Analysis IV
Uebung 09

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Ana 9
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$\boxed{1}$

(a) $f(z) = \frac{e^{izA}}{z}$ Polst. bei $z=0$ mit $\mathrm{Res}_0 f = \lim_{z\to 0} \frac{z e^{izA}}{z} = 1$

$\int_{U_\gamma} f(z)dz \underset{\uparrow}{=} \cancel{\ldots} = 0$
Res. Satz, ~~Indukt'lemma~~   Res. Satz

$\left|\int_{\gamma_R} f(z)\,dz\right| \underset{\uparrow}{=} \left|\int_0^\pi \frac{1}{R} e^{-i\varphi} e^{i(Re^{i\varphi})A} R i e^{i\varphi} d\varphi\right| \le$

$z = Re^{i\varphi} = R(\cos\varphi + i\sin\varphi)$

$\int_0^\pi \frac{1}{R} R e^{-R\sin\varphi A} d\varphi \xrightarrow{R\to\infty} 0$ da $\varphi \in (0,\pi) \Rightarrow \sin\varphi > 0$

$$\int_{-\infty}^{-0} \frac{e^{izA}}{z} dz + \underbrace{\int_{-0}^{+0} \frac{e^{izA}}{z} dz}_{\text{Indukt'lemma}} + \int_{+0}^{+\infty} \frac{e^{izA}}{z} dz = 0$$

$z \mapsto -z$

$\int_{+\infty}^{+0} + \frac{e^{-izA}}{z} dz \qquad -\pi i$

$$\Rightarrow \int_{+0}^{+\infty} -\frac{e^{-izA}}{z} + \frac{e^{izA}}{z}\, dz = +i\pi$$

$e^{i\varphi} = \cos\varphi + i\sin\varphi$

$e^{-i\varphi} = \cos\varphi - i\sin\varphi$

$2\cos\varphi = e^{i\varphi} + e^{-i\varphi}$

$2i\sin\varphi = e^{i\varphi} - e^{-i\varphi}$

$\parallel$

$$\underbrace{\frac{1}{2}\int_0^\infty 2\cancel{i} \frac{\sin zA}{z} dz}_{} = +\cancel{i}\pi \quad \leftarrow \text{Sym.}$$

$$2\frac{1}{2}\int_{-\infty}^{+\infty} \frac{\sin zA}{z} dz = \pi$$

$\square$

(b) $\int_{-\infty}^{+\infty} = \int_{-\infty}^{-N_\varepsilon} + \int_{-N_\varepsilon}^{N_\varepsilon} + \int_{N_\varepsilon}^{+\infty}$

g'

(ii) $\int_{-N_\varepsilon}^{+N_\varepsilon} \frac{f(t_0+x) - f(t_0)}{x} \sin(Ax)\, dx \xrightarrow{A\to\infty} 0$ Mit Lemma von Riemann weil

f mindestens ~~Hölderstetig und~~ Dini-Bed. erfüllt und damit $\int_{-N_\varepsilon}^{+N_\varepsilon} \frac{f(t_0+x) - f(t_0)}{x} dx$

~~...~~ $< \infty$ damit $\frac{f(t_0+x) - f(t_0)}{x} \in L^1$.

(i)

$\boxed{2}$ $S_n(t) = \int_{-\pi}^{\pi} f(x+t)\, D_n(x)\, dx$ $\qquad D_n(x) = \dfrac{\sin[(n+\frac{1}{2})x]}{2\pi \sin(\frac{x}{2})}$

$$\int_0^{\pi} D_n(x)\,dx = \int_{-\pi}^{0} D_n(x)\,dx = \frac{1}{2}$$

$$\left(S_n - \tfrac{1}{2}[f(t+0) + f(t-0)]\right) =$$

$$\left(\int_0^{\pi} [f(x+t) - f(t+0)]\, D_n(x)\,dx + \int_{-\pi}^{0} [f(x+t) - f(t-0)]\, D_n(x)\,dx\right)$$

$$\left(\int_0^{\pi} \frac{f(t+x) - f(t+0)}{x}\, x\, D_n(x)\,dx\right) =$$

$$\left(\int_0^{\pi} \underbrace{\underbrace{\frac{f(t+x) - f(t+0)}{x}}_{\in L^1} \underbrace{\frac{x}{\sin\frac{x}{2}}}_{\in L^1}}_{\tilde{F}(x,t)\, \in L^1} \sin[(n+\tfrac{1}{2})x]\right)$$

$\in L^1$ da $\left|\frac{x}{\sin\frac{x}{2}}\right| \leq \pi$ für $x \in [-\pi, \pi]$

und beschränkt! [Eigentlich ist Beschränktheit hinreichend notwendig, L^1 ist Eingabe...]

$\tilde{F}(x,t) \in L^1$

Wende Lemma von Riemann an. Für $n \to \infty$ werden

(∗). Analog 2. Integral

$\square$

$\boxed{3}$

(a) $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-i\lambda x}\, dx = \hat{f}(\lambda) = \lambda \cdot f(\lambda)$

$\boxed{\lambda \neq 0}$

$\mathcal{F} : f \mapsto \lambda \cdot f$

$f = f^*$ da $f \in \mathbb{R}$

$\left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-i\xi x}\, dx \right)^* = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\lambda} \hat{f}(x)\, e^{i\xi x}\, dx =$

$\mathcal{F}^{-1}\left[\frac{1}{\lambda}\hat{f}\right] = \frac{1}{\lambda} f = \hat{f}^*(x) = (\lambda f)^* = \lambda^* f$

$\Rightarrow \frac{1}{\lambda} = \lambda^*$

$\lambda = r e^{i\varphi} \qquad \lambda^* = r e^{-i\varphi} \qquad \frac{1}{\lambda} = \frac{1}{r} e^{-i\varphi}$

$\Rightarrow \frac{1}{r} e^{-i\varphi} = r e^{-i\varphi} \Rightarrow \boxed{r = 1}$

$\Rightarrow |\lambda| = 1$

(b) $\frac{1}{\sqrt{2\pi}} \int e^{-\frac{x^2}{2}} e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}[\overbrace{x^2 + 2i\xi x + (-\xi^2)}^{(x+i\xi)^2} + \xi^2]}\, dx$

$\stackrel{?}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{\zeta^2}{2}}\, d\zeta \;\ominus$

$\zeta = x + i\xi \quad d\zeta = dx$

$\int_{-\infty}^{+\infty} e^{-\frac{\zeta^2}{2}}\, d\zeta = 2\int_0^{\infty} e^{-\zeta^2/2}\, d\zeta = 2\int_0^{\infty} \frac{1}{\sqrt{2}} \cdot y^{\frac{1}{2}-1}\, dy = \sqrt{2} \cdot \Gamma\left(\frac{1}{2}\right) = \sqrt{2\pi}$

$\zeta^2/2 = y \quad dy = \zeta\, d\zeta \rightarrow d\zeta = \frac{1}{\zeta} dy = \frac{1}{\sqrt{2y}} dy = \frac{1}{\sqrt{2}} \cdot y^{-\frac{1}{2}}$

$\ominus \; \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \cdot \sqrt{2\pi} = e^{-\xi^2/2}$

$\square$

$\boxed{4}$

(a) (i) $f_\alpha > 0$ für $\alpha > 0$.

(ii) $\int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{\alpha}{\alpha^2+x^2} dx = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{1+\left(\frac{x}{\alpha}\right)^2} \frac{\cancel{\alpha}}{\alpha^{\cancel{2}}} \cancel{\alpha}\, d\left(\frac{x}{\alpha}\right) = \frac{1}{\pi} \operatorname{atan} \frac{x}{\alpha}\Big|_{-\infty}^{+\infty} = 1$ ✓

(iii) $\left(\int_{-\infty}^{-\delta} + \int_{\delta}^{+\infty}\right) \frac{1}{\pi} \frac{\alpha}{\alpha^2+x^2} dx < \varepsilon$

$\int_{\delta}^{+\infty} \frac{1}{\pi} \frac{\alpha}{\alpha^2+x^2} dx = \frac{1}{\pi} \operatorname{atan} \frac{x}{\alpha}\Big|_{\delta}^{+\infty} = \frac{1}{2} - \frac{1}{\pi} \operatorname{atan} \frac{\delta}{\alpha}$

~~bei~~ $\lim_{\alpha \to 0}$ darf man mit atan vertauschen, da atan stetig

$\frac{1}{\pi} \operatorname{atan}\left(\lim_{\alpha \to 0} \frac{\delta}{\alpha}\right) = \frac{1}{2}$.

$\downarrow$

0 wg. Symmetrie gilt auch $\int_{-\infty}^{-\varepsilon} \to 0$ $\square$

(b) (i) $g_\alpha > 0$ da $e^{\cdot} > 0$, ...

(ii) $\int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{x^2}{4\alpha}} dx = 2\int_0^{+\infty} \frac{1}{\cancel{2}\sqrt{\pi\alpha}} e^{-\frac{x^2}{4\alpha}} dx \ominus$

$\cancel{\xi = x^2 \quad d\xi = 2x\,dx \Rightarrow dx = \frac{1}{2}\frac{1}{x} d\xi = \frac{1}{2} \xi^{-1/2} d\xi = \frac{1}{2} \xi^{\ldots}}$

$\xi = \frac{x^2}{4\alpha} \Rightarrow d\xi = \frac{2x}{4\alpha} dx = \frac{1}{2\alpha}(4\alpha\xi)^{\frac{1}{2}} dx = \frac{1}{\sqrt{\alpha}} \cdot \xi^{-\left(\frac{1}{2}-1\right)} dx$

$\Updownarrow$

$(4\alpha\xi)^{\frac{1}{2}} = x$

$\ominus \int_0^{+\infty} \frac{1}{\sqrt{\pi\alpha}} \cdot \frac{1}{\sqrt{\alpha}} \cdot \xi^{\frac{1}{2}-1} e^{-\xi} d\xi = \frac{1}{\sqrt{\pi}} \cdot \Gamma\left(\frac{1}{2}\right) = \underline{1}$

(iii) Wieder Symmetrie beachten!

$\left(\int_{\delta}^{+\infty} \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{x^2}{4\alpha}} dx\right)^2 = \frac{1}{4\pi\alpha} \int_{\delta}^{+\infty} e^{-\frac{x^2}{4\alpha}} dx \cdot \int_{\delta}^{+\infty} e^{-\frac{y^2}{4\alpha}} dy =$

$\frac{1}{4\pi\alpha} \int_{\delta}^{+\infty}\int_{\delta}^{+\infty} e^{-\left(\frac{x^2}{4\alpha}+\frac{y^2}{4\alpha}\right)} \underbrace{dx\,dy}_{\text{Tonelli} \to\ d(x,y)} = \frac{1}{4\pi\alpha} \int_{\delta}^{+\infty} dr \underbrace{\int_0^{2\pi/4} d\varphi}_{2\pi/4}\ \underbrace{r}_{\text{Jacobian}} e^{-\frac{r^2}{4\alpha}}$

$= \cancel{\frac{1}{8\alpha}\left[\ldots\right]} = \frac{1}{8\alpha}\left[-\frac{4\alpha}{2} e^{-\frac{r^2}{4\alpha}}\right]_{\delta}^{\infty} = \frac{1}{4}\frac{2\alpha}{8\alpha} e^{-\frac{\delta^2}{4\alpha}}$

für $\lim_{\alpha \to 0}$ geht das gegen 0 $\Rightarrow \int_{\delta}^{+\infty} \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{x^2}{4\alpha}} dx \xrightarrow{\alpha \to 0} 0$. $\square$