

$$w_i = w(x, x_i) = \frac{1}{h} K\left(\frac{x_i - x}{h}\right)$$

$$f(x) = \sum_{i=1}^n w_i (y_i - a)^2$$

$$\frac{df}{da} = -2 \sum_{i=1}^n w_i (y_i - a) = 0 \quad -2 \sum w_i y_i + 2 \sum w_i a$$

$$\sum w_i y_i - \sum w_i a = 0$$

$$\sum w_i y_i = \sum w_i a$$

$$\hat{a} = \frac{\sum w_i y_i}{\sum w_i}$$

$$\frac{d^2 f}{da^2} = 2 \sum_{i=1}^n w_i > 0$$

, So  $\hat{a}$  is the minimizer

if we assume  $\sum_{i=1}^n w_i > 0$

(A1)

$$R_x = \begin{bmatrix} (x_1 - x)^0 & (x_1 - x)^1 & (x_1 - x)^2 & \dots & (x_1 - x)^D \\ (x_2 - x)^0 & (x_2 - x)^1 & (x_2 - x)^2 & \dots & (x_2 - x)^D \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (x_n - x)^0 & (x_n - x)^1 & (x_n - x)^2 & \dots & (x_n - x)^D \end{bmatrix}$$

$n \times (D+1)$

$$a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_D \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$n \times 1$

$$g(x; a) = R_x a$$

$$W = \begin{bmatrix} w_1 & & & \\ & w_2 & & 0 \\ & & \ddots & \\ 0 & & & w_n \end{bmatrix}$$

$$f(a) = (y - R_x a)^T W (y - R_x a)$$

$$\approx y^T W y - 2(R_x^T W y)^T a + a^T R_x^T W R_x a$$

$$\frac{\partial f}{\partial a} = -2(R_x^T W y) + 2R_x^T W R_x a \approx 0$$

$$R_x^T W R_x a \approx R_x^T W y$$

$$\hat{a} = (R_x^T W R_x)^{-1} R_x^T W y$$

$$(B) \quad D=1 \quad a \in \mathbb{R}^2$$

$$R_x = \begin{bmatrix} 1 & (x_1 - x)' \\ 1 & (x_2 - x)' \\ \vdots & \vdots \\ 1 & (x_n - x)' \end{bmatrix}_{n \times 2}$$

$$a = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}_{2 \times 1}$$

$$\begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix}$$

$$W = \begin{bmatrix} w_1 & & & 0 \\ & w_2 & & \\ & & \ddots & \\ 0 & & & w_n \end{bmatrix}$$

$$(R_x^T W R_x)_{ij} = \sum_{i=1}^n \sum_{k=1}^n (R_x^T)_{ik} (W)_{kk} (R_x)_{kj}$$

$$= \sum_{i=1}^n \sum_{k=1}^n (R_x)_{ki} w_{kk} (R_x)_{kj} = \sum_{k=1}^n (R_x)_{ki} w_{kk} (R_x)_{kj}$$

summand is 0 if

$k \neq i$

$$= \sum_{k=1}^n w_k (x_k - x)^{i-1} (x_k - x)^{j-1}$$

$$= \sum_{k=1}^n w_k (x_k - x)^{i+j-2}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\begin{aligned}
 S &= \det(R_X^T W R_X) = \left| \sum_{k=1}^n w_k \begin{vmatrix} \sum_{j=1}^n w_j (x_j - x)^2 & - \sum_{j=1}^n w_j (x_j - x) \\ - \sum_{k=1}^n w_k (x_k - x) & \sum_{l=1}^n w_l \end{vmatrix} \right| \\
 &= h^2 \left[ [S_0(x)] [S_2(x)] - [S_1(x)]^2 \right]
 \end{aligned}$$

$$\begin{aligned}
 (R_X^T W R_X)^{-1} &= \frac{1}{S} \begin{bmatrix} \sum_{i=1}^n w_i (x_i - x)^2 & - \sum_{j=1}^n w_j (x_j - x) \\ - \sum_{k=1}^n w_k (x_k - x) & \sum_{l=1}^n w_l \end{bmatrix} \\
 &= \frac{1}{S} \begin{bmatrix} h S_2(x) & -h S_1(x) \\ -h S_1(x) & S_0(x) \end{bmatrix}_{2 \times 2}
 \end{aligned}$$

$$\begin{aligned}
 (R_X^T W)_{ij} &= \sum_k (R_X^T)_{ik}(W)_{kj} \\
 &= \sum_k (R_X)_{ki}(W)_{kj} = (R_X)_{ji}(W)_{ij} \quad \text{summand is 0 if } k \neq j
 \end{aligned}$$

$$(R_X^T W) = \begin{bmatrix} h k_1 & h k_2 & \dots & h k_n \\ h k_1(x_1 - x) & h k_2(x_2 - x) & \dots & h k_n(x_n - x) \end{bmatrix}_{2 \times n}$$

$$(R_X^T W R_X)^{-1} (R_X^T W) = \frac{1}{S} \begin{bmatrix} h^2 k_1 (S_2(x) - S_1(x)(x_1 - x)) & \dots & h^2 k_n (S_2(x) - S_1(x)(x_n - x)) \\ S_0(x) h k_1 (x_1 - x) - h^2 S_1(x) k_1 & \dots & S_0(x) h k_n (x_n - x) - h^2 S_1(x) k_n \end{bmatrix}_{2 \times n}$$

$$\hat{f} = (R_x^T W R_x)^{-1} R_x^T W y$$

When  $w=x$  we get  $\hat{\alpha}_0$ , which is our target value.

$$\hat{\alpha}_{(0)} = \frac{1}{S} \begin{bmatrix} h^2 k_1 (s_2(x) - s_1(x)(x_1 - x)) & \dots & h^2 k_n (s_2(x) - s_1(x)(x_n - x)) \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\hat{\alpha}_{(0)} = \frac{\sum_{i=1}^n h^2 k_i (s_2(x) - s_1(x)(x_i - x)) y_i}{h^2 s_0(x) s_2(x) - [s_1(x)]^2}$$

$$= \frac{\sum_{i=1}^n k_i (s_2(x) - s_1(x)(x_i - x)) y_i}{s_0(x) s_2(x) - [s_1(x)]^2} = \frac{\sum_{i=1}^n \hat{w}_i(x) y_i}{\sum_{i=1}^n w_i}$$

$$s_0(x) s_2(x) - [s_1(x)]^2 = \sum_{j=1}^n k_j (s_2(x) - (s_1(x) \sum_{p=1}^n k_p (x_p - x)))$$

$$= \sum_{j=1}^n s_2(x) k_j - \sum_{p=1}^n s_1(x) k_p (x_p - x)$$

$$\sum_{i=1}^n \hat{w}_i = \sum_{i=1}^n k_i (s_2(x) - s_1(x)(x_i - x))$$

$$= \sum_{i=1}^n k_i (s_2(x) - s_1(x)(x_i - x))$$

$$= \sum_{i=1}^n s_2(x) k_i - \sum_{i=1}^n s_1(x)(x_i - x) k_i$$

C1

$$E[\hat{f}(x)] = e_i^T (R_x^T W R_x)^{-1} R_x^T W f(x)$$

$$\text{var}(\hat{f}(x)) = e_i^T (R_x^T W R_x)^{-1} R_x^T W W^T R_x (R_x^T W R_x)^{-1} e_i \sigma^2$$

D1

$$\hat{b}^2 = \frac{r^T r}{n - 2\text{tr}(H) + \text{tr}(H^T H)} \quad C = n - 2\text{tr}(H) + \text{tr}(H^T H)$$

$$E(\hat{b}^2) = \frac{E(r^T r)}{C} \quad r^T r = (y - Hy)^T (y - Hy) = y^T y - 2y^T H y + y^T H^T H y$$

$$= \frac{E(y^T y - 2y^T H y + y^T H^T H y)}{C}$$

$$= \frac{\text{tr}(b^2 I) + f(x)^T f(x) - 2\text{tr}(b^2 H^T) + f(x)^T H^T f(x) + \text{tr}(b^2 H^T H) + f(x)^T H^T H f(x)}{C}$$

$$= \frac{b^2 [n - 2\text{tr}(H^T) + \text{tr}(H^T H)]}{C} + \frac{[f(x)^T f(x) - 2f(x)^T H^T f(x) + f(x)^T H^T H f(x)]}{C}$$

$$= b^2 + \frac{[f(x) - H f(x)]^T [f(x) - H f(x)]}{n - [2\text{tr}(H) - \text{tr}(H^T H)]}$$

$$E(\hat{b}^2) \approx b^2 \quad \text{if} \quad \frac{[f(x) - H f(x)]^T [f(x) - H f(x)]}{n - [2\text{tr}(H) - \text{tr}(H^T H)]}$$

is relatively small, This happens if  $\|f(x) - H f(x)\|_2^2$  is small.