Elektrodynamik//Uebung 09

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$\boxed{1}$

(a) rot(3): $\underline{\nabla}\times(\underline{\nabla}\times\underline{E}) = -\frac{1}{c}\partial_t\,\underline{\nabla}\times\underline{B} \overset{(4)}{=} -\frac{n^2}{c^2}\partial_t^2\,\underline{E} = -\Delta\underline{E}$

$\left[\underline{\nabla}\times(\underline{\nabla}\times\underline{E})\right]_i = \varepsilon_{ijk}\partial_j\varepsilon_{k\mu\nu}\partial_\mu E_\nu = \varepsilon_{ijk}\varepsilon_{k\mu\nu}\partial_j\partial_\mu E_\nu =$

$(\delta_{i\mu}\delta_{j\nu}-\delta_{j\mu}\delta_{i\nu})\partial_j\partial_\mu E_\nu = \partial_j\partial_i E_j - \partial_j\partial_j E_i = \left[\underline{\nabla}(\underbrace{\underline{\nabla}\cdot\underline{E}}_{0\,\to(1)}) - \Delta E_i\right]$

(WG) $\left\{\begin{array}{l} (\Delta + \frac{n^2}{c^2}\partial_t^2)\underline{E} = 0 \quad \text{ist Wellengl. für } \underline{E} \\ \text{rot}(4):\ -\Delta\underline{B} = -\frac{n^2}{c^2}\partial_t^2\underline{B} \quad \text{"} \quad \underline{B} \end{array}\right.$ $\Rightarrow$ Eb. W. mögl.

$\underline{E} = \text{Re}\ \underline{E}^0 e^{i(\underline{k}\cdot\underline{x} - \omega t)}$

$\Delta\underline{E} = \text{Re}\ \underline{E}^0\ (\partial_x^2+\partial_y^2+\partial_z^2)\, e^{i(k_x x + k_y y + k_z z - \omega t)}$

$= \text{Re}\ \underline{E}^0\ \|\underline{k}\|^2 e^{i(\underline{k}\underline{x} - \omega t)}$

$\overset{(WG)}{=} \text{Re}\ \underline{E}^0\ -\omega^2\left(+\frac{n^2}{c^2}\right) e^{i(\underline{k}\underline{x} - \omega t)}$

$\Rightarrow \|\underline{k}\| = \frac{n^2\omega^2}{c^2} \Rightarrow \boxed{\omega = \pm\frac{c}{n}k}$

$\partial_x(\text{Re}\, f(x)) \leftarrow$

$\frac{\text{Re}\, f(x+h) - \text{Re}\, f(x)}{h} =$

$\text{Re}\ \frac{f(x+h)-f(x)}{h} \longrightarrow$

$\text{Re}\ f'(x)$

$\begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}\times\begin{pmatrix}E_x\\ E_y\\ E_z\end{pmatrix} = \begin{pmatrix}\partial_y E_z - \partial_z E_y\\ \partial_z E_x - \partial_x E_z\\ \partial_x E_y - \partial_y E_x\end{pmatrix} =$

$\begin{pmatrix}E_z\, ik_y - ik_z E_y\\ \vdots\end{pmatrix} = i\,\underline{k}\times\underline{E}$

$\underline{\nabla}\times\underline{E} = i\underline{k}\times\underline{E} \overset{(3)}{=} -\frac{1}{c}(-i\omega)\underline{B}$

$\underline{k}\times\underline{E} = \frac{\omega}{c}\underline{B} = \frac{k}{n}\underline{B}$

$\boxed{\hat{\underline{k}}\times\underline{E} = \frac{1}{n}\underline{B}}$

Kontrolle: $\underline{\nabla}\times\underline{B} = i\underline{k}\times\underline{B} \overset{(4)}{=} \frac{n^2}{c}(-i\omega)\underline{E}$

$\underline{k}\times\underline{B} = -n^2\frac{\omega}{c}\underline{E} = -n^2\frac{k}{n}\underline{E} \Rightarrow \frac{1}{n}\underline{B}\times\hat{\underline{k}} = \underline{E}$

$\Rightarrow \hat{\underline{k}}\times\underline{E} = \hat{\underline{k}}\times\left(\frac{1}{n}\underline{B}\times\hat{\underline{k}}\right) \overset{!}{=} \frac{1}{n}\underline{B}$

$\Rightarrow \underline{B}(\hat{\underline{k}}\cdot\hat{\underline{k}}) - \hat{\underline{k}}(\underbrace{\hat{\underline{k}}\cdot\underline{B}}_{0}) \overset{!}{=} \underline{B}$ ✓ da $\|\hat{\underline{k}}\| = 1$.

$\underline{k}\cdot\underline{B} = 0$ da $\underline{\nabla}\cdot\underline{B} = i\underline{k}\cdot\underline{B} \overset{(2)}{=} 0$.

(b)

$$\int_{\mathcal{F}} n^2 \underline{E}\, d\underline{f} \overset{\text{Gauß}}{=} \int_{\partial\mathcal{F}\times h} \operatorname{div} n^2 \underline{E}\, dV \overset{(*)}{=} 0$$

$$\| $$

$$\int_{\mathcal{F}} n^2 \underline{E}\,\underline{\nu}\, df = |\mathcal{F}|\left(n_2^2 E_2^\perp - n_1^2 E_1^\perp\right)$$

$$\Rightarrow \boxed{n_2^2 E_2^\perp = n_1^2 E_1^\perp} \qquad D^\perp \text{ stetig}$$

analog $\Rightarrow$ $\boxed{B_2^\perp = B_1^\perp}$ $B^\perp$ stetig | (*) h klein |

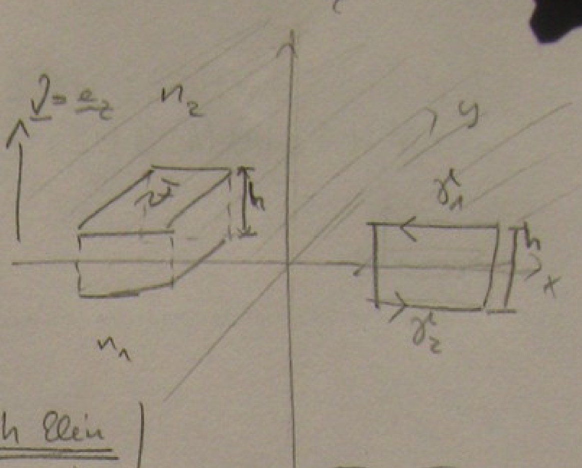
$$\int_{\partial\gamma} \operatorname{rot} \underline{E} \overset{df}{=} \int_\gamma \underline{E}\, d\underline{s} \overset{\text{Stokes}}{=} |\gamma_1|\cdot\left(E_2^\| - E_1^\|\right)$$

$$\overset{(a)}{=} \int_{\partial\gamma} -\frac{1}{c}\dot{\underline{B}}\, d\underline{f} \overset{(*)}{=} 0 \qquad (\dot{\underline{B}} \text{ ist beschr.}, \ |\partial\gamma| \to 0)$$

$$\Rightarrow \boxed{E_2^\| = E_1^\|} \qquad E^\| \text{ stetig}$$

$$\int_{\partial\gamma} \operatorname{rot}\left(\frac{1}{n^2}\underline{B}\right) d\underline{f} = \int_\gamma \frac{1}{n^2}\underline{B}\, ds = \frac{1}{n_2^2}B_2^\| - \frac{1}{n_1^2}B_1^\| \overset{(*)}{=} 0$$

$$\Rightarrow \boxed{\frac{B_2^\|}{n_2^2} = \frac{B_1^\|}{n_1^2}} \qquad H^\| \text{ stetig}$$



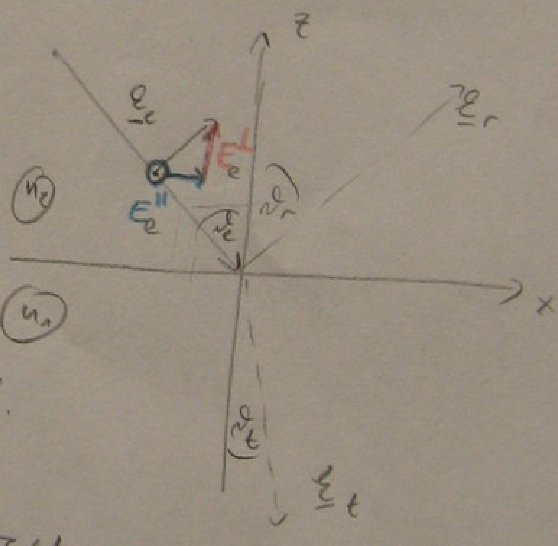
Gauß: $\int_V \operatorname{div}\vec{f}\, dV = \oint_{\partial V} \vec{f}\, d\underline{f}$

Stokes: $\int_{\mathcal{F}} \operatorname{rot}\vec{f}\, d\underline{f} = \oint_{\partial\mathcal{F}} \vec{f}\, d\underline{s}$

(c) Zerlege $\underline{E}, \underline{B}$ in $\underline{E}^\perp, \underline{E}^\|; \underline{B}^\perp, \underline{B}^\|$.

$$\left.\begin{aligned}
n_2^2 \underline{E}_e^\perp + n_2^2 \underline{E}_r^\perp &= n_1^2 \underline{E}_t^\perp \\
\underline{E}_e^\| + \underline{E}_r^\| &= \underline{E}_t^\| \\
\underline{B}_e^\perp + \underline{B}_r^\perp &= \underline{B}_t^\perp \\
\frac{\underline{B}_e^\|}{n_2^2} + \frac{\underline{B}_r^\|}{n_2^2} &= \frac{\underline{B}_t^\|}{n_1^2}
\end{aligned}\right\} \text{Stetigkeitsbed.}$$

| Achtung: $\perp, \|$ bed. $\perp$ bzw $\|$ zur Grenzfläche |



oBdA:

$$\underline{k}_e = k\,\hat{\underline{k}}_e = \frac{\omega n_2}{c}\cdot\begin{pmatrix}\sin\vartheta_e\\0\\-\cos\vartheta_e\end{pmatrix}$$

$$\underline{E}_e^\perp = (\underline{E}\cdot\underline{e}_z)\underline{e}_z \qquad \underline{E}_e^\| = \underline{E}_e - \underline{E}_e^\perp$$

$$\underline{k}_r = \frac{\omega n_2}{c}\begin{pmatrix}\sin\vartheta_r\\0\\\cos\vartheta_r\end{pmatrix}$$

$$\underline{k}_t = \frac{\omega n_1}{c}\cdot\begin{pmatrix}\sin\vartheta_t\\0\\-\cos\vartheta_t\end{pmatrix}$$

man darf alle $\underline{k}$ in xy-Ebene legen; vgl $\underline{k}_e^\| = \underline{k}_r^\| = \underline{k}_t^\|$

(c)

- In xy-Ebene muss an jedem Punkt die Phase von $\underline{E}$ gleich sein (dito $\underline{B}$), weil in der Ebene idealisiert genau eine Schicht Atome liegt:

$$\varphi(\underline{r},t) = \underline{k}\cdot\underline{r} - \omega t$$

$$k_e^x x + k_e^y y - \omega_e t = k_r^x x + k_r^y y - \omega_r t$$

muss für alle x, y, t gelten $\Rightarrow$

$$\boxed{\omega_e = \omega_r = \omega_t}, \quad \underbrace{k_e^x = k_r^x\ k_t^x,\ k_e^y = k_r^y = k_t^y}$$

$$\boxed{\underline{k}_e^{\parallel} = \underline{k}_r^{\parallel} = \underline{k}_t^{\parallel}}$$

Verw. konkrete Darst. von $\underline{k}$ mit Lotwinkel ϑ:

$$\underbrace{\frac{\omega_e n_2}{c}}_{k_e} \sin\vartheta_e = \frac{\omega_r n_2}{c}\sin\vartheta_r = \frac{\omega_t n_1}{c}\sin\vartheta_t$$

$$\boxed{\vartheta_e = \vartheta_r} \qquad \boxed{n_2 \sin\vartheta_e = n_1 \sin\vartheta_t}$$

- Damit Dispersionsrel. weiter gilt: $\|\underline{k}\| = \|\underline{k}^\perp + \underline{k}^\parallel\| = \frac{\omega n}{c}$:

$$\|\underline{k}_e^\parallel + \underline{k}_e^\perp\| = \|\underline{k}_r^\parallel + \underline{k}_r^\perp\| \quad \text{Da } k\text{-Vektoren senkrecht: Pythagoras}$$

$$\cancel{\|\underline{k}_e^\parallel\|} + \|\underline{k}_e^\perp\| = \cancel{\|\underline{k}_r^\parallel\|} + \|\underline{k}_r^\perp\| \Rightarrow \underline{k}_e^\perp = \pm\,\underline{k}_r^\perp,$$

physikalisch einzig sinnvoll: $\boxed{\underline{k}_r^\perp = -\underline{k}_e^\perp}$

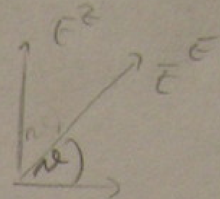
Für $\underline{k}_t^\perp$: $\underline{k}_t^\perp \parallel \underline{e}_z$, verw. Brechungsgesetz: $\vartheta_t = \arcsin \frac{n_2}{n_1}\sin\vartheta_e$

$$\|\underline{k}_t^\perp\| = \|\underline{k}_t\|\cdot\cos\vartheta_t = \|\underline{k}_t\|\cdot\underbrace{\cos\arcsin\left(\frac{n_2}{n_1}\sin\vartheta_e\right)}_{\sqrt{1-\left(\frac{n_2}{n_1}\right)^2\sin^2\vartheta_e}}$$

- Die Einfallsebene ist xz-Ebene; es ist $\underline{E}^E$ das E-Feld in und $\underline{E}^y$ der Vektor senkrecht zur Einfallseb.

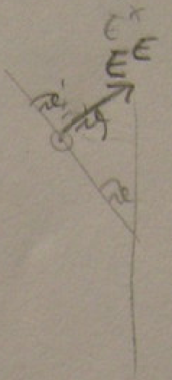
(c) [Forts.]

Es gilt die konkrete Darst.:



$$E^x = E^E \cdot \cos\vartheta \quad , \quad E^z = E^E \cdot \sin\vartheta$$

Verw. jetzt Stetigkeitsbed.:



$$E_e^x + E_r^x = E_t^x \Leftrightarrow E_e^E \cos\vartheta_e + E_r^E \cos\vartheta_r = E_t^E \cos\vartheta_t$$

$$\Updownarrow$$

$$E_e^y + E_r^y = E_t^y \; \boxed{\alpha} \qquad (E_e^E + E_r^E)\cos\vartheta_e = E_t^E \cdot \underset{\sqrt{1-\left(\frac{n_2}{n_1}\right)^2 \sin^2\vartheta_e}}{\underset{\parallel}{\cos\vartheta_t}} \; [\beta]$$

Verw. jetzt $\hat{\underline{k}} \times \underline{E} = \frac{1}{n}\underline{B}$ und Stetigkeitsbed. an B: Da $\underline{E}$ in Einfallsebene liegt ist $E^E = \frac{1}{n} B^y$ bzw $E^y = \frac{1}{n} B^E$.

$$B^x = B^E \cdot \cos\vartheta \; ;$$

$$\frac{B_e^x}{n_2^2} + \frac{B_r^x}{n_2^2} = \frac{B_t^x}{n_1^2} \Leftrightarrow \frac{B_e^E + B_r^E}{n_2^2}\cos\vartheta_e = \frac{B_t^E \cos\vartheta_t}{n_1^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{E_e^y + E_r^y}{n_2}\cos\vartheta_e = \frac{E_t^y \cos\vartheta_t}{n_1} \qquad [\gamma]$$

$$B_e^y + B_r^y = B_t^y \Leftrightarrow n_2(E_e^E + E_r^E) = n_1 E_t^E \quad [\delta]$$

Fasse jetzt Gl [α], [β], [γ], [δ] und Brechungsgesetz zusammen:

=> Fresnel

(d) $E_r^E = \frac{\tan(\vartheta_e - \vartheta_t)}{\tan(\vartheta_e + \vartheta_t)} E_e^E = 0 \leftarrow$

$$\tan(\vartheta_B - \vartheta_t) = 0 \Leftarrow \vartheta_B = \vartheta_t = \operatorname{asin}\left(\frac{n_2}{n_1} \sin\vartheta_B\right)$$

$$\Leftarrow \vartheta_B + \vartheta_t = \frac{\pi}{2}$$

$$\vartheta_B + \operatorname{asin}\left(\frac{n_2}{n_1}\sin\vartheta_B\right) = \frac{\pi}{2} \qquad \left(\begin{array}{c} \sin\vartheta_B + \frac{n_2}{n_1} \sin\vartheta_B = 1 \\ \left(1 + \frac{n_2}{n_1}\right) = \frac{1}{\sin\vartheta_B} \end{array}\right)$$

$$\vartheta_t = \frac{\pi}{2} - \vartheta_B$$

$$n_2 \sin\vartheta_B = n_1 \sin\left(\frac{\pi}{2} - \vartheta_B\right) = n_1 \cos(-\vartheta_B) = n_1 \cos\vartheta_B$$

$$\frac{n_1}{n_2} = \tan\vartheta_B$$

(e) Für $\vartheta_e > \vartheta^T$: $\quad \underline{k}_e^{\parallel} = \underline{k}_t^{\parallel}$, $\underline{k}_t^{\perp} = \|\underline{k}_t\| \, i \sqrt{\left(\frac{n_2}{n_1}\sin\vartheta_e\right)^2 - 1} = i\kappa$, $\kappa > 0$

$$\underline{E} = \underline{E}_0 \cdot e^{i(k_t^x x + k_t^y y - \omega t)} \cdot e^{-\kappa z}$$

Die Welle breitet weiterhin in z-Richtung aus, doch ist ihre Amplitude gedämpft und zerfällt exponentiell. Die eigentliche Phase hängt dabei nicht mehr von z ab.