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Theo Übung (9)

1 (a)

$$\Psi_{n,l,m}(r,\theta,\phi) = \underbrace{-N_{nl}(2\kappa r)^{l}\, e^{-\kappa r}\, L^{2l+1}_{n+l}(2\kappa r)}_{\text{Radialteil } R_{lm}(r)}\ \underbrace{Y_{lm}(\phi,\theta)}_{\text{Winkelteil}}$$

$$\Psi_{n,n-1,n-1}(r,\phi,\theta) = -N_{n,n-1}\cdot(2\kappa r)^{n-1}\cdot e^{-\kappa r}\cdot L^{2n-1}_{2n-1}(2\kappa r)\cdot Y_{n-1,n-1}(\phi,\theta)$$

$$N_{n,n-1} = \left(\frac{Z}{a_b}\right)^{3/2}\cdot\frac{2}{n^2\cdot(2n-1)!}\sqrt{\frac{0!}{(2n-1)!}}\,, \quad \kappa = \frac{Z}{a_b\cdot n}$$

$$L^{2n-1}_{2n-1}(2\kappa r) = \frac{(2n-1)!}{0!}\cdot e^{2\kappa r}\left(\frac{\partial}{\partial(2\kappa r)}\right)\left((2\kappa r)^0\cdot e^{-2\kappa r}\right)$$

$$= (2n-1)!\cdot\cancel{e^{2\kappa r}}\cdot(-1)^{2n-1}\,\cancel{e^{-2\kappa r}}$$

$2n$ ist stets gerade, $2n-1$ stets ungerade

$$Y_{n-1,n-1}(\phi,\theta) = \frac{(-1)^{n-1}}{2^{n-1}(n-1)!}\sqrt{\frac{2n-1}{4\pi}\,\frac{(2n-2)!}{0!}}\cdot\frac{e^{i\phi(n-1)}}{\sin^{n-1}\theta}\left(\frac{\partial}{\partial(\cos\theta)}\right)^{0}\sin^{2n-2}\theta$$

$$\Psi_{n,n-1,n-1}(r,\phi,\theta) = \cancel{-}\left(\frac{Z}{a_b}\right)^{3/2}\cdot\frac{2}{n^2\,\cancel{(2n-1)!}}\sqrt{\frac{1}{(2n-1)!}}\,(2\kappa r)^{n-1}\cdot e^{-\kappa r}\cdot\cancel{(2n-1)!}\cdot(-1)\cdot$$
$$\frac{(-1)^{n-1}}{2^{n-1}(n-1)!}\sqrt{\frac{2n-1}{4\pi}\cdot(2n-2)!}\cdot\frac{e^{i\phi(n-1)}}{\cancel{\sin^{n-1}\theta}}\cdot\sin^{2(n-1)}\theta$$

$$= (-1)^{n+1}\left(\frac{Z}{a_b}\right)^{3/2}\frac{1}{n^2}\,(2\kappa r)^{n-1}e^{-\kappa r}\ \frac{1}{(n-1)!}\sqrt{\frac{1}{\pi}}\ e^{i\phi(n-1)}\sin^{n-1}\theta$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{\kappa a_b n}{a_b}\right)^{3/2}\frac{1}{n^2\,(n-1)!}\,(-\kappa r)^{n-1}\,e^{-\kappa r+i\phi(n-1)}\sin^{n-1}\theta$$

$$= \frac{1}{\sqrt{\pi n}}\,\frac{1}{(n-1)!}\,\kappa^{3/2}\,(-\kappa r)^{n-1}\,e^{-\kappa r+i\phi(n-1)}\sin^{n-1}\theta$$

$d\tau = dr\, d\Omega$

Theo 9

1.) (b) $\langle r \rangle = \int \psi\, r\, \psi^* \, d\tau$

$$\psi = \cancel{-} + \left(\frac{Z}{a_b}\right)^{3/2} \frac{2}{n^2 \cancel{(2n-1)!}} \sqrt{\frac{1}{(2n-1)!!}} (2\kappa r)^{n-1} e^{-\kappa r} \cancel{(2n-1)!} \cancel{(-1)}\, Y_{n-1,n-1}(\phi,\theta)$$

$$= \frac{\kappa^{3/2}}{\sqrt{n}} 2^n (\kappa r)^{n-1} \sqrt{\frac{1}{(2n-1)!}}\, e^{-\kappa r}\, Y_{n-1,n-1}(\phi,\theta)$$

$$\psi^* = \text{—} \quad " \quad \text{—} \quad Y^*_{n-1,n-1}(\phi,\theta)$$

$$\langle r \rangle = \int R Y\, r\, R^* Y^* \, d\tau = \iint R\, r \underset{\uparrow}{R}\, d\tau \; Y Y^* d\Omega = \int R^2 r\, dr\, \mathcal{J} \underbrace{\int Y Y^* d\Omega\, \tilde{\mathcal{J}}}_{1}$$

da $R = R^*$ (r.o.)

weil Y normiert ist!

($\mathcal{J}, \tilde{\mathcal{J}}$ bilden die Jacobian der Integration! $\mathcal{J} \Rightarrow \mathcal{J} = r^2$, da in Kugelkoord. sonst $\mathcal{J} \cdot \tilde{\mathcal{J}} = r^2 \cos\theta$ ist)

$$\langle r \rangle = \int_0^\infty dr \, \frac{\kappa^3}{n} 2^{2n} (\kappa r)^{2n-2} \frac{1}{(2n-1)!} e^{-2\kappa r} r^3$$

$$= \frac{\cancel{\kappa^3}\, 2^{2n} \kappa^{2n-2+3}}{n\,(2n-1)!} \int r^{2n-2+3} e^{-2\kappa r} dr$$

setze $2\kappa r =: t$ $\Rightarrow \frac{dt}{dr} = 2\kappa$

$$= \;" \left(\frac{1}{2\kappa}\right)^{2n+1} \frac{1}{2\kappa} \underbrace{\int t^{2n+1} e^{-t} dt}_{\Gamma(2n+2) = (2n+1)!}$$

$$= \frac{1}{2\kappa} \frac{\cancel{2^{2n}}}{\cancel{n}} \cdot 2n\,(2n+1) \frac{1}{\cancel{2}^{\cancel{3}+1}} = \frac{2n+1}{2\kappa} = \frac{(2n+1)\, a_b\, n}{2Z}$$

$$= \frac{a_b n}{Z} \cdot (n + 1/2)$$

1 (c) $\Delta r = \sqrt{\langle r^2\rangle - \langle r\rangle^2}$

$\langle r\rangle^2 = \left[\frac{a_b n}{Z}(n+1/2)\right]^2$

$\underset{\text{analog zu (b)}}{\langle r^2\rangle =} \int_0^\infty R^2 r^4\, dr \underset{\text{siehe (b)}}{=} \frac{2^{2n} K^{2n+1}}{n\,(2n-1)!} \int_0^\infty r^{2n+2} e^{-2Kr}\, dr$

$\underset{\text{analog zu (b)}}{=} \frac{2^{2n} K^{2n+1}}{n\,(2n-1)!} \frac{1}{2K}\left(\frac{1}{2K}\right)^{2n+2} (2n+2)!$

$= \frac{1}{\cancel{4}48 n K^2}(\cancel{8}2n)(2n+1)(2n+2) = \frac{4n^2+6n+2}{4K^2} = \frac{(4n^2+6n+2)\,a_b^2 n^2}{4Z^2}$

$= \left(\frac{a_b n}{Z}\right)^2 (n^2 + \frac{3}{2}n + 1/2) = \left(\frac{a_b n}{Z}\right)^2 (n+1)(n+1/2)$

$\Delta r = \sqrt{\left(\frac{a_b n}{Z}\right)^2 (n+1/2)\left\{\cancel{}\,(n+1) - (n+1/2)\right\}}$

$= \left(\frac{a_b n}{Z}\right) \cdot \sqrt{\frac{n+1/2}{2}}$

$\Rightarrow \frac{\Delta r}{\langle r\rangle} = \sqrt{\frac{n+1/2}{2}} \Big/ (n+1/2) = \frac{1}{\sqrt{2n+1}}$

(d): „Bahnebene & orthogonal dazu" bedeutet: $\theta = \cancel{0}\ \pi/2$.
bzw. in ~~Polarkoord.~~ Zylinderkoord. mit (r,θ).

$|\psi|^2 = \left\{ \frac{1}{\pi n} \left(\frac{1}{(n-1)!}\right)^2 K^3 (-Kr)^{2n-2} e^{-2Kr} \sin^{2n-2}\theta \right\}$

$= \frac{1}{\pi n} \frac{1}{(n-1)!^2} \cancel{\frac{Z^3}{a_b^3}} \left(\frac{Z}{a_b}\right)^3 \frac{1}{n^3} \left(\frac{-Zr}{a_b \cdot n}\right)^{2n-2} e^{-2Zr/a_b \cdot n} \sin^{2n-2}\theta$

$\hookrightarrow$ von ϕ unabhängig $\rightarrow$ drehsymmetrisch zur z-Achse.