Ana 5

II

(a) $f(0) = 0\,, \quad g(0) = \int_0^1 \frac{1}{1+t^2}\,dt = \arctan\Big|_0^1 = \pi/4$

$$f'(x) = 2\cdot\left(\frac{d}{dx}\int_0^x e^{-t^2}dt\right)\cdot\int_0^x e^{-t^2}\,dt = 2e^{-x^2}\int_0^x e^{-t^2}dt$$

$\Rightarrow$ Abl. exist. $\Rightarrow f$ diff'bar.

$$g'(x) = \frac{d}{dx}\int_0^1 \frac{1}{1+t^2}\,e^{-x^2(1+t^2)}\,dt \overset{(*)}{\overset{!}{=}} \int_0^1 \frac{1}{1+t^2}\frac{\partial}{\partial x}e^{-x^2(1+t^2)}$$

Satz 2, 4.11.'05 (*)

$$= \int_0^1 -2x\, e^{-x^2(1+t^2)}\,dt = -2x\int_0^1 e^{-x^2(1+t^2)}\,dt$$

Subst.: $x\cdot t =: s \Rightarrow \frac{ds}{dt} = x \Rightarrow dt = \frac{1}{x}\cdot ds$

$$= \int_0^x -2\, e^{-s^2/t^2(1+t^2)}\,ds = -\int_0^x 2\,e^{-\cancel{x}^2}\,e^{-s^2}\,ds$$

(b) $\lim\limits_{x\to\infty} g(x)$

$f_\xi(t) = \frac{1}{t^2+1}\,e^{-\xi^2(1+t^2)}$ $\qquad g(t) = \frac{1}{1+t^2} \Rightarrow |f_\xi(t)| \le g(t)$,

$g\in L^1(\mathbb{R})$ da Riemann-int'bar $\Rightarrow$ mit Leb'konv'satz:

$$\lim_{\xi\to\infty}\int_0^1 \frac{1}{1+t^2}\,e^{-\xi^2(1+t^2)}\,dt = \int_0^1 \underbrace{\lim_{\xi\to\infty} e^{-\xi^2(1+t^2)}}_{0}\,\frac{1}{1+t^2}\,dt$$

$$= \int_0^1 0\;\frac{1}{1+t^2}\,dt = 0.$$

(b,c) $f(x) = \overset{\overset{!}{=}0}{f(0)} + \int_0^x f'(\tilde{x})\,d\tilde{x} \xrightarrow[x\to\infty]{} \pi/4$

$$g(x) = \underbrace{g(0)}_{\pi/4} + \underbrace{\int_0^x g'(\tilde{x})\,d\tilde{x}}_{-\pi/4} \xrightarrow[x\to\infty]{} 0$$

$$\Rightarrow \int_0^x f'(\tilde{x})\,d\tilde{x} \xrightarrow[x\to\infty]{} \pi/4$$

$$\int_{-\infty}^{+\infty} e^{-t^2}dt = 2\cdot\int_0^{+\infty} e^{-t^2}dt = 2\cdot\left(\int_0^{+\infty} e^{-t^2}dt\right)^{2/2} =$$

$$= 2\sqrt{\lim_{x\to\infty} f(x)} = 2\cdot\sqrt{\pi/4} = \sqrt{\pi}$$

2)

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a r \cos\theta \cos\varphi \\ b r \cos\theta \sin\varphi \\ c r \sin\theta \end{pmatrix} \quad , r \in (0,1),\ \varphi \in (0,2\pi),\ \theta \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

$$|\underline{\underline{J}}| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} a\cos\theta\cos\varphi, & -ar\sin\theta\cos\varphi, & -ar\cos\theta\sin\varphi \\ b\cos\theta\sin\varphi, & -br\sin\theta\sin\varphi, & br\cos\theta\cos\varphi \\ c\sin\theta, & cr\cos\theta, & 0 \end{vmatrix} r^2 abc$$

Entw.

$$\downarrow$$

$$= \sin\theta\,[-\sin\theta\cos\theta\cos^2\varphi - \sin\theta\cos\theta\sin^2\varphi] \cdot r^2abc$$

$$- \cos\theta\,[\cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi] \cdot r^2abc$$

$$= -\sin\theta\,(\sin\theta\cos\theta) - \cos\theta\,(\cos^2\theta) \cdot r^2abc$$

$$= -\cos\theta \cdot r^2abc$$

(a) $$\mathrm{vol}(E) = \left| \int_0^1 dr \int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/2} d\theta \quad 1 \cdot (-\cos\theta\, r^2abc) \right|$$

$$= \left| \int_0^1 dr \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \quad r^2abc \quad \underbrace{-\sin\theta\Big|_{-\pi/2}^{\pi/2}}_{-(1+1)} \right|$$

$$= \left| -4\pi abc \quad \tfrac{1}{3} r^3 \Big|_0^1 \right|$$

$$= \left| -4/3\,\pi abc \right| = 4/3\,\pi abc$$

(b) $$\int_B 1\, d\underline{x} = \frac{1}{8} \int_0^R dr \int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/2} d\theta \quad 1 \cdot +\cos\theta\, r^2$$

$$\frac{1}{8} \quad 2\pi \quad 2 \cdot \frac{1}{3} R^3 = \frac{1}{6}\pi R^3.$$

$$\int_B x_i\, d\underline{x} = \int_0^R dr \int_0^{\pi/2} d\varphi \int_0^{\pi/2} d\theta \ \underbrace{\sin\theta\cos\theta}_{*}\, r^2$$

$\int d\theta \sin\theta\cos\theta = \frac{1}{2}\sin^2\theta$

$$= \int_0^R dr \ \frac{1}{2}\pi \ r^3 \ \frac{1}{2}\sin^2\theta \Big|_0^{\pi/2}$$

$$= \frac{1}{4} R^4 \cdot \frac{1}{2}\pi \cdot \frac{1}{2}\, 1 = \frac{R^4}{16}\pi.$$

Wg. Symmetrie gilt dies für alle x_i!

$$\underline{S} = \frac{6^3}{\cancel{16}\,8} R\,(1,1,1)^T$$

$\boxed{V}$

(a) $\int x \, d\underline{x} = \int_0^1 \int_0^{(1-y^{2/3})^{3/2}} x \, dx \, dy = \int_0^1 \frac{1}{2} (1-y^{2/3})^3 \, dy$

$$y^{2/3} = \theta \qquad \frac{d\theta}{dy} = \frac{2}{3} \cdot y^{-1/3} = \frac{2}{3} \theta^{-1/2}$$

$$\Leftrightarrow dy = \frac{3}{2} \cdot \sqrt{\theta} \cdot d\theta$$

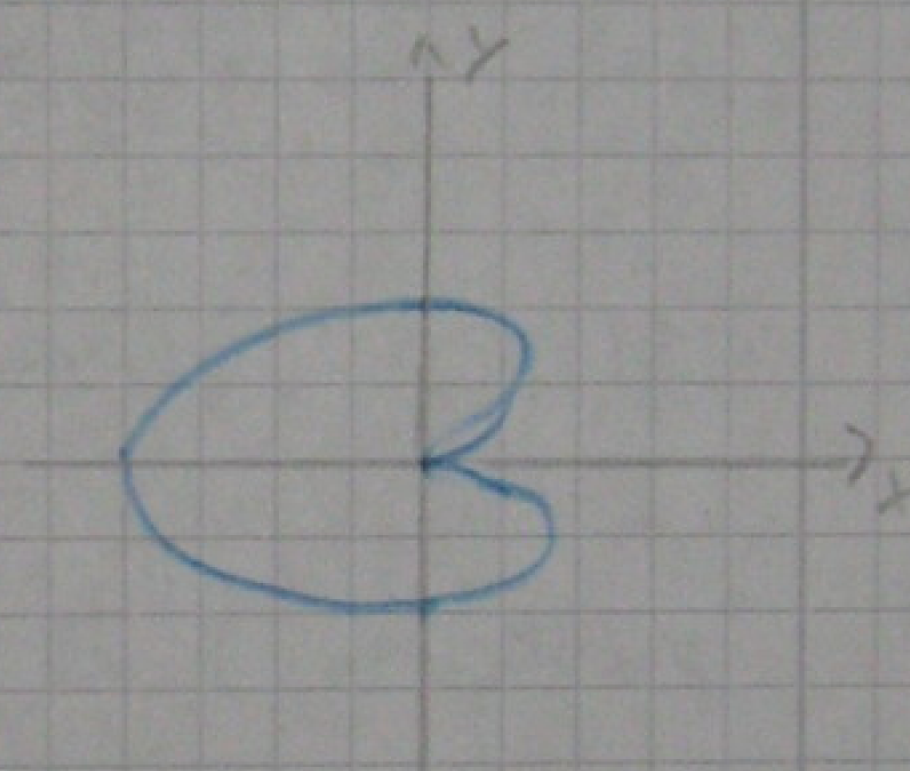
$$\int_0^1 \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot (1-\theta)^3 \sqrt{\theta} \, d\theta$$

$$(1-\theta)^3 = 1 \cdot 1 - 3 \cdot \theta + 3\theta^2 - 1\theta^3$$

$$\int_0^1 \frac{3}{4} \cdot \theta^{1/2} - 3\theta^{3/2} + 3\theta^{5/2} - \theta^{7/2} \, d\theta =$$

$$\frac{3}{4} \cdot \left[\frac{2}{3}\theta^{3/2} - \frac{6}{5}\theta^{5/2} + \frac{6}{7}\theta^{7/2} - \frac{2}{3}\theta^{9/2} \right]_0^1 = \frac{8}{105}.$$

(b) $\mathrm{vol}(K) =$



$$\int_0^{2\pi} d\varphi \int_0^{1-\cos\varphi} 1 \, r \, dr$$

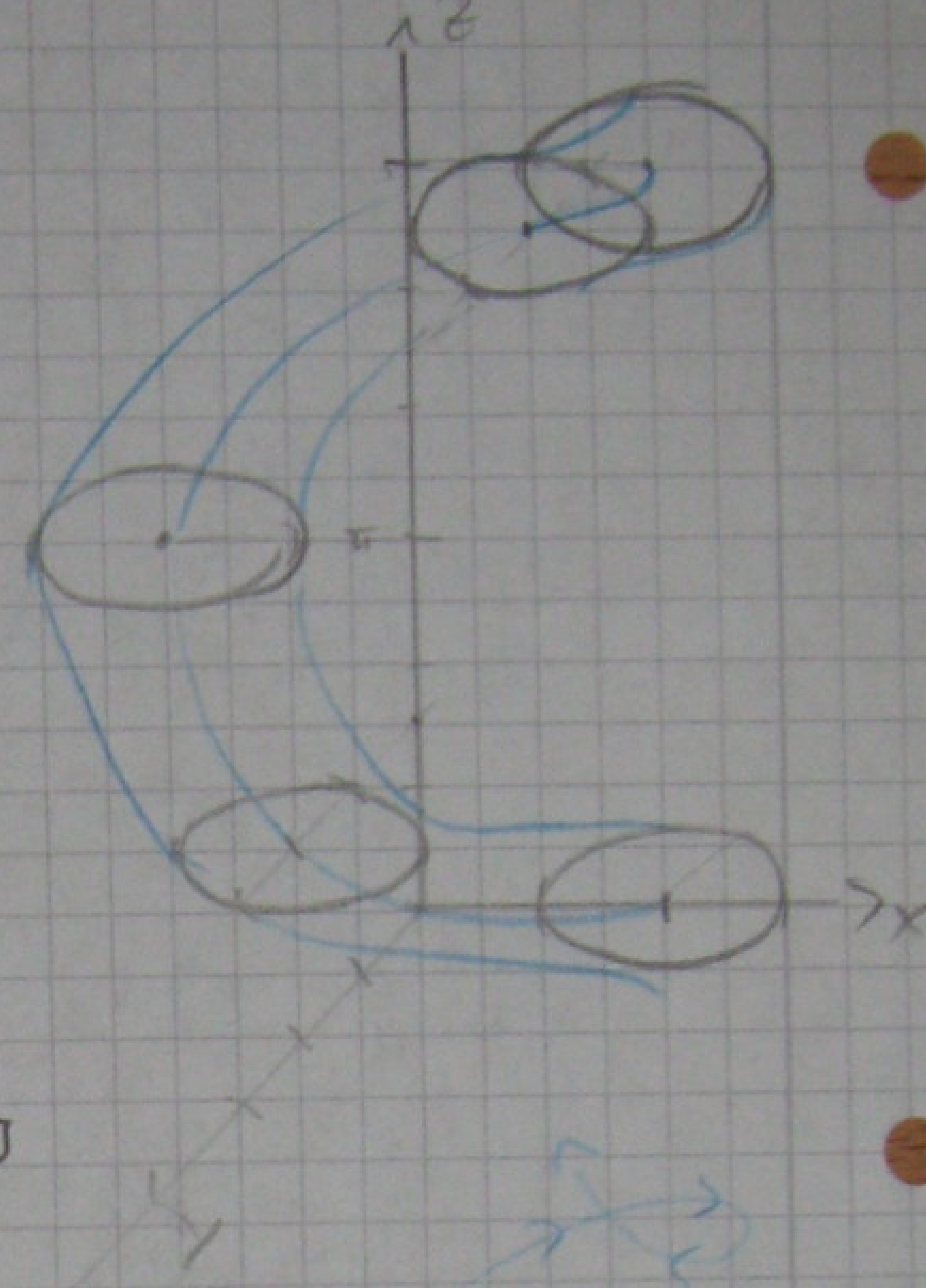
$$|\underline{Dg}| = \begin{vmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$$

$$= \int_0^{2\pi} d\varphi \; \frac{1}{2} (1-\cos\varphi)^2$$

$$= \frac{1}{2}\int_0^{2\pi} d\varphi \; \underbrace{1}_{2\pi} - \underbrace{2\cos\varphi}_{0} + \underbrace{\cos^2\varphi}_{\pi} = \pi + \frac{\pi}{2} = \frac{3}{2}\pi$$

(4)

$$M_h: \left\| \underline{x} - \begin{pmatrix} 2\cos x_3 \\ 2\sin x_3 \\ x_3 \end{pmatrix} \right\|_2 \le 1$$



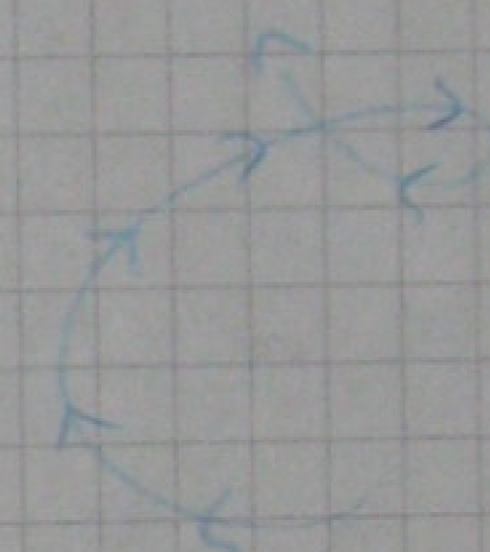
$$\underline{x} = \begin{pmatrix} 2\cos x_3 & + \tilde{x} \\ 2\sin x_3 & + \tilde{y} \\ x_3 & + \tilde{z} \end{pmatrix}$$

$$M_h = \{ \| (\tilde{x}, \tilde{y}, \tilde{z})^T \|_2^2 \le 1 \}$$

$\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 \le 1$

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ \tilde{z} \end{pmatrix}$$

$r \in (0, 1) \quad \theta \in [0, 2\pi), \ \tilde{z} \in [0, h]$



$$\underline{x} = \begin{pmatrix} 2\cos x_3 & + r\cos\theta \\ 2\sin x_3 & + r\sin\theta \\ x_3 & + \tilde{z} \end{pmatrix}$$

$|\underline{\underline{J}}| = r$ (Zylinderkoord.)

$$\mathrm{Vol}(M_h) = \int_0^h d\tilde{z} \int_0^{2\pi} d\varphi \int_0^1 dr \ r \ 1$$

$$= h \quad 2\pi \quad \frac{1}{2} = \pi \cdot h$$

($\rightsquigarrow$ S. v. Cavalieri: $\mathrm{Vol}(M_h)$ ident. mit Vol. eines Zyl.

d. Rad. $r = 1$, $h = h$.)