Digital Transmission through the Additive White Gaussian Noise Channel

In Chapter 6, we described methods for converting the output of a signal source into a sequence of binary digits. In this chapter, we consider the transmission of the digital information sequence over communication channels that are characterized as *additive white Gaussian noise (AWGN)* channels. The AWGN channel is one of the simplest mathematical models for various physical communication channels, including wirelines and some radio channels. Such channels are basically analog channels, which means that the digital information sequence to be transmitted must be mapped into analog signal waveforms.

We consider both *baseband channels;* i.e., channels having frequency passbands that usually include zero frequency ($f = 0$), and *bandpass channels;* i.e., channels having frequency passbands far removed from $f = 0$. When the digital information is transmitted through a baseband channel, there is no need to use a carrier frequency for transmission of the digitally modulated signals. On the other hand, there are many communication channels, including telephone channels, radio channels, and satellite channels, that are bandpass channels. In such channels, the information-bearing signal is impressed on a sinusoidal carrier, which shifts the frequency content of the information-bearing signal to the appropriate frequency band that is passed by the channel. Thus, the signal is transmitted by carrier modulation.

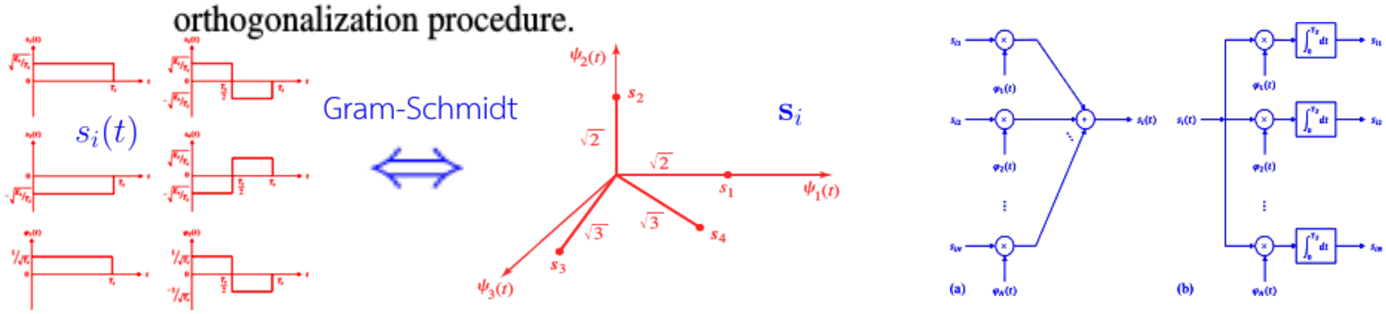
M waveforms ⇔ vector N มิติ

Linear Alg.

7.1 GEOMETRIC REPRESENTATION OF SIGNAL WAVEFORMS

The Gram-Schmidt orthogonalization procedure may be used to construct an orthonormal basis for a set of signals.

Suppose we have a set of M signal waveforms $s_m(t)$, $1 \le m \le M$ which are to be used for transmitting information over a communication channel. From the set of M waveforms, we first construct a set of $N \le M$ orthonormal waveforms, where N is the dimension of the signal space. For this purpose, we use the Gram-Schmidt orthogonalization procedure.



Gram-Schmidt Orthogonalization Procedure.

In general, the orthogonalization of the kth function leads to

$$\psi_k(t) = \frac{d_k(t)}{\sqrt{\mathcal{E}_k}} \tag{7.1.6}$$

where

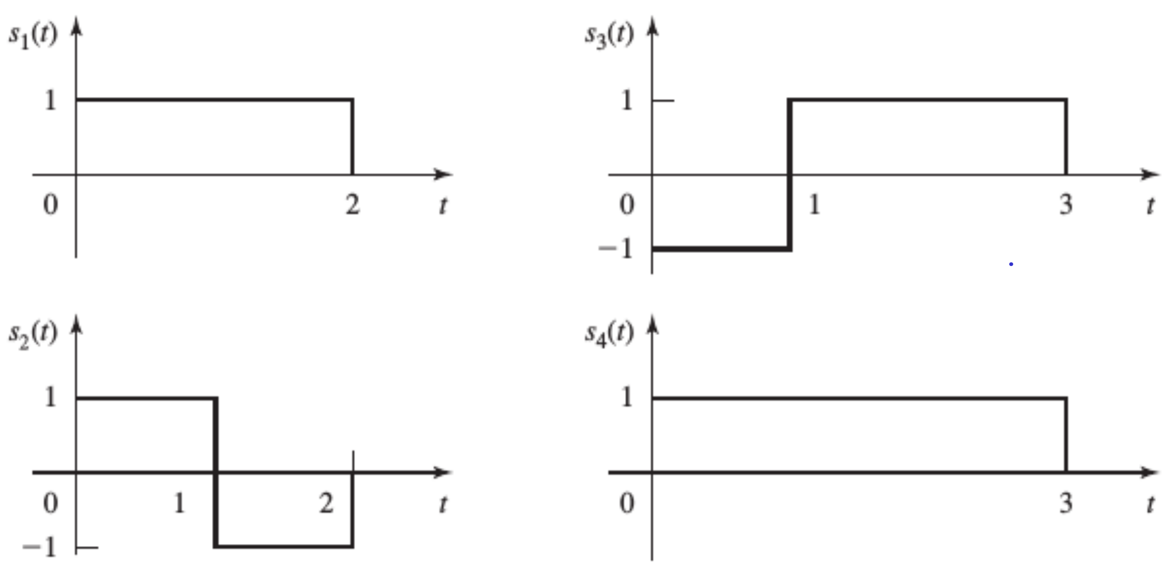
$$d_k(t) = s_k(t) - \sum_{i=1}^{k-1} c_{ki}\psi_i(t) \tag{7.1.7}$$

$$\mathcal{E}_k = \int_{-\infty}^{\infty} d_k^2(t)\, dt \quad \in \mathbb{R} \tag{7.1.8}$$

and

$$\mathbb{R} \ni \; c_{ki} = \int_{-\infty}^{\infty} s_k(t)\psi_i(t)\, dt, \quad i = 1, 2, \ldots, k-1 \tag{7.1.9}$$

Thus, the orthogonalization process is continued until all the M signal waveforms $\{s_m(t)\}$ have been exhausted and $N \leq M$ orthonormal waveforms have been constructed. The N orthonormal waveforms $\{\psi_n(t)\}$ form a *basis* in the N-dimensional signal space. The dimensionality N of the signal space will be equal to M if all the M signal waveforms are linearly independent; i.e., if none of the signal waveforms is a linear combination of the other signal waveforms.



(a) Original signal set

Figure 7.1 Application of Gram-Schmidt orthogonalization procedure to signals $\{s_i(t)\}$.

$$d_1(t) = s_1(t) \to \mathcal{E}_1 = \int_0^2 \cancelto{1}{s_1^2(t)}dt = 2 \qquad c_{21} = \int_0^2 s_2(t)\psi_1(t)dt = \int_0^1 \cancelto{1}{s_2(t)}\cancelto{1/\sqrt{2}}{\psi_1(t)}dt + \int_1^2 \cancelto{-1}{s_2(t)}\cancelto{1/\sqrt{2}}{\psi_1(t)}dt = 0$$

Example 7.1.1

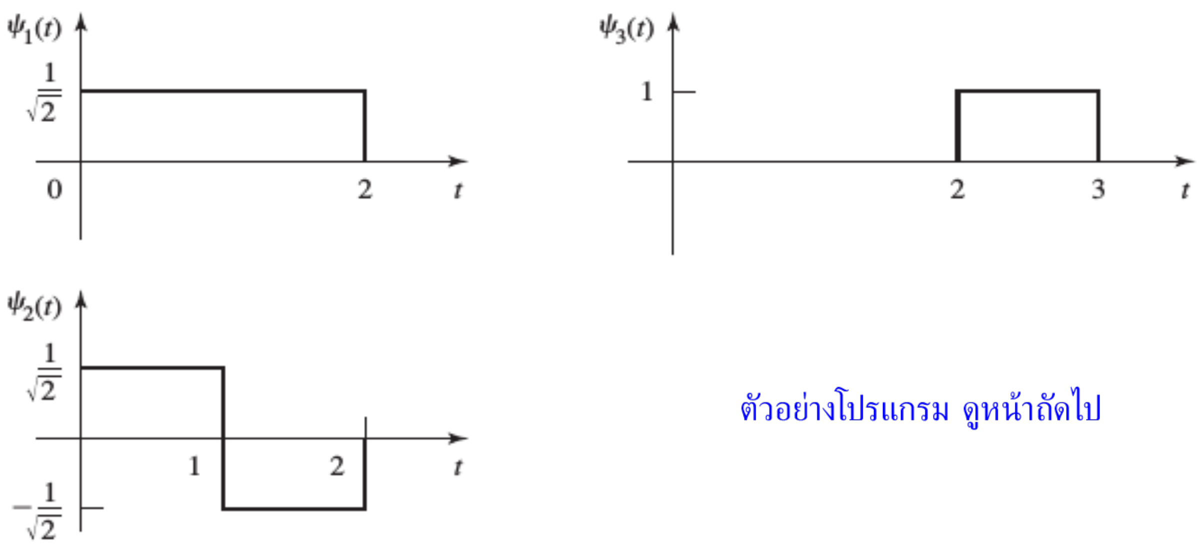
Let us apply the Gram-Schmidt procedure to the set of four waveforms illustrated in Figure 7.1(a). The waveform $s_1(t)$ has energy $\mathcal{E}_1 = 2$, so that $\psi_1(t) = s_1(t)/\sqrt{2}$. Next we observe that $c_{21} = 0$, so that $\psi_1(t)$ and $s_2(t)$ are orthogonal. Therefore, $\psi_2(t) = s_2(t)/\sqrt{\mathcal{E}_2} = s_2(t)/\sqrt{2}$. To obtain $\psi_3(t)$, we compute c_{31} and c_{32}, which are $c_{31} = 0$ and $c_{32} = -\sqrt{2}$. Hence,

$$d_3(t) = s_3(t) + \sqrt{2}\psi_2(t)$$

Since $d_3(t)$ has unit energy, it follows that $\psi_3(t) = d_3(t)$. Finally, we find that $c_{41} = \sqrt{2}$, $c_{42} = 0$, $c_{43} = 1$. Hence,

$$d_4(t) = s_4(t) - \sqrt{2}\psi_1(t) - \psi_3(t) = 0$$

Thus, $s_4(t)$ is a linear combination of $\psi_1(t)$ and $\psi_3(t)$ and, consequently, the dimensionality of the signal set is $N = 3$. The functions $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$ are shown in Figure 7.1(b)



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(b) Orthonormal waveforms

Figure 7.1 Application of Gram-Schmidt orthogonalization procedure to signals $\{s_i(t)\}$.

Example 7.1.1

```
s1(t)=heaviside(t)-heaviside(t-2)
s2(t)=heaviside(t)-heaviside(t-1)-heaviside(t-1)+heaviside(t-2)
s3(t)=-heaviside(t)+2*heaviside(t-1)-heaviside(t-3)
s4(t)=heaviside(t)-heaviside(t-3)
plot(s1(t),t,-1,4,figsize=(3,2))
```

```
d1(t)=s1(t)
E1=integral((d1(t))^2,t,0,2)
print("E1=",E1)
ψ1(t)=d1(t)/sqrt(E1)
plot(ψ1(t),t,-1,4,figsize=(3,2))
```

```
c21=integral(s2(t)*ψ1(t),t,0,2)
c21
```

```
d2(t)=s2(t) #c21=0
E2=integral((d2(t))^2,t,0,2)
print("E2=",E2)
ψ2(t)=d2(t)/sqrt(E2)
plot(ψ2(t),t,-1,4,figsize=(3,2))
```

```
c31=integral(s3(t)*ψ1(t),t,0,2)
print("c31=",c31)
c32=integral(s3(t)*ψ2(t),t,0,2)
print("c32=",c32)
d3(t)=s3(t)-c32*ψ2(t)
plot(d3(t),t,-1,4)
```

```
E3=integral((d3(t))^2,t,0,3)
E3
```

```
ψ3(t)=d3(t)/sqrt(E3)
plot(ψ3(t),t,-1,4,figsize=(3,2))
```

```
c41=integral(s4(t)*ψ1(t),t,0,2)
print("c41=",c41)
c42=integral(s4(t)*ψ2(t),t,0,2)
print("c42=",c42)
c43=integral(s4(t)*ψ3(t),t,0,3)
print("c43=",c43)
```

```
d4(t)=s4(t)-c41*ψ1(t)-c42*ψ2(t)-c43*ψ3(t)
E4=integral((d4(t))^2,t,0,3)
plot(d4(t),t,-1,5,figsize=(3,2))
```

Once we have constructed the set of orthogonal waveforms $\{\psi_n(t)\}$, we can express the M signals $\{s_m(t)\}$ as exact linear combinations of the $\{\psi_n(t)\}$. Hence, we may write

$$s_m(t) = \sum_{n=1}^{N} s_{mn}\,\psi_n(t), \quad m = 1, 2, \ldots, M \tag{7.1.10}$$

where

$$s_{mn} = \int_{-\infty}^{\infty} s_m(t)\psi_n(t)\,dt$$

and

$$\mathcal{E}_m = \int_{-\infty}^{\infty} s_m^2(t)\,dt = \sum_{n=1}^{N} s_{mn}^2 \tag{7.1.11}$$

Based on the expression in Equation (7.1.7), each signal waveform may be represented by the vector

$$s_m(t) \Leftrightarrow \quad \mathbf{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN}) \tag{7.1.12}$$

or equivalently, as a point in N-dimensional signal space with coordinates $\{s_{mi}, i = 1, 2, \ldots, N\}$. The energy of the mth signal waveform is simply the square of the length of the vector or, equivalently, the square of the Euclidean distance from the origin to the point in the N-dimensional space. We can also show that the inner product of two signals is equal to the inner product of their vector representations; i.e.,

$$\int_{-\infty}^{\infty} s_m(t)s_n(t)\,dt = \mathbf{s}_m \cdot \mathbf{s}_n \tag{7.1.13}$$

Thus, any N-dimensional signal can be represented geometrically as a point in the signal space spanned by the N orthonormal functions $\{\psi_n(t)\}$.

Example 7.1.2

Let us determine the vector representation of the four signals shown in Figure 7.1(a) by using the orthonormal set of functions in Figure 7.1(b). Since the dimensionality of the signal space is $N = 3$, each signal is described by three components, which are obtained by projecting each of the four signal waveforms on the three orthonormal basis functions $\psi_1(t), \psi_2(t), \psi_3(t)$. Thus, we obtain $\mathbf{s}_1 = (\sqrt{2}, 0, 0)$, $\mathbf{s}_2 = (0, \sqrt{2}, 0)$, $\mathbf{s}_3 = (0, -\sqrt{2}, 1)$, $\mathbf{s}_4 = (\sqrt{2}, 0, 1)$. These signal vectors are shown in Figure 7.2.

$$(s_{11}, s_{12}, s_{13})$$

$$s_{mn} = \int_{-\infty}^{\infty} s_m(t)\psi_n(t)\,dt$$

```
def smn(sm,ψm):
    return integral(sm(t)*ψm(t),t,-oo,oo)
```

```
ss1=vector([smn(s1,ψ1),smn(s1,ψ2),smn(s1,ψ3)])
show(ss1)
ss2=vector([smn(s2,ψ1),smn(s2,ψ2),smn(s2,ψ3)])
show(ss2)
ss3=vector([smn(s3,ψ1),smn(s3,ψ2),smn(s3,ψ3)])
show(ss3)
ss4=vector([smn(s4,ψ1),smn(s4,ψ2),smn(s4,ψ3)])
show(ss4)
```

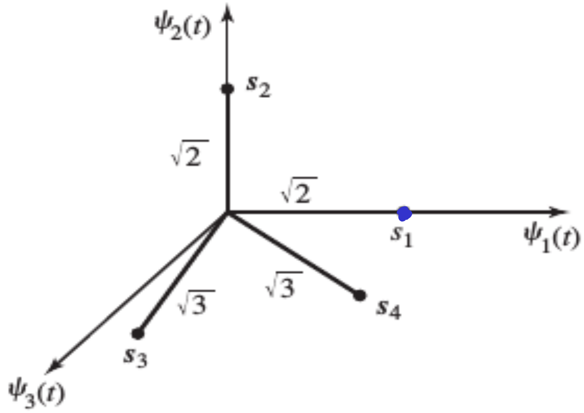


Figure 7.2 Signal vectors corresponding to the signals $s_i(t), i = 1, 2, 3, 4$.

Finally, we should observe that the set of basis functions $\{\psi_n(t)\}$ obtained by the Gram-Schmidt procedure is not unique. For example, another set of basis functions that span the three-dimensional space is shown in Figure 7.3. For this basis, the signal

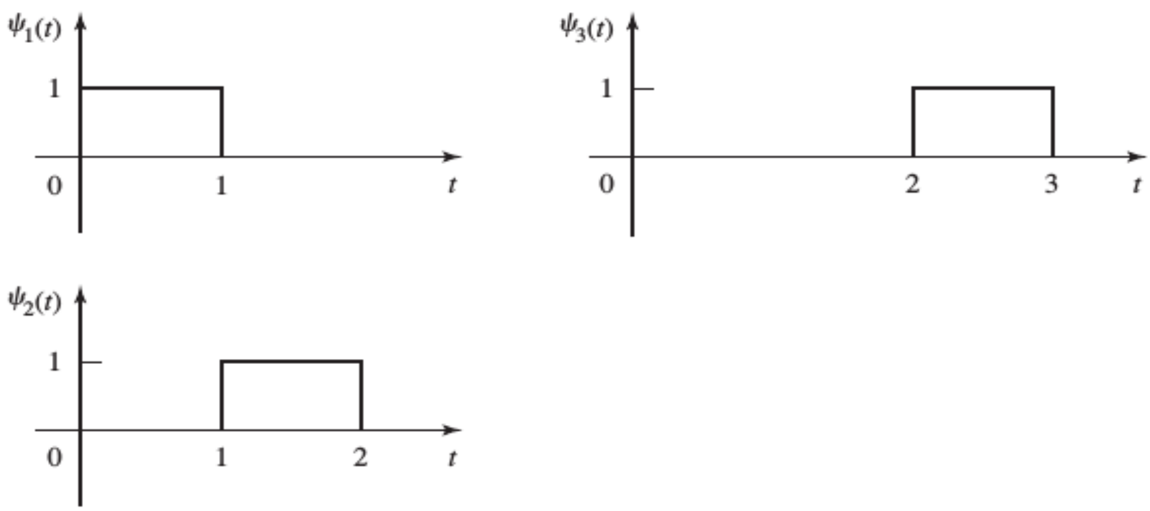


Figure 7.3 Alternate set of basis functions.

7.2 PULSE AMPLITUDE MODULATION

In *pulse amplitude modulation (PAM),* the information is conveyed by the amplitude of the transmitted signal. Let us first consider PAM signals that are appropriate for baseband channels.

$$1 \mapsto A$$

$$0 \mapsto -A$$

Baseband Signals. Binary PAM is the simplest digital modulation method. In binary PAM, the information bit 1 may be represented by a pulse of amplitude A and the information bit 0 is represented by a pulse of amplitude $-A$, as shown in Figure 7.4. This type of signaling is also called *binary antipodal signaling*. Pulses are transmitted at a bit rate $R_b = 1/T_b$ bits/sec, where T_b is called the bit interval. Although the pulses are shown as rectangular, in practical systems, the rise time and decay time are nonzero and the pulses are generally smoother. The pulse shape determines the spectral characteristics of the transmitted signal as described in Chapter 8.

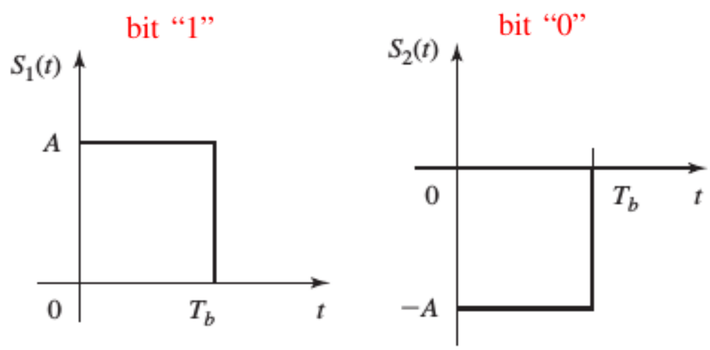
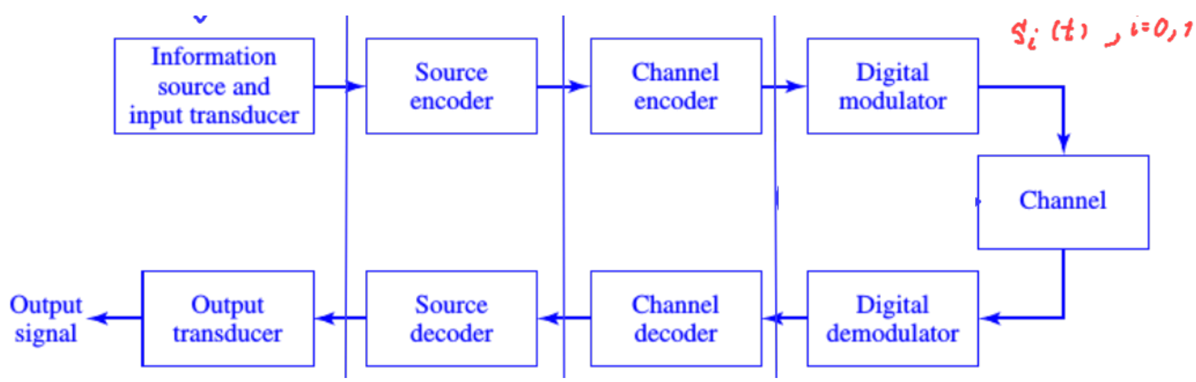


Figure 7.4 Binary PAM signals.



block = symbol

The generalization of PAM to nonbinary (M-ary) pulse transmission is relatively straightforward. Instead of transmitting one bit at a time, the binary information sequence is subdivided into blocks of k bits, called *symbols,* and each block, or symbol, is represented by one of $M = 2^k$ pulse amplitude values. Thus with $k = 2$, we have $M = 4$ pulse amplitude values. Figure 7.5 illustrates the PAM signals for $k = 2$, $M = 4$. Note that when the bit rate R_b is fixed, the symbol interval is

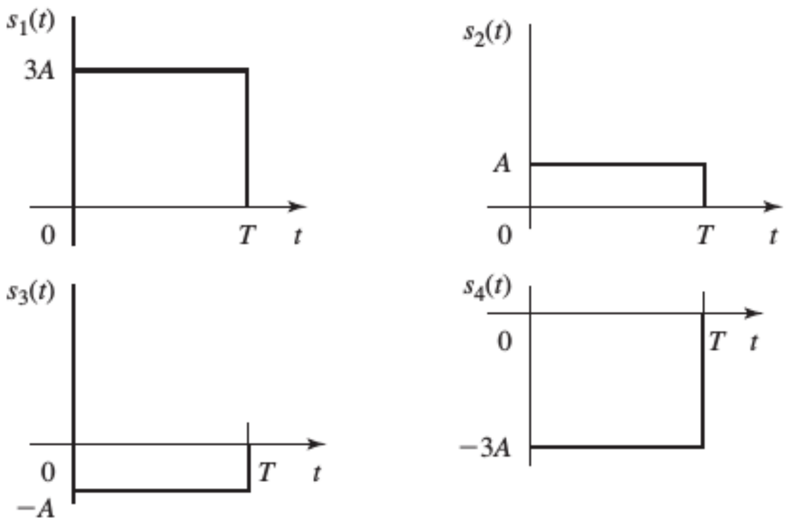


Figure 7.5 $M = 4$ PAM signal waveforms.

$$T = \frac{k}{R_b} = kT_b \tag{7.2.1}$$

as shown in Figure 7.6.

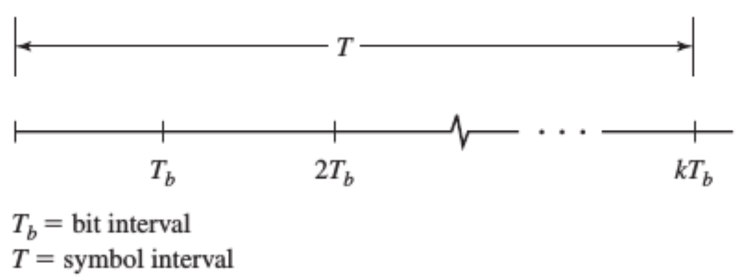
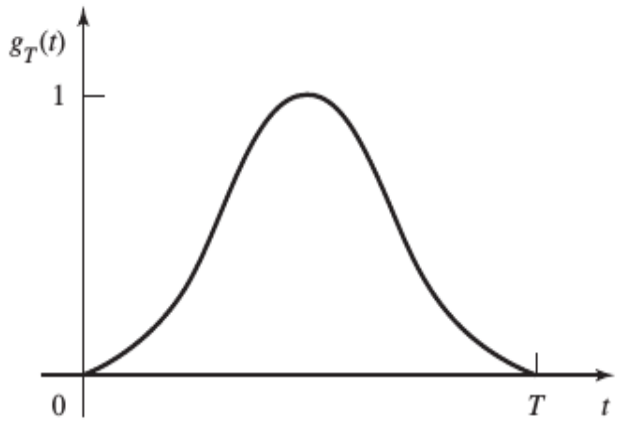


Figure 7.6 Relationship between the symbol interval and the bit interval.

We may characterize the PAM signals in terms of their basic properties. In general, the M-ary PAM signal waveforms may be expressed as

$$s_m(t) = A_m g_T(t), \quad m = 1, 2, \ldots, M, \quad 0 \le t \le T \tag{7.2.2}$$

where $g_T(t)$ is a pulse of some arbitrary shape as shown for example in Figure 7.7. We observe that the distinguishing feature among the M signals is the signal amplitude. All the M signals have the same pulse shape. Another important feature of these signals is



(a)

Figure 7.7 Signal pulse for PAM.

their energies. We note the signals have different energies; i.e.,

$$\mathcal{E}_m = \int_0^T s_m^2(t)\,dt = A_m^2 \int_0^T g_T^2(t)\,dt = A_m^2 \mathcal{E}_g, \quad m = 1, 2, \ldots, M \tag{7.2.3}$$

where $\mathcal{E}_g$ is the energy of the signal pulse $g_T(t)$.

Bandpass Signals. To transmit the digital signal waveforms through a bandpass channel by amplitude modulation, the baseband signal waveforms $s_m(t)$, $m = 1, 2, \ldots, M$ are multiplied by a sinusoidal carrier of the form $\cos 2\pi f_c t$, as shown in Figure 7.8, where f_c is the carrier frequency and corresponds to the center frequency in the passband of the channel. Thus, the transmitted signal waveforms may be expressed as

$$u_m(t) = A_m g_T(t) \cos 2\pi f_c t, \quad m = 1, 2, \ldots, M \tag{7.2.4}$$

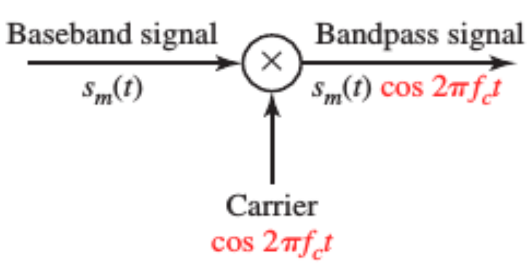


Figure 7.8 Amplitude modules sinusoidal carrier by the baseband signal.

the Fourier transform of

$$x(t)\cos(2\pi f_0 t)$$

is

$$\tfrac{1}{2}X(f-f_0)+\tfrac{1}{2}X(f+f_0)$$

Because multiplication of two signals in the time domain corresponds to the convolution of their spectra in the frequency domain, the spectrum of the amplitude-modulated signal given by Equation (7.2.4) is

$$U_m(f) = \frac{A_m}{2}[G_T(f-f_c) + G_T(f+f_c)] \tag{7.2.5}$$

Thus, the spectrum of the baseband signal $s_m(t) = A_m g_T(t)$, is shifted in frequency by an amount equal to the carrier frequency f_c. The result is a DSB-SC AM signal, as illustrated in Figure 7.9.

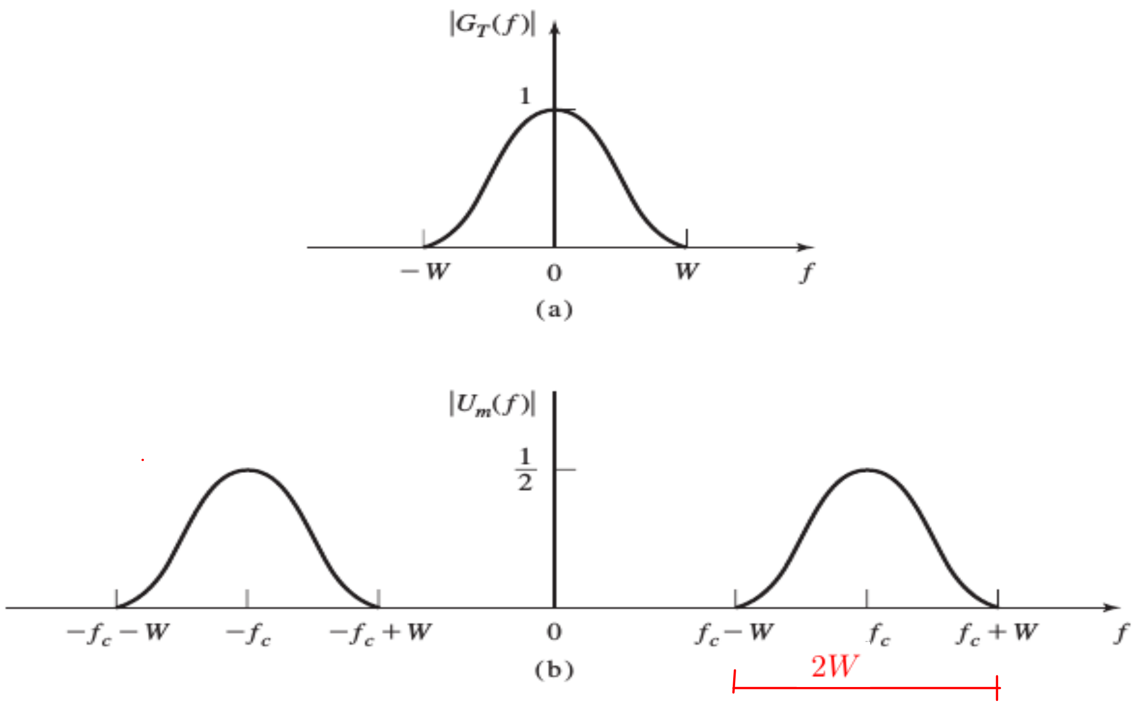


Figure 7.9 Spectra of (a) baseband and (b) amplitude-modulated signals.

The energy of the bandpass signal waveforms $u_m(t)$, $m = 1, 2, \ldots, M$, given by Equation (7.2.4) is defined as

$$\begin{aligned}\mathcal{E}_m &= \int_{-\infty}^{\infty} u_m^2(t)\,dt = \int_{-\infty}^{\infty} A_m^2 g_T^2(t)\cos^2 2\pi f_c t\,dt \\ &= \frac{A_m^2}{2}\int_{-\infty}^{\infty} g_T^2(t)\,dt + \frac{A_m^2}{2}\int_{-\infty}^{\infty} g_T^2(t)\cos 4\pi f_c t\,dt \end{aligned} \tag{7.2.6}$$

หมายเหตุ

$\cos^2 x = (1 + \cos 2x)/2$

We note that when $f_c \gg W$, the term

0

$$\int_{-\infty}^{\infty} g_T^2(t) \cos 4\pi f_c t \, dt \tag{7.2.7}$$

Consequently,

$$\mathcal{E}_m = \frac{A_m^2}{2} \int_{-\infty}^{\infty} g_T^2(t) \, dt = \frac{A_m^2}{2} \mathcal{E}_g \tag{7.2.8}$$

where $\mathcal{E}_g$ is the energy in the signal pulse $g_T(t)$. Thus, we have shown that the energy in the bandpass signal is one-half of the energy in the baseband signal. The scale factor of $\frac{1}{2}$ is due to the carrier component $\cos 2\pi f_c t$, which has an average power of $\frac{1}{2}$.

When the transmitted pulse shape $g_T(t)$ is rectangular; i.e.,

$$g_T(t) = \begin{cases} \sqrt{\frac{\mathcal{E}_g}{T}} & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{7.2.9}$$

the amplitude-modulated carrier signal is usually called *amplitude-shift keying (ASK)*.

Geometric Representation of PAM Signals.

The baseband signal waveforms for M-ary PAM are given in Equation (7.2.2), where $M = 2^k$, and $g_T(t)$ is a pulse with peak amplitude normalized to unity as previously illustrated in Figure 7.7.

The M-ary PAM waveforms are one-dimensional signals, which may be expressed as

$$\psi_k(t) = \frac{d_k(t)}{\sqrt{\mathcal{E}_k}}$$

$$s_m(t) = \sum_{n=1}^{N} s_{mn} \psi_n(t), \quad m = 1, 2, \ldots, M$$

$$s_m(t) = s_m \psi(t), \quad m = 1, 2, \ldots, M \tag{7.2.10}$$

where the basis function $\psi(t)$ is defined as

$$d_k(t) = s_k(t) - \sum_{i=1}^{k-1} c_{ki} \psi_i(t)$$

$$\psi(t) = \frac{1}{\sqrt{\mathcal{E}_g}} g_T(t), \quad 0 \le t \le T \tag{7.2.11}$$

$$\mathcal{E}_k = \int_{-\infty}^{\infty} d_k^2(t) \, dt$$

$\mathcal{E}_g$ is the energy of the signal pulse $g_T(t)$, and the signal coefficients (one-dimensional vectors) are simply

$$s_{mn} = \int_{-\infty}^{\infty} s_m(t) \psi_n(t) \, dt$$

$$s_m = \sqrt{\mathcal{E}_g} A_m, \quad m = 1, 2, \ldots, M \tag{7.2.12}$$

An important parameter is the Euclidean distance between two signal points, which is defined as

$$d_{mn} = \sqrt{|s_m - s_n|^2} = \sqrt{\mathcal{E}_g (A_m - A_n)^2} \tag{7.2.13}$$

If we select the signal amplitudes $\{A_m\}$ to be symmetrically spaced about zero and equally distant between adjacent signal amplitudes, we obtain the signal points for symmetric PAM, as shown in Figure 7.11.

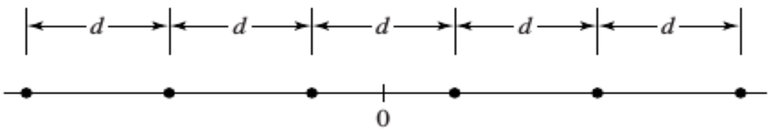


Figure 7.11 Signal points (constellation) for symmetric PAM.

We observe that the PAM signals have different energies. In particular, the energy of the mth signal is

$\mathcal{E}_m = \|\mathbf{s}_m\|^2 \rightarrow$

1

$$\mathcal{E}_m = s_m^2 = \mathcal{E}_g A_m^2, \quad m = 1, 2, \ldots, M \tag{7.2.14}$$

For equally probable signals, the average energy is

$$\mathcal{E}_{av} = \frac{1}{M} \sum_{m=1}^{M} \mathcal{E}_m = \frac{\mathcal{E}_g}{M} \sum_{m=1}^{M} A_m^2 \tag{7.2.15}$$

$P(s_i)$

2

If the signal amplitudes are symmetric about the origin, then

by mathematic induction:

$$A_m = (2m - 1 - M), \quad m = 1, 2, \ldots, M \quad : M \in 2\mathbb{Z} \tag{7.2.16}$$

and, hence, the average energy is

$$\mathcal{E}_{av} = \frac{\mathcal{E}_g}{M} \sum_{m=1}^{M} (2m - 1 - M)^2 = \mathcal{E}_g (M^2 - 1)/3 \tag{7.2.17}$$

When the baseband PAM signals are impressed on a carrier, the basic geometric representation of the digital PAM signal waveforms remains the same. The bandpass signal waveforms $u_m(t)$ may be expressed as

$$u_m(t) = s_m \psi(t) \tag{7.2.18}$$

where the basic signal waveform $\psi(t)$ is defined as

$$\psi(t) = \sqrt{\frac{2}{\mathcal{E}_g}}\, g_T(t) \cos 2\pi f_c t \tag{7.2.19}$$

and

$$s_m = \sqrt{\frac{\mathcal{E}_g}{2}} A_m, \quad m = 1, 2, \ldots, M \tag{7.2.20}$$

Note that the only change in the geometric representation of bandpass PAM signals, compared to baseband signals, is the scale factor $\sqrt{2}$, which appears in Equations (7.2.19) and (7.2.20).