Priors

$\mu \sim N(\mu, v)$

1. $i = 1, 2, \ldots, k$

$\lambda \sim Gamma(a, b)$

$j = 1, 2, \ldots, n_i$

$\psi \sim Gamma(c, d)$

$\lambda = \frac{1}{\sigma^2} \qquad \psi = \frac{1}{\tau^2}$

$$p(\mu, \lambda, \psi, \theta \mid Y) \propto \prod_{i=1}^{k} \prod_{j=1}^{n_i} p(Y_{ij} \mid \theta_i, \lambda) \times \prod_{i=1}^{k} p(\theta_i \mid \mu, \lambda, \psi) \times p(\mu)\, p(\lambda)\, p(\psi)$$

$$\propto \prod_{i=1}^{k} \prod_{j=1}^{n_i} \lambda^{\frac{1}{2}} e^{\frac{-\lambda(Y_{ij}-\theta_i)^2}{2}} \times \prod_{i=1}^{k} \lambda^{\frac{1}{2}} \psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i-\mu)^2}{2}} \times e^{\frac{-\phi(\mu-v)^2}{2}}\, \lambda^{a-1} e^{-\lambda b}\, \psi^{c-1} e^{-\psi d}$$

$$p(\mu \mid -) \propto \prod_{i=1}^{k} \lambda^{\frac{1}{2}} \psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i-\mu)^2}{2}} \times e^{\frac{-\phi(\mu-v)^2}{2}}$$

$$\propto e^{\frac{-\lambda\psi \sum_i (\theta_i-\mu)^2}{2}} \times e^{-\frac{\phi(\mu-v)^2}{2}}$$

$$\propto N\left((k\lambda\psi+\phi)^{-1}(\lambda\psi \sum_i \theta_i + \phi v),\ (k\lambda\psi+\phi)^{-1}\right)$$

$\mu^2(k\lambda\psi + \phi)$

$-2\mu(\lambda\psi \sum_i \theta_i + \phi v)$

$$p(\lambda|-) \propto \prod_{i=1}^{k}\prod_{j=1}^{n_i} \lambda^{\frac{1}{2}} e^{-\frac{\lambda(y_{ij}-\theta_i)^2}{2}} \times \prod_{i=1}^{k} \lambda^{\frac{1}{2}}\psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i-\mu)^2}{2}} \times \lambda^{a-1} e^{-\lambda b}$$

$$= \lambda^{\frac{\sum_i n_i}{2}+\frac{k}{2}+a-1} \; e^{-\lambda\left(\sum_j\sum_i \frac{(y_{ij}-\theta_i)^2}{2} + \psi\sum_i \frac{(\theta_i-\mu)^2}{2} + b\right)}$$

$$\propto \text{Gamma}\left(\frac{\sum_i n_i}{2}+\frac{k}{2}+a \,,\; \sum_j\sum_i \frac{(y_{ij}-\theta_i)^2}{2} + \psi\sum_i \frac{(\theta_i-\mu)^2}{2} + b\right)$$

$$p(\psi|-) \propto \prod_{i=1}^{k} \lambda^{\frac{1}{2}}\psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i-\mu)^2}{2}} \psi^{c-1} e^{-\psi d}$$

$$\propto \psi^{\frac{k}{2}+c-1} \; e^{-\psi\left(\lambda\sum_i \frac{(\theta_i-\mu)^2}{2} + d\right)}$$

$$\propto \text{Gamma}\left(\frac{k}{2}+c \,,\; \lambda\sum_i \frac{(\theta_i-\mu)^2}{2} + d\right)$$

$$p(\theta_m|-) \propto \prod_{i=1}^{k}\prod_{j=1}^{n_i} \lambda^{\frac{1}{2}} e^{-\frac{\lambda(y_{ij}-\theta_i)^2}{2}} \times \prod_{i=1}^{k} \lambda^{\frac{1}{2}}\psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i-\mu)^2}{2}}$$

$$\propto e^{-\lambda\sum_j \frac{(y_{mj}-\theta_m)^2}{2}} \times e^{-\frac{\lambda\psi(\theta_m-\mu)^2}{2}} \qquad -2\theta_m\mu\lambda\psi$$

$$\theta_m^2(n_i\lambda + \lambda\psi) - 2\theta_m\left(\lambda\sum_j y_{mj} + \lambda\psi\mu\right)$$

$$\propto N\left((n_i\lambda+\lambda\psi)^{-1}\left(\lambda\sum_j y_{mj} + \lambda\psi\mu\right),\; (n_i\lambda+\lambda\psi)^{-1}\right)$$

2. $i = 1, 2, \ldots, K$

$j = 1, 2, \ldots, n_i$

$p = 1, 2$

$X_{i_{n_i \times p}}$ = design matrix of size $n_i \times p$

$\tilde{Y}_i = \log Y_i$

$\beta_{i_{p \times 1}}$

$1 \times p$

$$\log Y_{ij} = \beta_{i0} + \beta_{i1} x_{i1} + \beta_{i2} x_{i2} + \beta_{i3} \log(x_{i1}) \times x_{i2}$$

$$\tilde{Y}_i = X_i \beta_i + \varepsilon_i$$

$$\theta \sim MVN(\mu_0, \Lambda_0^{-1})$$

$$\Sigma \sim \text{Invwish}(\eta_0, S_0)$$

$$\lambda \sim \text{Gamma}(a_0, b_0)$$

$$\beta_i \sim MVN(\theta, \Sigma)$$

$$\tilde{Y}_i \mid \beta_i, \lambda \sim MVN(X_i \beta_i, \lambda I_{n_i \times n_i})$$

$$p(\tilde{Y}_i, \beta_i, \lambda, \theta, \Sigma) \propto \prod_{i=1}^{K} [p(\tilde{Y}_i \mid \beta_i, \lambda)\, p(\beta_i \mid \theta, \Sigma)] \times p(\theta) \times p(\Sigma) \times p(\lambda)$$

$$\propto \left[\prod_{i=1}^{K} |\lambda I|^{\frac{1}{2}} e^{-\frac{\lambda}{2}(\tilde{Y}_i - X_i\beta_i)^T(\tilde{Y}_i - X_i\beta_i)} \times |\Sigma|^{-\frac{1}{2}} e^{-\frac{(\beta_i - \theta)^T \Sigma^{-1} (\beta_i - \theta)}{2}} \right]$$

$$\times e^{-\frac{(\theta - \mu_0)^T \Lambda_0 (\theta - \mu_0)}{2}} \times |\Sigma|^{-\frac{(\eta_0 + p + 1)}{2}} e^{-\frac{1}{2} tr(S_0 \Sigma^{-1})}$$

$$\times \lambda^{a_0 - 1} e^{-\lambda b_0}$$

$$p(\beta_i | -) \propto p(\tilde{y}_i | \beta_i, \lambda) \, p(\beta_i | \theta, \Sigma)$$

$$\propto e^{-\frac{\lambda}{2}(\tilde{y}_i - X_i\beta_i)^T(\tilde{y}_i - X_i\beta_i)} \; e^{-\frac{(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}}$$

$$\beta_i^T(\lambda X_i^T X_i + \Sigma^{-1})\beta_i - 2\beta_i^T(\lambda X_i^T \tilde{y}_i + \Sigma^{-1}\theta)$$

$$\propto MVN\left((\lambda X_i^T X_i + \Sigma^{-1})^{-1}(\lambda X_i^T\tilde{y}_i + \Sigma^{-1}\theta), \; (\lambda X_i^T X_i + \Sigma^{-1})^{-1}\right)$$

$$p(\theta | -) \propto \prod_{i=1}^{k} p(\beta_i | \theta, \Sigma) \times p(\theta)$$

$$\propto e^{-\sum_{i=1}^{k}\frac{(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}} \; e^{-\frac{(\theta-\mu_0)^T\Lambda_0^{-1}(\theta-\mu_0)}{2}}$$

$$\theta^T(k\Sigma^{-1} + \Lambda_0^{-1})\theta - 2\theta^T(\Sigma^{-1}\sum_{i=1}^{k}\beta_i + \Lambda_0^{-1}\mu_0)$$

$$\propto MVN\left((k\Sigma^{-1} + \Lambda_0^{-1})^{-1}(\Sigma^{-1}\sum_{i=1}^{k}\beta_i + \Lambda_0^{-1}\mu_0), \; (k\Sigma^{-1} + \Lambda_0^{-1})^{-1}\right)$$

$$p(\Sigma | -) \propto \prod_{i=1}^{k} p(\beta_i | \theta, \Sigma) \times p(\Sigma)$$

$$\propto |\Sigma|^{-\frac{k}{2}} e^{-\sum_{i=1}^{k}\frac{(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}} \; |\Sigma|^{-\frac{(\eta_0+p+1)}{2}} \; e^{-\frac{1}{2}tr(S_0\Sigma^{-1})}$$

$$\propto |\Sigma|^{-\frac{[(k+\eta_0)+p+1]}{2}} \; e^{-\frac{tr(\sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T\Sigma^{-1})}{2}} \; e^{-\frac{1}{2}tr(S_0\Sigma^{-1})}$$

$$= |\Sigma|^{-\frac{[(k+\eta_0)+p+1]}{2}} \; e^{-\frac{tr((\sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T + S_0)\Sigma^{-1})}{2}}$$

$$\propto InvWish\left(k+\eta_0, \; \sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T + S_0\right)$$

$$p(\lambda|-) \propto \prod_{i=1}^{K} p(\tilde{y}_i|\beta_i,\lambda) \times p(\lambda)$$

$$\propto \prod_{i=1}^{K} |\lambda I|^{\frac{1}{2}} e^{-\frac{\lambda}{2}(\tilde{y}_i - X_i\beta_i)^T(\tilde{y}_i - X_i\beta_i)} \times \lambda^{a_0-1} e^{-\lambda b_0}$$

$$\propto \lambda^{\frac{\sum_{i=1}^{K} n_i}{2} + a - 1} e^{-\lambda\left(\frac{\sum_{i=1}^{K}(\tilde{y}_i - X_i\beta_i)^T(\tilde{y}_i - X_i\beta_i)}{2} + b_0\right)}$$

$$\propto \text{Gamma}\left(\frac{\sum_{i=1}^{K} n_i}{2} + a, \frac{\sum_{i=1}^{K}(\tilde{y}_i - X_i\beta_i)^T(\tilde{y}_i - X_i\beta_i)}{2} + b_0\right)$$

Model

$$\theta \sim MVN(\mu_0, L_0)$$

$$\Sigma \sim Invwish(\eta_0, S_0)$$

$$\beta_i | \theta, \Sigma \sim MVN(\theta, \Sigma)$$

$$z_i | \beta_i \sim MVN(X_i \beta_i, I_{n_i})$$

$$Y_{ij} | z_{ij} \sim I(Y_{ij}=1) I(z_{ij}>0) + I(Y_{ij}=0) I(z_{ij}<0)$$

$$p(z, \beta, \theta, \Sigma | Y) \propto p(Y, z, \beta, \theta, \Sigma)$$

$$= \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} p(Y_{ij} | z_{ij}) \right] \times \prod_{i=1}^{k} p(z_i | \beta_i) \times \prod_{i=1}^{k} p(\beta_i | \theta, \Sigma) \times p(\theta) \times p(\Sigma)$$

Gibbs Sampling

$$p(z_{ij} | -) \propto p(Y_{ij} | z_{ij}) p(z_{ij} | \beta_i)$$

$$\propto \left[I(Y_{ij}=1) I(z_{ij}>0) + I(Y_{ij}=0) I(z_{ij} \leq 0) \right] N(z_{ij} | x_i^T \beta_i, 1)$$

$$\propto \begin{cases} TruncN(x_{ij}^T \beta_i, 1, 0, \infty) & \text{if } Y_{ij}=1 \\ TruncN(x_{ij}^T \beta_i, 1, -\infty, 0) & \text{if } Y_{ij}=0 \end{cases}$$

$$p(\beta_i|-) \propto \prod_{i=1}^{K} p(z_i|\beta_i) \times \prod_{i=1}^{K} p(\beta_i|\theta,\Sigma)$$

$$\propto e^{\frac{-(z_i - X_i\beta_i)^T(z_i - X_i\beta_i)}{2}} \times e^{\frac{-(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}}$$

$$\beta_i^T(X_i^TX_i + \Sigma^{-1}) - 2\beta_i^T(X_i^Tz_i + \Sigma^{-1}\theta)$$

$$\propto MVN((X_i^TX_i + \Sigma^{-1})^{-1}(X_i^Tz_i + \Sigma^{-1}\theta), (X_i^TX_i + \Sigma^{-1})^{-1})$$

$$p(\theta|-) \propto \prod_{i=1}^{K} p(\beta_i|\theta,\Sigma) \times p(\theta)$$

$$\propto \prod_{i=1}^{K} e^{\frac{-(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}} \times e^{\frac{-(\theta-\mu_0)^TL_0^{-1}(\theta-\mu_0)}{2}}$$

$$\theta^T(K\Sigma^{-1} + L_0^{-1})\theta - 2\theta^T(\Sigma^{-1}\sum_{i=1}^{K}\beta_i + L_0^{-1}\mu_0)$$

$$\propto MVN((K\Sigma^{-1} + L_0^{-1})^{-1}(\Sigma^{-1}\sum_{i=1}^{K}\beta_i + L_0^{-1}\mu_0), (K\Sigma^{-1} + L_0^{-1})^{-1})$$

$$p(\Sigma|-) \propto \prod_{i=1}^{K} p(\beta_i|\theta,\Sigma) \times p(\Sigma)$$

$$\propto \prod_{i=1}^{K} |\Sigma|^{-\frac{1}{2}} e^{\frac{-(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}} \times |\Sigma|^{\frac{-(\eta_0+p+1)}{2}} e^{-\frac{1}{2}tr(S_0\Sigma^{-1})}$$

$$\propto |\Sigma|^{\frac{-(K+\eta_0+p+1)}{2}} e^{-tr((\sum_{i=1}^{K}(\beta_i-\theta)(\beta_i-\theta)^T + S_0)\Sigma^{-1})}$$

$$\propto \text{InvWish}(K+\eta_0, (\sum_{i=1}^{K}(\beta_i-\theta)(\beta_i-\theta)^T + S_0))$$

$$\frac{\sum (x_i - \bar{x})^2}{(n-1)}$$

$$\frac{\sum x_i^2 - n\bar{x}^2}{(n-1)}$$

$$\frac{\frac{\sum x_i^2}{n} - \bar{x}^2}{\frac{n-1}{n}}$$

$$\frac{n}{n-1}\left(\frac{\sum x_i^2}{n} - \bar{x}^2\right)$$

$\bar{x} \to E(X) \to \bar{x}^2 \to E(X)^2$

$\frac{\sum_i x^2}{n} \to E(X^2)$

$\frac{\sum x_i^2}{n} - \bar{x}^2 \to E(X^2) - E(X)^2 = \sigma^2$

$\frac{n}{n-1} \to 1$

$\lim_{n\to\infty} P(|Y_n - \theta| > \varepsilon) = 0$

$Y_n = \max\{x_1, \ldots, x_n\}$

$P(Y_n \le y) = P(X_1 \le y, X_2 \le y, \ldots, X_n \le y)$

$= P(X_1 \le y)^n$

$= F_{X_1}(y)^n$

$$F_{X_1}(y) = \begin{cases} 0 & y < 0 \\ \frac{y}{\theta} & 0 < y \le \theta \\ 1 & y > \theta \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \left(\frac{y}{\theta}\right)^n & 0 < y \leq \theta \\ 1 & y > \theta \end{cases}$$

$$P(\theta - \varepsilon \leq Y_n \leq \theta + \varepsilon) = F_Y(\theta + \varepsilon) - F_Y(\theta - \varepsilon)$$

$$= 1 - \left(\frac{y}{\theta}\right)^n \qquad \varepsilon < \theta$$

$$\longrightarrow 1$$

$$P(\theta - \varepsilon \leq Y_n \leq \theta + \varepsilon) = 1 \qquad \varepsilon > \theta$$

$$P(|Y_n - \theta| \geq \varepsilon)$$

$$P(Y_n - \theta \geq \varepsilon) + P(Y_n - \theta \leq -\varepsilon)$$

$$1 - F_Y(\theta + \varepsilon) + P(Y_n \leq \theta - \varepsilon)$$

$$1 - F_Y(\theta + \varepsilon) + F_{Y_n}(\theta - \varepsilon)$$

$$1 - 1 + F_{Y_n}(\theta - \varepsilon)$$

$$f(x) = \begin{cases} e^{-(x-\theta)} & x > \theta \\ 0 & o.w. \end{cases}$$

$$F(x) = \begin{cases} 0 & x \leq \theta \\ 1 - e^{-(x-\theta)} & \theta < x \end{cases}$$

$$\int_\theta^x e^{-(t-\theta)} \, dt$$

$$\int_0^{x-\theta} e^{-u} \, du \qquad u = t-\theta \quad du = dt$$

$$\left(-e^{-u} \Big|_0^{x-\theta}\right)$$

$$-e^{-(x-\theta)} + 1$$

$$1 - e^{-(x-\theta)}$$

$$Y_n = \min\{X_1, \ldots, X_n\}$$

$$\begin{aligned} P(Y_n \leq y) &= 1 - P(Y_n \geq y) \\ &= 1 - P(X_1 \geq y, \ldots, X_n \geq y) \\ &= 1 - \prod_{i=1}^{n} P(X_i \geq y) \\ &= 1 - P(X_1 \geq y)^n \\ &= 1 - [1 - F_{X_1}(y)]^n \end{aligned}$$

$$P(Y_n \leq y) = \begin{cases} 0 & \\ 1 - e^{-n(y-\theta)} & \theta < y \end{cases}$$

$$F_Y(\theta + \varepsilon) - F_Y(\theta - \varepsilon)$$

$$1 - e^{-n\varepsilon} - 0$$

$$\sqrt{\frac{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$$

$$\sqrt{\frac{n_2 S_1^2 + n_1 S_2^2}{n_2 \sigma_1^2 + n_1 \sigma_2^2}}$$

$S_1^2 \xrightarrow{P} \sigma_1^2$

$S_2^2 \rightarrow \sigma_2^2$

$$F_{X_n}(x) = \begin{cases} 1 & x \geq \frac{1}{n} \\ 0 & x < \frac{1}{n} \end{cases}$$

$P(|X_n - X| \geq \varepsilon)$

$P(|-2X| \geq \varepsilon)$

$P(|X| \geq \frac{\varepsilon}{2})$

$X_n - \varepsilon \leq X \leq X_n + \varepsilon$

$X_n \leq X + \varepsilon$

$X_n \leq x$

$$\left[(1-p)+pe^{t}\right]^{n}$$

$$\left[1-\frac{\mu}{n}+\frac{\mu e^{t}}{n}\right]^{n}$$

$$\left[1+\frac{\mu(e^{t}-1)}{n}\right]^{n}$$

$$\sum_{k=0}^{\infty} e^{tk}\frac{\lambda^{k}e^{-\lambda}}{k!}$$

$$e^{-\lambda}\sum_{k=0}^{\infty}\frac{(e^{t}\lambda)^{k}}{k!}$$

$$e^{-\lambda}e^{e^{t}\lambda}$$

$$e^{\lambda(e^{t}-1)}$$

$$\frac{1}{\Gamma(\frac{p}{2})}\left(\frac{1}{2}\right)^{\frac{p}{2}}\int_{0}^{\infty}e^{tx}\,x^{\frac{p}{2}-1}\,e^{-\frac{x}{2}}\,dx$$

$$\frac{1}{\Gamma(\frac{p}{2})}\frac{1}{2^{\frac{p}{2}}}\int_{0}^{\infty}x^{\frac{p}{2}-1}\,e^{-x(\frac{1}{2}-t)}\,dx$$

$$\frac{\frac{1}{2^{\frac{p}{2}}}}{\cancel{\Gamma(\frac{p}{2})}}\frac{\cancel{\Gamma(\frac{p}{2})}}{(\frac{1}{2}-t)^{\frac{p}{2}}}$$

$$\frac{1}{2}-t>0$$

$$\frac{1}{(1-2t)^{\frac{p}{2}}} \qquad \frac{1}{2} > t$$

$$M_Y(t) = E(e^{tY}) = E\left(e^{t\left(\frac{Z-n}{\sqrt{2n}}\right)}\right)$$

$$= E\left(e^{\frac{tZ}{\sqrt{2n}}} e^{-\frac{tn}{\sqrt{2n}}}\right)$$

$$= e^{-\frac{tn}{\sqrt{2n}}} E\left(e^{\frac{tZ}{\sqrt{2n}}}\right)$$

$$= e^{-\frac{t\sqrt{n}}{\sqrt{2}}} \left(1-\frac{\sqrt{2}t}{\sqrt{n}}\right)^{-\frac{n}{2}}$$

$$= \left(e^{\frac{\sqrt{2}t}{\sqrt{n}}}\left(1-\frac{\sqrt{2}t}{\sqrt{n}}\right)\right)^{-\frac{n}{2}}$$

$$\approx \left(\left(1+\frac{\sqrt{2}t}{\sqrt{n}}+\frac{t^2}{n}\right)\left(1-\frac{\sqrt{2}t}{\sqrt{n}}\right)\right)^{-\frac{n}{2}}$$

$$\approx 1 - \cancel{\frac{\sqrt{2}t}{\sqrt{n}}} + \cancel{\frac{\sqrt{2}t}{\sqrt{n}}} - \frac{2t^2}{n} + \frac{t^2}{n} - \frac{\sqrt{2}t^3}{n^{\frac{3}{2}}}$$

$$\approx \left[1 - \frac{t^2}{n} + \psi(n)\right]^{-\frac{n}{2}}$$

$$\lim_{n\to\infty} M_Y(t) = e^{(-t^2)\left(-\frac{1}{2}\right)} = e^{\frac{t^2}{2}}$$

$\sqrt{Z_n}$

$$\frac{Z-n}{\sqrt{2n}}$$

$$\sqrt{n}\left(\frac{Z-n}{\sqrt{2n}}\right)$$

$$\sqrt{n}\left(\frac{Z}{\sqrt{2}n}-\frac{1}{\sqrt{2}}\right) \to N(0,1)$$

$$g'(t)=\frac{1}{2(t^{\frac{1}{2}})}$$

$$\sqrt{n}\left(\sqrt{\frac{Z}{\sqrt{2}n}}-\sqrt{\frac{1}{\sqrt{2}}}\right) \to N(0, 2^{-\frac{3}{2}})$$

$$g'(2^{-\frac{1}{2}})=\frac{1}{2(2^{-\frac{1}{4}})}$$

$$(g')^2 = 2^{-\frac{3}{2}} = \frac{1}{2^{\frac{3}{4}}}$$

$$\sqrt{n}\left(\frac{Z}{\sqrt{2}n}-\frac{1}{\sqrt{2}}\right) \to N(0,1)$$

$$\sqrt{n}\left(\sqrt{\frac{Z}{\sqrt{2}n}}-\sqrt{\frac{1}{\sqrt{2}}}\right) \to N(0, 2^{-\frac{3}{2}})$$

$$g(t)=\sqrt{t}$$

$$g'(t)=\frac{1}{2\sqrt{t}}$$

$$g'(2^{-\frac{1}{2}})=\frac{1}{2(2^{-\frac{1}{4}})}$$

$$=\frac{1}{2^{\frac{3}{4}}}=2^{-\frac{3}{4}}$$

$$\left[g'(2^{-\frac{1}{2}})\right]^2 = 2^{-\frac{3}{2}}$$

$$n\left(\frac{Y_n}{n}\right)\left(1-\frac{Y_n}{n}\right)$$

$$\frac{Y_n - np}{\sqrt{np(1-p)}} \sim N(0,1)$$

$$\frac{\sum_{i=1}^{n} I(Y_i \leq f(X_i))}{n}$$

WLLN

$$\rightarrow E(I(Y_i \leq f(X_i)))$$

$$= P(Y \leq f(X))$$

$$= \int_0^1 \int_0^{f(x)} dy\, dx$$

$$= \int_0^1 f(x)\, dx$$

$$E_{X,Y}(g(X,Y))$$

$$E_X E_{Y|X}(g(X,Y))$$

$$\int\int f_{X,Y}(x,y)\, g(x,y)\, dx\, dy$$

$$\int f_X \int f_{Y|X}(y|x)\, g(x,y)\, dy\, dx$$

$$E(Y)$$

$$E_Y E(X|Y)$$

$$\frac{1}{n^2} var\left(\sum_{i=1}^{n} I(y_i < f(x_i))\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} var(I(y_i < f(x_i)))$$

$$var(I(y < f(x)))$$

$$= P(Y < f(x))(1 - P(Y < f(x)))$$

$$= \left(\int_0^1 f(x)dx\right)\left(1 - \int_0^1 f(x)dx\right)$$

$$\frac{1}{n^2} var\left(\sum_{i=1}^{n} f(x_i)\right) \qquad = \int_0^1 f(x)dx - \left(\int_0^1 f(x)dx\right)^2$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} var(f(x_i))$$

$$var(f(x_i)) = E(f(x_i)^2) - E(f(x_i))^2$$

$$= \int_0^1 f(x)^2 dx - \left(\int_0^1 f(x)dx\right)^2$$

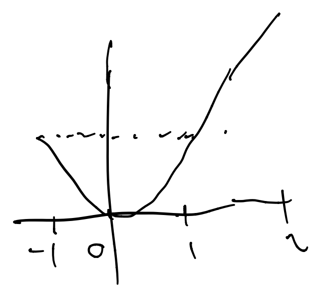
$$f(x)^2 < f(x)$$

$$E[Y] = E\,E[Y|X]$$

$$\int \int y\, f_{Y|X}(y|x)\, dy\, f(x)\, dx$$

$$\int y \int f_{Y,X}(y,x)\, dx\, dy$$

$$\int y\, f_Y(y)\, dy$$



$$f(x) = \begin{cases} \frac{1}{3} & -1 \le x < 2 \\ 0 & \text{o.w.} \end{cases}$$

$-1 < X < 0$ $\quad$ $0 < X < 1$ $\quad$ $1 < X < 2$

$u = x^2$

$|x| = \sqrt{u}$

$x = -\sqrt{u}$

$x = \sqrt{u}$

$\frac{dx}{du} = \frac{1}{2\sqrt{u}}$

$$\frac{dx}{du} = \frac{-1}{2\sqrt{u}}$$

$$f_u(u) = \frac{1}{3}\frac{1}{2\sqrt{u}} + \frac{1}{2\sqrt{u}}\frac{1}{3}$$

$$= \frac{1}{3\sqrt{u}} \qquad 0 < u < 1$$

$$f_u(u) = \frac{1}{6\sqrt{u}} \qquad 1 < u < 4$$

$$\int_0^1 \frac{1}{3\sqrt{u}}\,du + \int_1^4 \frac{1}{6\sqrt{u}}\,du$$

$$\left(\frac{2u^{\frac{1}{2}}}{3}\Big|_0^1\right) + \left(\frac{1}{3}u^{\frac{1}{2}}\Big|_1^4\right)$$

$$\frac{2}{3} + \left(\frac{2}{3} - \frac{1}{3}\right)$$

$$1$$

$$P(X = x) = \frac{1}{3} \qquad x = -1, 1, 0$$

$$Y = X^2$$

$$Y = \begin{cases} 1 & \text{w.p. } \frac{1}{3}+\frac{1}{3} = \frac{2}{3} \\ 0 & \text{w.p. } \frac{1}{3} \end{cases}$$

$\mathrm{Cov}(Y, X)$

$\mathrm{Cov}(X, X^2)$

$= E(X^3) - E(X)E(X^2)$

$= 0 \quad - 0$

$-\frac{1}{3} + \frac{1}{3}$

$\mathrm{Cov}(X_1, X_2)$

$\mathrm{Cov}(Z, Z^2)$

$E(Z^3) - E(Z)E(Z^2)$

$P(Z \leq t, Z^2 \leq s)$

$P(Z$

$$P(0<X<1, Y>1) = 0$$

$$\neq P(0<X<1)\,P(Y>1)$$

$$X = u^{\frac{1}{n}}$$

$$\left|\frac{dx}{du}\right| = \left|\frac{1}{n} u^{\frac{1}{n}-1}\right|$$

$$f_U(u) = f_X(x)\left|\frac{dx}{du}\right|$$

$$= \frac{1}{\alpha}\cancel{\alpha}\, u^{\frac{n-1}{n}} e^{-\frac{u}{\alpha}} \frac{1}{\cancel{\alpha}} u^{\frac{1}{n}-1}$$

$$= \frac{1}{\alpha} e^{-\frac{u}{\alpha}}$$

$$\frac{b}{a} x^{b-1} e^{-\frac{x^b}{a}}$$

$$\int \frac{b}{a}\, x^k\, x^{b-1} e^{-\frac{x^b}{a}}\, dx$$

$$\int_0^\infty (at)^{\frac{k}{b}} e^{-t}\, dt$$

$$\frac{x^b}{a} = t \qquad \frac{dt}{dx} = \frac{bx^{b-1}}{a}$$

$$a^{\frac{k}{b}} \int_{0}^{\infty} t^{\left(\frac{k}{b}+1\right)-1} e^{-t}\, dt \qquad x = (at)^{\frac{1}{b}} \qquad dt = \frac{b x^{b-1}}{a}\, dt$$

$$a^{\frac{k}{b}}\, \Gamma\left(\frac{k}{b}+1\right)$$