Theophys-Übung 4, Michael Bauer 16.05.09

Blatt 1/

1) a) $L(\vec{r},\dot{\vec{r}}) = \frac{m}{2}\dot{\vec{r}}^2$

$h_i^s : \vec{r} \longmapsto \vec{r} + s\vec{v}t = \vec{r}'$

$\dot{\vec{r}} + s\vec{v} \quad \dot{\vec{r}}'$

$L'(\vec{r}',\dot{\vec{r}}') = \frac{m}{2}\dot{\vec{r}}'^2 = \frac{m}{2}(\dot{\vec{r}} + s\vec{v})^2$

$= \frac{m}{2}(\dot{\vec{r}}^2 + 2s\vec{v}\dot{\vec{r}} + s^2\vec{v}^2)$

$= L(\vec{v},\dot{\vec{r}}) + ms\vec{v}\dot{\vec{r}} + \frac{m}{2}s^2\vec{v}^2$

$= L + \frac{d}{dt}\left(ms\vec{v}\vec{r} + \frac{1}{2}ms^2\vec{v}^2 t\right) = L + \frac{d}{dt}F$

b) $I(q^i,\dot{q}^i) = \left[\sum_i \frac{\partial L}{\partial \dot{q}^i}\frac{dh_s^i}{ds} - \frac{\partial F}{\partial s}\right]_{s=0} = \text{const}$

$I(\vec{r},\dot{\vec{r}}) = \left[\frac{\partial L}{\partial \dot{r}}\frac{dh_s^i}{ds} - \frac{\partial F}{\partial s}\right]_{s=0}$

$= \left[m\dot{\vec{r}}\cdot\vec{v}t - (m\vec{v}\vec{r} + ms\vec{v}^2 t)\right]_{s=0}$

$= m\dot{\vec{r}}\vec{v}t - m\vec{v}\vec{r} = m\vec{v}(\dot{\vec{r}}t - \vec{r}) = \text{const}$

$m\cdot\vec{v} = \text{const} \Rightarrow \dot{\vec{r}}t - \vec{r} = \text{const}, \quad \dot{\vec{r}} = \frac{p}{m}$

$\frac{p}{m}t - \vec{r} = \text{const.}$ Schwerpunktsatz

(nur 1 Teilchen)

<u>Beweis der Verallgemeinerung</u>

1) $\frac{d}{ds}L'(\hat{q}^i,\dot{\hat{q}}^i,t) = \sum_i\left[\frac{\partial L}{\partial q^i}\frac{d}{ds}\hat{q}^i + \frac{\partial L}{\partial \dot{q}^i}\frac{d}{ds}\dot{\hat{q}}^i\right]$

$= \sum_i\left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right)\frac{d}{ds}\hat{q}^i + \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}\left(\frac{d}{ds}\hat{q}^i\right)\right]$

$= \sum_i \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\frac{d}{ds}\hat{q}^i\right]$

(wie in der Vorlesung)

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$$2)\quad \frac{d}{ds} L'(\tilde{q}^i, \dot{\tilde{q}}^i, t) = \frac{d}{ds}\left[ L(q^i, \dot{q}^i, t) + \frac{d}{dt} F \right]$$

$$= \frac{d}{ds}\left(\frac{d}{dt} F\right) = \frac{d}{dt}\left(\frac{\partial F}{\partial s}\right)$$

$2) = 1)$ :

$$\frac{d}{dt}\left(\frac{\partial F}{\partial s}\right) = \frac{d}{dt} \sum_i \left( \frac{\partial L}{\partial \dot{q}^i} \frac{d}{ds} \tilde{q}^i \right)$$

$$0 = \frac{d}{dt}\left[ \sum_i \frac{\partial L}{\partial \dot{q}^i} \frac{d}{ds} \tilde{q}^i - \frac{\partial F}{\partial s} \right]$$

$$\Rightarrow \sum_i \frac{\partial L}{\partial \dot{q}^i} \frac{d h_s^i}{ds} - \frac{\partial F}{\partial s} = \text{const}$$

$$3)a)\quad \vec{B} = B \cdot \vec{e}_z \qquad \vec{B} = \nabla \times \vec{A}$$

$$\vec{A} = \begin{pmatrix} -yB \\ 0 \\ 0 \end{pmatrix} \qquad B = \begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \times \begin{pmatrix} -yB \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ -\partial z\, yB - 0 \\ 0 - (-\partial y\, yB) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} = B \cdot \vec{e}_z$$

$$0 \stackrel{!}{=} E = -\nabla \phi - \frac{1}{c} \underbrace{\frac{\partial A}{\partial t}}_{0} = -\nabla \phi$$

$$\Rightarrow \phi = 0$$

$$\vec{r}(t) = \begin{pmatrix} \cos(\omega_c t) \\ \sin(\omega_c t) \\ 0 \end{pmatrix} \qquad \dot{\vec{r}}(t) = \begin{pmatrix} -\omega_c \sin(\omega_c t) \\ \omega_c \cos(\omega_c t) \\ 0 \end{pmatrix}$$

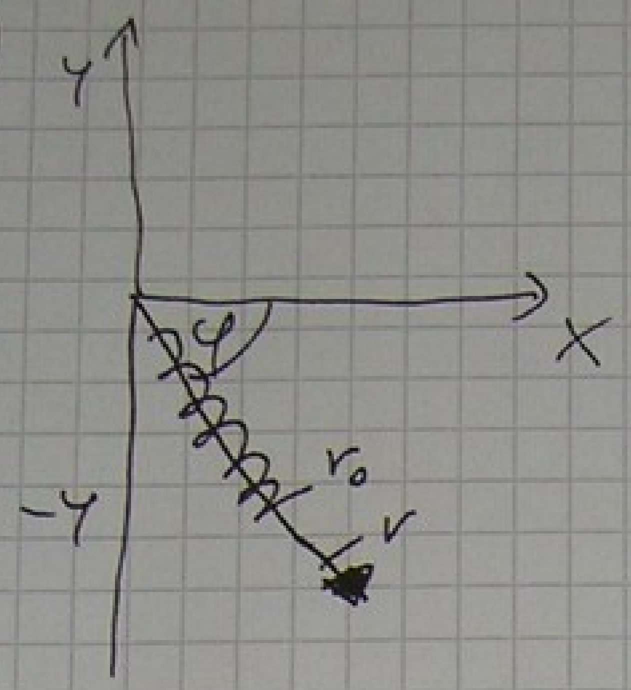
$$F = 0 - \frac{e\omega_c}{c} \begin{pmatrix} -\sin(\omega_c t) \\ \cos(\omega_c t) \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} \qquad \left[\ \omega_c = \frac{eB}{mc}\ \right]$$

$$= -\frac{e\omega_c}{c} \begin{pmatrix} B\cos(\omega_c t) - 0 \\ 0 + B\sin(\omega_c t) \\ 0 \end{pmatrix} = -\omega_c^2 m \begin{pmatrix} \cos \omega_c t \\ \sin \omega_c t \\ 0 \end{pmatrix}$$

$$F = m\ddot{\vec{r}} = -\omega_c^2 m \begin{pmatrix} \cos(\omega_c t) \\ \sin(\omega_c t) \\ 0 \end{pmatrix}$$ Bewegungsgleichung erfüllt

2)

a)




$\Rightarrow \quad x = r\cos\varphi$

$\quad\;\; y = -r\sin\varphi$

$V = mgy + \frac{1}{2}k(r - r_0)^2$

$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2)$

$V = -mgr\sin\varphi + \frac{1}{2}k(r-r_0)^2$

$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + mgr\sin\varphi - \frac{1}{2}k(r-r_0)^2$

$r: \quad 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \frac{d}{dt}(m\dot{r}) - \left(mr\dot{\varphi}^2 + mg\sin\varphi - k(r-r_0)\right)$

$\Rightarrow \quad m\ddot{r} = mr\dot{\varphi}^2 + mg\sin\varphi - k(r-r_0)$

$\varphi: \quad 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = \frac{d}{dt}(mr^2\dot{\varphi}) - (mgr\cos\varphi)$

$\Rightarrow \frac{d}{dt}(mr^2\dot{\varphi}) = mgr\cos\varphi$

$\qquad\qquad\qquad = m(2r\dot{r}\dot{\varphi} + r^2\ddot{\varphi})$

b) $\dot{\varphi} \approx 0, \quad \sin\varphi \approx 1 \quad \Leftrightarrow \quad \varphi \approx \frac{\pi}{2}$ — Pendel hängt nach unten

$r: \quad m\ddot{r} = 0 + mg\underbrace{\sin\frac{\pi}{2}}_{1} - k(r-r_0)$

$\varphi: \quad 0 = mgr\cos\frac{\pi}{2} = 0$

$\Rightarrow \quad m\ddot{r} + kr = m\cdot g + kr_0$ — inhomogene DGL

$\quad m\ddot{r} + kr = 0$ — homogene Lösung suchen

Ansatz $r_{hom} = \alpha e^{i\omega t} \Rightarrow \ddot{r} = \alpha\omega^2 i^2 e^{i\omega t}$

einsetzen $\Rightarrow \quad -\omega^2 m + k = 0 \quad \Leftrightarrow \quad \omega = \pm\sqrt{\frac{k}{m}}$

Partikulär Lösung : r = const $\Rightarrow \ddot{r} = 0$

$kr = mg + kr_0 \quad \Leftrightarrow \quad r_{part} = \frac{mg}{k} + r_0$

$r(t) = \alpha_1 e^{+i\sqrt{\frac{k}{m}}t} + \alpha_2 e^{-i\sqrt{\frac{k}{m}}t} + \frac{mg}{k} + r_0$