and

$$s_m = \sqrt{\frac{\mathcal{E}_g}{2}} A_m, \quad m = 1, 2, \ldots, M \tag{7.2.20}$$

Note that the only change in the geometric representation of bandpass PAM signals, compared to baseband signals, is the scale factor $\sqrt{2}$, which appears in Equations (7.2.19) and (7.2.20).

## 7.3 TWO-DIMENSIONAL SIGNAL WAVEFORMS

As we observed, PAM signal waveforms are basically one-dimensional signals. In this section, we consider the construction of two-dimensional signals.

### 7.3.1 Baseband Signals

Let us begin with the construction of two orthogonal signals. Two signal waveforms $s_1(t)$ and $s_2(t)$ are said to be orthogonal over the interval $(0, T)$ if

$$\int_0^T s_1(t)s_2(t)\,dt = 0 \tag{7.3.1}$$

Two examples of orthogonal signals are illustrated in Figure 7.12. Note that the two signals $s_1(t)$ and $s_2(t)$ completely overlap over the interval $(0, T)$, while the signals $s_1'(t)$ and $s_2'(t)$ are nonoverlapping in time. Both signal pairs satisfy the orthogonality property in Equation (7.3.1) and both signal pairs have identical energy; i.e.,

$$\mathcal{E} = \int_0^T s_1^2(t)\,dt = \int_0^T s_2^2(t)\,dt = \int_0^T [s_1'(t)]^2\,dt = \int_0^T [s_2'(t)]^2\,dt = A^2T \tag{7.3.2}$$

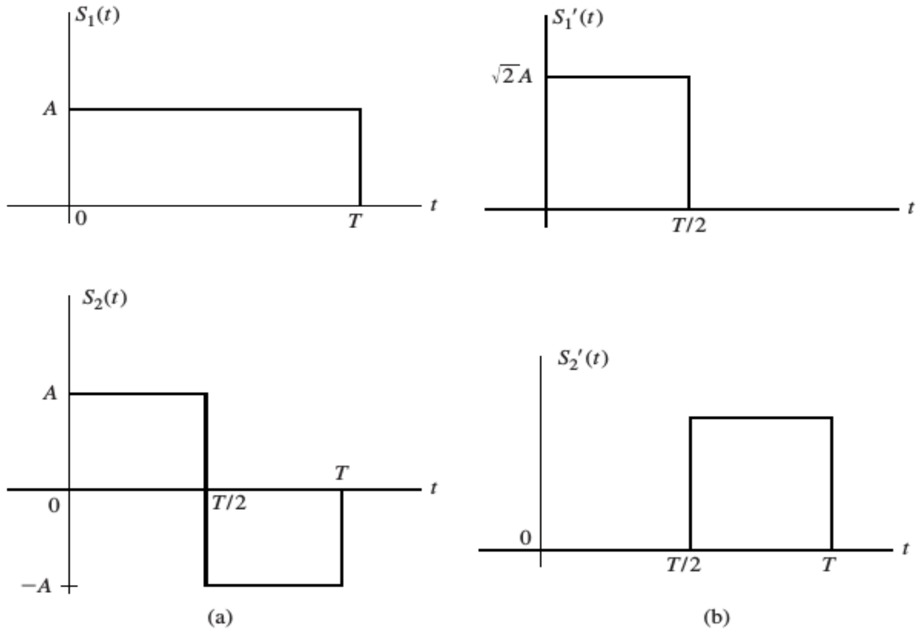



**Figure 7.12** Two sets of orthogonal signals.

Geometrically, these signal waveforms can be represented as signal vectors in two-dimensional space. As basis functions, we may select the unit energy, rectangular functions

$$\psi_1(t) = \begin{cases} \sqrt{2/T}, & 0 \le t \le T/2 \\ 0, & \text{otherwise} \end{cases}$$
$$\psi_2(t) = \begin{cases} \sqrt{2/T}, & T/2 < t \le T \\ 0, & \text{otherwise} \end{cases} \tag{7.3.3}$$

Then, the signal waveforms $s_1(t)$ and $s_2(t)$ shown in Figure 7.12(a) can be expressed as

$s_{mi} = \int_0^T s_m(t)\psi_i(t)dt$

$$s_1(t) = s_{11}\psi_1(t) + s_{12}\psi_2(t)$$
$$s_2(t) = s_{21}\psi_1(t) + s_{22}\psi_2(t) \tag{7.3.4}$$

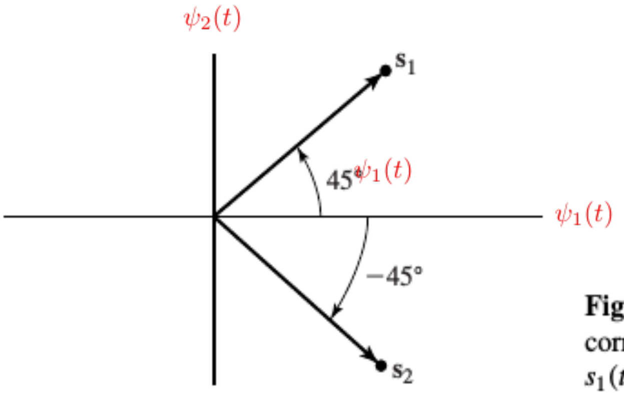



**Figure 7.13** The two signal vectors corresponding to the signals waveforms $s_1(t)$ and $s_2(t)$.

where

$$\mathbf{s}_1 = (s_{11}, s_{12}) = (A\sqrt{T/2}, A\sqrt{T/2})$$
$$\mathbf{s}_2 = (s_{21}, s_{22}) = (A\sqrt{T/2}, -A\sqrt{T/2}) \tag{7.3.5}$$

The signal vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ are illustrated in Figure 7.13. Note that the signal vectors are separated by $90°$, so that they are orthogonal. Furthermore, the square of the length of each vector gives the energy in each signal; i.e.,

$$\mathcal{E}_1 = \|\mathbf{s}_1\|^2 = A^2T$$
$$\mathcal{E}_2 = \|\mathbf{s}_2\|^2 = A^2T \tag{7.3.6}$$

The Euclidean distance between the two signals is

$$d_{12} = \sqrt{\|\mathbf{s}_1 - \mathbf{s}_2\|^2} = A\sqrt{2T} = \sqrt{2A^2T} = \sqrt{2\mathcal{E}} \tag{7.3.7}$$

where $\mathcal{E} = \mathcal{E}_1 = \mathcal{E}_2$ is the signal energy.

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Similarly, the pair of orthogonal signals shown in Figure 7.12(b) can be expressed as in Equation (7.3.4), where

$$\begin{aligned} \mathbf{s}_1' &= (A\sqrt{T}, 0) = (\sqrt{\mathcal{E}}, 0) \\ \mathbf{s}_2' &= (0, A\sqrt{T}) = (0, \sqrt{\mathcal{E}}) \end{aligned} \tag{7.3.8}$$

These two signal vectors are illustrated in Figure 7.14. Note that $\mathbf{s}_1'$ and $\mathbf{s}_2'$ are related to the signal vectors shown in Figure 7.13 by a simple 45° rotation. Hence, the Euclidean

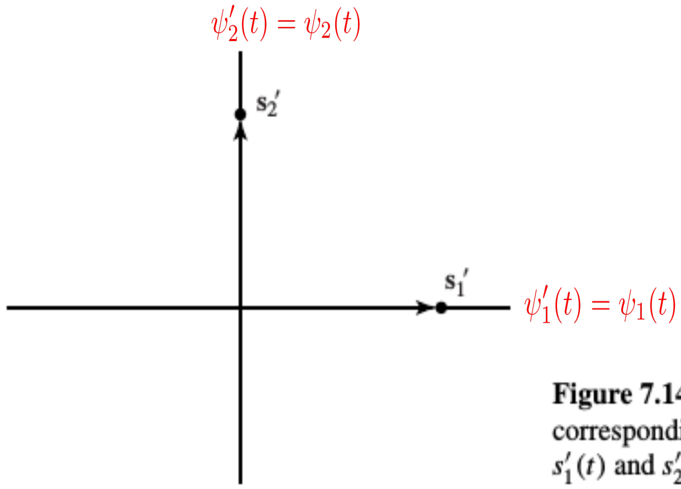



**Figure 7.14** The two signal vectors corresponding to the signal waveforms $s_1'(t)$ and $s_2'(t)$.

distance between the signal points $\mathbf{s}_1'$ and $\mathbf{s}_2'$ is identical to that for signal points $\mathbf{s}_1$ and $\mathbf{s}_2$.

Suppose that we wish to construct four signal waveforms in two dimensions. With four signal waveforms, we can transmit two information bits in a signaling interval of length $T$. If we begin with the two orthogonal signals $s_1(t)$ and $s_2(t)$, shown in Figure 7.12(a), and their corresponding vector representation in Figure 7.13, it is easy to see that a simple construction is one that adds two additional signal vectors, namely, $-\mathbf{s}_1$ and $-\mathbf{s}_2$. Thus, we obtain the 4-point signal-point constellation shown in Figure 7.15, which corresponds to the analog signal waveforms $s_1(t), s_2(t), -s_1(t)$, and $-s_2(t)$. Since the pair $s_1(t)$ and $s_2(t)$ are orthogonal and the pair $-s_1(t)$ and $-s_2(t)$ are orthogonal, the signal set consisting of the four signal waveforms is called a set of *biorthogonal signals*.

$$\mathbf{s}_1 \sim s_1(t) = s_{11}\psi_1(t) + s_{12}\psi_2(t) \Rightarrow -\mathbf{s}_1 \sim -s_{11}\psi_1(t) - s_{12}\psi_2(t) = -s_1(t)$$

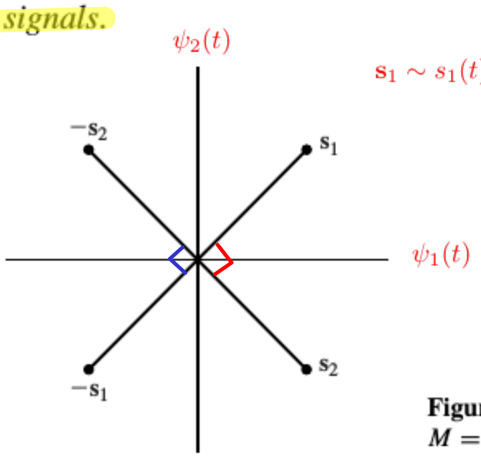
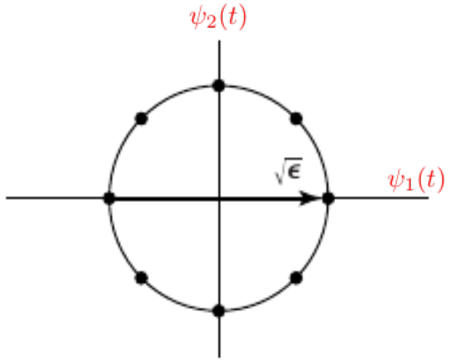



**Figure 7.15** Signal constellation for $M = 4$ biorthogonal signals.

The procedure for constructing a larger set of signal waveforms is relatively straightforward. Specifically, we can add additional signal points (signal vectors) in the two-dimensional plane, and construct the corresponding signal waveforms by using the two orthonormal basis functions $\psi_1(t)$ and $\psi_2(t)$ given by Equation (7.3.3). For example, suppose we wish to construct $M = 8$ two-dimensional signal waveforms, all of equal energy $\mathcal{E}$. These eight signal points are illustrated in Figure 7.16, and allow us to transmit three bits at a time. The corresponding eight signal waveforms are the two sets of biorthogonal signal waveforms constructed from the two pairs of orthogonal signals shown in Figure 7.12.




$$\mathcal{E}_i = \|s_i\|^2 = \mathcal{E}$$

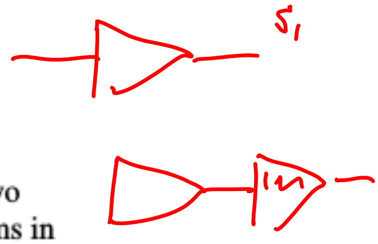
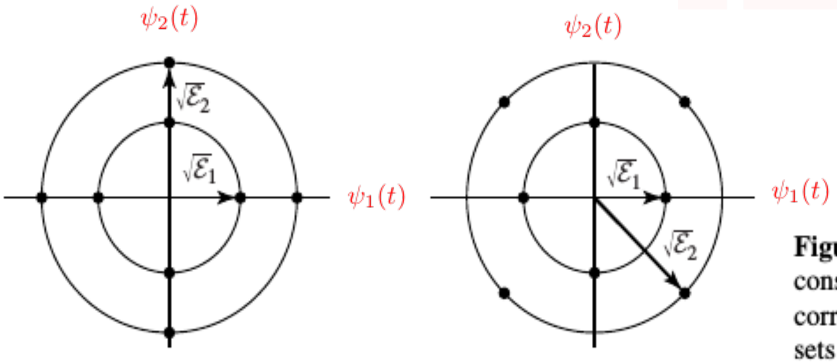



**Figure 7.16** $M = 8$ signal-point constellation corresponding to the two points of orthogonal signal waveforms in Figure 7.12 and their negatives, i.e., $s_1(t), s_2(t), s_1'(t), s_2'(t), -s_1(t), -s_2(t), -s_1'(t)$ and $-s_2'(t)$.

Alternatively, suppose that we remove the condition that all eight waveforms have equal energy. For example, suppose that we select four biorthogonal waveforms that have energy $\mathcal{E}_1$ and another four biorthogonal waveforms that have energy $\mathcal{E}_2$, where $\mathcal{E}_2 > \mathcal{E}_1$. Two possible eight signal-point constellations are shown in Figure 7.17, where the signal points are located on two concentric circles of radii $\sqrt{\mathcal{E}_1}$ and $\sqrt{\mathcal{E}_2}$. In Section 7.6.5 we show that the signal set in Figure 7.17(b) is preferable in an AWGN channel to that shown in Figure 7.17(a).

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$




(a) (b)

**Figure 7.17** Two $M = 8$ signal-point constellations in two dimensions, corresponding to a superposition of two sets of biorthogonal signal waveforms with different energies.

## 7.3.2 Two-dimensional Bandpass Signals—Carrier-Phase Modulation

$\{s_1(t), s_2(t), \ldots, s_M(t)\} \leftrightarrow \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_M$

if we have a set of $M$ two-dimensional signal waveforms, say $s_m(t)$, $m = 1, 2, \ldots, M$, we can generate a set of $M$ bandpass signal waveforms as

$$u_m(t) = s_m(t) \cos 2\pi f_c t, \quad m = 1, 2, \ldots, M, \quad 0 \le t \le T \tag{7.3.9}$$

In this section, we consider the special case in which the $M$ two-dimensional bandpass signals waveforms are constrained to have the same energy; i.e.,

$$\begin{aligned} \mathcal{E}_m &= \int_0^T u_m^2(t)\, dt = \int_0^T s_m^2(t) \cos^2 2\pi f_c t\, dt \\ &= \frac{1}{2} \int_0^T s_m^2(t)\, dt + \frac{1}{2} \int_0^T s_m^2(t) \cos 4\pi f_c t\, dt \end{aligned} \tag{7.3.10}$$

0

ถ้า $f_c \gg W \Rightarrow$

Hence,

$$\mathcal{E}_m = \frac{1}{2} \int_0^T s_m^2(t)\, dt = \mathcal{E}_s \quad \text{for all } m \tag{7.3.11}$$

where $\mathcal{E}_s$ denotes the energy/signal or/symbol.

As we indicated in the discussion following Equation (7.1.12), when all the signal waveforms have the same energy, the corresponding signal points in the geometric representation of the signal waveforms fall on a circle of radius $\sqrt{\mathcal{E}_s}$. For example, in the case of the four biorthogonal waveforms, the signal points are as shown in Figure 7.15 or, equivalently, any phase-rotated version of these signal points.

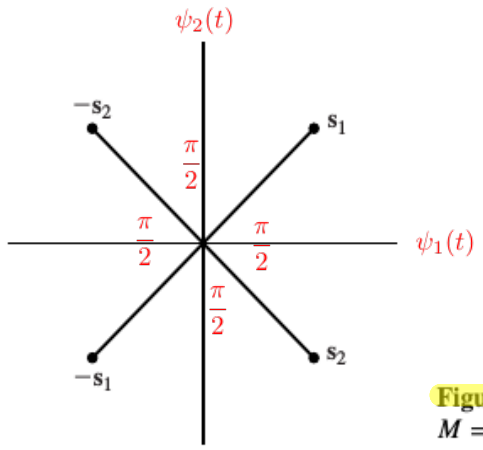



**Figure 7.15** Signal constellation for $M = 4$ biorthogonal signals.

Find equivalence relation function

$M = 4$ biorthogonal $\leftrightarrow s(t)\cos(2\pi f_c t + \pi m/2), m = 0, 1, 2, 3$

From this geometric representation for $M = 4$, we observe that the signal points are equivalent to a single signal whose phase is shifted by multiples of $\pi/2$. That is, a bandpass signal of the form $s(t)\cos(2\pi f_c t + \pi m/2)$, $m = 0, 1, 2, 3$, has the same geometric representation as an $M = 4$ general biorthogonal signal set. Therefore, a simple way to generate a set of $M$ bandpass signals having equal energy is to impress the information on the phase of the carrier. Thus, we have a carrier-phase modulated signal.

The general representation of a set of $M$ carrier-phase modulated signal waveforms is

$$u_m(t) = g_T(t)\cos\left(2\pi f_c t + \frac{2\pi m}{M}\right), \quad m = 0, 1, \ldots, M-1, \quad 0 \le t \le T \quad (7.3.12)$$

where $g_T(t)$ is a baseband pulse shape, which determines the spectral characteristics of the transmitted signal, as will be demonstrated in Chapter 8. When $g_T(t)$ is a rectangular pulse, defined as

$$g_T(t) = \sqrt{\frac{2\mathcal{E}_s}{T}}, \quad 0 \le t \le T \quad (7.3.13)$$

the corresponding transmitted signal waveforms

$$u_m(t) = \sqrt{\frac{2\mathcal{E}_s}{T}}\cos\left(2\pi f_c t + \frac{2\pi m}{M}\right), \quad m = 0, 1, \ldots, M-1, \quad 0 \le t \le T \quad (7.3.14)$$

have a constant envelope (the pulse shape $g_T(t)$ is a rectangular pulse) and the carrier phase changes abruptly at the beginning of each signal interval. This type of digital-phase modulation is called *phase-shift keying* (PSK). Figure 7.18 illustrates a four-phase ($M = 4$) PSK signal waveform, usually called a quadrature PSK (QPSK) signal.

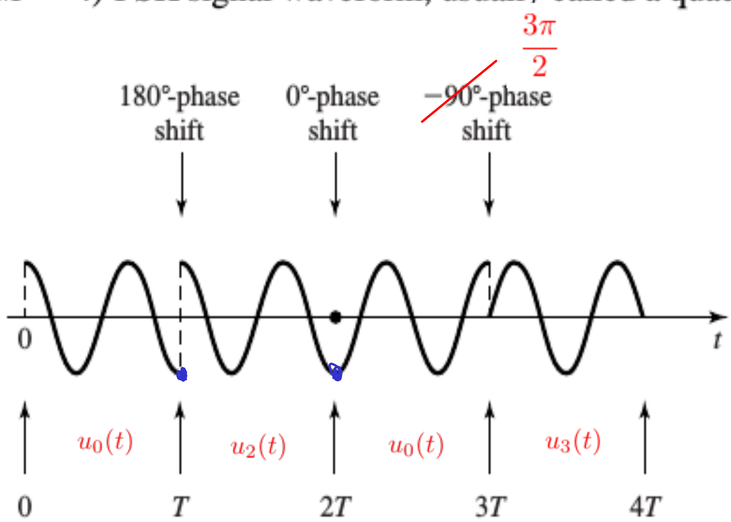



**Figure 7.18** Example of a four PSK signal.

$$u_m(t) = g_T(t)\underbrace{\cos\left(\frac{2\pi m}{M}\right)}_{A_{mc}}\cos(2\pi f_c t) - g_T(t)\underbrace{\sin\left(\frac{2\pi m}{M}\right)}_{A_{ms}}\sin(2\pi f_c t)$$

By viewing the angle of the cosine function in Equation (7.3.14) as the sum of two angles, we may express the waveforms in Equation (7.3.14) as

$$g_T(t) = \sqrt{\frac{2\mathcal{E}_s}{T}}, \qquad u_m(t) = g_T(t)\, A_{mc} \cos 2\pi f_c t - g_T(t)\, A_{ms} \sin 2\pi f_c t \tag{7.3.15}$$

where

$$\begin{aligned} A_{mc} &= \cos 2\pi m/M, \quad m = 0, 1, \ldots, M-1 \\ A_{ms} &= \sin 2\pi m/M, \quad m = 0, 1, \ldots, M-1 \end{aligned} \tag{7.3.16}$$

Thus, a phase-modulated signal may be viewed as two quadrature carriers with amplitudes $g_T(t)A_{mc}$ and $g_T(t)A_{ms}$ as shown in Figure 7.19, which depend on the transmitted phase in each signal interval.

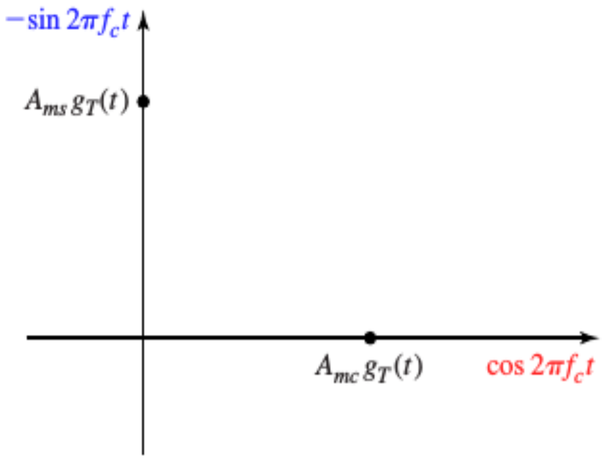



**Figure 7.19** Digital-phase modulation viewed as two amplitude-modulated quadrature carriers.

It follows from Equation (7.3.15) that digital phase-modulated signals can be represented geometrically as two-dimensional vectors with components $\sqrt{\mathcal{E}_s}\cos 2\pi m/M$, and $\sqrt{\mathcal{E}_s}\sin 2\pi m/M$, i.e.,

$$\mathbf{s}_m = \left(\sqrt{\mathcal{E}_s}\cos 2\pi m/M, \qquad \sqrt{\mathcal{E}_s}\sin 2\pi m/M\right) \tag{7.3.17}$$

Note that the orthogonal basis functions are $\psi_1(t) = \sqrt{\frac{2}{\mathcal{E}_g}}\, g_T(t)\cos 2\pi f_c t$, and $\psi_2(t) = -\sqrt{\frac{2}{\mathcal{E}_g}}\, g_T(t)\sin 2\pi f_c t$. Signal-point constellations for $M = 2, 4, 8$ are illustrated in Figure 7.20. We observe that binary-phase modulation is identical to binary PAM.

$0 \leftrightarrow \mathbf{s}_0$
$1 \leftrightarrow \mathbf{s}_1$

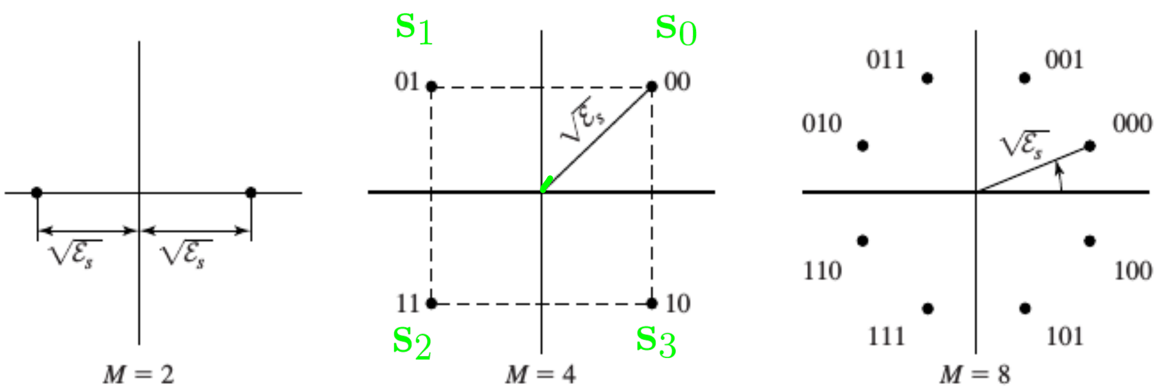



**Figure 7.20** PSK signal constellations.

The mapping or assignment of $k$ information bits into the $M = 2^k$ possible phases may be done in a number of ways. The preferred assignment is to use *Gray encoding,* in which adjacent phases differ by one binary digit as illustrated in Figure 7.20. Because the most likely errors caused by noise involve the erroneous selection of an adjacent phase to the transmitted phase, only a single bit error occurs in the $k$-bit sequence with Gray encoding.

$$\mathbf{s}_m = \left(\sqrt{\mathcal{E}_s}\cos 2\pi m/M, \quad \sqrt{\mathcal{E}_s}\sin 2\pi m/M\right) \qquad (7.3.17)$$

The Euclidean distance between any two signal points in the constellation is

$$d_{mn} = \sqrt{\|\mathbf{s}_m - \mathbf{s}_n\|^2}$$
$$= \sqrt{2\mathcal{E}_s\left(1 - \cos\frac{2\pi(m-n)}{M}\right)} \qquad (7.3.18)$$

$m - n = 1$

and the minimum Euclidean distance (distance between two adjacent signal points) is simply

$$d_{\min} = \sqrt{2\mathcal{E}_s\left(1 - \cos\frac{2\pi}{M}\right)} \qquad (7.3.19)$$

As we shall demonstrate in Equation (7.6.10), the minimum Euclidean distance $d_{\min}$ plays an important role in determining the error-rate performance of the receiver that demodulates and detects the information in the presence of additive Gaussian noise.