(A)

$$p(\theta) = \int_0^\infty p(\theta, w)\, dw$$

$$\propto \int w^{\frac{1}{2}} e^{-\frac{wk(\theta-\mu)^2}{2}} w^{\frac{d}{2}-1} e^{-w\left(\frac{\eta}{2}\right)} dw$$

$$= \int w^{\frac{d+1}{2}-1} e^{-w\left(\frac{k(\theta-\mu)^2+\eta}{2}\right)} dw$$

$\propto \text{Gamma}\left(\frac{d+1}{2}, \frac{k(\theta-\mu)^2}{2}+\eta\right)$

$$= \frac{\Gamma\left(\frac{d+1}{2}\right)}{\left(\frac{k(\theta-\mu)^2+\eta}{2}\right)^{\frac{d+1}{2}}}$$

$$\propto \frac{1}{\left(\frac{\eta}{2}\right)^{\frac{d+1}{2}}\left(1+\frac{k(\theta-\mu)^2}{\eta}\right)^{\frac{d+1}{2}}}$$

$$\propto \frac{1}{\left(1+\frac{1}{d}\frac{dk(\theta-\mu)^2}{\eta}\right)^{\frac{d+1}{2}}}$$

$M = \mu$

$v = d$

$s^2 = \frac{\eta}{dk}$

(B) $\vec{y} = (y_1, \ldots, y_n)$

$$p(\theta, w \mid \vec{y}) \propto p(\theta, w, \vec{y})$$

$$= p(\vec{y} \mid \theta, w)\, p(\theta \mid w)\, p(w)$$

$$\propto \prod_{i=1}^{n} p(y_i \mid \theta, w)\, p(\theta \mid w)\, p(w)$$

$$= \prod_{i=1}^{n} w^{\frac{1}{2}} e^{\frac{-w(y_i-\theta)^2}{2}} \; w^{\frac{d+1}{2}-1} e^{\frac{-wk(\theta-\mu)^2}{2}} \; e^{-\frac{w\eta}{2}}$$

$$* = w^{\frac{n+d+1}{2}-1} \; e^{-\frac{w\left(\sum_{i=1}^{n}(y_i-\theta)^2 + k(\theta-\mu)^2\right)}{2}} \; e^{-\frac{w\eta}{2}}$$

looking at the exponent

$$\sum_{i=1}^{n}(y_i^2 - 2\theta y_i + \theta^2) + k\theta^2 - 2\theta k\mu + k\mu^2$$

$$\sum_{i=1}^{n} y_i^2 - 2\theta n\bar{y} + n\theta^2 + k\theta^2 - 2\theta k\mu + k\mu^2$$

$$\theta^2(n+k) - 2\theta(n\bar{y}+k\mu) + \sum_{i=1}^{n} y_i^2 + k\mu^2$$

continue the work

$$* = w^{\frac{n+d+1}{2}-1} \; e^{-\frac{w(n+k)}{2}\left(\theta - \left(\frac{n\bar{y}+k\mu}{n+k}\right)\right)^2} \; e^{-\frac{w(n+k)}{2}\left(-\left(\frac{n\bar{y}+k\mu}{n+k}\right)^2 + \frac{\sum y_i^2 + k\mu^2}{(n+k)}\right)} \; e^{-\frac{w\eta}{2}}$$

$$= w^{\frac{n+d+1}{2}-1} \; e^{-\frac{w(n+k)}{2}\left(\theta - \left(\frac{n\bar{y}+k\mu}{n+k}\right)\right)^2} \; e^{-\frac{w}{2}\left((n+k)\left(\frac{\sum y_i^2 + k\mu^2}{(n+k)} - \left(\frac{n\bar{y}+k\mu}{n+k}\right)^2\right) + \eta\right)}$$

$d^* = n+d$

$k^* = n+k$

$\mu^* = \left(\frac{n\bar{y}+k\mu}{n+k} \right)$

$\eta^* = \Sigma y_i^2 + k\mu^2 - \frac{(n\bar{y}+k\mu)^2}{k^*}) + \eta$

c) By properties of normal - gamma distribution.

Conditioning on w and y, $\theta | w, y \sim N(\mu^*, (wk^*)^{-1})$

D) This also comes for free. Since marginally w is a gamma. $w \sim \text{Gamma}\left(\frac{d^*}{2}, \frac{\eta^*}{2}\right)$

Alright, I will do the integration.

$p(w|y) = \int_{-\infty}^{\infty} p(\theta, w|y)\, d\theta$

$$\propto \int_{-\infty}^{\infty} w^{\frac{d^*+1}{2}-1} e^{-w\frac{k^*(\theta-\mu)^2}{2}} e^{-w\frac{\eta^*}{2}}\, d\theta$$

$$= w^{\frac{d^*+1}{2}-1} e^{-w\frac{\eta^*}{2}} \int_{-\infty}^{\infty} e^{-w\frac{k^*(\theta-\mu)^2}{2}}\, d\theta$$

$N(\mu, (wk^*)^{-1})$

$$= w^{\frac{d^*+1}{2}-1} e^{-w\frac{\eta^*}{2}} (wk^*)^{-\frac{1}{2}}$$

$$\propto w^{\frac{d^*}{2}-1} e^{-\frac{w\eta^*}{2}}$$

$$\propto \text{Gamma}\left(\frac{d^*}{2}, \frac{\eta^*}{2}\right)$$

(E)

From C)

$$\theta \mid w, y \sim N(\mu^*, (wk^*)^{-1})$$

so from A)

$$m = \mu^*$$

$$v = d^*$$

$$\theta \mid y \sim t(\mu^*, s^2, v)$$

$$s^2 = \frac{\eta^*}{2d^* k^*}$$

$$t\left(\mu = \left(\frac{n\bar{y} + k\mu}{n+k}\right), v = n+d, s^2 = \frac{\sum y_i^2 + k\mu^2 - \frac{(n\bar{y} + k\mu)^2}{k^*} + \eta}{(n+d)(n+k)}\right)$$

F) As η, d, and k approach 0.

$p(w) \rightarrow \text{Gamma}(0, 0)$ which is not a valid prob dist

$p(\theta) \rightarrow t(0)$ which is not a valid prob dist

G)

$$\theta \mid y \sim t\left(\mu = \left(\frac{n\bar{y} + k\mu}{n+k}\right), v = n+d, s^2 = \frac{\sum y_i^2 + k\mu^2 - \frac{(n\bar{y} + k\mu)^2}{k^*} + \eta}{(n+d)(n+k)}\right)$$

$$\rightarrow t\left(\mu = \bar{y}, v = n, s^2 = \frac{\sum y_i^2 - n(\bar{y})^2}{n^2} = \frac{S_y}{n^2}\right)$$

$$w \mid y \sim \text{Gamma}\left(\frac{n+d}{2}, \frac{\sum y_i^2 + k\mu^2 - \frac{(n\bar{y} + k\mu)^2}{n+k} + \eta}{2}\right)$$

$$\rightarrow \text{Gamma}\left(a = \frac{n}{2}, b = \sum y_i^2 - n\bar{y}^2 = \frac{S_y}{2}\right)$$

(H) From (E)

$$\theta | y \sim t\left(\mu = \left(\frac{n\bar{y} + k\mu}{n+k}\right),\ v = n+d,\ s^2 = \frac{\Sigma y_i^2 + k\mu^2 - \frac{(n\bar{y}+k\mu)^2}{k^*} + \eta}{(n+d)(n+k)}\right)$$

$$s = \sqrt{\frac{\Sigma y_i^2 + k\mu^2 - \frac{(n\bar{y}+k\mu)^2}{k^*} + \eta}{(n+d)(n+k)}}$$

as η, d, k approch 0

$$\mu \to \bar{y}$$

$$s \to \sqrt{\frac{\sum_{i=1}^{n} y_i^2 - n\bar{y}^2}{n^2}}$$

$$= \sqrt{\frac{s_y}{n^2}}$$

so the credible interval is

$$\theta \in \bar{y} \pm t_{\frac{\alpha}{2}} \sqrt{\frac{s_y}{n^2}}$$

A frequentist interval is

$$\theta \in \bar{y} \pm t_{\frac{\alpha}{2}} \sqrt{\frac{s_y}{n(n-1)}}$$

We can see that as $n \to \infty$ the intervals approach the same quantity.

(A)

$K_{p \times p}$ $\Lambda_{n \times n}$

$$e^{-\frac{(\beta-m)^T wK(\beta-m)}{2}} \quad w^{\frac{p}{2}+\frac{d}{2}-1} \; e^{-w\left(\frac{\eta}{2}\right)}$$

$$p(\beta, w \mid y) \propto p(y \mid \beta, w)\, p(\beta \mid w)\, p(w)$$

$$\propto |w\Lambda|^{\frac{1}{2}} e^{-\frac{(y-X\beta)^T w\Lambda(y-X\beta)}{2}} \; |wK|^{\frac{1}{2}} e^{-\frac{(\beta-m)^T wK(\beta-m)}{2}} \; w^{\frac{d}{2}-1} e^{-w\left(\frac{\eta}{2}\right)}$$

$$= w^{\frac{n}{2}+\frac{p}{2}+\frac{d}{2}-1} \underbrace{e^{-\frac{(y-X\beta)^T w\Lambda(y-X\beta)}{2}} \; e^{-\frac{(\beta-m)^T wK(\beta-m)}{2}}} \; e^{-w\left(\frac{\eta}{2}\right)}$$

looking at the exponent for these terms

$$-\frac{w}{2}\left(y^T\Lambda y - 2\beta^T X^T\Lambda y + \beta^T X^T\Lambda X\beta + \beta^T K\beta - 2\beta^T K m + m^T K m \right)$$

$$-\frac{w}{2}\left(\beta^T (X^T\Lambda X + K)\beta - 2\beta^T (X^T\Lambda y + Km) + \tilde{m}^T\tilde{K}\tilde{m} \right) - \frac{w}{2}\left(y^T\Lambda y + m^T K m - \tilde{m}^T\tilde{K}\tilde{m} \right)$$

$$\tilde{K} = (X^T\Lambda X + K)$$

$$\tilde{m} = \tilde{K}^{-1}(X^T\Lambda y + Km)$$

$$-\frac{w}{2}\left((\beta-\tilde{m})^T\tilde{K}(\beta-\tilde{m}) \right) - \frac{w}{2}\left(y^T\Lambda y + m^T K m - \tilde{m}^T\tilde{K}\tilde{m} \right)$$

continue =

$$= w^{\frac{n}{2}+\frac{p}{2}+\frac{d}{2}-1} \; e^{-\frac{w}{2}(\beta-\tilde{m})^T\tilde{K}(\beta-\tilde{m})} \; e^{-\frac{w}{2}(y^T\Lambda y + m^T K m - \tilde{m}^T\tilde{K}\tilde{m})} \; e^{-w\left(\frac{\eta}{2}\right)}$$

$$w^{\frac{p}{2}+\frac{\tilde{d}}{2}-1} e^{-\frac{(\beta-\tilde{m})^T w \tilde{K}(\beta-\tilde{m})}{2}} e^{-w\left(\frac{\tilde{\eta}}{2}\right)}$$

$\propto$ NormalGamma$(\tilde{m}, \tilde{K}, \tilde{d}, \tilde{\eta})$

$$\tilde{d} = n+d \qquad \tilde{\eta} = y^T \Lambda y + m^T K m - \tilde{m}^T \tilde{K} \tilde{m} + \eta$$

$$\tilde{K} = (X^T \Lambda X + K) \qquad \tilde{m} = \tilde{K}^{-1}(X^T \Lambda y + Km)$$

Thus, $\quad \beta \mid y, w \sim N(\tilde{m}, (w\tilde{K})^{-1})$

(B) By properties of normal gamma

$$w \mid y \sim \text{Gamma}\left(\frac{\tilde{d}}{2}, \frac{\tilde{\eta}}{2}\right)$$

(C)

$$p(\beta \mid y) = \int_0^\infty p(\beta, w \mid y)\, dw$$

$$\propto \int_0^\infty w^{\frac{p}{2}+\frac{\tilde{d}}{2}-1} e^{-\frac{(\beta-\tilde{m})^T w \tilde{K}(\beta-\tilde{m})}{2}} e^{-w\left(\frac{\tilde{\eta}}{2}\right)} dw$$

$$= \int_0^\infty w^{\frac{p}{2}+\frac{\tilde{d}}{2}-1} e^{-w\left(\frac{(\beta-\tilde{m})^T \tilde{K}(\beta-\tilde{m}) + \tilde{\eta}}{2}\right)} dw$$

$$= \frac{\Gamma\left(\frac{p}{2}+\frac{\tilde{d}}{2}\right)}{\left(\frac{\tilde{\eta} + (\beta-\tilde{m})^T \tilde{K}(\beta-\tilde{m})}{2}\right)^{\frac{p}{2}+\frac{\tilde{d}}{2}}}$$

$$= \left(\frac{\tilde{\eta}}{2}\right)^{\frac{p+\tilde{d}}{2}} \frac{1}{\left(1 + \frac{1}{\tilde{\eta}}(\beta-\tilde{m})^T \tilde{K} (\beta - \tilde{m})\right)^{\frac{p}{2}+\frac{\tilde{d}}{2}}}$$

$$= \frac{1}{\left(1 + \frac{\tilde{d}}{\tilde{\eta}} \frac{(\beta-\tilde{m})^T \tilde{K} (\beta - \tilde{m})}{\tilde{d}}\right)^{\frac{p}{2}+\frac{\tilde{d}}{2}}}$$

$$\propto t(v, \mu, \Sigma^{-1})$$

$$\mu = \tilde{m}$$

$$\Sigma^{-1} = \frac{\tilde{d}}{\tilde{\eta}} \tilde{K}$$

$$v = \tilde{d}$$

(A)

$$p(y_i \mid \beta, w, \lambda_i) \propto |\Lambda|^{\frac{1}{2}} e^{-\frac{w\sum_{i=1}^{n} \lambda_i (y_i - \mu_i)^2}{2}} \qquad \mu_i = (X\beta)_i$$

$$\propto \lambda_i^{\frac{1}{2}} e^{-\frac{w\lambda_i (y_i - \mu_i)^2}{2}}$$

$$p(y_i \mid X, \beta, w) = \int_0^\infty p(y_i, \lambda_i \mid X, \beta, w)\, d\lambda_i$$

$$= \int_0^\infty p(y_i, \lambda_i, X, \beta, w)\, d\lambda_i$$

$$= \int_0^\infty p(y_i \mid \beta, w, \lambda_i)\, p(\lambda_i)\, p(\beta \mid w)\, p(w)\, d\lambda_i$$

$$\propto \int_0^\infty p(y_i \mid \beta, w, \lambda_i)\, p(\lambda_i)\, d\lambda_i$$

$$= \int_0^\infty \lambda_i^{\frac{1}{2}} e^{-\frac{w\lambda_i (y_i - \mu_i)^2}{2}} \lambda_i^{\frac{h}{2}-1} e^{-\lambda_i\left(\frac{h}{2}\right)}\, d\lambda_i$$

$$= \int_0^\infty \lambda_i^{\frac{h}{2}+\frac{1}{2}-1} e^{-\lambda_i\left(\frac{w(y_i-\mu_i)^2}{2}+\frac{h}{2}\right)}\, d\lambda_i$$

$$= \frac{\Gamma\left(\frac{h}{2}+\frac{1}{2}\right)}{\left(\frac{w(y_i-\mu_i)^2+h}{2}\right)^{\frac{h}{2}+\frac{1}{2}}}$$

$$\propto \frac{1}{\left(\frac{h}{2}\right)^{\frac{h}{2}+\frac{1}{2}}\left(1+\frac{w(y_i-\mu_i)^2}{h}\right)^{\frac{h}{2}+\frac{1}{2}}}$$

$$\propto \frac{1}{\left(1+\frac{w(y_i-\mu_i)^2}{h}\right)^{\frac{h}{2}+\frac{1}{2}}}$$

$$\propto t(v, m, s^2)$$

$v = h$

$m = \mu_i$

$s^2 = \frac{1}{w}$

(B)

$$p(\lambda_i \mid y, \beta, w) \propto p(y, \Lambda, \beta, w)$$

$$= p(y \mid \beta, w, \Lambda)\, p(\Lambda)\, p(\beta \mid w)\, p(w)$$

$$\propto p(y \mid \beta, w, \Lambda)\, p(\lambda_i)$$

$$= |\Lambda|^{\frac{1}{2}} e^{-\frac{(y - X\beta)^T w \Lambda (y - X\beta)}{2}} \lambda_i^{\frac{h}{2}-1} e^{-\lambda_i\left(\frac{h}{2}\right)}$$

$$\propto \lambda_i^{\frac{1}{2}} e^{-\frac{w \lambda_i (y_i - \mu_i)^2}{2}} \lambda_i^{\frac{h}{2}-1} e^{-\lambda_i\left(\frac{h}{2}\right)}$$

$\mu_i = (X\beta)_i$

$$= \lambda_i^{\frac{h}{2}+\frac{1}{2}-1} e^{-\lambda_i\left(\frac{w(y_i-\mu_i)^2}{2} + \frac{h}{2}\right)}$$

$$\propto \text{Gamma}\left(\frac{h+1}{2}, \frac{w(y_i-\mu_i)^2 + h}{2}\right)$$