$$w_i = w(x, x_i) = \frac{1}{h} K\left(\frac{x_i - x}{h}\right)$$

$$f(x) = \sum_{i=1}^{n} w_i (y_i - a)^2$$

$$\frac{df}{da} = -\sum_{i=1}^{n} 2 w_i (y_i - a) = 0$$

$-2 \sum w_i y_i + 2 \sum w a$

$$\sum w_i y_i - \sum w_i a = 0$$

$$\sum w_i y_i = \sum w_i a$$

$$\hat{a} = \frac{\sum w_i y_i}{\sum w_i}$$

$$\frac{df^2}{d^2 a} = 2 \sum_{i=1}^{n} w_i > 0$$

So $\hat{a}$ is the minimizer

if we assume $\sum_{i=1}^{n} w_i > 0$

(A)

$$R_x = \begin{bmatrix} (x_1-x)^0 & (x_1-x)^1 & (x_1-x)^2 & \cdots & (x_1-x)^D \\ (x_2-x)^0 & (x_2-x)^1 & (x_2-x)^2 & \cdots & (x_2-x)^D \\ \vdots & & & & \vdots \\ (x_n-x)^0 & (x_n-x)^2 & \cdots & & (x_n-x)^D \end{bmatrix}_{n\times(D+1)}$$

$$a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_D \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}$$

$$g(x;a) = R_x a \qquad W = \begin{bmatrix} w_1 & & & 0 \\ & w_2 & & \\ & & \ddots & \\ 0 & & & w_n \end{bmatrix}$$

$$f(a) = (y - R_x a)^T W (y - R_x a)$$

$$= y^T W y - 2(R_x^T W^T y)^T a + a^T R_x^T W R_x a$$

$$\frac{\partial f}{\partial a} = -2(R_x^T W^T y) + 2 R_x^T W R_x a = 0$$

$$R_x^T W R_x a = R_x^T W y$$

$$\hat{a} = (R_x^T W R_x)^{-1} R_x^T W y$$

$$\hat{y} = R_x \hat{a}$$

$$E(\hat{y}) = E(R_x \hat{a})$$

$$= R_x (R_x^T W R_x)^{-1} R_x^T W E(y)$$

(B) $\quad D=1 \quad a \in \mathbb{R}^2$

$$R_x = \begin{bmatrix} 1 & (x_1-x)^1 \\ 1 & (x_2-x)^1 \\ 1 & \vdots \\ \vdots & \vdots \\ 1 & (x_n-x)^1 \end{bmatrix}_{n\times 2}$$

$$a = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}_{2\times 1}$$

$$W = \begin{bmatrix} w_1 & & & 0 \\ & w_2 & & \\ 0 & & \ddots & \\ & & & w_n \end{bmatrix}$$

$$(R_x^T W R_x)_{ij} = \sum_{l}^{n} \sum_{k}^{n} (R_x^T)_{ik} (W)_{kl} (R_x)_{lj}$$

$$= \sum_{l}^{n} \sum_{k}^{n} (R_x)_{ki} W_{kl} (R_x)_{lj} = \sum_{k=1}^{n} (R_x)_{ki} W_{kk} (R_x)_{kj}$$

summand is 0 if $k \neq l$

$$= \sum_{k=1}^{n} w_k (x_k - x)^{i-1} (x_k - x)^{j-1}$$

$$= \sum_{k=1}^{n} w_k (x_k - x)^{i+j-2}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$S = \det(R_x^T W R_x) = (\sum_{k=1}^{n} w_k)(\sum_{j=1}^{n} w_j (x_j - x)^2 - \sum_{m=1}^{n} w_m (x_m - x) \sum_{s=1}^{n} w_s (x_s - x)$$

$$= h^2 [[S_0(x)][S_2(x)] - [S_1(x)]^2]$$

$$(R_x^T W R_x)^{-1} = \frac{1}{S} \begin{bmatrix} \sum_{i=1}^{n} w_i (x_i - x)^2 & -\sum_{j=1}^{n} w_j (x_j - x) \\ -\sum_{k=1}^{n} w_k (x_k - x) & \sum_{l=1}^{n} w_l \end{bmatrix}$$

$$= \frac{1}{S} \begin{bmatrix} h\, S_2(x) & -h S_1(x) \\ -h S_1(x) & S_0(x) \end{bmatrix}_{2\times 2}$$

$$(R_x^T W)_{ij} = \sum_k (R_x^T)_{ik} (W)_{kj}$$

$$= \sum_k^n (R_x)_{ki} (W)_{kj} = (R_x)_{ji} (W)_{jj} \qquad \text{summand is } 0 \text{ if } k \neq j$$

$$(R_x^T W) = \begin{bmatrix} h k_1 & h k_2 & \cdots & h k_n \\ h k_1 (x_1 - x) & h k_2 (x_2 - x) & \cdots & h k_n (x_n - x) \end{bmatrix}_{2\times n}$$

$$(R_x^T W R_x)^{-1} (R_x^T W) = \frac{1}{S} \begin{bmatrix} h^2 k_1 (S_2(x) - S_1(x)(x_1 - x)) & \cdots & h^2 k_n (S_2(x) - S_1(x)(x_n - x)) \\ S_0(x) h k_1 (x_1 - x) - h^2 S_1(x) k_1 & & S_0(x) h k_n (x_n - x) - h^2 S_1(x) k_n \end{bmatrix}_{2\times n}$$

$$\hat{r} = (R_x^T W R_x)^{-1} (R_x^T W) y$$

when $u = x$ we get $\hat{a}_0$, which is our target value.

$$\hat{a}(0) = \frac{1}{S} \left[h^2 k_1 (s_2(x) - s_1(x)(x_1 - x)) \; \cdots \; h^2 k_n (s_2(x) - s_1(x)(x_n - x)) \right] \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

$$\hat{a}(0) = \frac{\sum_{i=1}^{n} h^2 k_i (s_2(x) - s_1(x)(x_i - x)) y_i}{h^2 \; s_0(x) s_2(x) - [s_1(x)]^2}$$

$$= \frac{\sum_{i=1}^{n} k_i (s_2(x) - s_1(x)(x_i - x)) y_i}{s_0(x) s_2(x) - [s_1(x)]^2} = \frac{\sum_{i=1}^{n} \hat{w}_i(x) y_i}{\sum_{i=1}^{n} w_i}$$

$$s_0(x) s_2(x) - [s_1(x)]^2 = \sum_{j=1}^{n} k_j ((s_2(x) - (s_1(x) \sum_{p=1}^{n} k_p (x_p - x)))$$

$$= \sum_{j=1}^{n} s_2(x) k_j - \sum_{p=1}^{n} s_1(x) k_p (x_p - x)$$

$$\sum \hat{w}_i = \sum_{i=1}^{n} k_i (s_2(x) - s_1(x)(x_i - x))$$

$$= \sum_{i=1}^{n} k_i (s_2(x) - s_1(x)(x_i - x))$$

$$= \sum_{i=1}^{n} s_2(x) k_i - \sum_{i=1}^{n} s_1(x)(x_i - x) k_i$$

D1

$$\hat{\sigma}^2 = \frac{r^T r}{n - 2tr(H) + tr(H^T H)} \qquad C = n - 2tr(H) + tr(H^T H)$$

$$E(\hat{\sigma}^2) = \frac{E(r^T r)}{C} \qquad r^T r = (y - Hy)^T (y - Hy)$$

$$= y^T y - 2y^T H^T y + y^T H^T H y$$

$$= \frac{E(y^T y - 2y^T H^T y + y^T H^T H y)}{C}$$

$$= \frac{tr(\sigma^2 I) + f(x)^T f(x) - 2[tr(\sigma^2 H^T) + f(x)^T H^T f(x)] + tr(\sigma^2 H^T H) + f(x)^T H^T H f(x)}{C}$$

$$= \frac{\sigma^2 [n - 2tr(H^T) + tr(H^T H)]}{C} + \frac{[f(x)^T f(x) - 2f(x)^T H^T f(x) + f(x)^T H^T H f(x)]}{C}$$

$$= \sigma^2 + \frac{[f(x) - H f(x)]^T [f(x) - H f(x)]}{n - [2tr(H) - tr(H^T H)]}$$

$E(\hat{\sigma}^2) \approx \sigma^2$ if $\dfrac{[f(x) - H f(x)]^T [f(x) - H f(x)]}{n - [2tr(H) - tr(H^T H)]}$

is relatively small, This happens if $\|f(x) - Hf(x)\|_2^2$

is small