Ana II ②

2 (a) $a_n, b_n > 0$, $\frac{a_n}{a_{n+1}} \leq \frac{b_n}{b_{n+1}}$ $\sum_{n=1}^{\infty} b_n$ div $\Rightarrow \sum_{k=1}^{\infty} a_k$ div.

$\cancel{b_n = a_n = \frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdots \frac{a_{n-1}}{a_n} \cdot a_n}$

Kehrbrüche!

$$a_n = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_2}{a_1} \cdot a_1$$

$$b_n \cdot \underbrace{\frac{a_1}{b_1}}_{= C'} = \frac{b_n}{b_{n-1}} \cdots \frac{b_2}{b_1} \cdot a_1$$

$a_n \leq C' b_n$

(b) $n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1$ (für $n \geq N$)

$\frac{a_n}{a_{n+1}} - 1 < 1/n$

$\frac{a_n}{a_{n+1}} < \frac{1+n}{n}$ $b_n = \frac{1}{n}$

$\frac{a_n}{a_{n+1}} < \frac{b_n}{b_{n+1}}$

| $b_{n+1} = \frac{1}{n+1}$
| $b_n = \frac{1}{n}$
| $\frac{b_n}{b_{n+1}} = \frac{1}{n} : \frac{1}{n+1} = \frac{n+1}{n}$

(c) $\cancel{\frac{a_n}{a_{n+1}} = (n+1)!}$

Laut Quotientenkrit. muss $\frac{a_{n+1}}{a_n} \geq q > 1$

$$\frac{(n+1)!}{n!} \cdot \frac{(x+1)\cdots(x+n)}{(x+1)(x+2)\cdots(x+n)(x+n+1)} =$$

$\frac{n+1}{x+n+1} > 1$ $x > 0 \Rightarrow$

Quotientkrit. nicht günstig, weil für große n der Einfluss von x verschw.

Raabe $\lim n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{x+n+1}{n+1} - 1\right) = n\left(\frac{x}{n+1}\right) =$

$\frac{nx}{n+1} = \frac{x}{1+1/n} \xrightarrow{n\to\infty} \frac{x}{1} = x$

für $x < 1 \Rightarrow \lim < 1 \Rightarrow$ div

$x = 1$ $\tilde{a}_n = \frac{1}{n+1} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$ div

$x > 1$ Nach Satz: lim $> 1 \Rightarrow$ konv

3

(a) $\sum \frac{1000^n}{n!}$ $\sqrt[n]{\frac{1000^n}{n!}} = \frac{1000}{\sqrt[n]{n!}} \to 0 \le q < 1$

$\Rightarrow$ konv

(b) $\sum \frac{(n!)^2}{(2n)!}$ $\frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} =$

$\frac{(n+1)^2}{(2n+1)(2n+2)} < \frac{(n+1)^2}{(2n+1)^2} \to \left(\frac{1}{2}\right)^2 \le q \frac{3}{4} < 1$

konv

(c) $\sum \frac{n!}{n^n}$ $\sqrt[n]{\frac{n!}{n^n}} = \frac{\sqrt[n]{n!}}{n}$

$\frac{(n+1)!}{(n+1)^{(n+1)}} \cdot \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n = \frac{n^n}{(n+1)^n}$

$n \left(\left(\frac{n+1}{n}\right)^n - 1\right) > r/n + 1$

$\left(\frac{n+1}{n}\right)^n \to e$

$\left(\frac{n+1}{n}\right)^n > \frac{r+n}{n}$

$(n+1)^n \cdot n > (r+n) \cdot n^n = r n^n + n^{n+1}$

Quot: $\left(\frac{n}{n+1}\right)^n \to \frac{1}{e} \approx 0{,}36 < 1 \Rightarrow$ konv.

Raabe: $n\ (e-1) > 1{,}1\, n > 1$ konv

3.2

(a) $\left(\frac{n}{n+1}\right)^{n^2} = |\cdot| \Rightarrow$ bedingt $\hat{=}$ absolut!

$\left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$ $(\cdot)^2 \to \frac{1}{e^2} \ne q$

Wurzelkrit.

$\boxed{3.1}$

(a) $\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \frac{n}{n+1}^{n^2/n} \longrightarrow \frac{1}{e} < \cancel{\frac{1}{2}}\ \frac{1}{2} < 1 \Rightarrow$ konv.

(b) abs $\overset{?}{=}$ bed. ! Quot

$$\frac{((n+1)!)^3}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{(n!)^3} = \frac{(n+1)^3}{2^{2n+1}} \to 0 < 1$$

$(n!(n+1))^3 = \frac{(n!)^3 \cdot (n+1)^3}{2^{n^2+2n+1}} =$

konv.

(c) $\frac{1}{6^n}\binom{3n}{n} = \frac{1}{6^n}\,\frac{(3n)!}{n!\,(3n-n)!} = \frac{1}{6^n}\cdot\frac{(3n)!}{n!\,(2n)!}$

Quotienten:

$$\frac{(3n+3)!}{6^n \cdot 6 \cdot (n+1)!\,(2n+2)!} \cdot \frac{6^n\, n!\,(2n)!}{(3n)!} =$$

$$\frac{(3n+1)(3n+2)\cancel{(3n+3)}}{3\cdot 2\,(n+1)\,2\,(2n+1)(2n+2)} = \frac{9n^2+9n+2}{8n^2+12n+4} \longrightarrow \frac{9}{8} > 1$$

div

(d) $1 \cdot 1 = \frac{1}{n\sqrt[n]{n}} = \frac{1}{n}\cdot\frac{1}{\sqrt[n]{n}}$

$\sqrt[n]{\frac{1}{n}\cdot\frac{1}{\sqrt[n]{n}}} = \frac{1}{\sqrt[n]{n}\ \sqrt[n]{\sqrt[n]{n}}} = \frac{1}{n^{1/n}\cdot n^{1/n^2}}$

$(n^{1/n})^{1/n} = n^{1/n^2}$

konv. bedingt da $\frac{1}{n\sqrt[n]{n}} \downarrow$ (s. v. Leibn.)