varphi}{\partial y}\right)dy + \frac{\partial\phi}{\partial z}dz$

$$\operatorname{grad} = \begin{pmatrix} \frac{\partial r}{\partial x}\partial_r + \frac{\partial\varphi}{\partial x}\partial_\varphi \\ \frac{\partial r}{\partial y}\partial_r + \frac{\partial\varphi}{\partial y}\partial_\varphi \\ \partial_z \end{pmatrix} \underset{\substack{\uparrow \\ \text{Fkt. abl. } \hat{=} \text{ ein.}}}{=} \begin{pmatrix} \cos\varphi\,\partial_r - \frac{1}{r}\sin\varphi\,\partial_\varphi \\ \sin\varphi\,\partial_r + \frac{1}{r}\cos\varphi\,\partial_\varphi \\ \partial_z \end{pmatrix} = \underline{e}_r\partial_r + \frac{1}{r}\underline{e}_\varphi\partial_\varphi + \underline{e}_z\partial_z.$$

- $\underline{r} = \begin{pmatrix} r\cos\varphi\sin\vartheta \\ r\sin\varphi\sin\vartheta \\ r\cos\vartheta \end{pmatrix}$ $r = \sqrt{x^2+y^2+z^2}$, $\varphi = \operatorname{atan} y/x$, $\vartheta = \operatorname{acos}\frac{z}{\sqrt{x^2+y^2+z^2}}$ $\quad \operatorname{acos}'/b = -\frac{1}{\sqrt{1-y^2}}$

$d\phi = \left(\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\phi}{\partial\varphi}\frac{\partial\varphi}{\partial x} + \frac{\partial\phi}{\partial\vartheta}\frac{\partial\vartheta}{\partial x}\right)dx + \left(\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\phi}{\partial\varphi}\frac{\partial\varphi}{\partial y} + \frac{\partial\phi}{\partial\vartheta}\frac{\partial\vartheta}{\partial y}\right)dy$

$+ \left(\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial\phi}{\partial\varphi}\frac{\partial\varphi}{\partial z} + \frac{\partial\phi}{\partial\vartheta}\frac{\partial\vartheta}{\partial z}\right)dz$.

$\cancel{\operatorname{grad}^{(x)} = \frac{x}{\sqrt{x^2+y^2}}\partial_r - \frac{y}{x^2+y^2}\partial_\varphi + \frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2}}}\frac{zx}{\sqrt{x^2+y^2}^3}\partial_\vartheta}$

$\cancel{= \frac{x}{r}\partial_r - \frac{y}{r^2}\partial_\varphi + \frac{r}{\sqrt{r^2-z^2}}\frac{zx}{r^3}\partial_\vartheta}$

$\cancel{= \cos\varphi\sin\vartheta\,\partial_r - \frac{1}{r}\sin\varphi\sin\vartheta\,\partial_\varphi}$

$\frac{1}{r}\frac{r}{\sqrt{1-\cos^2\vartheta}}_{\sin\vartheta} \cdot \frac{r^2\cos\vartheta\cos\varphi\sin\vartheta}{r^3}$

$\operatorname{grad}^{(x)} = \frac{1}{r}x\,\partial_r - \frac{y}{x^2+y^2}\partial_\varphi + \frac{1}{\sqrt{1-\frac{z^2}{r^2}}}\cdot\frac{zx}{r^3}\partial_\vartheta =$ $\qquad \frac{1}{x^2+y^2} = \frac{1}{r^2-z^2}$

$= \cos\varphi\sin\vartheta\,\partial_r - \frac{1}{r}\frac{\sin\varphi}{\sin\vartheta}\partial_\varphi + \frac{1}{r}\cos\vartheta\cos\varphi\,\partial_\vartheta$ $\qquad = \frac{1}{r^2(1-\cos^2\vartheta)}$

$\operatorname{grad}^{(\varphi)} = \sin\varphi\sin\vartheta\,\partial_r + \cancel{\frac{x^2}{r\ldots}\partial_\varphi} + \frac{1}{r}\cos\vartheta\sin\varphi\,\partial_\vartheta$ $\qquad = \frac{1}{r^2\sin^2\vartheta}$

$\frac{x^2}{y^2+x^2}\frac{1}{x}\partial_\varphi = \frac{r\cos\varphi\sin\vartheta}{r^2(1-\cos^2\vartheta)}\partial_\varphi = \frac{1}{r}\frac{\cos\varphi}{\sin\vartheta}\partial_\varphi$

$\operatorname{grad}^{(y)} = \cos\vartheta\,\partial_r + 0\cdot\partial_\varphi - \frac{r}{r\sqrt{1-z^2}}\frac{r - \frac{z}{r}\cdot z}{r^2}\partial_\vartheta$

$= \cos\vartheta\,\partial_r - \frac{1}{\sin\vartheta}\left(\frac{1}{r} - \frac{\cos^2\vartheta}{r}\right)\partial_\vartheta = \cos\vartheta\,\partial_r - \frac{1}{r}\sin\vartheta\,\partial_\vartheta$

$\underline{e}_r = \begin{pmatrix} \cos\varphi\sin\vartheta \\ \sin\varphi\sin\vartheta \\ \cos\vartheta \end{pmatrix}$ $\underline{e}_\varphi = \begin{pmatrix} -r\sin\varphi\sin\vartheta \\ r\cos\varphi\sin\vartheta \\ 0 \end{pmatrix}$ $\underline{e}_\vartheta = \begin{pmatrix} r\cos\varphi\cos\vartheta \\ r\sin\varphi\cos\vartheta \\ -r\sin\vartheta \end{pmatrix}$

$\operatorname{grad} = \underline{e}_r\partial_r + \frac{1}{r^2}\frac{1}{\sin^2\vartheta}\underline{e}_\varphi\partial_\varphi + \frac{1}{r^2}\underline{e}_\vartheta\partial_\vartheta$

- Anschaulich: $\operatorname{grad}\phi$ zeigt dorthin wo ϕ größer wird; je länger $\operatorname{grad}\phi$, desto steiler wächst ϕ.

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(b)

Anschaulich: $\operatorname{div} \underline{v}$ ist Ergiebigkeit eines Raumpunkts: $\operatorname{div} \underline{v} > 0$: $\underline{v}$ hat Quellen, $\operatorname{div} \underline{v} < 0$: $\underline{v}$ hat Senken.

• $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ Allg. Lemma für später: (f stetig)

$$\lim_{\varepsilon \to 0} \int_{a-\varepsilon}^{a+\varepsilon} f(x)\,dx : \quad \min_{x \in U_{2\varepsilon}(a)} f(x) \cdot 2\varepsilon \leq \int_{a-\varepsilon}^{a+\varepsilon} f(x)\,dx \leq \max_{x \in U_{2\varepsilon}(a)} f(x) \cdot 2\varepsilon$$

für $\varepsilon \to 0$ gehen $\min f(x)$, $\max f(x) \to f(a)$; Integral folgt.

• $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$: $\quad |\Delta V| = (2\varepsilon)^3 = 8\varepsilon^3$.

$$\frac{1}{8\varepsilon^3}\Big[\iint v\begin{pmatrix} x \\ y \\ z-\varepsilon \end{pmatrix} \cdot \left(-\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) dx\,dy + \iint v\begin{pmatrix} x \\ y \\ z+\varepsilon \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dx\,dy$$
$$+ \iint v\begin{pmatrix} x \\ y-\varepsilon \\ z \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} dx\,dz + \iint v\begin{pmatrix} x \\ y+\varepsilon \\ z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dx\,dz$$
$$+ \iint v\begin{pmatrix} x-\varepsilon \\ y \\ z \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} dy\,dz + \iint v\begin{pmatrix} x+\varepsilon \\ y \\ z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dy\,dz\Big]$$

$$\to \frac{1}{8\varepsilon^3}\Big[\Big(v\begin{pmatrix} x \\ y \\ z+\varepsilon \end{pmatrix} - v\begin{pmatrix} x \\ y \\ z-\varepsilon \end{pmatrix}\Big) \cdot 4\varepsilon^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \Big(v\begin{pmatrix} x \\ y+\varepsilon \\ z \end{pmatrix} - v\begin{pmatrix} x \\ y-\varepsilon \\ z \end{pmatrix}\Big) 4\varepsilon^2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$+ \Big(v\begin{pmatrix} x+\varepsilon \\ y \\ z \end{pmatrix} - v\begin{pmatrix} x-\varepsilon \\ y \\ z \end{pmatrix}\Big) \cdot 4\varepsilon^2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\Big]$$

Hier verwendet man: $(f(x+\varepsilon) - f(x-\varepsilon))/2\varepsilon = (f(x+\varepsilon) - f(x))/2\varepsilon + (f(x) - f(x-\varepsilon))/2\varepsilon \xrightarrow{\varepsilon \to 0} f'(x)/2 + f'(x)/2 = f'(x)$

Außerdem die Notation $v(\underline{r}) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = v_z(\underline{r})$, ...

$$\to \frac{\partial v_z}{\partial z} + \frac{\partial v_y}{\partial y} + \frac{\partial v_x}{\partial x}$$

• In höheren Dimensionen ändert sich an dem Vorgehen nichts, da stets ein Würfel $dV = 2\varepsilon \cdot df$ sein wird und die Einheitsvektoren stets nur eine 1 beinhalten:

$$\operatorname{div}^d \underline{v} = \sum_{i=1}^{d} \frac{\partial v^i}{\partial x^i}.$$

• $\underline{r} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \quad |\Delta V| = \int_{r-\varepsilon}^{r+\varepsilon} dr \int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi \int_{z-\varepsilon}^{z+\varepsilon} dz \cdot \overbrace{r}^{\text{Jacobian}} = 4\varepsilon^2 \cdot \frac{1}{2}\left[(r+\varepsilon)^2 - (r-\varepsilon)^2\right]$

$= 8r\varepsilon^3$.

$\underline{r}_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \quad \underline{r}_\varphi = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} \quad \underline{r}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\mapsto$ Flächenelement in r-Richtung: $d\underline{f}^r = \underline{r}_\varphi \times \underline{r}_z = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ 0 \end{pmatrix} = r\,\underline{e}_r$

$v^r = \underline{v}\,\underline{e}_r$

$$\frac{1}{8r\varepsilon^3} \int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi \int_{z-\varepsilon}^{z+\varepsilon} dz\; \underline{v}\begin{pmatrix} r+\varepsilon \\ \varphi \\ z \end{pmatrix} \cdot (r+\varepsilon) \underbrace{\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}}_{\underline{e}_r} + \frac{1}{8r\varepsilon^3} \int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi \int_{z-\varepsilon}^{z+\varepsilon} dz\; \underline{v}\begin{pmatrix} r-\varepsilon \\ \varphi \\ z \end{pmatrix} \cdot (r-\varepsilon) \underbrace{\begin{pmatrix} -\cos\varphi \\ -\sin\varphi \end{pmatrix}}_{-\underline{e}_r}$$

$$\to \frac{1}{8r\varepsilon^3}\Big((r+\varepsilon)\, v^r\begin{pmatrix} r+\varepsilon \\ \varphi \\ z \end{pmatrix} \cdot \iint d\varphi\,dz - (r-\varepsilon)\, v^r\begin{pmatrix} r-\varepsilon \\ \varphi \end{pmatrix} \iint d\varphi\,dz\Big)$$

1 (b) – Forts. –

$$\iint d\varphi\, dz = 4\varepsilon^2$$

$$\to \frac{1}{r}\frac{1}{2\varepsilon}\Big[\underbrace{(r+\varepsilon)v^r\begin{pmatrix} r+\varepsilon\\ \varphi\\ z\end{pmatrix} - r v^r\begin{pmatrix} r\\ \varphi\\ z\end{pmatrix}}_{\partial(r\cdot v^r)/\partial r} + \underbrace{r v^r\begin{pmatrix} r\\ \varphi\\ z\end{pmatrix} - (r-\varepsilon)v^r\begin{pmatrix} r-\varepsilon\\ \varphi\\ z\end{pmatrix}}_{\text{dito}}\Big]$$

$r\ \varphi\ z$

$z\ r\ \varphi$

$\varphi\ z\ r$

$$\to \frac{1}{r}\frac{\partial(r v^r)}{\partial r}$$

– Fl'elem in φ-Richtung: $d\underline{f}^\varphi = -\underline{r}_z \times \underline{r}_r = +\begin{pmatrix}\sin\varphi\\ -\cos\varphi\\ 0\end{pmatrix} = \underline{e}_\varphi$

$$\frac{1}{8r\varepsilon^3}\left(\int_{z-\varepsilon}^{z+\varepsilon}dz\int_{r-\varepsilon}^{r+\varepsilon}dr\ \underline{v}\begin{pmatrix} r\\ \varphi+\varepsilon\\ z\end{pmatrix}\cdot\underline{e}_\varphi + \int_{z-\varepsilon}^{z+\varepsilon}dz\int_{r-\varepsilon}^{r+\varepsilon}dr\ \underline{v}\begin{pmatrix} r\\ \varphi-\varepsilon\\ z\end{pmatrix}(-\underline{e}_\varphi)\right) \qquad \iint dz\,dr = 4\varepsilon^2$$

$$\to \frac{1}{r}\frac{1}{2\varepsilon}\left(v^\varphi\begin{pmatrix} r\\ \varphi+\varepsilon\\ z\end{pmatrix} - v^\varphi\begin{pmatrix} r\\ \varphi\\ z\end{pmatrix} + v^\varphi\begin{pmatrix} r\\ \varphi\\ z\end{pmatrix} - v^\varphi\begin{pmatrix} r\\ \varphi-\varepsilon\\ z\end{pmatrix}\right) \to \frac{1}{r}\frac{\partial v^\varphi}{\partial\varphi}$$

– Flächen in z-Richt.: $d\underline{f}^z = \underline{r}_r \times \underline{r}_\varphi = r\begin{pmatrix}0\\0\\1\end{pmatrix} = r\cdot\underline{e}_z$

$$\frac{1}{8r\varepsilon^3}\left(\int_{r-\varepsilon}^{r+\varepsilon}dr\int_{\varphi-\varepsilon}^{\varphi+\varepsilon}d\varphi\ \underline{v}\begin{pmatrix} r\\ \varphi\\ z+\varepsilon\end{pmatrix}\cdot\underline{e}_z\, r + \int_{r-\varepsilon}^{r+\varepsilon}dr\int_{\varphi-\varepsilon}^{\varphi+\varepsilon}d\varphi\ \underline{v}\begin{pmatrix} r\\ \varphi\\ z-\varepsilon\end{pmatrix}\cdot(-\underline{e}_z)\, r\right)$$

$$\iint r\,dr\,d\varphi = 2\varepsilon\cdot\int r\,dr = 2\varepsilon\cdot\tfrac{1}{2}\left((r+\varepsilon)^2-(r-\varepsilon)^2\right) = \varepsilon\cdot 4r\varepsilon$$

$$\to \frac{1}{8r\varepsilon^3}\left(v^z\begin{pmatrix} r\\ \varphi\\ z+\varepsilon\end{pmatrix} - v^z\begin{pmatrix} r\\ \varphi\\ z\end{pmatrix} + v^z\begin{pmatrix} r\\ \varphi\\ z\end{pmatrix} - v^z\begin{pmatrix} r\\ \varphi\\ z-\varepsilon\end{pmatrix}\right)\cdot 4r\varepsilon^2$$

$$\to \frac{\partial v^z}{\partial z}$$

$$\Rightarrow \operatorname{div}\underline{v} = \frac{1}{r}\frac{\partial(r v^r)}{\partial r} + \frac{1}{r}\frac{\partial v^\varphi}{\partial\varphi} + \frac{\partial v^z}{\partial z} \quad \text{für } \underline{v} = v^r\underline{e}_r + v^\varphi\underline{e}_\varphi + v^z\underline{e}_z.$$

• $\underline{r} = \begin{pmatrix} r\cos\varphi\sin\vartheta\\ r\sin\varphi\sin\vartheta\\ r\cos\vartheta\end{pmatrix}$ $\quad |\Delta V| = \int_{r-\varepsilon}^{r+\varepsilon}dr\int_{\varphi-\varepsilon}^{\varphi+\varepsilon}d\varphi\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon}d\vartheta\ r^2\sin\vartheta = 2\varepsilon\cdot(2r^2\varepsilon + \tfrac{2}{3}\varepsilon^3)(\cos(\vartheta-\varepsilon) - \cos(\vartheta+\varepsilon))$

$$\tfrac{1}{3}\left[(r+\varepsilon)^3-(r-\varepsilon)^3\right] = \tfrac{1}{3}\left(r^3+3r^2\varepsilon+3r\varepsilon^2+\varepsilon^3 - r^3+3r^2\varepsilon-3r\varepsilon^2+\varepsilon^3\right) = 2r\varepsilon^2 + \tfrac{2}{3}\varepsilon^3$$

$$= 4\varepsilon^2\left(r^2+\tfrac{1}{3}\varepsilon^2\right)\left(\cos(\vartheta-\varepsilon) - \cos(\vartheta+\varepsilon)\right)$$

$$\underline{r}_r = \begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix} \quad \underline{r}_\varphi = \begin{pmatrix}-r\sin\varphi\sin\vartheta\\ r\cos\varphi\sin\vartheta\\ 0\end{pmatrix} \quad \underline{r}_\vartheta = \begin{pmatrix} r\cos\varphi\cos\vartheta\\ r\sin\varphi\cos\vartheta\\ -r\sin\vartheta\end{pmatrix}$$

– r-Richtung: $d\underline{f}^r = -\underline{r}_\varphi\times\underline{r}_\vartheta = \begin{pmatrix} + r^2\cos\varphi\sin^2\vartheta\\ + r^2\sin\varphi\sin^2\vartheta\\ + r^2\sin\vartheta\cos\vartheta\end{pmatrix} = r^2\sin\vartheta\ \underline{e}_r$

$$\frac{1}{|\Delta V|}\left(\int_{\varphi-\varepsilon}^{\varphi+\varepsilon}d\varphi\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon}d\vartheta\ \underline{v}\begin{pmatrix} r+\varepsilon\\ \varphi\\ \vartheta\end{pmatrix}(r+\varepsilon)^2\sin\vartheta\,\underline{e}_r - \int_{\varphi-\varepsilon}^{\varphi+\varepsilon}d\varphi\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon}d\vartheta\ \underline{v}\begin{pmatrix} r-\varepsilon\\ \varphi\\ \vartheta\end{pmatrix}(r-\varepsilon)^2\sin\vartheta\,\underline{e}_r\right)$$

$$\to \frac{1}{|\Delta V|}\left(v^r\begin{pmatrix} r+\varepsilon\\ \varphi\\ \vartheta\end{pmatrix}(r+\varepsilon)^2 - v^r\begin{pmatrix} r-\varepsilon\\ \varphi\\ \vartheta\end{pmatrix}(r-\varepsilon)^2\right)\cdot\left(\cos(\vartheta-\varepsilon)-\cos(\vartheta+\varepsilon)\right)2\varepsilon$$

$$\to \frac{1}{2\varepsilon\left(r^2+\tfrac{1}{3}\varepsilon^2\right)}\left(v^r\begin{pmatrix} r+\varepsilon\\ \varphi\\ \vartheta\end{pmatrix}(r+\varepsilon)^2 - v^r\begin{pmatrix} r-\varepsilon\\ \varphi\\ \vartheta\end{pmatrix}(r-\varepsilon)^2\right) \qquad \pm v^r\begin{pmatrix} r\\ \varphi\\ \vartheta\end{pmatrix} r$$

$\to 0$ –> geht in σ-Term ein

$$\to \frac{1}{r^2}\frac{\partial(r^2 v^r)}{\partial r}$$

– φ-Richtung: $d\underline{f}^\varphi = \underline{r}_\varphi\times\underline{r}_\vartheta = \begin{pmatrix} -r\sin\varphi\sin^2\vartheta - r\sin\varphi\cos^2\vartheta\\ r\cos\varphi\cos^2\vartheta + r\cos\varphi\sin^2\vartheta\\ r\cos\varphi\sin\varphi\cos\vartheta\sin\vartheta - r\cos\varphi\cos\vartheta\sin\varphi\sin\vartheta\end{pmatrix} = \begin{pmatrix}-r\sin\varphi\\ r\cos\varphi\\ 0\end{pmatrix} = \frac{r}{\sin\vartheta}\underline{e}_\varphi$

$$\frac{1}{|\Delta V|}\left(\int_{r-\varepsilon}^{r+\varepsilon}dr\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon}d\vartheta\ \frac{r}{\sin\vartheta}\ \underline{v}\begin{pmatrix} r\\ \varphi+\varepsilon\\ \vartheta\end{pmatrix}\underline{e}_\varphi - \int_{r-\varepsilon}^{r+\varepsilon}dr\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon}d\vartheta\ \frac{r}{\sin\vartheta}\ \underline{v}\begin{pmatrix} r\\ \varphi-\varepsilon\\ \vartheta\end{pmatrix}\underline{e}_\varphi\right)$$

(I) (b) – Forts. –

$$\frac{1}{|\Delta V|}\left(v^\varphi\begin{pmatrix}r\\ \varphi+\varepsilon\\ \vartheta\end{pmatrix} - v^\varphi\begin{pmatrix}r\\ \varphi-\varepsilon\\ \vartheta\end{pmatrix}\right)\cdot \frac{1}{2}\underbrace{[(r+\varepsilon)^2-(r-\varepsilon)^2]}_{4\varepsilon r}\,[\,\mathrm{artanh}\,\overbrace{\cos(\varphi-\varepsilon)}^{:=\tau-\varepsilon} - \mathrm{artanh}\,\overbrace{\cos(\varphi+\varepsilon)}^{\tau+\varepsilon}\,]$$

$$\int \frac{1}{\sin\vartheta}\,d\vartheta = \int \frac{1}{1-\cos^2\vartheta}\,d\vartheta = \quad \cos\vartheta = \chi \quad -\int\frac{1}{1-\chi^2}\,d\chi = -\mathrm{artanh}\,\chi = -\mathrm{artanh}\cos\varphi$$

$$\to \frac{1}{(2)\cdot(r^2+\varepsilon^2/3)\,[\overbrace{(\tau-\varepsilon)-(\tau+\varepsilon)}^{-2\varepsilon}]} \qquad \frac{\partial v^\varphi}{\partial\varphi}\cdot\frac{1}{r}\cdot \frac{\partial\,\mathrm{artanh}\,\varepsilon}{\partial\varepsilon} = \frac{\partial\,\mathrm{artanh}\,\tau}{\partial\tau}\cdot\frac{\partial\tau}{\partial\varepsilon}$$

$$\to \frac{1}{r}\cdot\frac{\partial v^\varphi}{\partial\varphi}\cdot\frac{1}{\sin\vartheta} \qquad = \frac{1}{1-\tau^2}\cdot(-\sin\varphi),\quad \tau^2=\cos^2\varphi$$

$$\frac{1}{1-\cos^2\varphi} = \frac{1}{\sin^2\varphi}$$

ϑ-Richtung:
$$d\underline{f}^\vartheta = -\underline{e}_r\, r \times \underline{e}_\varphi = \begin{pmatrix} + r\cos\varphi\sin\vartheta\cos\vartheta \\ + r\sin\varphi\sin\vartheta\cos\vartheta \\ -r\cos^2\sin^2\vartheta - \cancel{r\sin^2\varphi\sin^2\vartheta}\end{pmatrix} = \ldots\, r\sin\vartheta\,\underline{e}_\vartheta$$

$$\frac{1}{|\Delta V|}\Big(\int_{r-\varepsilon}^{r+\varepsilon}dr\int_{\varphi-\varepsilon}^{\varphi+\varepsilon}d\varphi\;\underline{v}\begin{pmatrix}r\\ \varphi\\ \vartheta+\varepsilon\end{pmatrix}\cdot\underline{e}_\vartheta\,\underbrace{r\sin(\vartheta+\varepsilon)}_{2\varepsilon r} - \int_{r-\varepsilon}^{r+\varepsilon}dr\int_{\varphi-\varepsilon}^{\varphi+\varepsilon}d\varphi\;\underline{v}\begin{pmatrix}r\\ \varphi\\ \vartheta-\varepsilon\end{pmatrix}\underline{e}_\vartheta\,\underbrace{r(\sin\vartheta-\varepsilon)}_{2\varepsilon r}$$

$$\to \frac{1}{|\Delta V|}[\,v^\vartheta\begin{pmatrix}r\\ \varphi\\ \vartheta+\varepsilon\end{pmatrix}(\sin\vartheta+\varepsilon) - v^\vartheta\begin{pmatrix}r\\ \varphi\\ \vartheta-\varepsilon\end{pmatrix}\sin(\vartheta-\varepsilon)\,]\cdot 4\varepsilon^2\, r\,\frac{2\varepsilon}{2\varepsilon}\quad \frac{1}{2\varepsilon(r^2+\varepsilon^2/3)\,[\cos(\vartheta-\varepsilon)-\cos(\vartheta+\varepsilon)]}$$

~~mit 1 erweitern, oberes ε für Abl. von unteres ε für $\frac{\partial(v^\vartheta\sin\vartheta)}{\partial\vartheta}$~~

$$\to \frac{\partial\left(v^\vartheta\begin{pmatrix}r\\ \varphi\\ \vartheta\end{pmatrix}\sin\vartheta\right)}{\partial\vartheta}\,\frac{1}{r}\cdot\frac{1}{\partial\cos\vartheta/\partial\vartheta} = \frac{1}{r\sin\vartheta}\cdot\frac{\partial v^\vartheta\begin{pmatrix}r\\ \varphi\\ \vartheta\end{pmatrix}}{\partial\vartheta}$$

$$\Rightarrow \operatorname{div}\underline{v} = \frac{1}{r^2}\frac{\partial(r^2\cdot v^r)}{\partial r} + \frac{1}{r\sin\vartheta}\cdot\frac{\partial v^\varphi}{\partial\varphi} + \frac{1}{r\sin\vartheta}\frac{\partial v^\vartheta}{\partial\vartheta}$$

für $\underline{v} = v^r\underline{e}_r + v^\varphi\underline{e}_\varphi + v^\vartheta\underline{e}_\vartheta$.

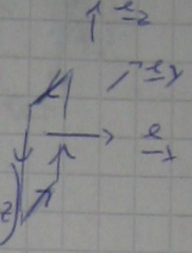
1 Theo 2

(a)

- ~~$\nabla\phi(\underline{r})$: Pfeil, der dorthin zeigt, wo ϕ größer wird; seine Länge gibt an, wie viel ϕ größer wird.~~

- ~~$\underline{r} = t\,\underline{e}_x \rightarrow d\underline{r} = \underline{e}_x\,dt$~~

(c)



- $\underline{n} = \underline{e}_x$: ~~$(\text{rot}\,\underline{v})^x = \lim_{\varepsilon\to 0} \frac{1}{4\varepsilon^2} \int_{y-\varepsilon}^{y+\varepsilon} dy \int_{z-\varepsilon}^{z+\varepsilon} dz\, \underline{v}\,d\underline{r}$~~

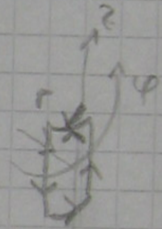
$(\text{rot}\,\underline{v})^x \leftarrow \frac{1}{4\varepsilon^2}\left(\int_{y-\varepsilon}^{y+\varepsilon} -dy\,\underline{e}_y \underline{v}\begin{pmatrix}x\\y\\z+\varepsilon\end{pmatrix} + \int_{y-\varepsilon}^{y+\varepsilon} dy\,\underline{e}_y \underline{v}\begin{pmatrix}x\\y\\z-\varepsilon\end{pmatrix} + \int_{z-\varepsilon}^{z+\varepsilon} dz\,\underline{v}\begin{pmatrix}x\\y+\varepsilon\\z\end{pmatrix}\underline{e}_z - \int dz\,\underline{v}\begin{pmatrix}x\\y-\varepsilon\\z\end{pmatrix}\underline{e}_z \right)$

$\rightarrow \frac{1}{4\varepsilon^2}\; -v^y\begin{pmatrix}x\\y\\z+\varepsilon\end{pmatrix}\cdot 2\varepsilon + v^y\begin{pmatrix}x\\y\\z-\varepsilon\end{pmatrix}2\varepsilon + \left(v^z\begin{pmatrix}x\\y+\varepsilon\\z\end{pmatrix} - v^z\begin{pmatrix}x\\y-\varepsilon\\z\end{pmatrix}\right)2\varepsilon$

$\rightarrow -\frac{\partial v^y}{\partial z} + \frac{\partial v^z}{\partial y}$

- Durch Vertauschen folgt: $\text{rot}\,\underline{v} = \left(\frac{\partial v^z}{\partial y} - \frac{\partial v^y}{\partial z}\right)\underline{e}_x + \ldots$ (identisch mit $\nabla \times \underline{v}$!)

- $\underline{r} = \begin{pmatrix}r\cos\varphi\\ r\sin\varphi\\ z\end{pmatrix}$ $\quad \underline{r}_{,r} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix}$ $\quad \underline{r}_{,\varphi} = \begin{pmatrix}-r\sin\varphi\\ r\cos\varphi\\ 0\end{pmatrix} = r\cdot\underline{e}_\varphi$ $\quad \underline{r}_{,z} = \begin{pmatrix}0\\0\\1\end{pmatrix}$



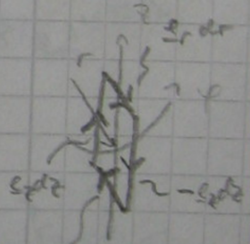
$\mapsto$ $\underline{n} = \underline{e}_r$; ~~$(\text{rot}\,\underline{v})^r = \int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi \int_{z-\varepsilon}^{z+\varepsilon} dz$~~

$|\Delta A| = \int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi \int_{z-\varepsilon}^{z+\varepsilon} dz \cdot r = r\cdot 4\varepsilon^2$ $\qquad d\underline{f} = r\,\underline{e}_\varphi \times \underline{e}_z$

$\frac{1}{r4\varepsilon^2}\left(\int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi\; \underline{v}\begin{pmatrix}r\\\varphi\\z-\varepsilon\end{pmatrix} r\underline{e}_\varphi + \int_{z-\varepsilon}^{z+\varepsilon} dz\; \underline{v}\begin{pmatrix}r\\\varphi+\varepsilon\\z\end{pmatrix}\underline{e}_z + \int_{\varphi+\varepsilon}^{\varphi-\varepsilon} d\varphi\; \underline{v}\begin{pmatrix}r\\\varphi\\z+\varepsilon\end{pmatrix} r\underline{e}_\varphi + \int_{z+\varepsilon}^{z-\varepsilon} dz\; \underline{v}\begin{pmatrix}r\\\varphi-\varepsilon\\z\end{pmatrix}\underline{e}_z \right)$

$\rightarrow \frac{1}{r4\varepsilon^2}\left[\left(r v^\varphi\begin{pmatrix}r\\\varphi\\z-\varepsilon\end{pmatrix} - r v^\varphi\begin{pmatrix}r\\\varphi\\z+\varepsilon\end{pmatrix}\right)2\varepsilon + \left(v^z\begin{pmatrix}r\\\varphi+\varepsilon\\z\end{pmatrix} - v^z\begin{pmatrix}r\\\varphi-\varepsilon\\z\end{pmatrix}\right)\cdot 2\varepsilon\right]$

$\rightarrow -\frac{1}{r}\left(+r\frac{\partial v^\varphi}{\partial z} - \frac{\partial v^z}{\partial \varphi}\right)$



$d\underline{f} \propto \underline{e}_r \times \underline{e}_z$

$\mapsto$ $\underline{n} = \underline{e}_\varphi$ $\qquad |\Delta A| = \int_{r-\varepsilon}^{r+\varepsilon} dr \int_{z-\varepsilon}^{z+\varepsilon} dz = 2\varepsilon\; 2\varepsilon = 4\varepsilon^2$

$\frac{1}{4\varepsilon^2}\left(\int_{z-\varepsilon}^{z+\varepsilon} \underline{v}\begin{pmatrix}r-\varepsilon\\\varphi\\z\end{pmatrix}\underline{e}_z dz + \int_{r-\varepsilon}^{r+\varepsilon} dr\; \underline{v}\begin{pmatrix}r\\\varphi\\z+\varepsilon\end{pmatrix}\underline{e}_r + \int_{z+\varepsilon}^{z-\varepsilon} dz\; \underline{v}\begin{pmatrix}r+\varepsilon\\\varphi\\z\end{pmatrix}\underline{e}_z + \int_{r+\varepsilon}^{r-\varepsilon} dr\; \underline{v}\begin{pmatrix}r\\\varphi\\z-\varepsilon\end{pmatrix}\underline{e}_r \right)$

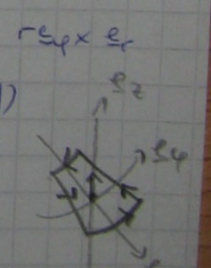
$\rightarrow \frac{1}{4\varepsilon^2}\left(v^z\begin{pmatrix}r-\varepsilon\\\varphi\\z\end{pmatrix}\cdot 2\varepsilon + v^r\begin{pmatrix}r\\\varphi\\z+\varepsilon\end{pmatrix}2\varepsilon - v^z\begin{pmatrix}r+\varepsilon\\\varphi\\z\end{pmatrix}2\varepsilon - v^r\begin{pmatrix}r\\\varphi\\z-\varepsilon\end{pmatrix}\right)$

$\rightarrow -\left(\frac{\partial v^z}{\partial r} - \frac{\partial v^r}{\partial z}\right)$

$\mapsto$ $\underline{n} = \underline{e}_z$ $\qquad |\Delta A| = \int_{r-\varepsilon}^{r+\varepsilon} dr \int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi\; r\,\|\underline{e}_z\| = 2\varepsilon\cdot 2r\varepsilon = 4r\varepsilon^2$ $\qquad r\underline{e}_\varphi \times \underline{e}_r$

$\frac{1}{4r\varepsilon^2}\left(\int_{\varphi+\varepsilon}^{\varphi-\varepsilon} (r-\varepsilon)\underline{e}_\varphi d\varphi\, \underline{v}\begin{pmatrix}r-\varepsilon\\\varphi\\z\end{pmatrix} + \int_{r-\varepsilon}^{r+\varepsilon} \underline{e}_r dr\cdot \underline{v}\begin{pmatrix}r\\\varphi-\varepsilon\\z\end{pmatrix} + \int_{\varphi-\varepsilon}^{\varphi+\varepsilon} (r+\varepsilon)\underline{e}_\varphi d\varphi\; \underline{v}\begin{pmatrix}r+\varepsilon\\\varphi\\z\end{pmatrix} + \int_{r+\varepsilon}^{r-\varepsilon} \underline{e}_r dr\, \underline{v}\begin{pmatrix}r\\\varphi+\varepsilon\\z\end{pmatrix}\right)$

$\rightarrow \frac{1}{4r\varepsilon^2}\left(-(r-\varepsilon)v^\varphi\begin{pmatrix}r-\varepsilon\\\varphi\\z\end{pmatrix}\cdot 2\varepsilon + v^r\begin{pmatrix}r\\\varphi-\varepsilon\\z\end{pmatrix}2\varepsilon + (r+\varepsilon)v^\varphi\begin{pmatrix}r+\varepsilon\\\varphi\\z\end{pmatrix}\cdot 2\varepsilon - v^r\begin{pmatrix}r\\\varphi+\varepsilon\\z\end{pmatrix}\right)$

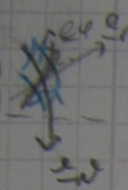


$\rightarrow -\frac{1}{r}\left(-\frac{\partial(rv^\varphi)}{\partial r} + \frac{\partial v^r}{\partial \varphi}\right)$

$\mapsto$ $\text{rot}\,\underline{v} = \frac{1}{r}\left(\frac{\partial v^z}{\partial\varphi} - r\frac{\partial v^\varphi}{\partial z}\right)\underline{e}_r + \left(\frac{\partial v^r}{\partial z} - \frac{\partial v^z}{\partial r}\right)\underline{e}_\varphi + \frac{1}{r}\left(\frac{\partial(v^\varphi r)}{\partial r} - \frac{\partial v^r}{\partial\varphi}\right)\underline{e}_z$

• $\underline{r} = \begin{pmatrix} r\cos\varphi\sin\vartheta \\ r\sin\varphi\sin\vartheta \\ r\cos\vartheta \end{pmatrix}$ $\quad \underline{r}_{,r} = \begin{pmatrix} \cos\varphi\sin\vartheta \\ \sin\varphi\sin\vartheta \\ \cos\vartheta \end{pmatrix} = \underline{e}_r \; ; \quad \underline{r}_{,\varphi} = \begin{pmatrix} -r\sin\varphi\sin\vartheta \\ r\cos\varphi\sin\vartheta \\ 0 \end{pmatrix} = r\cdot\sin\vartheta\,\underline{e}_\varphi$

$\underline{r}_{,\vartheta} = \begin{pmatrix} r\cos\varphi\cos\vartheta \\ r\sin\varphi\cos\vartheta \\ -r\sin\vartheta \end{pmatrix} = r\cdot\underline{e}_\vartheta$ $\qquad \underline{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$!



I $\cancel{\underline{n} = \underline{e}_r \quad d\underline{f} = -r^2\,\underline{e}_\varphi\times\underline{e}_\vartheta = r^2\sin\vartheta\,\underline{e}_r \qquad df = r^2\sin\vartheta}$

$\cancel{\int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi \int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon} d\vartheta\; r^2\sin\vartheta = 2r^2\varepsilon\,[\cos(\vartheta-\varepsilon)-\cos(\vartheta+\varepsilon)]}$

$\cancel{\frac{1}{|\Delta A|}\Big(\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon} d\vartheta\; r\,\underline{e}_\vartheta\,\underline{v}\begin{pmatrix} r \\ \varphi-\varepsilon \\ \vartheta \end{pmatrix} + \int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi\; r\,\underline{e}_\varphi\,\underline{v}\begin{pmatrix} r \\ \varphi \\ \vartheta+\varepsilon \end{pmatrix} + \int_{\vartheta+\varepsilon}^{\vartheta-\varepsilon} d\vartheta\; r\,\underline{e}_\vartheta\,\underline{v}\begin{pmatrix} r \\ \varphi+\varepsilon \\ \vartheta \end{pmatrix} + \int_{\varphi+\varepsilon}^{\varphi-\varepsilon} d\varphi\; r\,\underline{e}_\varphi\,\underline{v}\begin{pmatrix} r \\ \varphi \\ \vartheta-\varepsilon \end{pmatrix}\Big)}$

$\cancel{\to \frac{1}{2r^2\varepsilon[\cos(\vartheta-\varepsilon)-\cos(\vartheta+\varepsilon)]}\Big(r\,v^\vartheta\begin{pmatrix} r \\ \varphi-\varepsilon \\ \vartheta \end{pmatrix} 2\varepsilon + r\,v^\vartheta\begin{pmatrix} r \\ \varphi+\varepsilon \\ \vartheta \end{pmatrix} 2\varepsilon + r\,v^\varphi\begin{pmatrix} r \\ \varphi \\ \vartheta+\varepsilon \end{pmatrix} 2\varepsilon - r\,v^\varphi\begin{pmatrix} r \\ \varphi \\ \vartheta-\varepsilon \end{pmatrix} 2\varepsilon \Big)}$

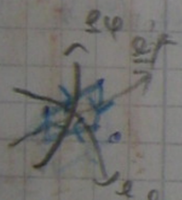
I $\underline{n} = \underline{e}_r \; ; \quad d\underline{f} = r^2\sin\vartheta\,\underline{e}_\vartheta\times\underline{e}_\varphi = r^2\sin\vartheta\;\underline{e}_r$

$\int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi \int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon} d\vartheta\; r^2\sin\vartheta\;\|\underline{e}_r\| = 2r^2\varepsilon\,[\cos(\vartheta-\varepsilon)-\cos(\vartheta+\varepsilon)]$

$\frac{1}{|\Delta A|}\Big(\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon} d\vartheta\; r\,\underline{e}_\vartheta\,\underline{v}\begin{pmatrix} r \\ \varphi-\varepsilon \\ \vartheta \end{pmatrix} + \int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi\; \underline{e}_\varphi\, r\sin(\vartheta+\varepsilon)\,\underline{v}\begin{pmatrix} r \\ \varphi \\ \vartheta+\varepsilon \end{pmatrix} + \int_{\vartheta+\varepsilon}^{\vartheta-\varepsilon} d\vartheta\; r\,\underline{e}_\vartheta\,\underline{v}\begin{pmatrix} r \\ \varphi+\varepsilon \\ \vartheta \end{pmatrix} + \int_{\varphi+\varepsilon}^{\varphi-\varepsilon} d\varphi\; r\sin(\vartheta-\varepsilon)\,\underline{e}_\varphi\,\underline{v}\begin{pmatrix} r \\ \varphi \\ \vartheta-\varepsilon \end{pmatrix}\Big)$

$\to \frac{2\varepsilon}{2\varepsilon}\frac{1}{r^2[\cos(\vartheta-\varepsilon)-\cos(\vartheta+\varepsilon)]}\Big[r\,v^\vartheta\begin{pmatrix} r \\ \varphi-\varepsilon \\ \vartheta \end{pmatrix} + r\sin(\vartheta+\varepsilon)\,v^\varphi\begin{pmatrix} r \\ \varphi \\ \vartheta+\varepsilon \end{pmatrix} - r\,v^\vartheta\begin{pmatrix} r \\ \varphi+\varepsilon \\ \vartheta \end{pmatrix} - r\sin(\vartheta-\varepsilon)\,v^\varphi\begin{pmatrix} r \\ \varphi \\ \vartheta-\varepsilon \end{pmatrix}\Big]$

$\to \frac{1}{r}\frac{1}{\sin\vartheta}\Big(-\frac{\partial v^\vartheta}{\partial\varphi}\Big) + \frac{1}{r}\frac{1}{\sin\vartheta}\frac{\partial(\sin\vartheta\, v^\varphi)}{\partial\vartheta}$



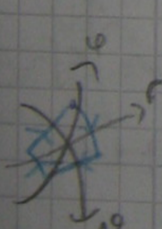
I $\underline{n} = \underline{e}_\vartheta \quad d\underline{f} = r\sin\vartheta\;\underline{e}_\varphi\times\underline{e}_r = r\sin\vartheta\;\underline{e}_\vartheta$

$\int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi \int_{r-\varepsilon}^{r+\varepsilon} dr\; r\sin\vartheta = 2r\varepsilon\cdot 2\varepsilon\cdot\sin\vartheta = 4r\varepsilon^2\sin\vartheta$

$\frac{1}{4r\varepsilon^2\sin\vartheta}\Big(\int_{r-\varepsilon}^{r+\varepsilon} dr\; \underline{e}_r\,\underline{v}\begin{pmatrix} r \\ \varphi+\varepsilon \\ \vartheta \end{pmatrix} + \int_{\varphi+\varepsilon}^{\varphi-\varepsilon} d\varphi\,\underline{v}\begin{pmatrix} r+\varepsilon \\ \varphi \\ \vartheta \end{pmatrix}(r+\varepsilon)\sin\vartheta\,\underline{e}_\varphi + \int_{r+\varepsilon}^{r-\varepsilon} dr\; \underline{e}_r\,\underline{v}\begin{pmatrix} r \\ \varphi-\varepsilon \\ \vartheta \end{pmatrix} + \int_{\varphi-\varepsilon}^{\varphi+\varepsilon} d\varphi\,(r-\varepsilon)\sin\vartheta\,\underline{e}_\varphi\,\underline{v}\begin{pmatrix} r-\varepsilon \\ \varphi \\ \vartheta \end{pmatrix}\Big)$

$\to \frac{1}{r\sin\vartheta}\underbrace{\frac{1}{2\varepsilon}\Big(v^r\begin{pmatrix} r \\ \varphi+\varepsilon \\ \vartheta \end{pmatrix} - v^r\begin{pmatrix} r \\ \varphi-\varepsilon \\ \vartheta \end{pmatrix}\Big)}_{\frac{\partial v^r}{\partial\varphi}} + \underbrace{\frac{1}{r\cancel{\sin\vartheta}}}_{\frac{1}{r}}\underbrace{\frac{1}{2\varepsilon}\Big(-(r+\varepsilon)\,v^\varphi\begin{pmatrix} r+\varepsilon \\ \varphi \\ \vartheta \end{pmatrix}\cancel{\sin\vartheta} + (r-\varepsilon)\,v^\varphi\begin{pmatrix} r-\varepsilon \\ \varphi \\ \vartheta \end{pmatrix}\cancel{\sin\vartheta}\Big)}_{\left(-\frac{\partial(r v^\varphi)}{\partial r}\right)}$

$\to \frac{1}{r\sin\vartheta}\frac{\partial v^r}{\partial\varphi} \qquad \frac{1}{r}\Big(-\frac{\partial(r v^\varphi)}{\partial r}\Big)$



I $\underline{n} = \underline{e}_\varphi \quad d\underline{f} = \underline{r}_{,r}\times\underline{r}_{,\vartheta} = \underline{e}_r\times r\underline{e}_\vartheta = r\,\underline{e}_\varphi$

$\int_{r-\varepsilon}^{r+\varepsilon} dr \int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon} d\vartheta\cdot r\,\underline{e}_\varphi = 2r\varepsilon\cdot 2\varepsilon\;\underline{e}_\varphi = 4r\varepsilon^2\,\underline{e}_\varphi\,.$

$\frac{1}{4r\varepsilon^2}\Big(\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon} d\vartheta\,(r+\varepsilon)\,\underline{e}_\vartheta\,\underline{v}\begin{pmatrix} r+\varepsilon \\ \varphi \\ \vartheta \end{pmatrix} + \int_{r+\varepsilon}^{r-\varepsilon} dr\; \underline{e}_r\,\underline{v}\begin{pmatrix} r \\ \varphi \\ \vartheta+\varepsilon \end{pmatrix} + \int_{\vartheta+\varepsilon}^{\vartheta-\varepsilon} d\vartheta\,(r-\varepsilon)\,\underline{e}_\vartheta\,\underline{v}\begin{pmatrix} r-\varepsilon \\ \varphi \\ \vartheta \end{pmatrix} + \int_{r-\varepsilon}^{r+\varepsilon} dr\; \underline{e}_r\,\underline{v}\begin{pmatrix} r \\ \varphi \\ \vartheta-\varepsilon \end{pmatrix}\Big)$

$\to \frac{1}{r}\frac{\partial(r v^\vartheta)}{\partial r} - \frac{1}{r}\frac{\partial v^r}{\partial\vartheta}$

$\Rightarrow \operatorname{rot}\underline{v} = \frac{1}{r\sin\vartheta}\Big(\frac{\partial(\sin\vartheta\, v^\varphi)}{\partial\vartheta} - \frac{\partial v^\vartheta}{\partial\varphi}\Big)\underline{e}_r + \frac{1}{r}\Big(\frac{1}{\sin\vartheta}\frac{\partial v^r}{\partial\varphi} - \frac{\partial(r v^\varphi)}{\partial r}\Big)\underline{e}_\vartheta$

$\quad + \frac{1}{r}\Big(\frac{\partial(r v^\vartheta)}{\partial r} - \frac{\partial v^r}{\partial\vartheta}\Big)\underline{e}_\varphi \,.$

2

Theo ②

(a) ϕ, $\gamma: [0,1] \to \mathbb{R}^d$ parametrisiere L

$\int \underline{\nabla}\phi\, d\underline{s} \overset{\text{Def}}{=} \int_0^1 \langle \underline{\nabla}\phi\; \gamma(t),\, \dot\gamma(t) \rangle\, dt$

$\frac{d\phi}{dt} = \langle \underline{\nabla}\phi, \dot\gamma \rangle \qquad \left(\frac{d\phi}{dt} = \frac{\partial\phi}{\partial\gamma} \cdot \frac{\partial\gamma}{\partial t} \right)$ $\Big| \Rightarrow \int_{\partial L} \underline{\nabla}\phi\, d\underline{s} = \int_0^1 \frac{d\phi}{dt}\, dt \ominus$

Nach Newton-Leibnitz $\ominus\ \phi \circ \gamma \big|_0^1$ $\rightsquigarrow$ nur von Anf./Ende abhängig!

(b) Zerteile Vol. V in ~~kl.~~ N kleiner Untervolumina; die Ränder der jew. ~~Unter~~ liegen aneinander. $\int_{\partial V} \underline{v}\, d\underline{f} = \Sigma_k \int \underline{v}\, d\underline{f}^k$ da Terme an Hilfsrand sich wegheben. Nach Def:

$\Sigma_k \int \underline{v}\, d\underline{f}^k = \Sigma_k |\delta V^k| \cdot \operatorname{div} \underline{v} \longleftarrow \int \operatorname{div} \underline{v}\, dV$ wenn man V ~~of~~ klein zerteilt.

(c) Zerteilung, denn sich Hilfslinien wegheben: $\int_{\partial A} \underline{v}\, d\underline{r} = \Sigma_k \int \underline{v}\, d\underline{r}^k$.

Nach Def. $\Sigma_k \int \underline{v}\, d\underline{r}^k = \Sigma_k \underline{n} \cdot \operatorname{rot} \underline{v} \cdot |\Delta A^k| \longleftarrow \int \operatorname{rot} \underline{v}\, d\underline{f}$ bei kleiner Zerteilung.

(d) $\underline{\nabla}(\phi \cdot \underline{\nabla}\psi) = \underline{\nabla}\phi \cdot \underline{\nabla}\psi + \phi \cdot \Delta\psi \qquad (*)$

$\int_V \phi\Delta\psi + \underline{\nabla}\phi \cdot \underline{\nabla}\psi\, d^d r = \int_V \underline{\nabla}(\phi\underline{\nabla}\psi)\, d^d r \overset{\text{Gauß}}{=} \int_{\partial V} \phi \cdot \underline{\nabla}\psi\, d\underline{f}. \qquad (7')$

$$\left.\begin{aligned} \int_V \phi\Delta\psi\, d^d r &\overset{(7)}{=} \int_{\partial V} \phi\underline{\nabla}\psi\, d\underline{f} - \int_V \underline{\nabla}\phi\, \underline{\nabla}\psi\, d^d r \\ \int_V \psi\Delta\phi\, d^d r &\overset{(7,*)}{=} \int_{\partial V} \psi\underline{\nabla}\phi\, d\underline{f} - \int_V \underline{\nabla}\phi\, \underline{\nabla}\psi\, d^d r \end{aligned}\right\} \Rightarrow$$

(*) Symm. d. Skal.produkts.

$\int_V \phi\Delta\psi - \psi\Delta\phi\, d^d r = \int_{\partial V} \phi\underline{\nabla}\psi - \psi\underline{\nabla}\phi\, d\underline{f}\ \boxed{\pm 0} \qquad (8')$

□

3 Durch Verschieben des Koord.systems kann man oBdA $\underline{r}' = \underline{0}$.

(ii) Mit S. v. Gauß: $\int_V \Delta \frac{-1}{4\pi|\underline{r}|}\, dV = \int_{\partial V} \underline{\nabla} \frac{-1}{4\pi|\underline{r}|}$.

Wähle Kugelkoord.: $\|\underline{r}\| = r$, $\underline{\nabla} \frac{-1}{4\pi r} = \frac{1}{4\pi r^2} \underline{e}_r$. Sei V ~~eine~~ Kugel mit Rad. R: $d\underline{f} = R^2 \sin\vartheta\, \underline{e}_r\, d\varphi\, d\vartheta$

$$\underbrace{\int_{[0,2\pi]} d\varphi}_{2\pi} \int_{[0,\pi]} d\vartheta\ \frac{1}{4\pi R^2} R^2 \underbrace{\sin\vartheta}_{2}\ \underbrace{\underline{e}_r \underline{e}_r}_{1} = \frac{4\pi}{4\pi} = 1.$$

(i) Mit ~~der~~ div in Kugelkoord.: $\Delta \frac{-1}{4\pi r} = \frac{1}{r^2} \frac{\partial\left(r^2 \nabla \frac{-1}{4\pi r}\right)}{\partial r} = \frac{1}{r^2} \cdot \frac{\partial \frac{-1}{4\pi}}{\partial r} = \frac{1}{r^2} \cdot 0$ für $r \neq 0$. (für $r = 0$ nicht def.)