Experimentalphysik IV
Uebung 03

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Die Aufgabe 13 findet man 1:1 im Demtroeder III, Kapitel 7, Uebungsaufgabe 2.

$\boxed{10}$ Kin. Energie: $\left\langle \frac{p^2}{2\mu} \right\rangle$ $\quad p = -i\hbar\partial_x$

$$\frac{p^2}{2\mu} = -\frac{\hbar^2 \Delta}{2\mu}$$

Exphys ③

In x-Koord.: $\psi = \psi(r) = 2\left(\frac{Z}{a}\right)^{\frac{3}{2}} e^{-\frac{Zr}{a}} \cdot \underbrace{\frac{1}{\sqrt{4\pi}}}_{Y_{00}}$

$$\left\langle \frac{p^2}{2\mu} \right\rangle = -\frac{\hbar^2}{2\mu} \int dr \boxed{\int d\vartheta \int d\varphi\, r^2 \sin\vartheta} \cdot \psi \cdot \Delta\psi$$

$N = 2\left(\frac{Z}{a}\right)^{\frac{3}{2}} \frac{1}{\sqrt{4\pi}}$

$\Rightarrow \psi = N e^{-\frac{Zr}{a}}$

$$\Delta\psi = \frac{1}{r^2}\partial_r\left(r^2 \partial_r N \cdot e^{-\frac{Zr}{a}}\right)$$

$$= \frac{1}{r^2}\partial_r\left(r^2 N\left(\frac{-Z}{a}\right) \cdot e^{-\frac{Zr}{a}}\right)$$

$$= \cancel{\frac{1}{r^2}}\left(2r\left(\frac{-Z}{a}\right) N e^{-\frac{Zr}{a}} + r^2 N\left(\frac{-Z}{a}\right)^2 e^{-\frac{Zr}{a}}\right)$$

$$\left\langle \frac{p^2}{2\mu} \right\rangle = -\frac{\hbar^2}{2\mu} \int dr \boxed{2\pi \cdot 2} \cdot N^2 \left(2r\left(\frac{-Z}{a}\right) e^{-\frac{2Zr}{a}} + r^2\left(\frac{-Z}{a}\right)^2 e^{-\frac{2Zr}{a}}\right)$$

$$= -\frac{\hbar^2}{2\mu}(4\pi N^2)\left[\int dr\left(\partial_\alpha e^{-\frac{2Zr}{a}\alpha}\right) + \int dr \frac{1}{4}\partial_\alpha^2 e^{-\frac{2Zr}{a}\alpha}\right]\Bigg|_{\alpha=1}$$

$$= \frac{-\hbar^2}{2\mu}(4\pi N^2)\left[\partial_\alpha\left(\int_0^{+\infty} dr\, e^{-\frac{2Zr\alpha}{a}}\right) + \frac{1}{4}\partial_\alpha^2\left(\int dr\, e^{-\frac{2Zr}{a}\alpha}\right)\right]\Bigg|_{\alpha=1}$$

$$\ldots \frac{1}{4}\partial_\alpha^2\left(\frac{a}{2Z\alpha}\right)\Big]\Big|$$

$$= \frac{-\hbar^2}{2\mu}(4\pi N^2)\left[\partial_\alpha\left(\int_0^{+\infty} dr\; e^{-\frac{2Zr\alpha}{a}}\right) + \frac{1}{4}\partial_\alpha^2\left(\int dr\; e^{-\frac{2Zr}{a}\alpha}\right)\right]\Big|_{\alpha=1}$$

$$= \frac{-\hbar^2}{2\mu}(4\pi N^2)\left[\partial_\alpha\left(+\frac{a}{2Z\alpha}\right) + \frac{1}{4}\partial_\alpha^2\left(\frac{a}{2Z\alpha}\right)\right]\Big|_{\alpha=1}$$

$$-\frac{a}{2Z\alpha^2} \qquad \frac{1}{4}\left(+2\frac{a}{2Z\alpha^3}\right)$$

! $$= \frac{-\hbar^2}{2\mu}(4\pi N^2)\left(-\frac{1}{4}\frac{a}{2Z}\right) = \frac{\hbar^2}{2\mu}4\pi\frac{a}{2Z}\frac{1}{4}\,4\,\frac{Z^3}{a^3}\cdot\frac{1}{4\pi}$$

$$= \frac{1}{2}\frac{\hbar^2 Z^2}{\mu a^2} \underset{Z=1}{=} \frac{1}{2}\frac{\hbar^2}{\mu a^2}$$

$$\psi(\underline{r})_{2,1,1} = \frac{1}{2\sqrt{6}}\left(\frac{Z}{a}\right)^{3/2}\frac{r}{a}\,e^{-\frac{r}{2a}}\cdot\begin{cases}\sqrt{\frac{3}{4\pi}}\cos\vartheta, & l=1, m=0\\ \mp\sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{\pm i\varphi} & l=1, m=\pm1\end{cases}$$

$$\langle E_{pot}\rangle = \left\langle\frac{-e^2}{4\pi\varepsilon_0 r}\right\rangle = \frac{-e^2}{4\pi\varepsilon_0}N^2\int dr\int e^{-\frac{2Zr}{a}}r\sin\vartheta = \frac{-e^2N^2}{4\pi\varepsilon_0}2\pi\cdot2\cdot\int r\,e^{-\frac{2Zr}{a}}dr$$

$$= \frac{e^2N^2}{4\pi\varepsilon_0}4\pi\int -\frac{a}{2Z}e^{-\frac{2Zr}{a}}dz = +\frac{e^2N^2a^2}{4Z^2\varepsilon_0}\left[e^{-\frac{2Zr}{a}}\right]_0^{+\infty} = -\frac{e^2N^2a^2}{4Z^2\varepsilon_0} = \frac{-e^2a^2\,4\,Z^3}{4Z^2\varepsilon_0\,a^3\,4\pi}$$

$$= -\frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{a}$$

$\boxed{11}$

(a) $$\begin{pmatrix} r\cos\varphi\sin\vartheta \\ r\sin\varphi\sin\vartheta \\ r\cos\vartheta \end{pmatrix} \xmapsto{\Pi} \begin{pmatrix} -r\cos\varphi\sin\vartheta \\ -r\sin\varphi\sin\vartheta \\ -r\cos\vartheta \end{pmatrix}$$

$r \longmapsto r$

$\vartheta \longmapsto \pi-\vartheta$

$\varphi \longmapsto +\varphi+\pi$

$\Pi Y_{00} = Y_{00}$

$\Pi Y_{10} = -Y_{10}$ $\qquad \cos\vartheta \longmapsto -\cos\vartheta$

$\Pi Y_{1\pm1} = -Y_{1\pm1}$ $\qquad \sin\vartheta \longmapsto -\sin\vartheta$

$\Pi Y_{20} = Y_{20}$ $\qquad \cos^2\vartheta \longmapsto \cos^2\vartheta$

$\Pi Y_{2\pm1} = Y_{2\pm1}$

$\Pi Y_{2\pm2} = Y_{2\pm2}$

$\sin\varphi = \frac{1}{2i}\left(e^{i\varphi}-e^{-i\varphi}\right)$

$\cos\varphi = \frac{1}{2}\left(e^{i\varphi}+e^{-i\varphi}\right)$

(b) $$\int_0^{2\pi} d\varphi \begin{pmatrix} \cos\varphi\sin\vartheta \\ \sin\varphi\sin\vartheta \\ \cos\vartheta \end{pmatrix} e^{i(m'-m)\varphi} = \underline{I}$$

$$\underline{I}\cdot e_1 = \int_0^{2\pi} d\varphi\, e^{i(m'-m)\varphi}\cdot\frac{1}{2}\left(e^{i\varphi}+e^{-i\varphi}\right)\sin\vartheta = \frac{\sin\vartheta}{2}\underbrace{\left[\int_0^{2\pi} d\varphi\, e^{i(m'-m+1)\varphi} + \int_0^{2\pi} d\varphi\, e^{i(m'-m-1)\varphi}\right]}_{2\pi\cdot\left[\delta_{m',m+1} + \delta_{m',m-1}\right]}$$

$$\underline{I}\, e_2 = \frac{\sin\vartheta}{2i}\int_0^{2\pi} d\varphi\left(\underbrace{e^{i(m'-m+1)\varphi}} - e^{i(m'-m-1)\varphi}\right)$$

$$= \frac{\sin\vartheta}{2i}\; 2\pi\left(\delta_{m',m+1} - \delta_{m',m-1}\right)$$

$$\underline{I}\, e_3 = \cos\vartheta\cdot 2\pi\cdot\delta_{m',m}$$

$$\underline{I} \cdot e_1 = \int_0^{2\pi} d\varphi \; e^{i(m'-m)\varphi} \cdot \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right) \sin\vartheta = \frac{\sin\vartheta}{2}\left[\underbrace{\int_0^{2\pi} d\varphi \; e^{i(m'-m+1)\varphi} + \int_0^{2\pi} d\varphi \; e^{i(m'-m-1)\varphi}}_{2\pi \cdot \left[\delta_{m',m+1} + \delta_{m',m-1}\right]}\right]$$

$$\underline{I} \cdot e_2 = \frac{\sin\vartheta}{2i} \int_0^{2\pi} d\varphi \left(\underbrace{e^{i(m'-m+1)\varphi}} - e^{i(m'-m-1)\varphi}\right)$$

$$= \frac{\sin\vartheta}{2i} \quad 2\pi\left(\delta_{m',m+1} - \delta_{m',m-1}\right)$$

$$\underline{I} \cdot e_3 = \cos\vartheta \cdot 2\pi \cdot \delta_{m',m}$$

$$\underline{I} \quad \rangle = \begin{pmatrix} \pi \sin\vartheta \, \delta_{m',m+1} + 0 + \pi \sin\vartheta \, \delta_{m',m-1} \\ -i\pi \sin\vartheta \, \delta_{m',m+1} + 0 + i\pi \sin\vartheta \, \delta_{m',m-1} \\ 0 + 2\pi \cos\vartheta \, \delta_{m',m} \end{pmatrix}$$

Vektor!

(c) $\underline{E} \parallel e_2 \Rightarrow \underline{E} \cdot \underline{r} = E \cdot z$

$\Rightarrow$ nur 3. Komponente von $\underline{I}$ entscheidend: $m = m'$!

(Y_{00}, Y_{10}) (Y_{00}, Y_{20}) (Y_{10}, Y_{20}) $(Y_{1\pm1}, Y_{2\pm1})$

Da nur das Integral über $\cos\vartheta$ auszuführen ist und $\cos\vartheta$ antisym., verschwindet die Summe, denn Produkt ~~sym~~ symmetrisch ist.

$\Rightarrow (Y_{00} \; Y_{10}) \rightarrow 0$

$\boxed{11}$ (c) – Forts –

Y_{00}, Y_{10} : $\cancel{\int d\varphi} \int d\vartheta \, \frac{\cancel{2}}{2} 2\pi \cos\vartheta \, \frac{1}{\sqrt{4\pi}} \frac{\sqrt{3}}{\sqrt{4\pi}} \cos\vartheta \, \cancel{r} \sin\vartheta =$

$-\frac{\sqrt{3}}{6} \cos^3\vartheta \Big|_0^\pi = -\frac{\sqrt{3}}{6} \cdot -2 = \frac{\sqrt{3}}{3}$

$\cos^2\vartheta = 1 - \sin^2\vartheta$

Y_{00}, Y_{20} : $\int d\vartheta \, 2\pi \sin\vartheta \, \frac{1}{\sqrt{4\pi}} \frac{\sqrt{5}}{\sqrt{16\pi}} (3\cos^2\vartheta - 1)\cos\vartheta =$

$\frac{\sqrt{5}\, 2\pi}{8\pi} \int_0^\pi 3\cos^2\vartheta \sin\vartheta - \sin\vartheta \, d\vartheta = \frac{\sqrt{5}}{4} \cdot \left[\underbrace{\left(-\cos^3\vartheta \Big|_0^\pi \right.}_{} + \underbrace{\left. \cos\vartheta \Big|_0^\pi \right)}_{} \right] = 0$

identisch.

$Y_{10} Y_{20}$: $\frac{2\pi \sqrt{15}}{8\pi} \int \underbrace{d\vartheta \sin\vartheta}_{-d(\cos\vartheta)} (3\cos^4\vartheta - \cos^2\vartheta) = -\frac{\sqrt{15}}{4} \left(\frac{1}{5} 3\cos^5\vartheta - \frac{1}{3}\cos^3\vartheta \right) \Big|_0^\pi$

$= -\frac{\sqrt{15}}{4} \left(-\frac{6}{5} + \frac{2}{3} \right) = + \frac{\sqrt{15}}{4} \frac{8}{15} = \frac{2}{\sqrt{15}}$

$\int \varphi$ liefert mit den $(e^{\pm i\varphi})^* \cdot e^{\pm i\varphi} = 1$.

$Y_{1\pm1} Y_{2\pm1}$: $\int_0^\pi d\vartheta \, 2\pi \cos\vartheta \sin\vartheta \, \frac{\sqrt{45}}{8\pi} \sin^2\vartheta \cos\vartheta =$

$\frac{\sqrt{45}}{4} \int \sin^3\vartheta \cos^2\vartheta \, d\vartheta = \frac{\sqrt{45}}{4} \int \sin\vartheta \, d\vartheta \, (\cos^2\vartheta - \cos^4\vartheta) =$

$\frac{\sqrt{45}}{4} \left(-\frac{1}{3}\cos^3\vartheta + \frac{1}{5}\cos^5\vartheta \right) \Big|_0^\pi = \frac{\sqrt{45}}{4} \left(\frac{2}{3} - \frac{2}{5} \right) = \frac{\sqrt{45} \cdot 4}{4 \cdot 15} = \frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\cancel{4}}{\cancel{4}}$

$\boxed{12}$ (a) $Y_l^m = P_l^m(\vartheta) \Phi_m(\varphi) = P_l^m(\vartheta) \cdot \frac{1}{\sqrt{2\pi}} e^{im\varphi}$ $\quad (Y_l^m, Y_{l'}^{m'})$:

$$\frac{\sqrt{4\pi}}{4}\left(-\frac{1}{3}\cos^3\vartheta + \frac{1}{5}\cos^5\vartheta\right)\Big|_0^\pi = \frac{\sqrt{4\pi}}{4}\left(\frac{2}{3} - \frac{2}{5}\right) = \frac{\sqrt{4\pi}\cdot 4}{4\cdot 15} = \frac{\sqrt{4\pi}}{15}$$

$\boxed{12}$ (a) $Y_l^m = P_l^m(\vartheta)\,\Phi_m(\varphi) = P_l^m(\vartheta)\cdot\frac{1}{\sqrt{2\pi}}e^{im\varphi}$ $\quad (Y_l^m, Y_{l'}^m)$:

$$\int d\vartheta \; 2\pi \cdot \cos\vartheta \cdot \sin\vartheta \; P_l^m \cdot P_{l'}^m \; d$$

$$\int -d(\cos\vartheta) \cdot \underbrace{\cos\vartheta \; P_l^m(\cos\vartheta)}\, P_{l'}^m(\cos\vartheta) \qquad z = \cos\vartheta$$

$$\int -dz \left[\left(\frac{(l+1-m)\,P_{l+1}^m(z)}{2l+1}\right) + \left(\frac{l+m}{2l+1}P_{l-1}^m\right)\right]\cdot P_{l'}^m(z)$$

Da die Legendre-Polynome in l orthogonal ~~sind~~ sind, entfallen alle $l' \neq l \pm 1 \Rightarrow \Delta l = \pm 1$.

(b)