Elektrodynamik
Uebung 11

Michael Kopp

July 14, 2010

(a) $$\frac{\partial}{\partial x_\alpha} E_\alpha = \Big(\underbrace{\left(\frac{\partial x'_\beta}{\partial x_\alpha}\right)}_{= R_{\beta\mu}\delta_{\mu\alpha}} \frac{\partial}{\partial x'_\beta} + \underbrace{\left(\frac{\partial t'}{\partial x_\alpha}\right)}_{=0} \frac{\partial}{\partial t'} \Big) E_\alpha$$

Forminvarianz!

$$= R_{\beta\alpha} \frac{\partial}{\partial x'_\beta} E_\alpha \overset{\downarrow}{=} \frac{\partial}{\partial x'_\alpha} E'_\alpha$$

$$= \frac{\partial}{\partial x'_\beta} \left(R_{\beta\alpha} E_\alpha \right) = \frac{\partial}{\partial x'_\alpha} E'_\alpha$$

$$\Rightarrow \quad R_{\alpha\beta} E_\beta = E'_\alpha \qquad \rho(r,t) = \rho'(r',t')$$

$$\overset{\text{analog}}{\Rightarrow} \quad R_{\alpha\beta} B_\beta = B'_\alpha$$

(b) $$\varepsilon_{\alpha\beta\gamma} \frac{\partial}{\partial x_\beta} E_\gamma = \varepsilon_{\alpha\beta\gamma} \underbrace{\left(\frac{\partial x'_\mu}{\partial x_\beta}\right)}_{= R_{\mu\kappa}\delta_{\kappa\beta}} \frac{\partial}{\partial x'_\mu} R^{-1}_{\gamma\nu} E'_\nu$$

$(\underline{v} \cdot \underline{\nabla})$

$$-\frac{1}{c}\frac{\partial}{\partial t} B_\alpha = -\frac{1}{c}\Big(\underbrace{\left(\frac{\partial t'}{\partial t}\right)}_{=1} \frac{\partial}{\partial t'} + \underbrace{\left(\frac{\partial x'_\mu}{\partial t}\right)}_{= v_\mu} \frac{\partial}{\partial x'_\mu} \Big) R^{-1}_{\alpha\nu} B'_\nu = -\frac{1}{c}\left(\frac{\partial}{\partial t'} + v_\mu \frac{\partial}{\partial x'_\mu} \right) R^{-1}_{\alpha\nu} B'_\nu$$

$\cdot R_{\alpha\tau}$

$$R^{-1}_{\alpha\nu} R_{\alpha\tau} = \delta_{\tau\nu}$$

$\nabla(\omega)$ Γ

$R_{\nu\beta} B_\beta = B'_\alpha \quad | \cdot R^{-1}_{\alpha\gamma} \qquad R^{-1}_{\gamma\alpha} R_{\nu\beta} B_\beta = \delta_{\gamma\beta} B_\beta = B_\gamma = R^{-1}_{\alpha\gamma} B'_\alpha$

$$\varepsilon_{\nu\mu\beta\gamma} R_{\alpha\tau} R_{\gamma\beta} \frac{\partial}{\partial x'_\mu} R^{-1}_{\gamma\nu} E'_\nu = -\frac{1}{c}\left(\frac{\partial}{\partial t'} + v_\mu \frac{\partial}{\partial x'_\mu}\right) B'_\tau$$

$$\underbrace{\varepsilon_{\nu\beta\gamma}\left(R_{\alpha\tau} R_{\mu\beta} \underset{\substack{\parallel \\ R_{\nu\gamma}}}{R^{-1}_{\gamma\nu}} \right)} \frac{\partial}{\partial x'_\mu} E'_\nu = -\frac{1}{c}\frac{\partial}{\partial t'} B'_\tau - \frac{1}{c} v_\mu \frac{\partial}{\partial x'_\mu} B'_\tau$$

$\longrightarrow$ da $R^{-1} = R^T$ für orthogonale Matrizen

$\det R \cdot \varepsilon_{\tau\mu\nu} = \varepsilon_{\tau\mu\nu}$ da $\det R = 1$

$$(\underline{\nabla}' \times \underline{E}')_\tau = -\frac{1}{c}\frac{\partial}{\partial t'} \underline{B}'_\tau - \frac{1}{c}(\underline{v}\cdot\underline{\nabla}')\underline{B}_\tau$$

wenn $(\underline{v}\cdot\underline{\nabla}')\underline{B} = 0$ ist (4) forminvariant.

2

(a) y, z bleiben Invariant, x wird verkürzt:

ΔS^2 bleibt invariant:

$$\underbrace{(c\Delta t)^2}_{\frac{v^2}{c^2}(2r)^2} - \underbrace{(\Delta x)^2}_{(2r)^2} = \underbrace{(c\Delta t')^2}_{0} - \overbrace{(\Delta x')^2}^{\text{gesucht}}$$

0 da in x' gemessen wird

$$c\Delta t = c(t_1 - t_2) = c\left(\frac{t_1'}{\gamma} + \frac{v}{c^2}x_1 - \left(\frac{t_2'}{\gamma} + \frac{v}{c^2}x_2\right)\right)$$

$$= c\left(\frac{\Delta t'}{\gamma} + \frac{v}{c^2}\Delta x\right) = \frac{v}{c}\Delta x$$

$$\left(1 - \frac{v^2}{c^2}\right)(2r^2)^2 = (\Delta x')^2 = (2r')^2$$

$$r' = \frac{1}{\gamma} r$$

Für $v \to \infty$: $\gamma \to \infty$ $r' \to 0$

Für $v \to 0$: $\gamma \to 1$ $r' \to r$.

(b) $x_r' = \gamma(wt - vt)$, $t' = \gamma\left(t - \frac{vw}{c^2}t\right)$

Bew. nach rechts

$= \gamma(w - v)t$ $= \gamma\left(1 - \frac{vw}{c^2}\right)t$

$$x' = \frac{\gamma(w-v)}{\gamma\left(1 - \frac{vw}{c^2}\right)} t' \longrightarrow w\left(\frac{1 - \frac{v}{w}}{1 - \frac{vw}{c^2}}\right) < w$$

(b) $x_r' = \gamma(wt - vt)$, $\qquad t' = \gamma\left(t - \frac{vw}{c^2}t\right)$

Bew. nach rechts

$\qquad = \gamma(w-v)t \qquad\qquad = \gamma\left(1-\frac{vw}{c^2}\right)t$

$$x_r' = \frac{\gamma(w-v)}{\gamma\left(1-\frac{vw}{c^2}\right)}\, t' \longrightarrow \quad w\left(\frac{1-\frac{v}{w}}{1-\frac{vw}{c^2}}\right) < w$$

$w \to 0 \qquad x' = -v\,t' \qquad\qquad 1-\frac{v}{w} < 1-\frac{vw}{c^2}$

$w \to c \qquad x' = \frac{c-v}{1-\frac{v}{c}}\,t' = c\,t' \cancel{\frac{1-\frac{v}{c}}{1-\frac{v}{c}}} \qquad\qquad \frac{v}{w} > \frac{vw}{c^2}$

$\qquad\qquad\qquad\qquad\qquad\qquad c^2 > w^2$ ✓

→ Für „normale" Geschw. breitet sich eine Ellipse in S' aus, für $w \to 0$ bewegt sich nur der „Wellenursprung" gleichförmig, für $w \to \infty$ breitet sich eine Kugelwelle mit Geschw. c aus.

Bew. nach links: $x' = \gamma(-wt - vt) = -\gamma(w+v)t \qquad t' = \gamma\left(1+\frac{vw}{c^2}\right)t$

$$x_l' = \frac{-(w+v)}{1+\frac{vw}{c^2}}\, t'$$ diese Geschw. ist größer als oben!

$$x_z' = \frac{x_l' + x_r'}{2} = \left(\frac{w-v}{1-\frac{vw}{c^2}} + \frac{-w-v}{1+\frac{vw}{c^2}}\right)\frac{t'}{2} = \frac{w-v+\frac{vw^2}{c^2}-\cancel{\frac{v^2w}{c^2}}-w-v+\frac{vw^2}{c^2}+\cancel{\frac{v^2w}{c^2}}}{1-\frac{v^2w^2}{c^4}}$$

$$= \frac{t'}{2}\,\frac{-2v+2\frac{vw^2}{c^2}}{1-\frac{v^2w^2}{c^4}} = -t'\,\frac{v\left(1-\frac{w^2}{c^2}\right)}{1-\frac{v^2w^2}{c^4}} \xrightarrow{w\to 0} -v\,t'$$
$$\xrightarrow{w\to c} 0$$

$\boxed{3}$

(b) Aus Invarianz der MWGl folgt Inv. der Wellengl.:

$$\psi'(x',t') = \psi(x(x',t'), t(x',t'))$$

$$\psi(x',t') = \mathrm{Re}\, A \exp(ik(\gamma x' + \gamma v t' - \gamma c t' - \gamma \tfrac{v}{c} x'))$$
$$= \mathrm{Re}\, A \exp(ik(\gamma(1-\tfrac{v}{c}) x' - c\gamma(1-\tfrac{v}{c}) t'))$$

$$\gamma(1-\tfrac{v}{c}) = \frac{1-v/c}{[(1-v/c)(1+v/c)]^{1/2}} = \left(\frac{1-\frac{v}{c}}{1+\frac{v}{c}}\right)^{1/2} = \left(\frac{c-v}{c+v}\right)^{1/2}$$

$$\Rightarrow k' = k\left(\frac{c-v}{c+v}\right)^{1/2}, \quad c' = c\left(\frac{c-v}{c+v}\right)^{1/2}$$

$$x(x',t') = \frac{x'}{\gamma} + v\,t(x',t')$$
$$t(x',t') = \frac{t'}{\gamma} + \frac{v}{c^2} x(x',t')$$
$$\Rightarrow t(x',t') = \frac{x(x',t') - \frac{x'}{\gamma}}{v}$$
$$\Rightarrow \frac{t'}{\gamma} + \frac{v}{c^2} x = \frac{x - \frac{x'}{\gamma}}{v}$$
$$\frac{vt'}{\gamma} + \frac{v^2}{c^2} x = x - \frac{x'}{\gamma}$$
$$\frac{1}{\gamma}(x' + vt') = (1 - \tfrac{v^2}{c^2}) x = \frac{1}{\gamma^2} x$$

$$\Rightarrow x(x',t') = \gamma(x' + vt')$$
$$\Rightarrow t(x',t') = \gamma(t' + \tfrac{v}{c^2} x')$$

(a) Quick & Dirty:

$$\psi'(x',t') = \psi(x' + vt', t')$$
$$= \mathrm{Re}\, A \exp(ik(x' + vt' - ct'))$$
$$= \mathrm{Re}\, A \exp(ik(x' + (v-c) t'))$$

$$k' = k, \quad c' = c - v$$