Elektrodynamik
Uebung 06

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☒ Lemma: vgl Jackson... Kap. 5.6

Für $f(\underline{r}')$, $g(\underline{r}')$; ~~$\underline{j}(\underline{r}')$~~

$\underline{j}(\underline{r}')$ quellenfrei, nur auf endl. Gebiet def.

$$\int \left(f \cdot \underline{j} \cdot \underline{\nabla}' g + g\, \underline{j} \cdot \underline{\nabla}' f\right) dV = 0 \qquad (*)$$

Denn:

$$\int f\, \underline{j} \underline{\nabla}' g\, dV = \int f\, j^\alpha \partial'_\alpha (g)\, dV \underset{\text{p.I.}}{=} f j^\alpha g\Big|_\partial - \int g \cdot \partial'_\alpha (f j^\alpha)\, dV$$

$$= \int g\, \underbrace{\underline{\nabla}'(f \underline{j})}_{= (\underline{\nabla}' f)\underline{j} + f \cdot \underbrace{(\underline{\nabla} \cdot \underline{j})}_{0 \text{ in Magnetostatik}}}\, dV$$

$$\int g\, \underline{j} \underline{\nabla}' f\, dV = f j^\alpha g\Big|_\partial - \int f (\underline{\nabla}' g) \underline{j}\, dV$$

Die Integrale verschwinden weil $\underline{j}$ auf dem „Rand" nicht mehr def ist (bzw. verschwindet):

$$\int * \, dV = -\int g(\underline{\nabla}' f)\underline{j} + f(\underline{\nabla}' g)\underline{j}\, dV$$

$$= -\int \underline{j} \cdot \underline{\nabla}'(gf)\, dV = -\underbrace{\underline{j} \cdot gf\Big|_\partial}_{0} + \int gf \cdot \underbrace{\underline{\nabla} \cdot \underline{j}}_{0}\, dV = 0$$

(a) $$\frac{1}{\|\underline{r} - \underline{r}'\|} \underset{\text{Taylor}}{=} \frac{1}{r} + \frac{\underline{r}' \cdot \underline{r}}{r^3} - \frac{r'^2}{r^3} + \frac{3(\underline{r} \cdot \underline{r}')^2}{r^5} - \frac{4\pi}{3} \delta^{(3)}(\underline{r}) \frac{\underline{r}'^2}{2}$$

$$\Rightarrow \underline{A}_1 = \frac{1}{c} \int \frac{\underline{j}(\underline{r}')}{r}\, dV \qquad \underline{A}_2 = \frac{1}{c} \int \frac{\underline{j}(\underline{r}')(\underline{r} \cdot \underline{r}')}{r^3}\, dV$$

(b) $\int \frac{\underline{j}(\underline{r}')}{r}\, dV = 0$ da $\int \underline{j}(\underline{r}')\, dV = 0$ und über $\frac{1}{r}$ nicht integriert wird; wähle für das Lemma $f = 1$, $g = r_i$:

$$\int \cancel{1}\, \underline{j}\, \underbrace{\underline{\nabla}' r_i}_{\delta e_i} + \cancel{r_i\, \underline{j} \cdot \underline{\nabla}\, 1}\, dV = \int j^i\, dV \underset{\text{Lemma}}{=} 0$$

(c) Das Lemma mit $f = x^\alpha$, $g = x^\beta$ liefert $(x = \underline{r}')$

$$\int x^\alpha j^\mu \overbrace{\partial_\mu x^\beta}^{\delta_{\mu\beta}} + x^\beta j^\nu \overbrace{\partial_\nu x^\alpha}^{\delta_{\nu\alpha}}\, dV = \int x^\alpha j^\beta + x^\beta j^\alpha\, dV;$$

damit ist der zweite Entwicklungsterm

$$\frac{1}{c}\int \frac{(\underline{r}'\cdot\underline{r})}{r^3}\,\underline{j}\,dV = \frac{1}{c}\int \frac{r^\alpha}{r^3} r'^\alpha j^\beta\, dV = \frac{1}{c}\frac{r^\alpha}{r^3}\int r'^\alpha j^\beta\, dV =$$

$$+\frac{1}{2}\frac{r^\alpha}{r^3}\int r'^\alpha j^\beta - r'^\beta j^\alpha\, dV = \frac{1}{2}\frac{1}{r^3}\int (\underline{r}'\underline{r})\underline{j} - \underline{r}'(\underline{r}\underline{j})\, dV =$$

Mit BAC-CAB: $\underline{a}\times(\underline{b}\times\underline{c}) = \underline{b}(\underline{a}\cdot\underline{c}) - \underline{c}(\underline{a}\cdot\underline{b})$ ~~und~~ $\Rightarrow$

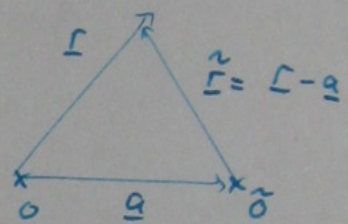
$$\underline{c}(\underline{a}\cdot\underline{b}) = \underline{b}(\underline{a}\underline{c}) - \underline{a}\times(\underline{b}\times\underline{c})$$
$$\underline{j}(\underline{r}\cdot\underline{r}') = \underline{r}'(\underline{r}\cdot\underline{j}) - \underline{r}\times(\underline{r}'\times\underline{j})$$

$$= -\frac{1}{2}\frac{1}{c}\frac{1}{r^3}\int \underline{r}\times(\underline{r}'\times\underline{j})\,dV \underset{\uparrow}{=} \underbrace{\frac{1}{2c}\int\underline{r}'\times\underline{j}\,dV}_{\underline{m}} \times \frac{\underline{r}}{r^3}. \qquad \square$$

Schiefsym. v. $\times$

(c'') Für $\underline{B}_2$ siehe Aufg. 2.

(d) Die Stromdichte transformiert sich mit



$$\underline{j}(\underline{r}) = \underline{j}(\tilde{\underline{r}}+\underline{a}) =: \tilde{\underline{j}}(\tilde{\underline{r}}).$$

$$\tilde{\underline{m}} = \frac{1}{2c}\int\tilde{\underline{r}}\times\tilde{\underline{j}}(\tilde{r})\,d\tilde{V} = \frac{1}{2c}\int(\underline{r}-\underline{a})\times\underline{j}(\underline{r})\,dV$$

$$= \underbrace{\frac{1}{2c}\int\underline{r}\times\underline{j}(\underline{r})\,dV}_{\underline{m}} - \underbrace{\frac{1}{2c}\,\underline{a}\times\int\underline{j}(\underline{r})\,dV}_{0}$$

$d\tilde{V} = dV$ da $\underline{\underline{J}}(\underline{r}-\underline{a}) = \mathbb{1}$

Der zweite Term verschwindet nach Lemma; vgl. dazu (b).

(e) Die Schleife liege in $z=0$ Ebene; verw. Zylinderkoord.;
$\underline{r} = r\underline{e}_r$ wobei $r = r(\varphi)$ die Schleife berandet.
$\underline{j}$ soll stets tangential liegen, $j = \|\underline{j}\|$ ist überall gleich groß:
$\underline{j} = j_\varphi\underline{e}_\varphi + j_r\underline{e}_r$ der j_r-Anteil fällt beim $\times$-Prod. weg.

Da kein Strom weggelangen soll ist $j_\varphi = J$.

$$\underline{m} = \frac{1}{2c}\int r\underline{e}_r\times j_\varphi\underline{e}_\varphi \underbrace{r}_{\text{Jacobian}}\,dr\,d\varphi\,\underbrace{\delta(z)\,dz}_{1} \underset{\uparrow}{=} \frac{1}{2c}\,j_\varphi\underline{e}_z\int_{\partial U} r^2\,dr\wedge d\varphi =$$

da $\underline{e}_r\times\underline{e}_\varphi = \underline{e}_z$

~~U ist Parametergebiet der Fläche der Leiterschleife, mit S.v.St. ist~~ ~~$\int_U r^2\,dr\wedge d\varphi = \int_{\partial U} r^3\,d\varphi$, da $d(r^3\,d\varphi) =$~~

Eine Stammfkt. von $r^2\,dr\wedge d\varphi$ ist $\frac{1}{3}r^3\,d\varphi$: $d(\frac{1}{3}r^3\,d\varphi) = r^2\,dr\wedge d\varphi$, damit ist die Integration lokal exakt und somit kann man (→ Homotopieinvariant) über einen Kreis mit Rad. R integrieren:

$$= \frac{1}{2c}\,j_\varphi\underline{e}_z\int\delta(r-R)\,r^2\,dr\,d\varphi = \frac{1}{2c}\,j_\varphi\underline{e}_z\,R^2\cdot 2\pi = \frac{j_\varphi\underline{e}_z}{c}A.$$

$\boxed{2}$

(a)

$$\underline{E}_z = \left[\underline{\nabla}\left(\underline{p} \left(\underline{\nabla} \frac{1}{r} \right) \right) \right]_\alpha = \partial_\alpha \left[p_\beta \left(\partial_\beta \frac{1}{r} \right) \right] = p_\beta \left(\partial_\alpha \partial_\beta \frac{1}{r} \right)$$

$$\underset{\uparrow}{=} p_\beta \left(- \frac{\delta_{\alpha\beta}}{r^3} + 3 \frac{x_\alpha x_\beta}{r^5} - \frac{4}{3}\pi \delta_{\alpha\beta} \delta(\underline{r}) \right) = - \frac{\underline{p}}{r^3} + 3 \frac{(\underline{p}\,\underline{r}) \cdot \underline{r}}{r^5} - \frac{4\pi}{3} \underline{p}\, \delta(\underline{r})$$

(8) $\boxed{4}$

$\boxed{\underline{p}, \underline{m} \text{ sind konst.}}$

$$\underline{B}_z = \left[- \operatorname{rot}\left(\underline{m} \times \underline{\nabla} \frac{1}{r} \right) \right]_\alpha = - \varepsilon_{\alpha\beta\gamma}\, \partial_\beta\, \varepsilon_{\gamma\mu\nu}\, m_\mu \left(\partial_\nu \frac{1}{r} \right)$$

$$= + \underbrace{\varepsilon_{\alpha\beta\gamma}\, \varepsilon_{\gamma\mu\nu}} \left(- \frac{\delta_{\beta\nu}}{r^3} + 3 \frac{x_\beta x_\nu}{r^5} - \frac{4\pi}{3} \delta_{\beta\nu}\, \delta^{(3)}(\underline{r}) \right) m_\mu$$

$$+ (\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu})$$

$$= m_\alpha \left(\frac{\overset{3}{\delta_{\beta\beta}}}{r^3} - 3 \frac{\overset{r^2}{x_\beta x_\beta}}{r^5} + \frac{4\pi}{3} \underbrace{\delta_{\beta\beta}}_{3} \delta^{(3)}(\underline{r}) \right) + m_\mu \left(\frac{\delta_{\mu\alpha}}{r^3} + 3 \frac{x_\alpha x_\mu}{r^5} - \frac{4\pi}{3} \delta_{\alpha\mu} \delta^{(3)}(\underline{r}) \right)$$

$$= \cancel{\frac{3\underline{m}}{r^3}} - 3 \cancel{\frac{\underline{m}}{r^3}} + \overset{\frac{2}{3}\cdot 4\pi}{\cancel{4\pi}}\, \underline{m}\, \delta^{(3)}(\underline{r}) - \frac{\underline{m}}{r^3} + 3 \frac{(\underline{r}\,\underline{m})\,\underline{r}}{r^5} - \cancel{\frac{4\pi}{3} \underline{m}\, \delta^{(3)}(\underline{r})}$$

$\square$

(b) Unterschiede für $\underline{r} = 0$: $\underline{E}$ hat $\downarrow$, $\underline{B}$ $\uparrow$

$\underline{p}$ bzw. $\underline{m}$: z-Richtung: $\underline{p} = \underline{e}_z$

$$\underline{E}(\underline{r}) = \frac{3 z\, \underline{r}}{r^5} - \frac{\underline{e}_z}{r^3} - \frac{4\pi}{3} \delta(r)\, \underline{e}_z$$

(c) $\underline{B}_z$ hat keine Quellen, weil der δ-Term in die falsche Richtung zeigt: $-\frac{\underline{m}}{r^3}$ zeigt nach „unten", $\frac{8\pi}{3}\delta\, \underline{m}$ aber nach oben. Eine Quelle wie bei $\underline{E}_z$ müsste richtig zeigen, also auch nach „unten".

$\boxed{B}$ $\underline{p}_A, \underline{p}_B$ bei $\underline{R}_A, \underline{R}_B$ $\underline{K}_{el}^{Dipol} = (\underline{p}\cdot\underline{\nabla})\underline{E}_{el,A}$

$\underline{N}_{el}^{Dipol} = \underline{p}\times\underline{E}_{el}$

$$\underline{K}_{el}^{Dipol} = (\underline{p}_B\cdot\underline{\nabla})\left.\frac{3(\underline{p}_A\cdot(\underline{r}-\underline{R}_A))(\underline{r}-\underline{R}_A)}{\|\underline{r}-\underline{R}_A\|^5} - \frac{\underline{p}_A}{\|\underline{r}-\underline{R}_A\|^3} - \cancel{\frac{4\pi}{3}\delta(\underline{r}-\underline{R}_A)\underline{p}_A}\right|_{\underline{r}=\underline{R}_B}$$

$$[\ ]_{\gamma} = p_B^{\alpha}\partial^{\alpha}\left(3\, p_A^{\beta}(r^{\beta}-R_A^{\beta})(r^{\gamma}-R_A^{\gamma})\left[(r^{\delta}-R_A^{\delta})(r^{\delta}-R_A^{\delta})\right]^{-5/2} - p_A^{\gamma}\left[(r^{\delta}-R_A^{\delta})(r^{\delta}-R_A^{\delta})\right]^{-3/2}\right)$$

$$= p_B^{\alpha}\partial^{\alpha}\Big(3\big(p_A^{\beta}r^{\beta}r^{\gamma} - p_A^{\beta}r^{\beta}R_A^{\gamma} - p_A^{\beta}R_A^{\beta}r^{\gamma} + \underbrace{p_A^{\beta}R_A^{\beta}R_A^{\gamma}}_{\partial^{\alpha}=0}\big)\left[r^{\delta}r^{\delta} - 2r^{\delta}R_A^{\delta} - R_A^{\delta}R_A^{\delta}\right]^{-5/2} - p_A^{\gamma}[\ \cdot\ ]^{-3/2}$$

$$= \left[3(\underline{p}_B\underline{p}_A)r^{\gamma} + 3(\underline{p}_A\underline{r})p_B^{\gamma} - 3p_B^{\gamma}p_A^{\beta}R_A^{\beta} - 3p_B^{\gamma}(\underline{p}_A\underline{R}_A)\right]\frac{1}{\|\underline{r}-\underline{R}_A\|^5} - 5\frac{1}{\|\underline{r}-\underline{R}_A\|^7}(r^{\alpha}-R_A^{\alpha})p_B^{\alpha}$$

$$\left(3[\underline{p}_A\cdot(\underline{r}-\underline{R}_A)](\underline{r}-\underline{R}_A)\right) + \frac{3}{3}p_A^{\alpha}p_B^{\alpha}(r^{\alpha}-R_A^{\alpha})\frac{1}{\|\underline{r}-\underline{R}_A\|^5} \ominus \Big|_{\underline{r}=\underline{R}_B}$$

Aber hier $\frac{1}{r^3}$

$$\partial^{\alpha}(r^{\delta}r^{\delta}+R_A^{\delta}R_A^{\delta}-2r^{\delta}R_A^{\delta})^{-k/2} = -\frac{k}{2}(\ \cdot\ )^{-\frac{k+2}{2}}\cdot(2r^{\alpha}-2R_A^{\alpha})$$

$\underline{\Delta} = \underline{R}_B - \underline{R}_A$

$\ominus$

$$\left[3(\underline{p}_A\cdot\underline{p}_B)\underline{R}_B + 3(\underline{p}_A\underline{R}_B)\underline{p}_B - 3(\underline{p}_A\cdot\underline{p}_B)\underline{R}_A - 3(\underline{p}_A\cdot\underline{R}_A)\underline{p}_B\right]\frac{1}{\|\underline{\Delta}\|^5}$$

$$-5\frac{1}{\|\underline{\Delta}\|^7}(\underline{\Delta}\cdot\underline{p}_B)\,3(\underline{p}_A\cdot\underline{\Delta})\underline{\Delta} + 3(\underline{p}_B\cdot\underline{\Delta})\underline{p}_A\frac{1}{\|\underline{\Delta}\|^5}$$

$$\underline{K}_{el} = \frac{3(\underline{p}_A\underline{p}_B)\underline{\Delta} + 3(\underline{p}_A\underline{\Delta})\underline{p}_B + 3(\underline{p}_B\cdot\underline{\Delta})\underline{p}_A}{\|\underline{\Delta}\|^5} - \frac{15(\underline{\Delta}\underline{p}_B)(\underline{\Delta}\underline{p}_A)\underline{\Delta}}{\|\underline{\Delta}\|^7}$$

$$\underline{M}_{el} = \underline{p}_B\times\underline{E}_{el} = \frac{3(\underline{p}_A\underline{\Delta})\,\underline{p}_B\times\underline{\Delta} - \|\underline{\Delta}\|^2(\underline{p}_B\times\underline{p}_A)}{\|\underline{\Delta}\|^5}$$

keine Unterschiede da der δ-Term verschwindet