Elektrodynamik
Uebung 03

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$\boxed{1}$ $\dot{\underline{v}} = \omega_B \cdot \underline{v} \times \underline{e}_z$, $\underline{v} = \begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix}$ $\underline{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} \dot{v}^x \\ \dot{v}^y \\ \dot{v}^z \end{pmatrix} = \omega_B \cdot \begin{pmatrix} v^y \\ -v^x \\ 0 \end{pmatrix} = \omega_B \cdot \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix}.$$

$\longrightarrow$ $v^z = \text{const.}$ $\longrightarrow$ $\begin{pmatrix} \dot{v}^x \\ \dot{v}^y \end{pmatrix} = \omega_B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v^x \\ v^y \end{pmatrix}.$

Löse Matrixgl.:

$$\begin{vmatrix} -\lambda & \omega_B \\ -\omega_B & -\lambda \end{vmatrix} = \lambda^2 + \omega_B^2 = 0 \Rightarrow \lambda = \pm i \omega_B.$$

$$\omega_B \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \cdot \underline{v}_i = 0 \Rightarrow \underline{v}_i = \begin{pmatrix} i \\ -1 \end{pmatrix} \quad \text{analog} \quad \underline{v}_{-i} = \begin{pmatrix} i \\ +1 \end{pmatrix} \propto \begin{pmatrix} -i \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} v^x \\ v^y \end{pmatrix} = A \cdot \begin{pmatrix} i \\ -1 \end{pmatrix} e^{i\omega_B t} + B \cdot \begin{pmatrix} -i \\ -1 \end{pmatrix} e^{-i\omega_B t}$$

Damit die Lösung reell ist, muss $A = B \in \mathbb{R}$.

Die Verkettung der Bewegungen vernichtet einen Freiheitsgrad.

$$\Rightarrow \begin{pmatrix} v^x \\ v^y \end{pmatrix} = \tilde{A} \cdot \begin{pmatrix} i & 2i & \sin(\omega_B t) \\ -1 & 2 & \cos(\omega_B t) \end{pmatrix} \overset{\tilde{A} = -A/2}{=} +A \begin{pmatrix} \sin(\omega_B t) \\ \cos(\omega_B t) \end{pmatrix}.$$

$$\Rightarrow \underline{v} = \begin{pmatrix} A \sin(\omega_B t) \\ A \cos(\omega_B t) \\ C \end{pmatrix} \qquad \underline{v}(0) = \begin{pmatrix} 0 \\ A \\ C \end{pmatrix} = \underline{v}_0$$

$$\Rightarrow \underline{r} = \begin{pmatrix} -A \frac{1}{\omega_0} \cos(\omega_B t) \\ A \frac{1}{\omega_0} \sin(\omega_B t) \\ Ct + D \end{pmatrix} \qquad \underline{r}(0) = \begin{pmatrix} -A \frac{1}{\omega_0} \\ 0 \\ D \end{pmatrix} = \underline{r}_0 .$$

Theo (3)

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2 (a) $(\text{grad}\,\phi)_\alpha = \partial_\alpha \phi \qquad [\text{rot}\,\underline{v}]_\alpha = \varepsilon_{\alpha\beta\gamma}\,\partial_\beta v_\gamma$

$$(\text{rot grad}\,\phi)_\alpha = \varepsilon_{\alpha\beta\gamma}\,\partial_\beta(\partial_\gamma\phi) \underset{\uparrow}{=} \varepsilon_{\alpha\beta\gamma}\,\partial_\gamma\partial_\beta\phi = -\varepsilon_{\alpha\gamma\beta}\,\partial_\beta\partial_\gamma\phi$$

$\Rightarrow \varepsilon_{\alpha\gamma\beta}\,\partial_\beta\partial_\gamma\phi = 0$ [wenn C^2 und Part. Abl. vertauschbar]

(b) $\partial_\alpha\,\varepsilon_{\alpha\beta\gamma}\,\partial_\beta v_\gamma = \varepsilon_{\alpha\beta\gamma}\,\partial_\alpha\partial_\beta v_\gamma \underset{C^2}{=} \varepsilon_{\alpha\beta\gamma}\,\partial_\beta\partial_\alpha v_\gamma = -\varepsilon_{\beta\alpha\gamma}\,\partial_\alpha\partial_\beta v_\gamma$

$\Rightarrow \varepsilon_{\alpha\beta\gamma}\,\partial_\alpha\partial_\beta v_\gamma = 0$.

(c) $(\text{rot rot}\,\underline{v})_\alpha = \varepsilon_{\alpha\beta\gamma}\,\partial_\beta(\varepsilon_{\gamma\mu\nu}\,\partial_\mu v_\nu) = \varepsilon_{\alpha\beta\gamma}\,\varepsilon_{\gamma\mu\nu}\,\partial_\beta\partial_\mu v_\nu$

$$= (\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu})\,\partial_\beta\partial_\mu v_\nu$$

$$= \partial_\beta\partial_\alpha v_\beta - \partial_\beta\partial_\beta v_\alpha = \partial_\alpha \underbrace{\partial_\beta v_\beta}_{(\underline{\nabla}\,\underline{v})} - \underbrace{\partial_\beta\partial_\beta}_{\Delta} v_\alpha$$

$$\partial_\alpha(\underline{\nabla}\,\underline{v}) - \Delta v_\alpha$$

$$= \underline{\nabla}\circ(\underline{\nabla}\,\underline{v}) - \Delta\underline{v}$$

3 (a) Es ist $\underline{e}_i(\overset{\text{grad}}{\underline{\nabla}} f) = \underline{e}_i\cdot(\partial_j f)\,\underline{e}_j = (\partial_j f)\,\delta_{ji} = \partial_i f$

gleichzeitig $\underset{\text{div}}{\underline{\nabla}}(\underline{e}_i f) = (\partial_j \underline{e}_j)\cdot(\underline{e}_i f) = \partial_j\,\delta_{ji}\,f = \partial_i f$

$\Rightarrow$ ~~also~~ Gradient und Divergenz sind hier gleich.

Multipl. beide Seiten von (2) mit $\underline{e}_i$:

$$\underline{e}_i\cdot\int_V \partial_i \langle a|b\rangle\, d^d r = \int_V \text{div}(\underline{e}_i\langle a|b\rangle)\,d^d r \underset{\text{Gauß}}{=} \int_{\partial V}(\underline{e}_i\langle a|b\rangle)\cdot d\underline{f}$$

$$= \int_{\partial V}\langle a|b\rangle\cdot d\underline{f}_i$$

(b) ~~$\int \partial_j a_i b_j\, d^d r = \int \text{div}(\underline{b})\, a_i\, d^d r$~~

~~$\int a_i b_j\, df_j \quad \int a_i\, \underline{b}\, d\underline{f}$~~

$$\int_V \partial_j(\underbrace{a_i b_j}_{v_j})\, d^d r \underset{(1)}{=} \int_{\partial V} v_j\, df_j = \int_{\partial V}(a_i b_j)\, df_j.$$

[4]

(a) $\mathrm{rot}\, \underline{v}_{\parallel} = \underline{0}$ gilt da $\underline{v}_{\parallel} = \mathrm{grad}\, \phi$ und $\mathrm{rot}\,\mathrm{grad} = 0$ (vgl. 2a) ; $\mathrm{div}\, \underline{v}_{\perp} = 0$ da $\underline{v}_{\perp} = \mathrm{rot}\, \underline{A}$ und $\mathrm{div}\,\mathrm{rot} = 0$ (vgl 2 b).

(c) $\underline{v}$ besteht aus den Teilen $\underline{v}_{\perp}$, $\underline{v}_{\parallel}$ aus Gl. (4), (5); diese werden gebildet alleine aus der Kenntnis von $\mathrm{div}\, \underline{v}$ (4) und $\mathrm{rot}\, \underline{v}$ (5).

(d) $\mathrm{rot}\, \underline{v} = \underline{0} \Rightarrow$ $\underline{v}_{\perp} = 0$ da $\mathrm{rot}\, \underline{0} = 0$. $\Rightarrow \underline{v} = \underline{v}_{\parallel}$ (weg. (b)) $\Rightarrow$ mit (4)

$$\phi = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\mathrm{div}_{r'}\, \underline{v}(\underline{r}')}{\|\underline{r} - \underline{r}'\|} dr' + C \qquad C \in \mathbb{R}$$

(e) ~~div~~ $\mathrm{div}\, \underline{v} = 0 \Rightarrow \underline{v}_{\parallel} = 0 \Rightarrow \underline{v} = \underline{v}_{\perp} \Rightarrow$

$$\underline{A} = \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\mathrm{rot}\, \underline{v}(\underline{r}')}{\|\underline{r} - \underline{r}'\|} dr' + \underline{\psi} \qquad \underline{\psi} = \mathrm{grad}\, \varphi$$

bel. Skalarfeld

(b)

$$\underline{v} \cdot \underline{e}_i \overset{!}{=} \partial_i \int_V \partial_j' v_j(r') G\, dr' - \varepsilon_{ijk}\, \partial_j \int_V \varepsilon_{klm} (\partial_l' v_m(r')) G\, dr'$$

$$= \partial_i \int_V \partial_j' v_j(r') G\, dr' - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \int \cdots \quad \textcircled{=}$$

$$\int \partial_j' v_j(r') G\, dr' = \int \partial_j' (v_j G)\, dr - \int v_j(r') \partial_j' G\, dr' = + \int v_j(r') \partial_j G\, dr$$

$\overset{\shortparallel}{0}$ → ~~dann~~ ist div_j S. v. Gauß anwenden! ↖ $\partial_j G = -\partial_j' G$

$$\textcircled{=}\ \partial_i \int v_j(r') (\partial_j G)\, dr' - \partial_j \int v_j(r') \partial_i G\, dr' + \partial_j \int v_i(r')(\partial_j G)\, dr' \quad \textcircled{=}$$

Die part. Abl. darf man ins Integral ziehen; dort gilt

$$\partial_i [v_j(r')(\partial_j G)] = \underbrace{(\partial_i v_j(r'))(\partial_j G)}_{0 \text{ da } v_j(r'),\ \text{nicht } v_j(r)!} + v_j(r')\, \partial_i \partial_j G$$

$$\textcircled{=}\ \underbrace{\int v_j(r')(\partial_i\partial_j G)\, dr' - \int v_j(r')(\partial_j \partial_i G)\, dr'}_{0 \text{ da } \partial_i\partial_j = \partial_j\partial_i} + \int v_i(r')\, \underbrace{(\partial_j\partial_j G)}_{\Delta G}\, dr'$$

$$= \int v_i(r') \cdot \delta(r - r')\, dr' = v_i(r) \qquad \square$$