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(a) $|\psi_\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |\psi_n\rangle$

$= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \cdot \frac{1}{\sqrt{n!}} (a^\dagger)^n |\psi_0\rangle$ } weil $a^\dagger$ Creator

$= e^{-|\alpha|^2/2} \left(\sum_n \frac{\alpha^n}{n!} (a^\dagger)^n\right) |\psi_0\rangle$ wg. Lin'tät

$= e^{-\alpha\alpha^*/2} \, e^{\alpha a^\dagger} \, |\psi_0\rangle$

$= e^{-\alpha\alpha^*/2} \, e^{\alpha a^\dagger} \, e^{-\alpha^* a} \, |\psi_0\rangle$ } Tip

$= e^{\alpha a^\dagger} \cdot e^{-\alpha^* a} \cdot e^{-\alpha\alpha^*/2} \, |\psi_0\rangle$ } vertauschen da $e^{-|\alpha|^2/2} \in \mathbb{R}$

$= e^{\alpha a^\dagger - \alpha^* a} \cdot |\psi_0\rangle$ } Baker-Campbell... (*)

(*) z.z.: $[\alpha a^\dagger, -\alpha^* a] \underset{\text{Lin'tät}}{=} -\alpha\alpha^* [a^\dagger, a] = \alpha\alpha^* [a, a^\dagger] = \alpha\alpha^*$

(b) • $x = \frac{1}{\sqrt{2}} (a + a^\dagger)$

$\langle x\rangle = \langle \psi_\alpha | \frac{1}{\sqrt{2}} (a + a^\dagger) | \psi_\alpha \rangle$

$= \frac{1}{\sqrt{2}} \langle \psi_\alpha | a | \psi_\alpha \rangle + \frac{1}{\sqrt{2}} \langle \psi_\alpha | a^\dagger | \psi_\alpha \rangle$

$= \frac{1}{\sqrt{2}} \langle \psi_\alpha | \alpha | \psi_\alpha \rangle + \frac{1}{\sqrt{2}} \left(\langle \psi_\alpha | a | \psi_\alpha \rangle\right)^*$

$= \frac{1}{\sqrt{2}} \alpha \langle \psi_\alpha | \psi_\alpha \rangle + \frac{1}{\sqrt{2}} \alpha^* \langle \psi_\alpha | \psi_\alpha \rangle^*$ $\qquad \langle \psi_\alpha | \psi_\alpha \rangle^* \underset{\text{Symmetrie}}{=} \langle \psi_\alpha | \psi_\alpha \rangle$

$= \frac{1}{\sqrt{2}} (\alpha + \alpha^*) \langle \psi_\alpha | \psi_\alpha \rangle = \frac{1}{\sqrt{2}} 2 \,\mathrm{Re}(\alpha) \cdot \langle \psi_\alpha | \psi_\alpha \rangle$

• $\partial_x = \frac{1}{\sqrt{2}} (a - a^\dagger) \quad \rightsquigarrow \quad p = -i\hbar \partial_x = -i\hbar (a - a^\dagger)$

$\langle p\rangle = -i\hbar \langle \psi_\alpha | \frac{1}{\sqrt{2}} (a - a^\dagger) | \psi_\alpha \rangle$

$= -i\hbar \frac{1}{\sqrt{2}} \left(\langle \psi_\alpha | a | \psi_\alpha \rangle - \langle \psi_\alpha | a^\dagger | \psi_\alpha \rangle \right)$

$= -i\hbar \frac{1}{\sqrt{2}} (\alpha - \alpha^*) \langle \psi_\alpha | \psi_\alpha \rangle$

$= \frac{1}{\sqrt{2}} (-i\hbar) \, 2 \,\mathrm{Im}(\alpha) \langle \psi_\alpha | \psi_\alpha \rangle$

• $x^2 = \frac{1}{2} (a + a^\dagger)^2 = \frac{1}{2} (a^2 + a a^\dagger + a^\dagger a + a^{\dagger 2})$

$\langle x^2\rangle = \frac{1}{2} \left(\langle \psi_\alpha | a^2 | \psi_\alpha \rangle + \langle \psi_\alpha | a a^\dagger \underset{(*)}{+} a^\dagger a | \psi_\alpha \rangle + \langle \psi_\alpha | a^{\dagger 2} | \psi_\alpha \rangle \right)$

$= \frac{1}{2} (\alpha^2 + \langle \psi_\alpha | 2 a^\dagger a | \psi_\alpha \rangle + 1 \cdot + \alpha^{*2}) \langle \psi_\alpha | \psi_\alpha \rangle$ $\qquad a a^\dagger + a^\dagger a \overset{(*)}{=} 2a^\dagger a - a^\dagger a + a a^\dagger = 2a^\dagger a + [a, a^\dagger] = 2a^\dagger a + 1$

$= \frac{1}{2} (\alpha^2 + 2\alpha\alpha^* + 1 + \alpha^{*2}) \langle \psi_\alpha | \psi_\alpha \rangle$

$= \left[\frac{1}{2} (\alpha + \alpha^*)^2 + 1/2 \right] \langle \psi_\alpha | \psi_\alpha \rangle$

$= \left[2 (\mathrm{Re}(\alpha))^2 + 1/2 \right] \langle \psi_\alpha | \psi_\alpha \rangle$

- $p^2 = (-i\hbar\partial_x)^2 = -\hbar^2\partial_x^2 = -\hbar^2 \frac{1}{2}(a - a^+)^2 = -\frac{1}{2}\hbar^2 (aa - aa^+ - a^+a + a^+a^+)$

$= -\frac{1}{2}\hbar^2 (aa - (2a^+a - a^+a + aa^+) + a^+a^+)$

$= -\frac{1}{2}\hbar^2 (aa - 2a^+a - [a, a^+] + a^+a^+)$

$\langle p^2\rangle = -\frac{1}{2}\hbar^2 \langle\psi_\alpha| aa - 2a^+a - 1 + a^+a^+ |\psi_\alpha\rangle$

$= -\frac{1}{2}\hbar^2 (\alpha^2 - 2\alpha^*\alpha - 1 + \alpha^{*2}) \langle\psi_\alpha|\psi_\alpha\rangle$

$= [-\frac{1}{2}\hbar^2 (\alpha - \alpha^*)^2 + \frac{1}{2}\hbar^2] \langle\psi_\alpha|\psi_\alpha\rangle$

$= [-\hbar^2 2 (\mathrm{Im}(\alpha))^2 + \frac{1}{2}\hbar^2] \langle\psi_\alpha|\psi_\alpha\rangle$

Unter der Annahme, dass ψ_α normiert ist, folgt

$$\langle\psi_\alpha|\psi_\alpha\rangle \equiv 1 .$$

Diese Annahme ist sinnvoll, weil die Wellenfunkt. stets normiert werden sollen.

- $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$

$= 2\mathrm{Re}^2(\alpha) + 1/2 - 2\mathrm{Re}^2(\alpha) = 1/2$

- $(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2$

$= -\hbar^2 2\,\mathrm{Im}^2(\alpha) + 1/2\,\hbar^2 + 2\hbar^2\,\mathrm{Im}^2(\alpha) = 1/2\,\hbar^2$

$\rightsquigarrow \Delta x^2 \cdot \Delta p^2 = \hbar^2/4 \quad \rightsquigarrow \quad \Delta x \cdot \Delta p = \hbar/2$ $\square$

(c) Analog zu (a):

$$|\psi_\beta\rangle = e^{-|\beta|^2/2} \sum_n \frac{\beta^n}{\sqrt{n!}} \frac{1}{\sqrt{n!}} (a^+)^n |\psi_0\rangle = e^{-|\beta|^2/2} e^{\beta a^+} |\psi_0\rangle$$

$\langle\psi_\alpha|$ ist dualer Vekt. zu $|\psi_\alpha\rangle$, also nach (a):

$$\langle\psi_\alpha| = e^{-|\alpha|^2/2} \langle\psi_0| e^{\alpha^* a} .$$

$$\langle\psi_\alpha|\psi_\beta\rangle = e^{-|\alpha|^2/2} \langle\psi_0| e^{\alpha^* a} e^{\beta a^+} |\psi_0\rangle e^{-|\beta|^2/2}$$

Tip aus (a)

$$= e^{-|\alpha|^2/2} \langle\psi_0| \underbrace{\underbrace{e^{-\beta a^+} e^{\alpha^* a}}_{e^{-\beta a^+ + \alpha^* a} \cdot e^{-[-\beta a^+, \alpha^* a]}} \underbrace{e^{\beta a^+} e^{-\alpha^* a}}_{e^{\beta a^+ - \alpha^* a} \cdot e^{-[\beta a^+, -\alpha^* a]/2}}}_{\mathbb{1}\ (*)} |\psi_0\rangle e^{-|\beta|^2/2}$$

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$e^{-[-\beta a^+, \alpha^* a]} = e^{-\alpha^*\beta/2}$, $\quad e^{-[\beta a^+, -\alpha^* a]/2} = e^{-\alpha^*\beta/2}$

$$= e^{-|\alpha|^2/2} \cdot e^{-\alpha^*\beta/2} \cdot 1 \cdot e^{-\alpha^*\beta/2} \cdot e^{-|\beta|^2/2}$$

$\square$

(*) da identisch bis auf Vorzeichen im Exponent: $e^{-A} \cdot e^{A} = \mathbb{1}$.