Theo Phys I (4)

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(a) $\frac{y}{x} = \tan\alpha \qquad y = x \cdot \tan\alpha \qquad \dot{y} = \dot{x} \cdot \tan\alpha$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m \; \dot{y}^2 (1 + \tan^{-2}\alpha)$$

$$V = mgy$$

$$L = T - V = \frac{1}{2} m \; \dot{y}^2 (1 + \tan^{-2}\alpha) - mgy$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \frac{d}{dt} \cancel{\tfrac{1}{2}} m \dot{y} (1 + \tan^{-2}\alpha) + mg$$

$$= \cancel{m} \ddot{y} \underbrace{(1 + \tan^{-2}\alpha)}_{\gamma} + \underbrace{\cancel{m}g}_{-c} = 0$$

$$\ddot{y} = \frac{c}{\gamma} \qquad | \int$$

$$\dot{y} = \frac{c}{\gamma} t + \delta \qquad | \int$$

$$y = \frac{1}{2}\frac{c}{\gamma} t^2 + \delta t + \varepsilon$$

$$y(t) = \frac{-g}{2(1 + \tan^{-2}\alpha)} t^2 + v_0 t + y_0$$

(b) $y = c \cdot x^2 \qquad \dot{y} = 2c \, x \dot{x} \iff \dot{x} = \frac{\dot{y}}{2cx}$

$\cancel{\frac{dy}{dt} = 2c \, x \frac{dy}{dx}}$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m \left(4c^2 x^2 \dot{x}^2 + \dot{x}^2 \right) =$$

$$= \frac{1}{2} m \; \dot{x}^2 (4c^2 x^2 + 1)$$

$$V = mgy = mgcx^2 \quad \left[= \frac{1}{2} m \; \dot{y} \left(\frac{1}{4c^2x^2} + 1 \right) \right]$$

$$L = \frac{1}{2} m \dot{x}^2 (4c^2x^2 + 1) - mgc\,x^2$$

$$\cancel{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt} \; m\dot{x}(4c^2x^2+1) - 4m\dot{x}^2 c x \; 2mgcx}$$

$$L \approx \frac{1}{2} m\dot{x}^2 - mgx^2 c \quad \rightarrow \text{Schwingung (harmon.)} \Rightarrow \omega = \sqrt{2gc}$$

$$\approx \frac{1}{2} m \dot{x}^2 \qquad \rightarrow \text{gleichförm. Bewegung}$$

(c) $\underline{L} = \underline{\underline{I}} \cdot \underline{\omega}$ $E_R = \frac{1}{2} I \omega^2$

$E_p = mgy$

$y = x \tan\alpha$ $\dot{y} = \dot{x} \tan\alpha$

~~$T = \frac{1}{2} I \omega^2 ...$~~

"−": wird φ größer wird y, x kleiner

$-s = R \cdot \varphi \Rightarrow -s \cdot \sin\alpha = y = -R\varphi \sin\alpha$

$x = -R\varphi \frac{\sin\alpha}{\tan\alpha}$

$\dot{y} = -R\dot{\varphi} \sin\alpha$

$\dot{x} = -R\dot{\varphi} \underbrace{\frac{\sin\alpha}{\tan\alpha}}_{\cos\alpha}$

$\dot{\varphi} = \omega$

$T = \frac{1}{2} I\omega^2 + \frac{1}{2} m \left(R^2 \dot{\varphi}^2 \sin^2\alpha + R^2 \dot{\varphi}^2 \underbrace{\frac{\sin^2\alpha}{\tan^2\alpha}}_{\cos^2\alpha} \right)$

$= \frac{1}{2} \underbrace{I}_{mR^2} \omega^2 + \frac{1}{2} m R^2 \dot{\varphi}^2$

$= mR^2 \dot{\varphi}^2$

$V = mgy = mgx \tan\alpha = -mgR\varphi \sin\alpha$

$L = mR^2 \dot{\varphi}^2 + \cancel{mgx\tan\alpha}\ mgR\varphi \sin\alpha$

$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = \frac{d}{dt} 2mR^2 \dot{\varphi} - mgR \sin\alpha$

$= 2mR^2 \ddot{\varphi} - mgR \sin\alpha = 0$

$\ddot{\varphi} = \frac{g \sin\alpha}{2R}$

$\dot{\varphi} = \frac{g \sin\alpha}{2R} t + \dot{\varphi}_0$

$\varphi = \frac{g \sin\alpha}{4R} t^2 + \dot{\varphi}_0 t + \varphi_0$

(c) Parabel nach Bogenlänge parametrisieren:

$$\cancel{\Delta s = \int \frac{d(cx^2)}{dx}\,dx = \int 2cx\,dx =}$$

$$r = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ cx^2 \end{pmatrix} \qquad \frac{dr}{dx} = \begin{pmatrix} 1 \\ 2cx \end{pmatrix}$$

$$\left\| \frac{dr}{dx} \right\| = \sqrt{1+4c^2x^2}$$

$$\Delta s = \int_{x_0}^{x_1} \sqrt{1+4c^2x^2}\,dx \qquad \Big|\ u = 2cx \qquad \frac{du}{dx} = 2c$$

$$= \int \sqrt{1+u^2}\ \frac{1}{2c}\,du \qquad u = \sinh v \qquad \frac{du}{dv} = \cosh v$$

$$= \frac{1}{2c}\int \sqrt{1+\sinh^2 v}\cdot \cosh v\,dv$$

$$= \frac{1}{2c}\int \cosh^2 v\,dv = \frac{1}{2c}\left(\frac{1}{2}\cosh v \sinh v + \frac{1}{2} v\right)\Big|$$

$$= \frac{1}{2}\left[\sqrt{\frac{1}{4c^2} + x^2}\cdot x + \frac{1}{2c}\operatorname{arcsinh} 2cx\right]_{x_0}^{x_1}$$

$$= R\cdot\Delta\varphi = R\varphi \quad \Rightarrow \quad \varphi(x)$$

$$\cancel{\Rightarrow y(x) \Rightarrow y\,x\,\varphi}$$

$$\Rightarrow \quad y(x) = cx^2 \ \Rightarrow\ \dot y = 2cx\dot x$$

$$\dot\varphi(x) = \frac{1}{R}\frac{1}{2}\left[\sqrt{\frac{1}{4c^2}+x^2} + \text{[illegible]}\ \frac{1\,x}{2\sqrt{\frac{1}{4c^2}+x^2}} + \frac{1}{2c\sqrt{1+4c^2x^2}}\right]\dot x$$

$$= \frac{x}{2R}\sqrt{1+4c^2x^2}\left(2 + \frac{4c^2x^2}{\sqrt{1+4c^2x^2}}\right) = \omega(x)$$

$$T = \frac{1}{2}I\omega^2 + \frac{1}{2}(\dot x^2 + \dot y^2)m$$

$$= \frac{1}{8R^2}\left(\dot x\sqrt{1+4c^2x^2}\left(2+\frac{4c^2x^2}{\sqrt{1+4c^2x^2}}\right)\right)^2 mR^2 + \frac{1}{2}m\left(4c^2x^2\dot x^2 + \dot x^2\right)$$

$$V = mgcx^2$$

$$L = \frac{1}{8R^2}\left(\dot x\left\{\sqrt{1+4c^2x^2}\left(2+\frac{4c^2x^2}{\sqrt{1+4c^2x^2}}\right)\right)^2 mR^2 + \frac{1}{2}m\left(4c^2x^2\dot x^2 + \dot x^2\right) - mgcx^2$$

$$\approx \cancel{\frac{1}{4R^2}\,2\,mR^2 + \frac{1}{2}m\dot x^2 - mgcx^2}$$

$$L \approx \frac{1}{8} \cdot \dot{x}\, 4 \cdot m + \frac{1}{2}\, \dot{x}^2 \cdot m - mgcx^2$$

$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{x}^2 - mgcx^2$$

$$= m\dot{x}^2 - mgcx^2$$

E-L-Gl.:

$$2m\ddot{x} = 2mgcx \qquad \omega = \sqrt{gc}$$

$\boxed{2}$ $$\varphi = \pi - 2b\int_{r_{min}}^{\infty} \frac{1}{r^2\sqrt{1-\frac{b^2}{r^2}-\frac{V(r)}{E}}}\,dr$$

(a) [1] $\underline{r>a} \Rightarrow V=0$

$$\varphi = \pi - 2b\int_{r_m}^{\infty}\frac{1}{r\sqrt{r^2-b^2}}\,dr$$

$$= \pi - 2b\cdot\frac{1}{b}\arccos\frac{b}{r}\Big|_{r_m}^{\infty} = 2\arccos\frac{b}{r_{min}}$$

Da $\frac{b}{r_m} \in [-1;1] \Rightarrow |r_m| \in [b,\infty[$

$r_{min} = b$

$b<a \Rightarrow r_{min} = a$

$\underline{r<a}$

[2] $$\varphi = \pi - 2\int_{r_m}^{\infty} b\,\frac{1}{r^2}\frac{1}{\sqrt{1-\frac{b^2}{r^2}+\frac{V_0}{E}}}\,dr$$

$$= \pi - 2\int_{r_m}^{\infty} b\cdot\frac{1}{r^2\sqrt{n^2-b^2/r^2}}\,dr$$

$$\cancel{= \frac{\sqrt{1+V_0/E}}{(\sqrt{1+V_0/E})}\frac{1}{r\sqrt{(rn)^2-b^2}}\,dr}$$

$$\cancel{= \pi - 2b\frac{\sqrt{1+V_0/E}}{b}\arccos\frac{b}{\sqrt{1+V_0/E}\,r}\Big|_{r_m}^{\infty}}$$

$$\cancel{= \pi - \sqrt{1+V_0/E}\,\pi +}$$

$$\varphi = \pi - 2\int_{r_m}^{a} b\cdot\frac{1}{r^2\sqrt{n^2-b^2/r^2}}\,dr - 2\int_a^{\infty} b\,\frac{1}{r^2\sqrt{1-b^2/r^2}}\,dr$$

$nr = y$

$\frac{dy}{dr} = n$

$\frac{dy}{n} = dr$

$$= \pi - 2\int_{r_m}^{a} b\,\frac{n}{(nr)\sqrt{(rn)^2-b^2}}\,dr - 2\int_a^{\infty} b\,\frac{1}{r\sqrt{r^2-b^2}}\,dr$$

$$= \pi - 2\cancel{b}\frac{1}{\cancel{b}}\arccos\frac{b}{rn}\Big|_{r_m}^{a} - 2\cancel{b}\frac{1}{\cancel{b}}\arccos\frac{b}{r}\Big|_a^{\infty}$$

$$= \cancel{\pi} - 2\arccos\frac{b}{an} + 2\arccos\frac{b}{nr_m} - \underbrace{2\arccos 0}_{-\pi} + 2\arccos\frac{b}{a}$$

$$\varphi = 2\left(-\arccos\frac{b}{an} + \arccos\frac{b}{nr_m} + \arccos\frac{b}{a}\right)$$

Theophys I (5)

3) ~~Weil~~ $\underline{\omega} \times \underline{r} = \underline{v}$, $\underline{\omega} \perp \underline{r}$

$$\underline{L} = m(\underline{r} \times \underline{v}) = m(\underline{r} \times \underline{\omega} \times \underline{r})$$
$$= m\, \underline{r}^2\, \underline{\omega}$$

Da $\underline{r} = (x\,y\,z)$, $\underline{\omega} = (u\,v\,w)$

$$\begin{matrix} x \\ y \\ z \end{matrix} \times \left(\begin{matrix} u \\ v \\ w \end{matrix} \times \begin{matrix} x \\ y \\ z \end{matrix} \right) =$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} vz - wy \\ wx - uz \\ uy - vx \end{pmatrix} = \begin{pmatrix} uy^2 - vxy - zwx + uz^2 \\ vz^2 - wyz - xuy + vx^2 \\ wx^2 - uzw - vzy + wy^2 \end{pmatrix}$$

$$= \begin{pmatrix} u(y^2 + z^2) + x(-vy - zw) \\ v(z^2 + x^2) + y(-wz - xu) \\ w(x^2 + y^2) + z(-uw - vy) \end{pmatrix}$$

Da $\underline{r} \perp \underline{\omega} \Rightarrow ux + vy + wz = 0 \Rightarrow$

$$= \begin{pmatrix} u\,(x^2 + y^2 + z^2) \\ v \quad " \\ w \quad " \end{pmatrix} = \underline{r}^2 \cdot \underline{\omega}$$

$$\underline{M} = \dot{\underline{L}} = \frac{d}{dt} m\underline{r}^2\underline{\omega} = 2m\underline{r}\dot{\underline{r}}\,\underline{\omega} + m\underline{r}^2\dot{\underline{\omega}}$$

Da $\dot{\underline{L}} = \overset{0}{\cancel{\text{const.}}} \Rightarrow 2\cancel{m}\underline{r}\dot{\underline{r}}\,\underline{\omega} = -\cancel{m}\underline{r}^2\dot{\underline{\omega}}$

Da $\dot{r} < 0 \Rightarrow \dot{\omega} > 0 \Rightarrow$ ~~Die Bewegung wird~~

Die ~~Kreisel~~ Winkelgeschw. nimmt zu.

~~Da~~ $m\underline{r}^2\underline{\omega} = \underline{\gamma} \Rightarrow \underline{\omega} = \frac{\underline{\gamma}}{m\underline{r}^2}$

~~&~~ $v = \omega\, r \Rightarrow$

$$v \sim \frac{1}{\cancel{m}\, r}$$