Elektrodynamik
Uebung 05

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7 (a)

$$\phi_1(\underline{r}) = \frac{Q^{tot}}{r}$$

$$\phi_2(\underline{r}) = \frac{\underline{p}\,\underline{r}}{r^3}, \quad \underline{p} = \int \rho(\underline{r}')\,\underline{r}'\,dV'$$

$$\phi_3(\underline{r}) = \frac{1}{2} Q_{\alpha\beta}\frac{x_\alpha x_\beta}{r^5}, \quad Q_{\alpha\beta} = \int \rho(\underline{r}')\left(3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta}\right) dV'$$

(a)

$$\rho(\underline{r}) = Q\Big(\delta(\underline{r}-(0,a,0)) + \delta(\underline{r}-(0,-a,0)) + \delta(\underline{r}-(0,0,a)) + \delta(\underline{r}-(0,0,-a)) - \delta(\underline{r}-(-a,0,0)) - \delta(\underline{r}-(-a/2,0,0)) - \delta(\underline{r}-(a,0,0)) - \delta(\underline{r}-(2a,0,0))\Big)$$

$\rightsquigarrow Q_{tot} = 0. \quad \Rightarrow \phi_1 = 0$

$$\rightsquigarrow \underline{p} = Q\left[\begin{pmatrix}0\\a\\0\end{pmatrix} + \begin{pmatrix}0\\-a\\0\end{pmatrix} + \begin{pmatrix}0\\0\\a\end{pmatrix} + \begin{pmatrix}0\\0\\-a\end{pmatrix} - \begin{pmatrix}-a\\0\\0\end{pmatrix} - \begin{pmatrix}-a/2\\0\\0\end{pmatrix} - \begin{pmatrix}a\\0\\0\end{pmatrix} - \begin{pmatrix}2a\\0\\0\end{pmatrix}\right]$$

$$= Q\begin{pmatrix}-3/2\\0\\0\end{pmatrix} \quad \Rightarrow \quad \phi_2 = -\frac{\frac{3}{2}Q x}{r^3}$$

$$\rightsquigarrow Q_{xx} = \int \rho(\underline{r}')\, 2x'^2 - y'^2 - z'^2\, dV'$$

$$= Q\left(-a^2 - a^2 - a^2 - a^2 - 2a^2 - \tfrac{1}{2}a^2 - 2a^2 - 8a^2\right) = -\frac{33}{2}Qa^2$$

$$Q_{yy} = \int \rho(\underline{r}')\, 2y'^2 - x'^2 - z'^2\, dV'$$

$$= Q\left(2a^2 + 2a^2 - a^2 - a^2 + a^2 + \tfrac{a^2}{4} + a^2 + 4a^2\right) = +\frac{33}{4}a^2 Q$$

$$Q_{zz} = Q\left(-a^2 - a^2 + 2a^2 + 2a^2 + a^2 + \tfrac{a^2}{4} + a^2 + 4a^2\right) = +\frac{33}{4}a^2 Q$$

$$Q_{xy} = \int \rho(\underline{r}')\, 3x'y'\, dV' = 0, \quad Q_{yz} = Q_{xz} = 0$$

$$\Rightarrow \phi_3 = \frac{1}{2}\left(-\frac{33}{2}a^2Q\frac{x^2}{r^5} + \frac{33}{4}a^2Q\frac{y^2}{r^5} + \frac{33}{4}a^2Q\frac{z^2}{r^5}\right) = a^2Q\,\frac{33}{8}\left(\frac{y^2+z^2-2x^2}{r^5}\right)$$

(b) ρ: Q verteilt auf $\left\{\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1\right\}$, außerhalb 0.

Volumen: $\frac{4}{3}\pi a b^2 \quad \rightsquigarrow \phi_1 = \frac{Q}{r}$ $\qquad \underline{r} = \begin{pmatrix} br\cos\varphi\sin\vartheta \\ br\sin\varphi\sin\vartheta \\ ar\cos\vartheta\end{pmatrix}$

$$\rightsquigarrow \underline{p} = \frac{Q}{V}\int_V \underline{r}\,dV = \frac{Q}{V}\cdot\int_0^1 dr\int_0^{2\pi}d\varphi\int_0^{\pi}d\vartheta\; ab^2r^2\sin\vartheta\begin{pmatrix} br\cos\varphi\sin\vartheta \\ br\sin\varphi\sin\vartheta \\ ar\cos\vartheta\end{pmatrix} \qquad |\underline{J}| = ab^2 r^2\sin\vartheta$$

$= \begin{pmatrix}0\\0\\0\end{pmatrix}$ $\to \int \sin\varphi\, d\varphi = 0$ etc. aber eine Periode; $\to$ 3. Komp. $\qquad \rightsquigarrow \phi_2 \equiv 0$

$$\int_0^\pi \sin\vartheta\cos\vartheta\,d\vartheta = \frac{1}{2}\sin^2\vartheta\Big|_0^\pi$$

$$\rightsquigarrow Q_{xx} = \frac{Q}{V}\int \underbrace{2x^2 - y^2 - z^2}\ \underbrace{dV}_{ab^2r^2\sin\vartheta\, dr\, d\varphi\, d\vartheta}$$

$$2x^2 - y^2 - z^2 = 2b^2r^2\cos^2\varphi\sin^2\vartheta - b^2r^2\sin^2\varphi\sin^2\vartheta - a^2r^2\cos^2\vartheta$$

$$= \underbrace{b^2r^2}_{\frac{1}{4}}\underbrace{\sin^2\vartheta}_{\frac{4}{3}}\underbrace{(\underbrace{2\cos^2\varphi}_{\pi} - \underbrace{\sin^2\varphi}_{\pi})}_{\pi} - \underbrace{a^2r^2}_{\frac{1}{4}}\underbrace{\cos^2\vartheta}_{\frac{2}{3}}$$

Bei $\int_0^\pi d\vartheta$ ist cos antisymm., sin symm.

$$Q_{xx} = \frac{Q}{V}\cdot\left(ab^4\cdot\frac{\pi}{3} - a^3b^2\frac{1}{6}\right) = Q\left(\frac{1}{4}b^2 - \frac{1}{8\pi}a^2\right)$$

(1) (b) – Forts. – Theo (2)

$$Q_{yy} = \frac{Q}{V} \int 2y^2 - x^2 - z^2 \, dV$$

$$= \frac{Q}{V} \int_0^1 dr \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \left(2b^2 \underbrace{r^2}_{\frac14} \underbrace{\sin^2\varphi}_{\pi} \underbrace{\sin^2\vartheta}_{\frac43} - b^2 \underbrace{r^2}_{\frac14} \underbrace{\cos^2\varphi}_{\pi} \underbrace{\sin^2\vartheta}_{\frac43} - a^2 \underbrace{r^2}_{\frac14} \underbrace{\cos^2\vartheta}_{\frac23} \right) ab^2 r \sin\vartheta$$

$$= \frac{Q}{V} \left(\cancel{2} a b^4 \frac{\pi}{3} - \cancel{ab^4 \frac{\pi}{3}} - a^3 b^2 \frac{1}{6} \right) = Q_{xx} \qquad -b^2 r^2 \underbrace{\sin^2\vartheta}_{\frac43}$$

$$Q_{zz} = \frac{Q}{V} \int 2z^2 - x^2 - y^2 \, dV = \frac{Q}{V} \int_0^1 dr \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \left(2a^2 r^2 \underbrace{\cos^2\vartheta}_{\frac23} \overbrace{- b^2 r^2 \cos^2\varphi \sin^2\vartheta - b^2 r^2 \sin^2\varphi \underbrace{\sin^2\vartheta}_{\frac43}} \right) ab^2 r \sin\vartheta$$

$$= \frac{Q}{V} \left(2a^3 b^2 \frac16 - ab^4 \frac13 \right) = Q \cdot \left(\frac{1}{4\pi} a^2 - \frac{1}{4\pi} b^2 \right)$$

$$Q_{xy} = \frac{Q}{V} \int 3xy \, dV = \frac{Q}{V} \iiint dr\, d\vartheta\, d\varphi \; 3 \cdot b r^2 \underbrace{\cos\varphi \sin\varphi}_{0} \sin^2\vartheta \cdot ab^2 r^2 \sin\vartheta = 0$$

$\rightarrow$ da periodisch

$$Q_{xz} = \frac{Q}{V} \int 3xz \, dV = \frac{Q}{V} \iiint dr\, d\vartheta\, d\varphi \; 3 \cdot ab r^2 \underbrace{\cos\varphi}_{0} \sin\vartheta \cos\vartheta \; ab^2 r^2 \sin\vartheta = 0$$

$$Q_{zy} = \frac{Q}{V} \int 3zy \, dV = \frac{Q}{V} \iiint dr\, d\vartheta\, d\varphi \; 3 \cdot ab r^2 \underbrace{\sin\varphi}_{0} \sin\vartheta \cos\vartheta \; ab^2 r^2 \sin\vartheta = 0$$

$$\Rightarrow \phi_3 = \frac12 \left(Q_{xx} \frac{x^2}{r^5} + Q_{yy} \frac{y^2}{r^5} + Q_{zz} \frac{z^2}{r^5} \right).$$

(c) ρ: Q verteilt auf $\left\{ \underline{r} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \Big| \; r \in (0,R),\ \varphi \in (0, 2\pi),\ z \in (0, L) \right\}$ $|\underline{r}| = r$

$$V = \pi R^2 \cdot L \qquad \rightsquigarrow \phi_1 = \frac{Q}{r}$$

$$\Rightarrow \underline{P} = \frac{Q}{V} \int \underline{r} \, dV = \frac{Q}{V} \int_0^R dr \int_0^{2\pi} d\varphi \int_0^L dz \begin{pmatrix} r^2 \cos\varphi \\ r^2 \sin\varphi \\ r z \end{pmatrix} = \frac{Q}{V} \cdot \begin{pmatrix} 0 \\ 0 \\ \frac12 R^2 2\pi \frac12 L^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac12 \pi R^2 L^2 \end{pmatrix}$$

$$\underline{P} = \begin{pmatrix} 0 \\ 0 \\ Q \cdot L/2 \end{pmatrix} \qquad \phi_2 = \frac{QLz}{2r^3}$$

$$Q_{xx} = \frac{Q}{V} \int 2x^2 - y^2 - z^2 \, dV = \frac{Q}{V} \int dr \int d\varphi \int dz \left(2 \underbrace{r^2}_{\frac14 R^4} \underbrace{\cos^2\varphi}_{\pi} - r^2 \underbrace{\sin^2\varphi}_{\pi} \underbrace{- z^2}_{\frac13 L^3} \right) \cdot r$$

$$= \frac{Q}{V} \cdot \left(\frac14 R^4 \pi L - \frac16 2\pi R^2 L^3 \right) = Q \left(\frac{R^2}{4} - \frac{L^2}{3} \right)$$

$$Q_{yy} = \frac{Q}{V} \int 2y^2 - x^2 - z^2 \, dV = \frac{Q}{V} \int dr \int d\varphi \int dz \left(2 \underbrace{r^2}_{\frac14 R^4} \underbrace{\sin^2\varphi}_{\pi} - r^2 \underbrace{\cos^2\varphi}_{\pi} - z^2 \right) \cdot r$$

$$= \frac{Q}{V} \left(\frac14 R^4 \pi L - \frac{\pi}{3} R^2 L^3 \right) = Q \left(\frac{R^2}{4} - \frac{L^2}{3} \right) = Q_{xx}$$

$$Q_{zz} = \frac{Q}{V} \int 2z^2 - x^2 - y^2 \, dV = \frac{Q}{V} \int dr \int d\varphi \int dz \left(2z^2 \overbrace{- r^2 \cos^2\varphi - r^2 \sin^2\varphi}^{-r^2} \right) r$$

$$= \frac{Q}{V} \cdot \left(\frac{1}{\cancel{2}} R^2 \cdot \cancel{2}\pi \cdot \frac23 L^3 - \frac14 R^4 2\pi L \right) = Q \left(\frac23 L^2 - \frac12 R^2 \right)$$

2

(a) $f(z,x) = (\underbrace{1+z^2-2zx}_{\omega})^{-1/2}$ $\qquad f_{,z}(z,x) = -\frac{1}{2}\cdot(1+z^2-2zx)^{-3/2}\cdot\underbrace{(2z-2x)}_{I}$

$= -\frac{1}{2}\,\omega^{-3/2}\cdot I$

Leitet man ω ab, so liefert dies Ableitung in I. Die I tauchen in einer Ableitung nur mit geraden oder ungeraden Potenzen auf.

Beweis (gerade → ungerade)

(IA) 0. Abl. $f \propto \omega^{-1/2}\cdot I^0$ ✓

(IV) enthält die eine Abl. nur gerade Potenzen von I, so enthält die nächste nur ungerade.

(IS) $(\omega^k\cdot I^{2n})_{,z} = \underbrace{(k\,\omega^{k-1}\cdot I)\cdot 2n\,I^{2n}}_{I^{2n+1}} + \omega^k\cdot 2n\,\underbrace{I^{2n-1}}_{}\cdot\underbrace{I_{,z}}_{2}$

$n\in\mathbb{N}$ $\qquad I^{2n+1}$ ← ungerade → I^{2n-1} □

Beweis (ungerade → gerade):

wie oben; (IS): $(\omega^k I^{2n+1})_{,z} = \underbrace{k(k\,\omega^{k-1} I) I^{2n+1}}_{I^{2n+2}} + \omega^k(2n+1)\,\underbrace{I^{2n}}_{}\cdot\underbrace{I_{,z}}_{2}$

I^{2n+2} ← gerade → I^{2n} □

Für z=0 – wie in der Entwicklung von f in z benötigt – ist ω=1 und I = −2x. Die Potenz in I ist also die Potenz in x.

⇒ P_l ist gerade für gerades l, ungerade für unger. l. ■

Leitkoeffizient: • Wiederholtes ableiten von f liefert von ω jeweils die äußere Ableitung $-\frac{1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\cdot\frac{-7}{2}\cdots = (-1)^l(2l-1)!!\,\frac{1}{2^l}$

• Die Taylorentw. liefert $\frac{1}{l!}$

• Der Koeff. I^l liefert $(-2)^l$ da $I(z=0) = -2x$.

⇒ $\text{Koeff} = \frac{\cancel{(-1)^l}\cdot(2l-1)!!\,\cancel{(-2)^l}}{\cancel{2^l}\,l!}$ mit $(2l-1)!! = \frac{(2l)!}{2^l\cdot l!}$: $\text{Koeff} = \frac{(2l)!}{2^l\cdot(l!)^2}$ ■

Explizit:

(b) $l=0$: $P_0 = 1$

$l=1$: $P_1 = \frac{1}{1}\left(-\frac{1}{2}\,1(-2x)\right) = x$

$l=2$: $P_2 = \frac{1}{2}\left(\frac{1}{2}\cdot\frac{3}{2}\cdot 1\cdot(-2x)^2 - \frac{1}{2}\cdot 2\cdot 1\right) = \frac{1}{2}(3x^2-1)$

(e) $l=0$: $P_{00} = 1$

$l=1$: $P_{11} = -\sqrt{1-x^2}\,1$ $\qquad P_{10} = x$ $\qquad P_{1,-1} = \sqrt{1-x^2}$

$l=2$: $P_{22} = (1-x^2)\cdot 3$ $\qquad P_{21} = -1\sqrt{1-x^2}\,3x$ $\qquad P_{20} = P_2 = \frac{1}{2}(3x^2-1)$

$P_{2,-2} = \frac{1}{24}\,3\,(1-x^2) = \frac{1}{8}(1-x^2)$ $\qquad P_{2,-1} = \frac{1}{6}\sqrt{1-x^2}\,3x = \frac{x}{2}\sqrt{1-x^2}$

Theo (5)

$\boxed{3}$

(a) $$Y_{l,-m} = \sqrt{\frac{(2l+1)(l-m)!}{4\pi\,(l+m)!}} \cdot \underbrace{(-1)^{-m}}_{(-1)^m} \frac{(l-m)!}{(l+m)!} P_{l,m} \cdot \underbrace{\exp(-im\varphi)}_{\exp(im\varphi)^*}$$ $\square$

(b) $l=0$: $Y_{00} = \frac{1}{\sqrt{4\pi}} 1$

$l=1$ $Y_{1,-1} = +\sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{-i\varphi}$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cdot 1 \cdot \cos\vartheta \cdot e^0 \quad (2l+1)$$

$$Y_{1,1} = \sqrt{\frac{3}{8\pi}} \cdot (-1)\underbrace{\sqrt{1-x^2}}_{\sin\vartheta} \cdot 1 \cdot e^{i\varphi} = -\sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{i\varphi}$$

$l=2$ $$Y_{2,2} = \sqrt{\frac{5}{4\pi\cdot 24}} \cdot \underbrace{(1-x^2)}_{\sin^2\vartheta} \cdot 3 \cdot e^{2i\varphi} = \sqrt{\frac{15}{32\pi}} \sin^2\vartheta\, e^{2i\varphi}$$

$$Y_{2,1} = \sqrt{\frac{5}{4\pi\, 6}} (-1) \sin\vartheta\, 3\cos\vartheta\, e^{i\varphi} = -\sqrt{\frac{15}{8\pi}} \sin\vartheta\cos\vartheta\, e^{i\varphi}$$

$$Y_{2,0} = \sqrt{\frac{5\cdot 2}{4\pi\cdot 2}} \frac{1}{2}(3x^2-1) = \sqrt{\frac{5}{16\pi}} (3\cos^2\vartheta - 1)$$

(c) $x = \cos\vartheta$

$$\text{z.z.: } \int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} d\varphi\, \underbrace{N_{lm}}_{\text{Normierung}} Y_{lm}\, e^{im\varphi} \cdot N_{l'm'} Y_{l'm'}\, e^{im'\varphi} = \delta_{ll'} \cdot \delta_{m,m'}$$

$\boxed{2}(f)$ S.u.

$$\int_0^{2\pi} \frac{1}{2\pi} \cdot e^{im\varphi} \cdot e^{-im'\varphi}\, d\varphi = \int_0^{2\pi} \frac{1}{2\pi} e^{i(m-m')\varphi}\, d\varphi \begin{cases} \frac{1}{2\pi} \frac{1}{i(m-m')} e^{i(m-m')\varphi} \Big|_0^{2\pi} = 0, & m \neq m' \\ \frac{1}{2\pi} \cdot e^0 \Big|_0^{2\pi} = 1, & m = m' \end{cases}$$

(d) Die Gl ist identisch zu 2(g). $x = \cos\vartheta$, N_{lm}: Normierung von Y_{lm} $\square$

$$\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} = \frac{1}{\sin\vartheta}\left(\cos\vartheta\frac{\partial}{\partial\vartheta} + \sin\vartheta\frac{\partial^2}{\partial\vartheta^2}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} =: D$$

$$D\, Y_{lm} = D\, N_{lm} \cdot P_{lm} \cdot e^{im\varphi} =$$

$$N_{lm}\left(\frac{1}{\cancel{\sin\vartheta}} \underbrace{\cos\vartheta}_{x} \underbrace{\left(\frac{\partial}{\partial\vartheta} P_{lm}\right)}_{\underbrace{\frac{\partial P_{lm}}{\partial x}}_{P'_{lm}} \underbrace{\frac{\partial x}{\partial\vartheta}}_{-\sin\vartheta}} e^{im\varphi} + \underbrace{\left(\frac{\partial^2}{\partial\vartheta^2} P_{lm}\right)}_{\frac{\partial}{\partial\vartheta}[P'(-\sin\vartheta)] = P''\underbrace{(-\sin\vartheta)^2}_{1-x^2} + P'(-\cos\vartheta)} e^{im\varphi} + \frac{1}{\sin^2\vartheta} P_{lm} \underbrace{\frac{\partial^2}{\partial\varphi^2} e^{im\varphi}}_{-m^2 e^{im\varphi}}\right)$$

$$= N_{lm}\left(-x\cdot P'_{lm} + P''_{lm}(1-x^2) - xP'_{lm} - \frac{m^2}{1-x^2} P_{lm}\right) \cdot e^{im\varphi} \overset{[2]g)}{=} -l(l+1) \underbrace{N_{lm} P_{lm} e^{im\varphi}}_{Y_{lm}}$$ $\square$

$\boxed{4}$ $\frac{\partial}{\partial x_\alpha \partial x_\beta} (x_\gamma x_\gamma)^{-\frac{1}{2}} = \frac{\partial}{\partial x_\alpha}\left(-\frac{1}{2}(x_\gamma x_\gamma)^{-\frac{3}{2}} \frac{\partial}{\partial x_\beta}(x_\gamma x_\gamma)\right)$

Theo ③

$\frac{\partial}{\partial x_\alpha}(x_\gamma x_\gamma) = \delta_{\alpha\gamma} x_\gamma + \delta_{\alpha\gamma} x_\gamma = 2 x_\gamma \delta_{\alpha\gamma}$

$= -\frac{\partial}{\partial x_\alpha}\left((x_\gamma x_\gamma)^{-\frac{3}{2}} \delta_{\beta\gamma} x_\beta\right) = -\frac{\partial}{\partial x_\alpha}\left((x_\gamma x_\gamma)^{-\frac{3}{2}} x_\gamma\right)$

$= \frac{3}{2}(x_\gamma x_\gamma)^{-\frac{5}{2}} \cdot 2\delta_{\gamma\gamma} x_\alpha \cdot x_\gamma - (x_\gamma x_\gamma)^{-\frac{3}{2}} \delta_{\alpha\gamma}$

$= \frac{3r^2}{r^5} -$

$\partial_\alpha \partial_\beta (x_\gamma x_\gamma)^{-1/2} = \partial_\alpha\left(-\frac{1}{2}(x_\gamma x_\gamma)^{-3/2} \cdot \underbrace{2\delta_{\beta\gamma} x_\gamma}_{x_\beta}\right)$

$= 3(x_\gamma x_\gamma)^{-5/2} \cdot \delta_{\alpha\nu} x_\nu \cdot x_\beta - (x_\gamma x_\gamma)^{-3/2} \delta_{\alpha\beta}$

$= 3\frac{x_\alpha x_\beta}{r^5} - \frac{\delta_{\alpha\beta}}{r^3}$

für $r \neq 0$!

$\partial_\alpha \partial_\beta = \boxed{11} + 12 + 13 + 21 + \boxed{22} + 23 + 31 + 32 + \boxed{33}$

$\delta_{\alpha\beta} = 1$: $\frac{\partial^2}{\partial x_\alpha^2} \frac{1}{|r|} \overset{!}{=} \cancel{-\frac{1}{r^3} + 3\frac{1}{r^3}} - \frac{4\pi}{3}\delta^{(3)}(r)$

Bekannt: $\frac{\partial^2}{\partial x_\alpha \partial x_\alpha} \frac{1}{|r|} = -\frac{4\pi}{3}\delta^{(3)}(r)$

$(\partial_x^2 + \partial_y^2 + \partial_z^2)\frac{1}{r} = -\frac{4\pi}{3}\delta^{(3)}(r)$

$\frac{1}{r^3} - \frac{4\pi}{3}\delta^{(3)}(r) + \frac{2}{r^3}$

∂_x^2