Laag II ③

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$\boxed{13}$ $Q(c)$ bz: $2x_1x_2 + cx_3^2 + 2(c-1)x_3 = 0$ $\Leftrightarrow$

- In Matrizenform: $\underline{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\underline{\underline{A}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}$; $\underline{b} = \begin{pmatrix} 0 \\ 0 \\ 2(c-1) \end{pmatrix}$

$$\underline{v}^T \cdot \underline{\underline{A}} \cdot \underline{v} + \underline{b}^T \cdot \underline{v} = 0$$

- Eigenwerte v. $\underline{\underline{A}}$:

$$\chi(\lambda) = \det \begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & c-\lambda \end{pmatrix} = \lambda^2(c-\lambda) - (c-\lambda) = 0$$

$\Rightarrow$ EW. $\lambda_1 = 1$; $\lambda_2 = -1$; $\lambda_3 = c$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{matrix} a = b \\ c = 0 \end{matrix} \Rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \underline{v}_1' = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \cdot \begin{pmatrix} r \\ s \\ t \end{pmatrix} = -\begin{pmatrix} r \\ s \\ t \end{pmatrix} \Rightarrow \begin{matrix} r = -s \\ t = 0 \end{matrix} \Rightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \underline{v}_2' = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \cdot \begin{pmatrix} r \\ s \\ t \end{pmatrix} = c \begin{pmatrix} r \\ s \\ t \end{pmatrix} \Rightarrow \begin{matrix} r = s = 0 \\ t = 1 \end{matrix} \Rightarrow \underline{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \underline{v}_3' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Transformationsmatrix: $\underline{\underline{B}}^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \in SO(3)$ $\underline{\underline{B}} = \underline{\underline{B}}^T = \underline{\underline{B}}^{-1}$

$$\underline{v}^T \cdot \underline{\underline{A}} \cdot \underline{v} + \underline{b}^T \cdot \underline{v} =$$

$$\underline{v}^T \cdot \underline{\underline{B}}^T \underline{\underline{B}} \underline{\underline{A}} \underline{\underline{B}}^T \underline{\underline{B}} \cdot \underline{v} + \underline{b}^T \underline{\underline{B}}^T \cdot \underline{\underline{B}} \underline{v} = \qquad \text{sei } \underline{w} = \begin{pmatrix} \xi \\ \eta \\ \rho \end{pmatrix} = \underline{\underline{B}} \underline{v}$$

$$\underline{w}^T \cdot \hat{\underline{\underline{A}}} \cdot \underline{w} + \tilde{\underline{b}}^T \cdot \underline{w} = \qquad \text{mit } \tilde{A} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}, \tilde{\underline{b}} = \underline{\underline{B}} \underline{b}$$

$$(\xi\ \eta\ \rho) \cdot \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \rho \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2(c-1) \end{pmatrix}^T \cdot \begin{pmatrix} \xi \\ \eta \\ \rho \end{pmatrix} =$$

$$\xi^2 - \eta^2 + c\rho^2 - 2(c-1)\rho = 0 \qquad \text{(Rotierte Quadrik)}$$

- Quadrat. Ergänzen:

$$\xi^2 - \eta^2 + \left(\frac{(c-1)^2}{c} - 2(c-1)\rho + c\rho^2\right) - \frac{(c-1)^2}{c} =$$

$$\xi^2 - \eta^2 + \Big(\underbrace{\sqrt{c}\rho - \tfrac{c-1}{\sqrt{c}}}\Big)^2 - \frac{(c-1)^2}{c} = 0 \qquad \left[\sqrt{c}\rho - \frac{c-1}{\sqrt{c}} = \tau\right]$$

$$\xi^2 - \eta^2 + \tau^2 - \frac{(c-1)^2}{c} = 0$$

$$\xi^2 - \eta^2 + \tau^2 = \frac{(c-1)^2}{c} \qquad \text{(Rotiert \& verschoben)}$$

- $c > 0$, $c \neq 1$: Einschal. Hyperboloid
- $c = 1$ Ellipsenkegel
- $c < 0$ Zweischal. Hyperboloid
- $c = 0$

$$Q(0): 2x_1x_2 - 2x_3 = 0$$

$$\underline{v}^T \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underline{v} + \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}^T \cdot \underline{v} = 0$$

$$\underline{\underline{B}}^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{v}^T \cdot B^T \cdot B \cdot A \cdot B^T \cdot B \cdot \underline{v} + \underline{b}^T B^T B \underline{v} =$$

$$\underline{w}^T \quad \tilde{A} \quad \cdot \underline{w} \quad + \quad \tilde{\underline{b}} \cdot \underline{w}$$

$$\underline{w} = \begin{pmatrix} \xi \\ \eta \\ \rho \end{pmatrix} \; , \quad \tilde{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; , \quad \tilde{\underline{b}} = \underline{\underline{B}}\, \underline{b} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$

$$-\xi^2 + \eta^2 - 2\rho = 0$$

$$-\xi^2 + \eta^2 = 2\rho$$

Hyperbolisches Paraboloid

Aufgabe 14

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(2) $\langle x+iy, x+iy\rangle_{\mathbb{C}} = \langle x,x\rangle + \langle y,y\rangle + i(\underbrace{\langle y,x\rangle - \langle x,y\rangle}_{=0})$

$= \langle x,x\rangle + \langle y,y\rangle$

≥ 0, da $\langle x,x\rangle \geq 0$ und $\langle y,y\rangle \geq 0$

$\langle x+iy, x+iy\rangle_{\mathbb{C}} = 0 \iff \underbrace{\langle x,x\rangle}_{\geq 0} + \underbrace{\langle y,y\rangle}_{\geq 0} = 0 \iff x=0 \wedge y=0$

$\iff x+iy = 0$

$\overline{\langle x+iy, z+iw\rangle_{\mathbb{C}}} = \overline{\langle x,z\rangle + \langle y,w\rangle + i(\langle y,z\rangle - \langle x,w\rangle)}$

$= \langle x,z\rangle + \langle y,w\rangle - i(\langle y,z\rangle - \langle x,w\rangle)$

$= \langle z,x\rangle + \langle w,y\rangle + i(\langle w,x\rangle - \langle z,y\rangle)$

$= \langle z+iw, x+iy\rangle_{\mathbb{C}}$

$\langle(\alpha+i\beta)(x+iy), z+iw\rangle_{\mathbb{C}} = \langle(\alpha x-\beta y)+i(\alpha y+\beta x), z+iw\rangle$

$= \langle \alpha x-\beta y, z\rangle + \langle \alpha y+\beta x, w\rangle + i(\langle \alpha y+\beta x, z\rangle - \langle \alpha x-\beta y, w\rangle)$

$= \alpha\langle x,z\rangle - \beta\langle y,z\rangle + \alpha\langle y,w\rangle + \beta\langle x,w\rangle + i(\alpha\langle y,z\rangle + \beta\langle x,z\rangle - \alpha\langle x,w\rangle + \beta\langle y,w\rangle)$

$= (\alpha+i\beta)\langle x,z\rangle + (\alpha+i\beta)\langle y,w\rangle + i((\alpha+i\beta)\langle y,z\rangle - (\alpha+i\beta)\langle x,w\rangle)$

$= (\alpha+i\beta)\langle x+iy, z+iw\rangle_{\mathbb{C}}$

$\langle x+iy+u+iv, z+iw\rangle_{\mathbb{C}} = \langle (x+u)+i(y+v), z+iw\rangle$

$= \langle x+u, z\rangle + \langle y+v, w\rangle + i(\langle y+v, z\rangle - \langle x+u, w\rangle)$

$= \langle x,z\rangle + \langle u,z\rangle + \langle y,w\rangle + \langle v,w\rangle + i(\langle y,z\rangle + \langle v,z\rangle - \langle x,w\rangle - \langle u,w\rangle)$

$= \langle x+iy, z+iw\rangle_{\mathbb{C}} + \langle u+iv, z+iw\rangle_{\mathbb{C}}$

(a)
$$
\begin{aligned}
f_{\mathbb{C}}((\alpha+i\beta)(x+iy)) &= f_{\mathbb{C}}((\alpha x-\beta y)+i(\alpha y+\beta x)) \\
&= f(\alpha x-\beta y)+if(\alpha y+\beta x) \\
&= \alpha f(x)-\beta f(y)+i\alpha f(y)+i\beta f(x) \\
&= \alpha\,(f(x)+if(y))+i\beta\,(f(x)+if(y)) \\
&= (\alpha+i\beta)(f_{\mathbb{C}}(x+iy))
\end{aligned}
$$

$$
\begin{aligned}
f_{\mathbb{C}}(x+iy+u+iv) &= f_{\mathbb{C}}(x+u+i(y+v)) \\
&= f(x+u)+if(y+v) \\
&= f(x)+f(u)+if(y)+if(v) \\
&= f_{\mathbb{C}}(x+iy)+f_{\mathbb{C}}(u+iv)
\end{aligned}
$$

(b)
$$
\begin{aligned}
\|x+iy\|_{\mathbb{C}}^2 = \langle x+iy, x+iy\rangle_{\mathbb{C}} &= \langle x,x\rangle+\langle y,y\rangle+0 \\
&= \|x\|^2+\|y\|^2
\end{aligned}
$$

(c)
$$
\begin{aligned}
\langle f_{\mathbb{C}}(x+iy), z+iw\rangle_{\mathbb{C}} &= \langle f(x)+if(y), z+iw\rangle_{\mathbb{C}} \\
&= \langle f(x),z\rangle+\langle f(y),w\rangle+i\,(\langle f(y),z\rangle-\langle f(x),w\rangle) \\
&= \langle x,f(z)\rangle+\langle y,f(w)\rangle+i\,(\langle y,f(z)\rangle-\langle x,f(w)\rangle) \\
&= \langle x+iy, f(z)+if(w)\rangle_{\mathbb{C}} \\
&= \langle x+iy, f_{\mathbb{C}}(z+iw)\rangle_{\mathbb{C}}
\end{aligned}
$$

(d) $(\mathbb{R}^n)_{\mathbb{C}} \cong \mathbb{C}^n$, da es eine Bijektion $f: (\mathbb{R}^n)_{\mathbb{C}} \to \mathbb{C}^n$ gibt.

$$
\begin{array}{ccccc}
f: \mathbb{R}^n & \oplus & i\mathbb{R}^n & \longrightarrow & \mathbb{C}^n \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
x & + & iy & \longmapsto & \overbrace{x+iy}
\end{array}
$$

Das kanonische Skalarprodukt im $\mathbb{C}^n$ ist: $\langle x+iy, z+iw\rangle = \sum_{i=1}^{n} (x_i+iy_i)(z_i-iw_i)$

$$
\begin{aligned}
&= \sum_{i=1}^{n} x_i z_i + y_i w_i + i(y_i z_i - x_i w_i) \\
&= \sum_{i=1}^{n} x_i z_i + \sum_{i=1}^{n} y_i w_i + i\left(\sum_{i=1}^{n} y_i z_i - \sum_{i=1}^{n} x_i w_i\right) \\
&= \langle x|z\rangle + \langle y|w\rangle + i\,(\langle y|z\rangle - \langle x|w\rangle)
\end{aligned}
$$

$\Rightarrow$ Es ist identisch mit dem komplexifizierten Skalarprodukt.