1. $i = 1, 2, \ldots, k$

$j = 1, 2, \ldots, n_i$

Priors

$\mu \sim N(\mu, v)$

$\lambda \sim \text{Gamma}(a, b)$

$\psi \sim \text{Gamma}(c, d)$

$\lambda = \frac{1}{\sigma^2} \qquad \psi = \frac{1}{\tau^2}$

$$p(\mu, \lambda, \psi, \theta \mid Y) \propto \prod_{i=1}^{k} \prod_{j=1}^{n_i} p(y_{ij} \mid \theta_i, \lambda) \times \prod_{i=1}^{k} p(\theta_i \mid \mu, \lambda, \psi) \times p(\mu)\, p(\lambda)\, p(\psi)$$

$$\propto \prod_{i=1}^{k} \prod_{j=1}^{n_i} \lambda^{\frac{1}{2}} e^{\frac{-\lambda(y_{ij} - \theta_i)^2}{2}} \times \prod_{i=1}^{k} \lambda^{\frac{1}{2}} \psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i - \mu)^2}{2}} \times e^{\frac{-\phi(\mu - v)^2}{2}} \lambda^{a-1} e^{-\lambda b} \psi^{c-1} e^{-\psi d}$$

$$p(\mu \mid -) \propto \prod_{i=1}^{k} \lambda^{\frac{1}{2}} \psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i - \mu)^2}{2}} \times e^{\frac{-\phi(\mu - v)^2}{2}}$$

$$\propto e^{\frac{-\lambda\psi \sum_i (\theta_i - \mu)^2}{2}} \times e^{\frac{-\phi(\mu - v)^2}{2}}$$

$$\propto N\left((k\lambda\psi + \phi)^{-1}(\lambda\psi \textstyle\sum_i \theta_i + \phi v),\ (k\lambda\psi + \phi)^{-1}\right)$$

$\mu^2 (k\lambda\psi + \phi)$

$-2\mu(\lambda\psi \sum_i \theta_i + \phi v)$

$$p(\lambda|-) \propto \prod_{i=1}^{k}\prod_{j=1}^{n_i} \lambda^{\frac{1}{2}} e^{\frac{-\lambda(y_{ij}-\theta_i)^2}{2}} \times \prod_{i=1}^{k} \lambda^{\frac{1}{2}}\psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i-\mu)^2}{2}} \times \lambda^{a-1} e^{-\lambda b}$$

$$= \lambda^{\frac{\sum_i n_i}{2} + \frac{k}{2} + a - 1} \; e^{-\lambda\left(\sum_j \sum_i \frac{(y_{ij}-\theta_i)^2}{2} + \psi \sum_i \frac{(\theta_i-\mu)^2}{2} + b\right)}$$

$$\propto \text{Gamma}\left(\frac{\sum_i n_i}{2} + \frac{k}{2} + a, \; \sum_j \sum_i \frac{(y_{ij}-\theta_i)^2}{2} + \psi \sum_i \frac{(\theta_i-\mu)^2}{2} + b\right)$$

$$p(\psi|-) \propto \prod_{i=1}^{k} \lambda^{\frac{1}{2}}\psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i-\mu)^2}{2}} \psi^{c-1} e^{-\psi d}$$

$$\propto \psi^{\frac{k}{2}+c-1} \; e^{-\psi\left(\lambda \sum_i \frac{(\theta_i-\mu)^2}{2} + d\right)}$$

$$\propto \text{Gamma}\left(\frac{k}{2}+c, \; \lambda \sum_i \frac{(\theta_i-\mu)^2}{2} + d\right)$$

$$p(\theta_m|-) \propto \prod_{i=1}^{k}\prod_{j=1}^{n_i} \lambda^{\frac{1}{2}} e^{\frac{-\lambda(y_{ij}-\theta_i)^2}{2}} \times \prod_{i=1}^{k} \lambda^{\frac{1}{2}}\psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i-\mu)^2}{2}}$$

$$\propto e^{-\lambda \sum_j \frac{(y_{mj}-\theta_m)^2}{2}} \times e^{-\frac{\lambda\psi(\theta_m-\mu)^2}{2}} \qquad -2\theta_m \mu \lambda \psi$$

$$\theta_m^2(n_i\lambda + \lambda\psi) - 2\theta_m\left(\lambda \sum_j y_{mj} + \lambda\psi\mu\right)$$

$$\propto N\left((n_i\lambda + \lambda\psi)^{-1}\left(\lambda \sum_j y_{mj} + \lambda\psi\mu\right), \; (n_i\lambda + \lambda\psi)^{-1}\right)$$

2.

$i = 1, 2, \ldots, K$

$j = 1, 2, \ldots, n_i$

$p = 1, 2$

$X_{i_{n_i \times p}}$ = design matrix of size $n_i \times p$

$\tilde{Y}_i = \log Y_i$

$\beta_{i_{p \times 1}}$

$1 \times p$

$$\log Y_{ij} = \beta_{i0} + \beta_{i1} x_{i1} + \beta_{i2} x_{i2} + \beta_{i3} \log(x_{i1}) \times x_{i2}$$

$$\tilde{Y}_i = X_i \beta_i + \varepsilon_i$$

$$\theta \sim MVN(\mu_0, \Lambda_0^{-1})$$

$$\Sigma \sim Invwish(\eta_0, S_0)$$

$$\lambda \sim Gamma(a_0, b_0)$$

$$\beta_i \sim MVN(\theta, \Sigma)$$

$$\tilde{Y}_i \mid \beta_i, \lambda \sim MVN(X_i \beta_i, \lambda I_{n_i \times n_i})$$

$$p(\tilde{Y}_i, \beta_i, \lambda, \theta, \Sigma) \propto \prod_{i=1}^{K} [p(\tilde{Y}_i \mid \beta_i, \lambda)\, p(\beta_i \mid \theta, \Sigma)] \times p(\theta) \times p(\Sigma) \times p(\lambda)$$

$$\propto \left[\prod_{i=1}^{K} |\lambda I|^{\frac{1}{2}} e^{-\frac{\lambda}{2}(\tilde{Y}_i - X_i\beta_i)^T(\tilde{Y}_i - X_i\beta_i)} \times |\Sigma|^{-\frac{1}{2}} e^{-\frac{(\beta_i-\theta)^T \Sigma^{-1} (\beta_i - \theta)}{2}} \right]$$

$$\times\, e^{-\frac{(\theta-\mu_0)^T \Lambda_0 (\theta-\mu_0)}{2}} \times |\Sigma|^{-\frac{(\eta_0+p+1)}{2}} e^{-\frac{1}{2} tr(S_0 \Sigma^{-1})}$$

$$\times\, \lambda^{a_0-1} e^{-\lambda b_0}$$

$$p(\beta_i \mid -) \propto p(\tilde{y}_i \mid \beta_i, \lambda)\, p(\beta_i \mid \theta, \Sigma)$$

$$\propto e^{-\frac{\lambda}{2}(\tilde{y}_i - X_i\beta_i)^T(\tilde{y}_i - X_i\beta_i)}\; e^{-\frac{(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}}$$

$$\beta_i^T(\lambda X_i^T X_i + \Sigma^{-1})\beta_i - 2\beta_i^T(\lambda X_i^T\tilde{y}_i + \Sigma^{-1}\theta)$$

$$\propto MVN\left((\lambda X_i^T X_i + \Sigma^{-1})^{-1}(\lambda X_i^T\tilde{y}_i + \Sigma^{-1}\theta),\ (\lambda X_i^T X_i + \Sigma^{-1})^{-1}\right)$$

$$p(\theta \mid -) \propto \prod_{i=1}^{k} p(\beta_i \mid \theta, \Sigma)] \times p(\theta)$$

$$\propto e^{-\frac{\sum_{i=1}^{k}(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}}\; e^{-\frac{(\theta-\mu_0)^T\Lambda_0^{-1}(\theta-\mu_0)}{2}}$$

$$\theta^T(k\Sigma^{-1} + \Lambda_0^{-1})\theta - 2\theta^T(\Sigma^{-1}\sum_{i=1}^{k}\beta_i + \Lambda_0^{-1}\mu_0)$$

$$\propto MVN\left((k\Sigma^{-1} + \Lambda_0^{-1})^{-1}(\Sigma^{-1}\sum_{i=1}^{k}\beta_i + \Lambda_0^{-1}\mu_0),\ (k\Sigma^{-1} + \Lambda_0^{-1})^{-1}\right)$$

$$p(\Sigma \mid -) \propto \prod_{i=1}^{k} p(\beta_i \mid \theta, \Sigma)] \times p(\Sigma)$$

$$\propto |\Sigma|^{-\frac{k}{2}}\, e^{-\frac{\sum_{i=1}^{k}(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}}\; |\Sigma|^{-\frac{(\eta_0+p+1)}{2}}\; e^{-\frac{1}{2}tr(S_0\Sigma^{-1})}$$

$$\propto |\Sigma|^{-\frac{[(k+\eta_0)+p+1]}{2}}\; e^{-\frac{tr(\sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T\Sigma^{-1})}{2}}\; e^{-\frac{1}{2}tr(S_0\Sigma^{-1})}$$

$$= |\Sigma|^{-\frac{[(k+\eta_0)+p+1]}{2}}\; e^{-\frac{tr((\sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T + S_0)\Sigma^{-1})}{2}}$$

$$\propto \text{InvWish}\left(k+\eta_0,\ \sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T + S_0\right)$$

$$p(\lambda|-) \propto \prod_{i=1}^{K} p(\tilde{y}_i|\beta_i, \lambda) \times p(\lambda)$$

$$\propto \prod_{i=1}^{K} |\lambda I|^{\frac{1}{2}} e^{-\frac{\lambda}{2}(\tilde{y}_i - x_i\beta_i)^T(\tilde{y}_i - x_i\beta_i)} \times \lambda^{a_0-1} e^{-\lambda b_0}$$

$$\propto \lambda^{\frac{\sum_{i=1}^{K} n_i}{2} + a - 1} e^{-\lambda\left(\frac{\sum_{i=1}^{K}(\tilde{y}_i - x_i\beta_i)^T(\tilde{y}_i - x_i\beta_i)}{2} + b_0\right)}$$

$$\propto \text{Gamma}\left(\frac{\sum_{i=1}^{K} n_i}{2} + a, \frac{\sum_{i=1}^{K}(\tilde{y}_i - x_i\beta_i)^T(\tilde{y}_i - x_i\beta_i)}{2} + b_0\right)$$

Model

$$\theta \sim MVN(\mu_0, L_0^{-1})$$

$$\Sigma \sim \text{Invwish}(\eta_0, S_0)$$

$$\beta_i | \theta, \Sigma \sim MVN(\theta, \Sigma)$$

$$z_{ij} | \beta_i \sim MVN(X_i \beta_i, I_{n_i})$$

$$Y_{ij} | z_{ij} \sim I(Y_{ij}=1) I(z_{ij} > 0) + I(Y_{ij}=0) I(z_{ij} < 0)$$

$$p(z, \beta, \theta, \Sigma | Y) \propto p(Y, z, \beta, \theta, \Sigma)$$

$$= \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} p(Y_{ij} | z_{ij}) \right] \times \prod_{i=1}^{k} \prod_{j=1}^{n_i} p(z_{ij} | \beta_i) \times \prod_{i=1}^{k} p(\beta_i | \theta, \Sigma) \times p(\theta) \times p(\Sigma)$$

Gibbs Sampling

$$p(z_{ij} | -) \propto p(Y_{ij} | z_{ij}) p(z_{ij} | \beta_i)$$

$$\propto \left[I(Y_{ij}=1) I(z_{ij} > 0) + I(Y_{ij}=0) I(z_{ij} \leq 0) \right] N(z_{ij} | x_i^T \beta_i, 1)$$

$$\propto \begin{cases} \text{TruncN}(x_{ij}^T \beta_i, 1, 0, \infty) & \text{if } Y_{ij} = 1 \\ \text{TruncN}(x_{ij}^T \beta_i, 1, -\infty, 0) & \text{if } Y_{ij} = 0 \end{cases}$$

$$p(\beta_i|-) \propto \prod_{i=1}^{k} p(z_i|\beta_i) \times \prod_{i=1}^{k} p(\beta_i|\theta,\Sigma)$$

$$\propto \prod_{i=1}^{k} e^{-\frac{(z_i - X_i\beta_i)^T(z_i - X_i\beta_i)}{2}} \times \prod_{i=1}^{k} e^{-\frac{(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}}$$

$$\beta_i^T(X_i^T X_i + \Sigma^{-1}) - 2\beta_i^T\left(\sum_{i=1}^{k} X_i^T z_i + \Sigma^{-1}\sum_{i=1}^{k}\beta_i\right)$$

$$\propto MVN\left((X_i^T X_i + \Sigma^{-1})^{-1}\left(\sum_{i=1}^{k} X_i^T z_i + \Sigma^{-1}\sum_{i=1}^{k}\beta_i\right), (X_i^T X_i + \Sigma^{-1})^{-1}\right)$$

$$p(\theta|-) \propto \prod_{i=1}^{k} p(\beta_i|\theta,\Sigma) \times p(\theta)$$

$$\propto \prod_{i=1}^{k} e^{-\frac{(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}} \times e^{-\frac{(\theta-\mu_0)^T L_0^{-1}(\theta-\mu_0)}{2}}$$

$$\theta^T(k\Sigma^{-1} + L_0^{-1})\theta - 2\theta^T\left(\Sigma^{-1}\sum_{i=1}^{k}\beta_i + L_0^{-1}\mu_0\right)$$

$$\propto MVN\left((k\Sigma^{-1} + L_0^{-1})^{-1}\left(\Sigma^{-1}\sum_{i=1}^{k}\beta_i + L_0^{-1}\mu_0\right), (k\Sigma^{-1} + L_0^{-1})^{-1}\right)$$

$$p(\Sigma|-) \propto \prod_{i=1}^{k} p(\beta_i|\theta,\Sigma) \times p(\Sigma)$$

$$\propto \prod_{i=1}^{k} |\Sigma|^{-\frac{1}{2}} e^{-\frac{(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}} \times |\Sigma|^{-\frac{(\eta_0+p+1)}{2}} e^{-\frac{1}{2}tr(S_0\Sigma^{-1})}$$

$$\propto |\Sigma|^{-\frac{(k+\eta_0+p+1)}{2}} e^{-tr\left(\left(\sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T + S_0\right)\Sigma^{-1}\right)}$$

$$\propto \mathrm{InvWish}\left(k+\eta_0, \left(\sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T + S_0\right)\right)$$