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[1]

(b) Bekannt: $\psi(x,t)$ lässt sich schreiben als

$$\psi(x,t) = \int \frac{1}{2\pi} \cdot \varphi(\xi) \cdot e^{-i\frac{\varepsilon_k t}{\hbar}} e^{i\xi x} \, d\xi \qquad (*)$$

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m}$$

Für $t=0$:

$$\psi(x,0) = \int \frac{1}{2\pi} \varphi(\xi) \, e^{i\xi x} \, d\xi$$

Die $\varphi(\xi)$ kann man durch Fouriertransform. bestimmen:

$$\varphi(\xi) = \int \psi(x,0) \, e^{-i\xi x} \, dx$$

Mit $\psi(x,0)$ aus (1) gilt:

$$\varphi(\xi) = \int A \, e^{ik_0 x} e^{-\frac{(x-x_0)^2}{4\sigma}} \cdot e^{-i\xi x} \, d\xi = \int A \, e^{-\frac{(x-x_0)^2}{4\sigma}} e^{-i\overbrace{(\xi-\xi_0)}^{\Delta\xi} x} \, dx$$

Exponent: $-\frac{1}{4\sigma}(x^2 - 2xx_0 + x_0^2) - \frac{1}{4\sigma} i\Delta\xi 4\sigma x =$

$$-\frac{1}{4\sigma}\left(x^2 - 2xx_0 + x_0^2 + i\Delta\xi 4\sigma x\right) = -\frac{1}{4\sigma}\left[x^2 + (i\Delta\xi 4\sigma - 2x_0)x\right] - \frac{x_0^2}{4\sigma} =$$

$$-\frac{1}{4\sigma}\left(x + \frac{i\Delta\xi 4\sigma - 2x_0}{2}\right)^2 + \frac{1}{4\sigma}\left(\frac{i\Delta\xi 4\sigma - 2x_0}{2}\right)^2 - \frac{x_0^2}{4\sigma}$$

$$= A \cdot \sqrt{4\sigma\pi} \; e^{+\frac{1}{4\sigma}\left(\frac{i\Delta\xi 4\sigma - 2x_0}{2}\right)^2 - \frac{x_0^2}{4\sigma}}$$

Exponent: $+\frac{1}{4\sigma}\left(-\Delta\xi^2 4\sigma^2 - i\Delta\xi 4\sigma x_0 + \cancel{x_0^2} - \cancel{x_0^2}\right) =$

$$-\Delta\xi^2\sigma - i\Delta\xi x_0$$

$$= \underbrace{\sqrt[4]{\frac{1}{2\pi\sigma}} \sqrt{4\sigma\pi}}_{B} \cdot e^{-\Delta\xi(\Delta\xi\sigma + ix_0)}$$

Ins. $\psi(x,t)$ mit $(*)$:

$$\psi(x,t) = \int \frac{1}{2\pi} B \cdot e^{-\Delta\xi(\Delta\xi\sigma + ix_0)} \cdot e^{-i\frac{\hbar\xi^2 t}{2m}} e^{i\xi x} \, d\xi$$

Exponent: $-(\xi-\xi_0)^2\sigma - i(\xi-\xi_0)x_0 - i\frac{\hbar t}{2m}\xi^2 + i\xi x =$

$$-i\sigma\left[-i\xi^2 + i2\xi\xi_0 - i\xi_0^2 + \frac{x_0}{\sigma}\xi - \frac{x_0}{\sigma}\xi_0 + \frac{\hbar t}{2m\sigma}\xi^2 - \frac{x}{\sigma}\xi\right] =$$

$$-i\sigma\left[\left(-i + \frac{\hbar t}{2m\sigma}\right)\xi^2 + \left(2i\xi_0 + \frac{x_0}{\sigma} - \frac{x}{\sigma}\right)\xi + \left(i\xi_0^2 - \frac{x_0}{\sigma}\xi_0\right)\right] =$$

$$\underbrace{-i\sigma\left[\left(-i + \frac{\hbar t}{2m\sigma}\right)\right.}\left\{\xi^2 + \left(\frac{2i\xi_0 + \frac{x_0}{\sigma} - \frac{x}{\sigma}}{-i + \frac{\hbar t}{2m\sigma}}\right)\xi\right\}\Big] - i\sigma\left(i\xi_0^2 - \frac{x_0}{\sigma}\xi_0\right) =$$

$$-\sigma_t\left(\xi + \frac{2i\xi_0 + (x_0 - x)/\sigma}{2\left(-i + \frac{\hbar t}{2m\sigma}\right)}\right)^2 + \underbrace{\sigma_t\left(\frac{2i\xi_0 + (x_0 - x)/\sigma}{2\left(-i + \frac{\hbar t}{2m\sigma}\right)}\right)^2 - \sigma\xi_0^2 + ix_0\xi_0}_{\omega}$$

$$= \frac{1}{2\pi} B \cdot \sqrt{\frac{\pi}{\sigma_t}} \cdot e^{\omega}$$

$$\left(-i + \frac{\hbar t}{2m\sigma}\right) = -i\frac{1}{\sigma}\left(\sigma + i\frac{\hbar t}{2m}\right) = -i\,\sigma_t/\sigma$$

$$= \frac{1}{2\pi} \cdot \left(\frac{1}{2\pi\sigma}\right)^{1/4} (4\sigma\pi)^{1/2} \left(\pi/\sigma_t\right)^{1/2} \cdot e^{\omega}$$

$$= \left(\frac{4^2\sigma^2\pi^2\pi^2}{2^4\pi^4 2\pi\sigma\,\sigma_t^2}\right)^{1/4} e^{\omega} = \left(\frac{\sigma}{2\pi\sigma_t^2}\right)^{1/4}$$

Exponent: $\sigma_t \cdot \frac{\sigma^2}{4\sigma_t^2} \left(2 i \xi_0 + \frac{x_0 - x}{\sigma}\right)^2 - \sigma \xi_0^2 + i x_0 \xi_0 =$

$- \xi_0^2 \frac{\sigma^2}{\sigma_t} + \frac{(x_0-x)^2}{\cancel{\sigma^2}} \frac{\sigma^2}{4\sigma_t} + i\xi_0 \frac{x_0 - x}{\cancel{\sigma}} \frac{\sigma}{\sigma_t} - \sigma\xi_0^2 + i x_0 \xi_0$

Laut Maxima ist dies das selbe wie in (2).

(c)

$\langle x \rangle = \int \psi^* \cdot \psi \cdot x \, dx = \int |\psi|^2 x \, dx$

Bei $|\psi|$ spielen die e^i-Terme keine Rolle, da $|e^{i\varphi}| = 1$.

$= \int \left(\frac{\sigma}{2\pi}\right)^{1/2} \left((\sigma_t^{-2})^* \cdot (\sigma_t^{-2})\right)^{1/4} \cdot e^{-\frac{\left(x - \overbrace{(x_0 + \frac{t\hbar \xi_0}{m})}^{\lambda}\right)^2}{4} \left[\frac{1}{\sigma_t} + \frac{1}{\sigma_t^*}\right]} dx$

$\sigma_t := |\sigma_t| e^{i\varphi} : (\sigma_t^{-2})^* = \left(\frac{1}{|\sigma_t|^2} e^{-i2\varphi}\right)^* = \frac{1}{|\sigma_t|^2} e^{i2\varphi}$

$(\sigma_t^{-2}) = \frac{1}{|\sigma_t|^2} e^{-i2\varphi}$

$\left((\sigma_t^{-2})^* (\sigma_t^{-2})\right)^{1/4} = \left(\frac{1}{|\sigma_t|^4}\right)^{1/4} = \frac{1}{|\sigma_t|}$

$\frac{1}{\sigma_t} \cdot \frac{\sigma_t^*}{\sigma_t^*} = \frac{\sigma_t^*}{|\sigma_t|^2}$ $\quad \frac{1}{\sigma_t^*} = \frac{\sigma_t}{|\sigma_t|^2}$ $\quad \frac{1}{\sigma_t} + \frac{1}{\sigma_t^*} = \frac{\sigma_t + \sigma_t^*}{|\sigma_t|^2} = \frac{2\sigma}{|\sigma_t|^2}$

$\langle x \rangle = \int \frac{1}{|\sigma_t|} \left(\frac{\sigma}{2\pi}\right)^{1/2} x \, e^{-\frac{(x-\lambda)^2}{4} \frac{2\sigma}{|\sigma_t|^2}} dx$ $\qquad (x-\lambda) =: y$, $\frac{dy}{dx} = 1$

$= \int \frac{1}{|\sigma_t|} \sqrt{\frac{\sigma}{2\pi}} (y + \lambda) \, e^{-\frac{y^2}{2} \frac{\sigma}{|\sigma_t|^2}}$

$\int$ damit verschw. weg. Punktsymmetrie

$= \frac{1}{\cancel{|\sigma_t|}} \sqrt{\frac{\sigma}{2\pi}} \, \lambda \sqrt{\frac{\cancel{2\pi |\sigma_t|^2}}{\sigma}} = \lambda = x_0 + \frac{t\hbar \xi_0}{m} t$

$\langle v \rangle = \partial_t \langle x \rangle = \frac{t\xi_0}{m}$

(d) $\langle x^2 \rangle = \int |\psi|^2 x^2 \, dx$

$= \frac{1}{|\sigma_t|} \sqrt{\frac{\sigma}{2\pi}} \int x^2 e^{-\frac{(x-\lambda)^2}{2} \frac{\sigma}{|\sigma_t|^2}} dx$ $\qquad (x - \lambda) =: y$

$= \quad " \quad \int (y^2 + 2\underbrace{y}_{\text{verschwindet}}\lambda + \lambda^2) \, e^{-\frac{y^2}{2}\frac{\sigma}{|\sigma_t|^2}} dy$ $\qquad \frac{dy}{dx} = 1$, $x = y + \lambda$, $x^2 = y^2 + 2y\lambda + \lambda^2$

$= \underbrace{\lambda^2}_{s.o.} + \frac{1}{|\sigma_t|} \sqrt{\frac{\sigma}{2\pi}} \int y^2 e^{-\frac{y^2}{2} \frac{\sigma}{|\sigma_t|^2}} dy$

Aus $\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$

folgt mit $y^2 = t$ u.a. $y = \frac{z}{\sqrt{b}}$:

$$\Gamma\left(\tfrac{3}{2}\right) = 2\int_0^\infty b^{-3/2}\, z^2\, e^{-z^2/b} = \frac{1}{2}\underset{\sqrt{\pi}}{\underset{\shortparallel}{\Gamma\left(\tfrac{1}{2}\right)}} \qquad \text{Da } \left(\tfrac{3}{2}\right) = \left(1 + \tfrac{1}{2}\right)$$

$$\Rightarrow \int_0^\infty z^2\, e^{-z^2/b}\, dz = \frac{1}{4}\sqrt{\pi}\; b^{3/2}$$

$$\Rightarrow \langle x^2 \rangle = \lambda^2 + \frac{1}{|\sigma_t|}\sqrt{\frac{\sigma}{2\pi}} \cdot \frac{1}{4}\sqrt{\pi} \cdot \left(\frac{2|\sigma_t|}{\sigma}\right)^{3/2}$$

$$= \lambda^2 + |\sigma_t|^2\, \frac{1}{\sigma}\, \cancel{\frac{\sqrt{2}\,4}{\sqrt{2}\,4}}$$

$$\Rightarrow \langle x \rangle^2 = \lambda^2$$

$$\Rightarrow (\Delta x)^2 = \lambda^2 + \frac{|\sigma_t|^2}{\sigma} - \lambda^2 = \frac{|\sigma_t|^2}{\sigma}$$

$$|\sigma_t|^2 = \sigma^2 + \frac{\hbar^2}{4m^2}t^2 \Rightarrow (\Delta x)^2 = \sigma + \frac{\hbar^2}{4\sigma m^2}t^2$$

$$= \sigma\Big(\underset{\sigma_0}{\underset{\downarrow}{1}} + \underbrace{\frac{\hbar^2}{4\sigma^2 m^2}t^2}_{\sigma_0}\Big)$$

e) $\psi \xrightarrow{\delta_0 = 0,\ \omega \in \mathbb{R}} T$

$$T(x,t) = \left(\frac{\sigma}{2\pi\sigma_t^2}\right)^{1/4} e^{-\frac{(x-x_0)^2}{4\sigma_t}} \qquad \sigma_t = \sigma + \frac{\hbar}{2m}t$$

$$\langle x \rangle = \int \text{— " —}\; x\, dx \qquad (x - x_0) =: y, \quad \frac{dy}{dx} = 1$$

$$= \int \left(\frac{\sigma}{2\pi\sigma_t^2}\right)^{1/4} \underset{\text{verschw.}}{\underset{\uparrow}{(y + x_0)}}\, e^{-\frac{y^2}{4\sigma_t}}\, dy$$

$$= \left(\frac{\sigma}{2\pi\,\sigma_t^2}\right)^{1/4} \cdot x_0 \cdot \sqrt{4\pi\sigma_t} = (8\pi\sigma)^{1/4}\, x_0$$

$$\langle x^2 \rangle = \int \left(\frac{\sigma}{2\pi\sigma_t^2}\right)^{1/4} \cdot x^2\, e^{-\frac{(x-x_0)^2}{4\sigma_t}}\, dx \qquad x - x_0 =: y$$

$$= \left(\frac{\sigma}{2\pi\sigma_t^2}\right)^{1/4} \int \underset{\text{verschw.}}{\underset{\uparrow}{\underset{y^2 + 2yx_0 + x_0^2}{(y + x_0)^2}}}\, e^{-\frac{y^2}{4\sigma_t}}\, dy$$

$$= (8\pi\sigma)^{1/4}\, x_0^2 + \left(\frac{\sigma}{2\pi\sigma_t^2}\right)^{1/4} \int y^2\, e^{-\frac{y^2}{4\sigma_t}}\, dy \qquad 8\sigma\sigma_t^4$$

↓ ws Int-oben ob

$$= \text{— " —} + \left(\frac{\sigma}{2\pi\sigma_t^2}\right)^{1/4} 2\,\frac{1}{4}\sqrt{\pi}\,(4\sigma_t)^{3/2} \qquad \frac{\sigma}{2\pi\sigma_t^2}\,\pi^2\,\frac{4^6}{4^4}\,\sigma_t^6$$

$$= x_0^2 \sqrt[4]{8\pi\sigma} + 2\sigma_t \sqrt[4]{8\pi\sigma}$$

$$= \sqrt[4]{8\pi\sigma}\,(x_0^2 + 2\sigma_t)$$

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \sqrt[4]{8\pi\sigma}\,(x_0^2 + 2\sigma_t) - \sqrt{8\pi\sigma} \cdot x_0^2$$

$$= x_0^2\left(\sqrt[4]{8\pi\sigma} - \sqrt{8\pi\sigma}\right) + 2\sigma_t \sqrt[4]{8\pi\sigma}$$

$$= \text{—} \quad '' \quad \text{—} \quad 2\left(\sigma + \frac{\hbar}{2m}t\right)\sqrt[4]{8\pi\sigma}$$

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$$= \sigma\Big(\underbrace{\frac{x_0^2}{\sigma}\left(\sqrt[4]{8\pi\sigma} - \sqrt{8\pi\sigma}\right) + 2\sqrt[4]{8\pi\sigma}}_{b_0} + \underbrace{2\frac{\hbar}{2m}\sqrt[4]{8\pi\sigma}\,\frac{1}{\sigma}}_{b_1} t\Big)$$

(7)
$$\psi(x,t) = \int \frac{1}{2\pi}\varphi(\xi) \cdot e^{i\xi x - i\omega t}\, d\xi$$

$$= \int \frac{1}{2\pi}\varphi(\xi) \cdot e^{i(\xi x - c_0 t - c_1 \xi t)}\, d\xi$$

$$= \int \frac{1}{2\pi} A\sqrt{4\pi\sigma}\; e^{-(\xi-\xi_0)x_0 - \sigma(\xi-\xi_0)^2}\; e^{i(\xi(x-c_1 t) - c_0 t)}\, d\xi$$

Exponent: $-\xi x + \xi_0 x_0 - \sigma\xi^2 + 2\sigma\xi\xi_0 - \sigma\xi_0^2 + i\xi(x - c_1 t) - ic_0 t =$

$$(-\sigma)\xi^2 + (-x + 2\sigma\xi_0 + i(x - c_1 t))\xi + (\xi_0 x_0 - \sigma\xi_0^2 - ic_0 t) =$$

$$-\sigma\left(\xi + \left(\frac{x}{2\sigma} - \xi_0 - \frac{i(x-c_1 t)}{2\sigma}\right)\right)^2 + \underbrace{\sigma\left(\frac{x}{2\sigma} - \xi_0 - \frac{i(x-c_1 t)}{2\sigma}\right)^2 + (\xi_0 x_0 - \sigma\xi_0^2 - ic_0 t)}$$

$$\psi(x,t) = \sqrt{\frac{\pi}{\cancel{\sigma}}} \cdot \frac{1}{\cancel{2\pi}} A \sqrt{\cancel{4\pi\sigma}} \cdot \exp\big(\qquad \big)$$

$$= A \cdot \exp\big(\quad '' \quad \big)$$

$$= A\exp\left(\frac{x^2}{4\sigma} + \sigma\xi_0^2 - x\xi_0 - \frac{(x-c_1 t)^2}{4\sigma} - i(x - c_1 t)\left(\frac{x}{2\sigma} - \xi_0\right) + \xi_0 x_0 - \sigma\xi_0^2 - ic_0 t\right)$$

$$= A\exp\left(-\frac{(x-c_1 t)^2}{4\sigma}\right)\exp\left(\frac{x^2}{4\sigma} - x\xi_0 + \cancel{\sigma\xi_0^2} - \xi_0 x_0 - \cancel{\sigma\xi_0^2}\right)\underbrace{\exp\left(i\left((x-c_1 t)\left(\tfrac{x}{\sigma} - \xi_0\right) - c_0 t\right)\right)}$$

$$= A\exp\Big(-\underbrace{\frac{(x-c_1 t)^2}{4\sigma}}\Big)\exp\left(\frac{x^2}{4\sigma} - \xi_0(x - x_0)\right) \cdot \text{mod Sch. Kompl. Schwg.}$$

Gauß-Glocke mit mit Breite σ

"wandert" durch $c_1 t$.

(8) $\int |\psi|^2 dx \equiv 1$

$$\int \psi^* \psi\, dx = |A|^2 \int \cancel{e^{-i\xi_0 x}}\; \cancel{e^{i\xi_0 x}}\; e^{-\frac{(x-x_0)^2}{4\sigma}\cdot 2}$$

$$= |A|^2 \cdot \sqrt{2\pi\sigma} \overset{!}{=} 1$$

$$\Rightarrow \quad A = \left(\frac{1}{2\pi\sigma}\right)^{1/4}$$