Quantum Gates and Circuits

Prakashan Korambath
OARC, UCLA

Quantum Computing

Kets (Dirac Notation) and Vectors

Ket column vectors represent the states of a Quantum system

$|0\rangle$ and $|1\rangle$

Other notation is called Bra notation which is a complex conjugate of Ket vectors

$\langle 0|$ $\langle 1|$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathit{and} \; |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Superposition (qubits can exist in both 0 and 1 states simultaneously) and entanglement (state of one qubit directly affects the state of another) enable quantum computers to perform computations in ways that classical computers cannot for certain problems.

Quantum Bit or Qubit

Qubits can be in either $|0\rangle$ or $|1\rangle$ or linear superposition of both states

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \textit{ where } |\alpha|^2 + |\beta|^2 = 1$$

α and β are amplitudes. Square of their sum is called Quantum probability and should be 1.
The superposition is what makes Quantum computing different from classical computing.
Ground state is represented by $|0\rangle$
Excited state is represented by $|1\rangle$
Qubits in superposed state occupies all states $|0\rangle$ and $|1\rangle$ simultaneously, but collapses into $|0\rangle$ or $|1\rangle$ when an observation is made. When we do a measurement we can only get one answer and not all answers to all states in the superposition state.

Multi Qubit

Two bits have four possible states

$$00 \quad 01 \quad 10 \quad 11$$

The state of a two-qubit system can be represented as

$$\alpha_{00}\,|00\rangle + \alpha_{01}\,|01\rangle + \alpha_{10}\,|10\rangle + \alpha_{11}\,|11\rangle$$

$$p(|00\rangle) = |\langle 00|\alpha\rangle|^2 = |\alpha_{00}|^2$$

$$\Sigma\,|\alpha|^2 = 1$$

Qubit Basis set

Two orthogonal x-basis sets are: (Eigenstate σ_x)

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Two orthogonal y-basis sets are: (Eigenstate σ_y)

$$|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$$

$$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$$

Measurement

Measure the Qubit state below in the bases $(|0\rangle, |1\rangle)$

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}\,|1\rangle)$$

$$P(0) = |\langle 0| \frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}\,|1\rangle)|^2 = |\frac{1}{\sqrt{3}}\langle 00\rangle + \frac{\sqrt{2}}{\sqrt{3}}\langle 01\rangle|^2 = \frac{1}{3}$$

$$P(1) = |\langle 1| \frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}\,|1\rangle)|^2 = |\frac{1}{\sqrt{3}}\langle 10\rangle + \frac{\sqrt{2}}{\sqrt{3}}\langle 11\rangle|^2 = \frac{2}{3}$$

Measure the Qubit state below in the bases $(|+\rangle, |-\rangle)$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$P(+) = |\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|^2 = \frac{1}{4}|\langle 00\rangle - \langle 11\rangle|^2 = 0$$

Measurement Contd.

Measure the Qubit state below in the bases $(|+\rangle, |-\rangle)$

$$|\psi\rangle = \frac{1}{2}(|0\rangle + \frac{\sqrt{3}}{2}|1\rangle)$$

$$P(+) = |\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|\frac{1}{2}(|0\rangle + \frac{\sqrt{3}}{2}|1\rangle)|^2 = \frac{1}{8}|\langle 00\rangle + \sqrt{3}\langle 11\rangle|^2$$

$$P(+) = \frac{(1+\sqrt{3})^2}{8} = \frac{4+2\sqrt{3}}{8}$$

$$P(-) = \frac{4-2\sqrt{3}}{8}$$

Operations on Qubit

Quantum logic gates are represented by unitary matrices

$$U^\dagger U = UU^\dagger = I$$

$$\alpha|0\rangle + \beta|1\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

$$\alpha|0\rangle + \beta|1\rangle = \alpha\begin{bmatrix}1 \\ 0\end{bmatrix} + \beta\begin{bmatrix}0 \\ 1\end{bmatrix} = \begin{bmatrix}\alpha \\ 0\end{bmatrix} + \begin{bmatrix}0 \\ \beta\end{bmatrix} = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

Quantum Gates must be reversible and there is no way to perfectly copy a state. This is called no-cloning theorem.

Quantum Gates: NOT Gate

Quantum equivalent of NOT Gate is also called Pauli X Gate:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

act on a single qubit. Transforms $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$

$$X|0\rangle = |1\rangle \quad \textit{and} \quad X|1\rangle = |0\rangle$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

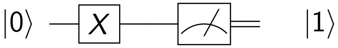
$$X(\alpha|0\rangle + \beta|1\rangle) = X \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \alpha|1\rangle + \beta|0\rangle$$

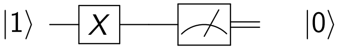
INPUT	OUTPUT
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\alpha\|0\rangle + \beta\|1\rangle$	$\alpha\|1\rangle + \beta\|0\rangle$

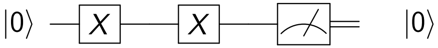
Execution: X Gate

$$X\left|+\right\rangle = \frac{1}{\sqrt{2}}(X\left|0\right\rangle + X\left|1\right\rangle) = \frac{1}{\sqrt{2}}(\left|1\right\rangle + \left|0\right\rangle) = \left|+\right\rangle$$

$$X\left|-\right\rangle = \frac{1}{\sqrt{2}}(X\left|0\right\rangle - X\left|1\right\rangle) = \frac{1}{\sqrt{2}}(\left|1\right\rangle - \left|0\right\rangle) = -\left|-\right\rangle$$







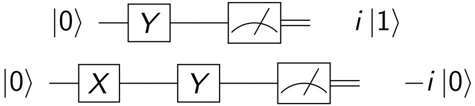
Quantum Gates: Pauli Y Gate

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad and \quad \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

act on a single qubit. Transforms $|0\rangle$ to i$|1\rangle$ and $|1\rangle$ to -i$|0\rangle$

$$Y\,|0\rangle = i\,|1\rangle \quad and \quad Y\,|1\rangle = -i\,|0\rangle$$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix} \quad and \quad \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -i \\ 0 \end{bmatrix}$$



Quantum Gates: Pauli Z Gate

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

act on a single qubit. $|0\rangle$ remains unchanged, but transforms $|1\rangle$ to -$|1\rangle$

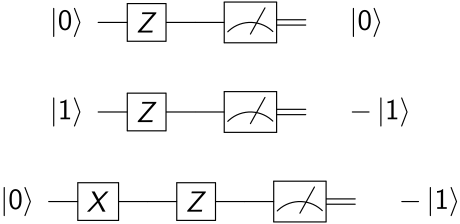
$$Z\,|0\rangle = |0\rangle \quad \textit{and} \quad Z\,|1\rangle = -\,|1\rangle$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$Z(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha Z\,|0\rangle + \beta Z\,|1\rangle = \alpha\,|0\rangle - \beta\,|1\rangle$$

INPUT	OUTPUT
$|0\rangle$	$|0\rangle$
$|1\rangle$	-$|1\rangle$
$\alpha\,|0\rangle + \beta\,|1\rangle$	$\alpha\,|1\rangle - \beta\,|0\rangle$

Execution: Z Gate



Quantum Gates: Hadamard Gate

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

act on a single qubit. Transform $|0\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and $|1\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \;\; \textit{and} \;\; H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right] = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

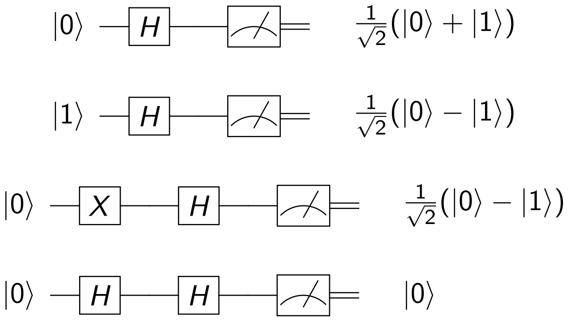
Quantum Gates: Hadamard Gate Contd.

Hadamard gate takes a zero and turns it into an equally weighted superposition of zero and one. This state itself has a name, the plus state. When measured, there is a 50% chance of observing a zero or one. Hadamard apply to the one state creates the minor state, which is still 50% zero and one, but with the one state having a negative amplitude.

$$H(\alpha|0\rangle + \beta|1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha+\beta\\ \alpha-\beta\end{bmatrix}$$

INPUT	OUTPUT
$\|0\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$
$\|1\rangle$	$\frac{1}{\sqrt{2}}\|0\rangle - \|1\rangle$
$\alpha\|0\rangle + \beta\|1\rangle$	$\frac{\alpha}{\sqrt{2}}(\|0\rangle + \|1\rangle) + \frac{\beta}{\sqrt{2}}(\|0\rangle - \|1\rangle)$
	$\frac{\alpha+\beta}{\sqrt{2}}\|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}\|1\rangle$

Execution: Hadamard Gate



Execution: Hadamard Gate Contd.

$$|0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

$$|1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

$$|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

Quantum Gates: Controlled NOT Gate.

This Gate operates on 2 qubits, control qubit and target qubit. If the control qubit is 0, target is unchanged. If the control qubit is 1 then the target is inverted.

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

INPUT	OUTPUT
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

Quantum Gates: Controlled NOT Gate. contd..

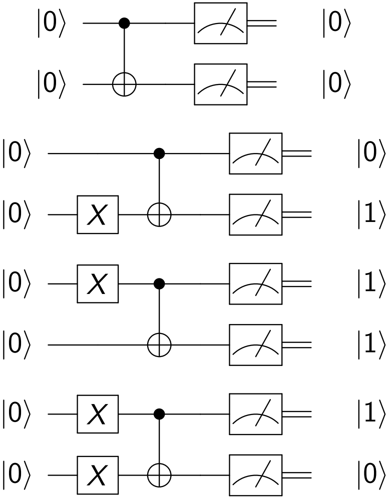
$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|11\rangle = |1\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Execution: CNOT Gate



Tensor Products

Tensor Product of $|0\rangle$ and $|1\rangle$

$$|0\rangle|1\rangle = |01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Tensor Product of $|0\rangle$ and $|+\rangle$

$$|0\rangle|+\rangle = |0+\rangle = |0\rangle \otimes |+\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) =$$
$$\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

Tensor Products Contd.

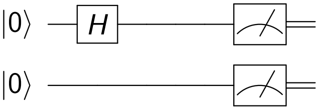
Tensor Product of $|+\rangle$ and $|1\rangle$

$|+\rangle|1\rangle = |+1\rangle = |+\rangle \otimes |1\rangle = \left(\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |1\rangle\right) = \frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right)$

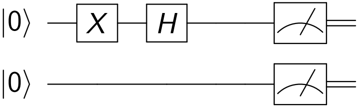
Tensor Product of $|-\rangle$ and $|+\rangle$

$|-\rangle|+\rangle = |-+\rangle = |-\rangle \otimes |+\rangle = \left(\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right) \otimes \left(\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\right) = \left(\frac{1}{2}\left(|00\rangle + |01\rangle - |10\rangle - |11\rangle\right)\right)$

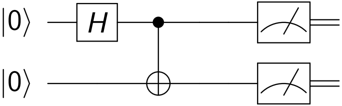
Execution: EPR State and Entanglement



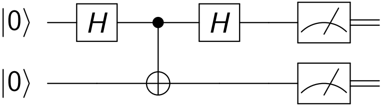
$$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$



$$\frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$$



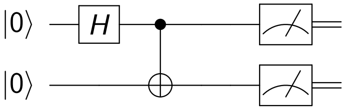
$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$



$$\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) + \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)\right]$$

Entanglement

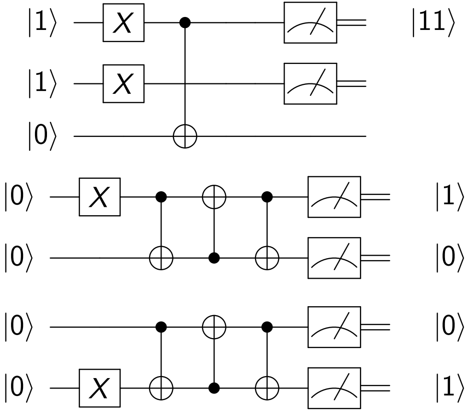
The Bell State (Maximally Entangled State)



$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

This state is an equal superposition of both qubits being in the $|0\rangle$state and both qubits being in the $|1\rangle$ state, and the important thing is that the qubits are entangled. The key feature of this state is that if you measure one qubit, you instantly know the state of the other qubit, no matter how far apart they are.

Execution: AND Gate and SWAP Gate

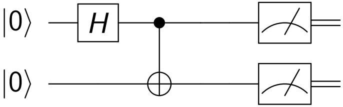


Bell State

Tensor Product of $|+\rangle$ and $|0\rangle$

$$|+\rangle|0\rangle = |+0\rangle = |+\rangle \otimes |0\rangle = \left(\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |0\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right) \xrightarrow{CNOT} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = |\phi^+\rangle$$



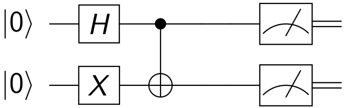
$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Bell State Contd..

Tensor Product of $|+\rangle$ and $|1\rangle$

$$|+\rangle|1\rangle = |+1\rangle = |+\rangle \otimes |1\rangle = \left(\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |1\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right) \xrightarrow{CNOT} = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) = |\psi^+\rangle$$



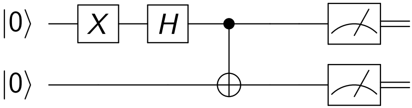
$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

Bell State Contd..

Tensor Product of $|-\rangle$ and $|0\rangle$

$|-\rangle|0\rangle = |-0\rangle = |-\rangle \otimes |0\rangle = \left(\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \otimes |0\rangle\right) =$

$\frac{1}{\sqrt{2}}\left(|00\rangle - |10\rangle\right) \xrightarrow{CNOT} = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) = |\phi^-\rangle$

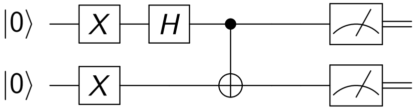


$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

Bell State Contd..

Tensor Product of $|-\rangle$ and $|1\rangle$

$|-\rangle|1\rangle = |-1\rangle = |-\rangle \otimes |1\rangle = \left(\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \otimes |1\rangle\right) =$
$\frac{1}{\sqrt{2}}\left(|01\rangle - |11\rangle\right) \xrightarrow{CNOT} = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) = |\psi^-\rangle$

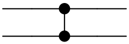


$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle\,;\,\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle\,;\,\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle\,;\,\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle$$

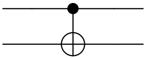
Controlled-Z and Controlled-X

Controlled-Z



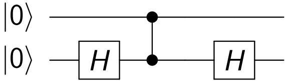
$$\begin{aligned} CZ(1,0)|00\rangle &= |00\rangle \\ CZ(1,0)|01\rangle &= |01\rangle \\ CZ(1,0)|10\rangle &= |10\rangle \\ CZ(1,0)|11\rangle &= -|11\rangle \end{aligned}$$

Controlled-X



$$\begin{aligned} CX(1,0)|00\rangle &= |00\rangle \\ CX1,0)|01\rangle &= |01\rangle \\ CX(1,0)|10\rangle &= |11\rangle \\ CX(1,0)|11\rangle &= |10\rangle \end{aligned}$$

More Gates and Circuits



$$|00\rangle \xrightarrow{H} |0\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

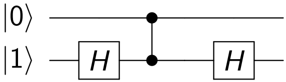
$$= \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

$$\xrightarrow{CZ} \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

$$\xrightarrow{H} \frac{1}{\sqrt{2}}\left(|0\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + (|0\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{2}2(|00\rangle) = |00\rangle$$

More Gates and Circuits



$$|00\rangle \xrightarrow{H} |0\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

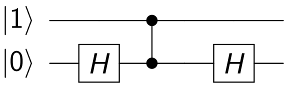
$$= \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle)$$

$$\xrightarrow{CZ} \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle)$$

$$\xrightarrow{H} \frac{1}{\sqrt{2}}\left(|0\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) - (|0\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle))\right)$$

$$= \frac{1}{2}2(|01\rangle) = |01\rangle$$

More Gates and Circuits



$$|10\rangle \xrightarrow{H} |1\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

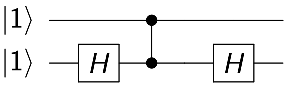
$$= \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle)$$

$$\xrightarrow{CZ} \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)$$

$$\xrightarrow{H} \frac{1}{\sqrt{2}}\left(|1\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) - (|1\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{2}2(|11\rangle) = |11\rangle$$

More Gates and Circuits



$$|10\rangle \xrightarrow{H} |1\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)$$

$$\xrightarrow{CZ} \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle)$$

$$\xrightarrow{H} \frac{1}{\sqrt{2}}\left(|1\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + (|1\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{2}2(|10\rangle) = |10\rangle$$

Gate Operation Summary

$$X|0\rangle = |1\rangle\,;X|1\rangle = |0\rangle\,;X|+\rangle = |+\rangle\,;X|-\rangle = -|-\rangle$$

$$Y|0\rangle = i|1\rangle \quad and \quad Y|1\rangle = -i|0\rangle$$

$$Z|0\rangle = |0\rangle\,;Z|1\rangle = -|1\rangle\,;Z|+\rangle = |-\rangle\,;Z|-\rangle = |+\rangle$$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle\,;H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$

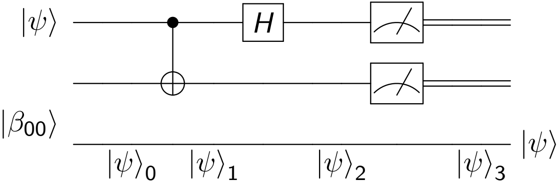
Quantum Teleportation

The state to be transported by Alice is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are unknown amplitudes.
The state input into the circuit $|\psi_0\rangle$ is $|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle$ The entangled state that is known to both Alice and Bob is

$$\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1_A\rangle|1\rangle_B)$$



Quantum circuit for teleporting a qubit. The two top lines represent Alice's system, while the bottom line is Bob's system. The meters represent measurement, and the double lines coming out of them carry classical bits (recall that single lines denote qubits).

Quantum Teleportation contd...

The combined state of three qubits is given by
$\alpha |0\rangle + \beta |1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1_A\rangle |1\rangle_B)$

$$= \frac{1}{\sqrt{2}}\left[\alpha |0\rangle (|0\rangle_A |0\rangle_B + |1_A\rangle |1\rangle_B) + \beta |1\rangle (|0\rangle_A |0\rangle_B + |1_A\rangle |1\rangle_B)\right]$$

After CNOT on the first qubit where we use the convention that the first two qubits (on the left) belong to Alice, and the third qubit to Bob. $CX(1,0)|10\rangle = |11\rangle$ and $CX(1,0)|11\rangle = |01\rangle$

$$= \frac{1}{\sqrt{2}}\left[\alpha |0\rangle (|0\rangle_A |0\rangle_B + |1_A\rangle |1\rangle_B) + \beta |1\rangle (|1\rangle_A |0\rangle_B + |0_A\rangle |1\rangle_B)\right]$$

After Hadamard gate on Alice's first qubit we get .

$$= \frac{1}{2}\left[\alpha(|0\rangle+|1\rangle)(|0\rangle_A |0\rangle_B+|1_A\rangle |1\rangle_B)+\beta(|0\rangle-|1\rangle)(|1\rangle_A |0\rangle_B+|0_A\rangle |1\rangle_B)\right]$$

Quantum Teleportation contd...

$$= \frac{1}{2}\left[\alpha(|00\rangle_A |0\rangle_B + |01\rangle_A |1\rangle_B + |10\rangle_A |0\rangle_B + |11\rangle_A |1\rangle_B\,]\right)$$

$$+\frac{1}{2}\left[\beta(|01\rangle_A |0\rangle_B + |00\rangle_A |1\rangle_B - |10\rangle_A |1\rangle_B) - |11\rangle_A |0\rangle_B)\right]$$

The new state is re-written by regrouping terms as follows:

$$= \frac{1}{2}\left[\,|00\rangle_A (\alpha |0\rangle_B + \beta |1\rangle_B) + |01\rangle_A (\alpha |1\rangle_B + \beta |0\rangle_B)\right.$$

$$\left.+\frac{1}{2}\left[\,|10\rangle_A (\alpha |0\rangle_B - \beta |1\rangle_B) + |11\rangle_A (\alpha |1_B\rangle - \beta |0\rangle_B)\right]\right.$$

The above expression breaks down into four terms. The first term has Alice's qubits in the state $|00\rangle$, and Bob has to do nothing If Alice performs a measurement and obtains the result 00. In that case Bob's qubit is in the state $\alpha |0\rangle + \beta |1\rangle$, which is the original state $|\psi\rangle$. If Bob receives 01 from Alice then Bob has to swap the coeffients around by applying the X gate. If the measurement is 10 then Bob can fix up his state by applying the Z gate. If the measurement is 11 then Bob can fix up his state by applying first an X andthen a Z gate. All four different measurement scenario and operations that Bob has to perform are below.

Quantum Teleportation contd...

$$
\begin{aligned}
00 \rightarrow |\psi_3(00)\rangle &= \alpha |0\rangle_B + \beta |1\rangle_B \\
01 \rightarrow |\psi_3(01)\rangle &= \alpha |1\rangle_B + \beta |0\rangle_B \xrightarrow{X} \alpha |0\rangle_B + \beta |1\rangle_B \\
10 \rightarrow |\psi_3(10)\rangle &= \alpha |0\rangle_B - \beta |1\rangle_B \xrightarrow{Z} \alpha |0\rangle_B + \beta |1\rangle_B \\
11 \rightarrow |\psi_3(11)\rangle &= \alpha |1_B\rangle - \beta |0\rangle_B \xrightarrow{X} \alpha |0\rangle_B - \beta |1\rangle_B \xrightarrow{Z} \\
&= \alpha |0\rangle_B + \beta |1\rangle_B
\end{aligned}
$$

Depending on Alice's measurement outcome, Bob's qubit will end up in one of these four possible states. Of course, to know which state it is in, Bob must be told the result of Alice's measurement which is done through classical communication channel.
Once Bob has learned the measurement outcome, he can 'fix up' his state, recovering $|\psi\rangle$, by applying the appropriate quantum gate operations as explained above. After the teleportation process only the target qubit is left in the state $|\psi\rangle$, and the original data qubit ends up in one of the computational basis states $|0\rangle$ or $|1\rangle$ depending up on the measurement.