$$MSE(\hat{f}, f) = E[(f - \hat{f})^2]$$

$$= E(f^2 - 2f\hat{f} + \hat{f}^2)$$

$$= f^2 - 2fE(\hat{f}) + E(\hat{f})^2 - E(\hat{f})^2 + E(\hat{f}^2)$$

$$= (f - E(\hat{f}))^2 + E(\hat{f}^2) - [E(\hat{f})]^2$$

$$= Bias(\hat{f})^2 + var(\hat{f})$$

(A)

Write $Y = \sum_{i=1}^{n} I(-\frac{h}{2} < x_i < \frac{h}{2})$

Define $\hat{f}(0) = \frac{Y}{nh}$

$\frac{Y}{n} \xrightarrow{P} P(-\frac{h}{2} < x < \frac{h}{2}) = \pi_h$ by weak law of large number

$\hat{f}(0) = \frac{Y}{nh} \xrightarrow{P} \frac{\pi_h}{h} \approx f(0)$ by slutsky theorem

(B)

$$f(x) \approx f(0) + x f'(0) + \frac{x^2}{2} f''(0)$$

$$MSE(\hat{f}(0), f(0)) = Bias(\hat{f}(0))^2 + var(\hat{f}(0))$$

$$E(\hat{f}(0)) = E\left(\frac{Y}{nh}\right)$$

$$= E\left(\frac{\sum_{i=1}^{n} I(-\frac{h}{2} < X_i < \frac{h}{2})}{nh}\right)$$

$$= \frac{n\pi_h}{nh}$$

$$= \frac{\pi_h}{h}$$

$$Bias(\hat{f}(0))$$

$$= E[\hat{f}(0)] - f(0)$$

$$= \frac{\pi_h}{h} - f(0)$$

$$var(\hat{f}(0)) = var\left(\frac{\sum_{i=1}^{n} (-\frac{h}{2} < X_i < \frac{h}{2})}{nh}\right)$$

$$= \frac{1}{n^2h^2} \sum_{i=1}^{n} var(I(-\frac{h}{2} < X_i < \frac{h}{2}))$$

$$= \frac{1}{n^2h^2} n\pi_h(1-\pi_h)$$

$$= \frac{\pi_h(1-\pi_h)}{nh^2}$$

$$MSE(\hat{f}(0), f(0)) = Bias(\hat{f}(0))^2 + var(\hat{f}(0))$$

$$= \left[\frac{\pi_h}{h} - f(0)\right]^2 + \frac{\pi_h(1-\pi_h)}{nh^2}$$

⊛ plugging in π_h

$$= \left[f(0) + \frac{f''(0)h^2}{24} - f(0)\right]^2 + \frac{\left[hf(0) + \frac{f''(0)h^3}{24}\right]\left[1 - hf(0) - \frac{f''(0)h^3}{24}\right]}{nh^2}$$

$$= \frac{f''(0)^2h^4}{24^2} + \frac{\frac{hf(0)}{h} - \frac{h^2f(0)^2}{h} - \frac{f''(0)f(0)h^4}{24h} + \frac{f''(0)h^3}{24h} - \frac{f''(0)f(0)h^4}{24h} - \frac{f''(0)^2h^6}{24^2h}}{nh}$$

$$= \frac{f''(0)^2h^4}{24^2} + \frac{f(0) - h(f(0))^2 - \frac{2f''(0)f(0)h^3}{24} + \frac{f''(0)h^2}{24} - \frac{[f''(0)]^2h^5}{24^2}}{nh}$$

$$= \left[\frac{f''(0)^2}{24^2} - \frac{f''(0)^2}{24^2n}\right]h^4 + \frac{f(0)}{nh}$$

$$A = \left[\frac{f''(0)^2}{24^2} - \frac{f''(0)}{24^2n}\right] \quad B = f(0)$$

⊛ $$\pi_h \approx \int_{-\frac{h}{2}}^{\frac{h}{2}} f(0) + xf'(0) + \frac{x^2}{2}f''(0)\,dx$$

$$= hf(0) + f'(0)\left(\frac{x^2}{2}\Big|_{-\frac{h}{2}}^{\frac{h}{2}}\right) + \frac{f''(0)}{2}\left(\frac{x^3}{3}\Big|_{-\frac{h}{2}}^{\frac{h}{2}}\right)$$

$$= h(f(0)) + f'(0)\left(\frac{h^2}{8} - \frac{h^2}{8}\right) + \frac{f''(0)}{2}\left(\frac{h^3}{24} + \frac{h^3}{24}\right)$$

$$= h(f(0)) + f''(0)\frac{h^3}{24}$$

(c)

$$f(h) \approx Ah^4 + \frac{B}{nh}$$

$$\underset{h}{\operatorname{argmin}}\, f(h)$$

$$\frac{df}{dh} = 4Ah^3 - \frac{B}{nh^2} = 0$$

$$\rightarrow \quad 4Ah^3 = \frac{B}{nh^2}$$

$$h^5 = \frac{B}{4An}$$

$$h^* = \left(\frac{B}{4An}\right)^{\frac{1}{5}}$$

$$\hat{\beta}_0 + \hat{\beta}_1 x^*$$

$$\hat{\beta}_1 x^*$$

$$\frac{\Sigma x_i y_i}{\Sigma x_i^2} x^*$$

$$\sum_{i=1}^{n} \frac{x_i x^*}{\Sigma x_i^2} y_i$$

$$w(x_i, x^*) = \frac{x_i x^*}{\Sigma x_i^2}$$

Curve fitting by linear smoothing

(A) Given new observation x^*, we can get a predicted value y^* using the equation

$$y^* = \hat{\beta}_0 + \hat{\beta}_1 x^*$$

since Xs are standardized

this reduce to

$$y^* = \hat{\beta}_1 x^* \qquad \text{where} \qquad \hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$= \frac{\sum_{i=1}^{n} x_i y_i \, x^*}{\sum_{i=1}^{n} x_i^2} \qquad\qquad = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

$$= \sum_{i=1}^{n} \frac{x_i x^*}{\sum_{i=1}^{n} x_i^2} y_i$$

$$= \sum_{i=1}^{n} w(x_i, x^*) y_i \qquad w(x_i, x^*) = \frac{x_i x^*}{\sum_{i=1}^{n} x_i^2}$$