(B)

$$f \sim GP(m, C)$$

$$\rightarrow f = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix} \sim MVN(m_x, C_{x \times x})$$

$$\tilde{f} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \\ f(x^*) \end{bmatrix} \sim MVN(\tilde{m}, C_{x, x^*})$$

$$\tilde{m} = \begin{bmatrix} m_{x_1} \\ m_{x_2} \\ \vdots \\ m_{x_N} \\ m_{x^*} \end{bmatrix} \qquad \tilde{C} = \begin{bmatrix} C_{x \times x} & C_{x \times x^*} \\ C_{x \times x^*} & C_{x^* \times x^*} \end{bmatrix}$$

$$f(x^* | x) \sim MVN(\mu, \Sigma)$$

$$\Sigma = C_{x^* \times x^*} - C_{x^* \times x} C_{x \times x}^{-1} C_{x \times x^*}$$

$$\mu = m_{x^*} + C_{x^* \times x} C_{x \times x}^{-1} (f(x) - m_x)$$

Conditional derivation

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$$

$$f(x_2 | x_1) \propto f(x_1, x_2)$$

$$\propto e^{-\frac{(x-\mu)^T \Lambda (x-\mu)}{2}}$$

looking at the exponent

$$-\frac{1}{2} \left(\begin{bmatrix} (x_1-\mu_1)^T & (x_2-\mu_2)^T \end{bmatrix} \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} \right)$$

$$-\frac{1}{2} [(x_1-\mu_1)^T \Lambda_{11} (x_1-\mu_1) + (x_2-\mu_2)^T \Lambda_{21} (x_1-\mu_1)$$

$$+ (x_1-\mu_1)^T \Lambda_{12} (x_2-\mu_2) + (x_2-\mu_2)^T \Lambda_{22} (x_2-\mu_2)]$$

$$\propto -\frac{1}{2} [x_2^T \Lambda_{22} x_2 - 2 x_2^T \Lambda_{22} \mu_2 + x_1^T \Lambda_{12} x_2 - \mu_1^T \Lambda_{12} x_2$$

$$+ x_2^T \Lambda_{21} x_1 - x_2^T \Lambda_{21} \mu_1]$$

$$= \frac{1}{2} [x_2^T \Lambda_{22} x_2 - 2 x_2^T \Lambda_{22} \mu_2 + 2 x_2^T \Lambda_{21} x_1 - 2 x_2^T \Lambda_{21} \mu_1)$$

$$= -\frac{1}{2} [x_2^T \Lambda_{22} x_2 - 2 x_2^T (\Lambda_{22} \mu_2 - \Lambda_{21} (x_1-\mu_1))]$$

$$\tilde{\Lambda} = \Lambda_{22} = (\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})^{-1}$$

$$\tilde{\Lambda}\tilde{\mu} = (\Lambda_{22}\mu_2 - \Lambda_{21}(x_1 - \mu_1))$$

$$\tilde{\mu} = \mu_2 - \Lambda_{22}^{-1}\Lambda_{21}(x_1 - \mu_1)$$

$$= \mu_2 + (\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})(\Sigma_{21} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})^{-1}\Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1)$$

$$= \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1)$$

$$\tilde{\Sigma} = (\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$$

(c)

$$Y \mid \theta \sim N(R\theta, \Sigma)$$

$$\theta \sim N(M, V)$$

$$p(Y, \theta) = p(Y \mid \theta)\, p(\theta)$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} e^{\frac{-(Y-R\theta)^T \Sigma^{-1} (Y-R\theta)}{2}} \frac{1}{(2\pi)^{\frac{p}{2}} |V|^{\frac{1}{2}}} e^{\frac{-(\theta-M)^T V^{-1} (\theta-M)}{2}}$$

looking at the exponent only

$$-\frac{1}{2}\left[Y^T \Sigma^{-1} Y - 2 Y^T \Sigma^{-1} R\theta + \theta^T R^T \Sigma^{-1} R\theta + \theta^T V^{-1} \theta - 2\theta^T V^{-1} M + M^T V^{-1} M \right]$$

We recognize this as kernel of MVN

Now, we will find the precision matrix by looking at the second order terms only.

$$Y^T \Sigma^{-1} Y - 2Y^T \Sigma^{-1} R\theta + \theta^T (R^T \Sigma^{-1} R + V^{-1}) \theta$$

$$\begin{bmatrix} Y^T & \theta^T \end{bmatrix} \underbrace{\begin{bmatrix} \Sigma^{-1} & -\Sigma^{-1} R \\ -R^T \Sigma^{-1} & R^T \Sigma^{-1} R + V^{-1} \end{bmatrix}}_{S} \begin{bmatrix} Y \\ \theta \end{bmatrix}$$

Now, since we recognize this as kernel of MVN, the mean will simply be marginal mean. of each parameter.

$$\begin{bmatrix} Y \\ \theta \end{bmatrix} \sim N\left(\begin{bmatrix} Rm \\ m \end{bmatrix}, \begin{bmatrix} \Sigma^{-1} & -\Sigma^{-1}R \\ -R^T\Sigma^{-1} & R^T\Sigma^{-1}R + V^{-1} \end{bmatrix}^{-1} \right)$$

$$\text{or } N\left(\begin{bmatrix} Rm \\ m \end{bmatrix}, \begin{bmatrix} \Sigma + RVR^T & RV \\ V^TR^T & V \end{bmatrix} \right)$$

$$\begin{aligned} E(Y) &= E(E(Y|\theta)) \\ &= E(R\theta) \\ &= RE(\theta) \\ &= Rm \end{aligned}$$

$$\begin{aligned} E(YY^T) &= E(E(YY^T|\theta)) \\ &= E(\Sigma + E(Y|\theta)E(Y|\theta)^T) \\ &= \Sigma + E(R\theta\theta^TR^T) \\ &= \Sigma + RE(\theta\theta^T)R^T \\ &= \Sigma + R(V + mm^T)R^T \end{aligned}$$

$$\begin{aligned} E(Y\theta^T) &= E(E(Y\theta^T|\theta)) \\ &= E(E(Y|\theta)\theta^T) \\ &= E(R\theta\theta^T) \\ &= RE(\theta\theta^T) \\ &= R(V + mm^T) \end{aligned}$$

$$\mathrm{Cov}(\theta) = V$$

$$\begin{aligned}
\mathrm{Cov}(Y) &= E(YY^T) - E(Y)E(Y)^T \\
&= \Sigma + R(V + MM^T)R^T - RMM^TR^T \\
&= \Sigma + RVR^T
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(Y,\theta) &= E(Y\theta^T) - E(Y)E(\theta)^T \\
&= R(V + MM^T) - RMM^T \\
&= RV
\end{aligned}$$