Priors

$\mu \sim N(\mu, v)$

$\lambda \sim Gamma(a, b)$

$\Psi \sim Gamma(c, d)$

1. $i = 1, 2, \ldots, k$

$j = 1, 2, \ldots, n_i$

$\lambda = \frac{1}{\sigma^2}$ $\quad \psi = \frac{1}{\tau^2}$

$$p(\mu, \lambda, \psi, \theta \mid Y) \propto \prod_{i=1}^{k} \prod_{j=1}^{n_i} p(Y_{ij} \mid \theta_i, \lambda) \times \prod_{i=1}^{k} p(\theta_i \mid \mu, \lambda, \psi) \times p(\mu)\, p(\lambda)\, p(\psi)$$

$$\propto \prod_{i=1}^{k} \prod_{j=1}^{n_i} \lambda^{\frac{1}{2}} e^{\frac{-\lambda(Y_{ij} - \theta_i)^2}{2}} \times \prod_{i=1}^{k} \lambda^{\frac{1}{2}} \psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i - \mu)^2}{2}} \times e^{\frac{-\phi(\mu - v)^2}{2}} \lambda^{a-1} e^{-\lambda b} \psi^{c-1} e^{-\psi d}$$

$$p(\mu \mid -) \propto \prod_{i=1}^{k} \lambda^{\frac{1}{2}} \psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i - \mu)^2}{2}} \times e^{\frac{-\phi(\mu - v)^2}{2}}$$

$$\propto e^{\frac{-\lambda\psi \sum_i (\theta_i - \mu)^2}{2}} \times e^{-\frac{\phi(\mu - v)^2}{2}}$$

$$\propto N\left((k\lambda\psi + \phi)^{-1}(\lambda\psi \sum_i \theta_i + \phi v),\ (k\lambda\psi + \phi)^{-1}\right)$$

$\mu^2(k\lambda\psi + \phi)$

$-2\mu(\lambda\psi \sum_i \theta_i + \phi v)$

$$p(\lambda|-) \propto \prod_{i=1}^{k}\prod_{j=1}^{n_i} \lambda^{\frac{1}{2}} e^{\frac{-\lambda(y_{ij}-\theta_i)^2}{2}} \times \prod_{i=1}^{k} \lambda^{\frac{1}{2}}\psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i-\mu)^2}{2}} \times \lambda^{a-1} e^{-\lambda b}$$

$$= \lambda^{\frac{\sum_i n_i}{2}+\frac{k}{2}+a-1} \; e^{-\lambda\left(\sum_j\sum_i \frac{(y_{ij}-\theta_i)^2}{2} + \psi\sum_i \frac{(\theta_i-\mu)^2}{2} + b\right)}$$

$$\propto \text{Gamma}\left(\frac{\sum_i n_i}{2}+\frac{k}{2}+a,\ \sum_j\sum_i \frac{(y_{ij}-\theta_i)^2}{2} + \psi\sum_i \frac{(\theta_i-\mu)^2}{2} + b\right)$$

$$p(\psi|-) \propto \prod_{i=1}^{k} \lambda^{\frac{1}{2}}\psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i-\mu)^2}{2}} \psi^{c-1} e^{-\psi d}$$

$$\propto \psi^{\frac{k}{2}+c-1} \; e^{-\psi\left(\lambda\sum_i \frac{(\theta_i-\mu)^2}{2} + d\right)}$$

$$\propto \text{Gamma}\left(\frac{k}{2}+c,\ \lambda\sum_i \frac{(\theta_i-\mu)^2}{2} + d\right)$$

$$p(\theta_m|-) \propto \prod_{i=1}^{k}\prod_{j=1}^{n_i} \lambda^{\frac{1}{2}} e^{\frac{-\lambda(y_{ij}-\theta_i)^2}{2}} \times \prod_{i=1}^{k} \lambda^{\frac{1}{2}}\psi^{\frac{1}{2}} e^{-\frac{\lambda\psi(\theta_i-\mu)^2}{2}}$$

$$\propto e^{-\lambda\sum_j \frac{(y_{mj}-\theta_m)^2}{2}} \times e^{-\frac{\lambda\psi(\theta_m-\mu)^2}{2}} \qquad -2\theta_m\mu\lambda\psi$$

$$\theta_m^2(n_i\lambda + \lambda\psi) - 2\theta_m\left(\lambda\sum_j y_{mj} + \lambda\psi\mu\right)$$

$$\propto N\left((n_i\lambda+\lambda\psi)^{-1}\left(\lambda\sum_j y_{mj} + \lambda\psi\mu\right),\ (n_i\lambda+\lambda\psi)^{-1}\right)$$

2. $i = 1, 2, \ldots, K$

$j = 1, 2, \ldots, n_i$

$p = 1, 2$

$X_{i_{n_i \times p}}$ = design matrix of size $n_i \times p$

$\tilde{Y}_i = \log Y_i$

$\beta_{i_{p\times 1}}$

$1 \times p$

$$\log Y_{ij} = \beta_{i0} + \beta_{i1} x_{i1} + \beta_{i2} x_{i2} + \beta_{i3} \log(x_{i1}) \times x_{i2}$$

$$\tilde{Y}_i = X_i \beta_i + \varepsilon_i$$

$$\theta \sim MVN(\mu_0, \Lambda_0^{-1})$$

$$\Sigma \sim \text{Invwish}(\eta_0, S_0)$$

$$\lambda \sim \text{Gamma}(a_0, b_0)$$

$$\beta_i \sim MVN(\theta, \Sigma)$$

$$\tilde{Y}_i \mid \beta_i, \lambda \sim MVN(X_i\beta_i, \lambda I_{n_i \times n_i})$$

$$p(\tilde{Y}_i, \beta_i, \lambda, \theta, \Sigma) \propto \prod_{i=1}^{K} [p(\tilde{Y}_i \mid \beta_i, \lambda)\, p(\beta_i \mid \theta, \Sigma)] \times p(\theta) \times p(\Sigma) \times p(\lambda)$$

$$\propto \left[\prod_{i=1}^{K} |\lambda I|^{\frac{1}{2}} e^{-\frac{\lambda}{2}(\tilde{Y}_i - X_i\beta_i)^T(\tilde{Y}_i - X_i\beta_i)} \times |\Sigma|^{-\frac{1}{2}} e^{-\frac{(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}} \right]$$

$$\times\, e^{-\frac{(\theta-\mu_0)^T\Lambda_0(\theta-\mu_0)}{2}} \times |\Sigma|^{-\frac{(\eta_0+p+1)}{2}} e^{-\frac{1}{2}tr(S_0\Sigma^{-1})}$$

$$\times\, \lambda^{a_0-1} e^{-\lambda b_0}$$

$$p(\beta_i|-) \propto p(\tilde{y}_i|\beta_i,\lambda)\,p(\beta_i|\theta,\Sigma)$$

$$\propto e^{-\frac{\lambda}{2}(\tilde{y}_i - X_i\beta_i)^T(\tilde{y}_i - X_i\beta_i)}\ e^{-\frac{(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}}$$

$$\beta_i^T(\lambda X_i^T X_i + \Sigma^{-1})\beta_i - 2\beta_i^T(\lambda X_i^T\tilde{y}_i + \Sigma^{-1}\theta)$$

$$\propto MVN\left((\lambda X_i^T X_i + \Sigma^{-1})^{-1}(\lambda X_i^T\tilde{y}_i + \Sigma^{-1}\theta),\ (\lambda X_i^T X_i + \Sigma^{-1})^{-1}\right)$$

$$p(\theta|-) \propto \prod_{i=1}^{k} p(\beta_i|\theta,\Sigma) \times p(\theta)$$

$$\propto e^{-\sum_{i=1}^{k}\frac{(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}}\ e^{-\frac{(\theta-\mu_0)^T\Lambda_0^{-1}(\theta-\mu_0)}{2}}$$

$$\theta^T(k\Sigma^{-1} + \Lambda_0^{-1})\theta - 2\theta^T(\Sigma^{-1}\sum_{i=1}^{k}\beta_i + \Lambda_0^{-1}\mu_0)$$

$$\propto MVN\left((k\Sigma^{-1} + \Lambda_0^{-1})^{-1}(\Sigma^{-1}\sum_{i=1}^{k}\beta_i + \Lambda_0^{-1}\mu_0),\ (k\Sigma^{-1} + \Lambda_0^{-1})^{-1}\right)$$

$$p(\Sigma|-) \propto \prod_{i=1}^{k} p(\beta_i|\theta,\Sigma) \times p(\Sigma)$$

$$\propto |\Sigma|^{-\frac{k}{2}} e^{-\sum_{i=1}^{k}\frac{(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}}\ |\Sigma|^{-\frac{(\eta_0+p+1)}{2}}\ e^{-\frac{1}{2}tr(S_0\Sigma^{-1})}$$

$$\propto |\Sigma|^{-\frac{[(k+\eta_0)+p+1]}{2}}\ e^{-\frac{tr(\sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T\Sigma^{-1})}{2}}\ e^{-\frac{1}{2}tr(S_0\Sigma^{-1})}$$

$$= |\Sigma|^{-\frac{[(k+\eta_0)+p+1]}{2}}\ e^{-\frac{tr((\sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T + S_0)\Sigma^{-1})}{2}}$$

$$\propto InvWish\left(k+\eta_0,\ \sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T + S_0\right)$$

$$p(\lambda|-) \propto \prod_{i=1}^{K} p(\tilde{y}_i|\beta_i,\lambda) \times p(\lambda)$$

$$\propto \prod_{i=1}^{K} |\lambda I|^{\frac{1}{2}} e^{-\frac{\lambda}{2}(\tilde{y}_i - x_i\beta_i)^T(\tilde{y}_i - x_i\beta_i)} \times \lambda^{a_0 - 1} e^{-\lambda b_0}$$

$$\propto \lambda^{\frac{\sum_{i=1}^{K} n_i}{2} + a - 1} e^{-\lambda\left(\frac{\sum_{i=1}^{K}(\tilde{y}_i - x_i\beta_i)^T(\tilde{y}_i - x_i\beta_i)}{2} + b_0\right)}$$

$$\propto \mathrm{Gamma}\left(\frac{\sum_{i=1}^{K} n_i}{2} + a,\ \frac{\sum_{i=1}^{K}(\tilde{y}_i - x_i\beta_i)^T(\tilde{y}_i - x_i\beta_i)}{2} + b_0\right)$$

Model

$$\theta \sim MVN(\mu_0, L_0^{-1})$$

$$\Sigma \sim Invwish(\eta_0, S_0)$$

$$\beta_i | \theta, \Sigma \sim MVN(\theta, \Sigma)$$

$$z_{ij} | \beta_i \sim MVN(X_i \beta_i, I_{n_i})$$

$$y_{ij} | z_{ij} \sim I(y_{ij}=1) I(z_{ij} > 0) + I(y_{ij}=0) I(z_{ij} < 0)$$

$$p(z, \beta, \theta, \Sigma | y) \propto p(y, z, \beta, \theta, \Sigma)$$

$$= \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} p(y_{ij} | z_{ij}) \right] \times \prod_{i=1}^{k} \prod_{j=1}^{n_i} p(z_{ij} | \beta_i) \times \prod_{i=1}^{k} p(\beta_i | \theta, \Sigma) \times p(\theta) \times p(\Sigma)$$

Gibbs Sampling

$$p(z_{ij} | -) \propto p(y_{ij} | z_{ij}) p(z_{ij} | \beta_i)$$

$$\propto \left[I(y_{ij}=1) I(z_{ij} > 0) + I(y_{ij}=0) I(z_{ij} \leq 0) \right] N(z_{ij} | x_i^T \beta_i, 1)$$

$$\propto \begin{cases} TruncN(x_{ij}^T \beta_i, 1, 0, \infty) & \text{if } y_{ij} = 1 \\ TruncN(x_{ij}^T \beta_i, 1, -\infty, 0) & \text{if } y_{ij} = 0 \end{cases}$$

$$p(\beta_i|-) \propto \prod_{i=1}^{k} p(z_i|\beta_i) \times \prod_{i=1}^{k} p(\beta_i|\theta,\Sigma)$$

$$\propto \prod_{i=1}^{k} e^{\frac{-(z_i - X_i\beta_i)^T(z_i - X_i\beta_i)}{2}} \times \prod_{i=1}^{k} e^{\frac{-(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}}$$

$$\beta_i^T(X_i^TX_i + \Sigma^{-1}) - 2\beta_i^T(\sum_{i=1}^{k} X_i^Tz_i + \Sigma^{-1}\sum_{i=1}^{k}\beta_i)$$

$$\propto MVN((X_i^TX_i + \Sigma^{-1})^{-1}(\sum_{i=1}^{k} X_i^Tz_i + \Sigma^{-1}\sum_{i=1}^{k}\beta_i), (X_i^TX_i + \Sigma^{-1})^{-1})$$

$$p(\theta|-) \propto \prod_{i=1}^{k} p(\beta_i|\theta,\Sigma) \times p(\theta)$$

$$\propto \prod_{i=1}^{k} e^{\frac{-(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}} \times e^{\frac{-(\theta-\mu_0)^TL_0^{-1}(\theta-\mu_0)}{2}}$$

$$\theta^T(k\Sigma^{-1} + L_0^{-1})\theta - 2\theta^T(\Sigma^{-1}\sum_{i=1}^{k}\beta_i + L_0^{-1}\mu_0)$$

$$\propto MVN((k\Sigma^{-1} + L_0^{-1})^{-1}(\Sigma^{-1}\sum_{i=1}^{k}\beta_i + L_0^{-1}\mu_0), (k\Sigma^{-1} + L_0^{-1})^{-1})$$

$$p(\Sigma|-) \propto \prod_{i=1}^{k} p(\beta_i|\theta,\Sigma) \times p(\Sigma)$$

$$\propto \prod_{i=1}^{k} |\Sigma|^{-\frac{1}{2}} e^{\frac{-(\beta_i-\theta)^T\Sigma^{-1}(\beta_i-\theta)}{2}} \times |\Sigma|^{\frac{-(\eta_0+p+1)}{2}} e^{-\frac{1}{2}tr(S_0\Sigma^{-1})}$$

$$\propto |\Sigma|^{\frac{-(k+\eta_0+p+1)}{2}} e^{-tr((\sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T + S_0)\Sigma^{-1})}$$

$$\propto InvWish(k+\eta_0, (\sum_{i=1}^{k}(\beta_i-\theta)(\beta_i-\theta)^T + S_0))$$

$$R = f(p)(p-c)$$

$$\frac{dR}{dp} = f(p) + (p-c)f'(p) = 0$$

case 1 : $p = \frac{\log a - \beta_0}{\beta_1}$

$$: \beta_0 + \beta_1 \log p + (p-c)\frac{\beta_1}{p} = 0$$

$$\beta_0 + \beta_1 = -\beta_1 \log p + \frac{c\beta_1}{p}$$

$$\log\alpha + \beta\log p + \log(p-c) = 0$$

Case 2 :

$$(\beta_0 + \beta_2) + (\beta_1 + \beta_3)\log p + (p-c)\frac{(\beta_1+\beta_3)}{p} = 0$$

$$(\beta_0 + \beta_2) + (\beta_1+\beta_3)\log p + (\beta_1+\beta_3) - \frac{c}{p}(\beta_1+\beta_3) = 0$$

$$R(p) = Q(p-c)$$

$$= \underline{\alpha p^{\beta}}(p-c)$$

$$\log R = \log\alpha + \beta\log p + \log(p-c) = 0$$

$$\frac{\beta}{p} + \frac{1}{p-c} = 0$$

$$\frac{\beta}{p} = \frac{1}{c-p}$$

$$Q = e^{\beta_0 + \log p^{\beta_1}}$$

$$= e^{\beta_0} p^{\beta_1}$$

$$\beta c - \beta p = p$$

$$\frac{\beta c}{\beta + 1} = p$$

$p(y, \theta, k)$

$p(y \mid \theta, k)\, p(\theta, k)$

$p(y \mid \theta_k, k)\, p(\theta_k, \theta_{-k}, k)$

$p(y \mid \theta_k, k)\quad p(\theta_{-k} \mid \theta_k, k) \cdot p(\theta_k, k)$

$p(y \mid \theta_k, k)\quad p(\theta_k \mid k)\, p(k)\ p(\theta_{-k} \mid \theta_k, k)$

$$\frac{b^a}{\Gamma(a)} \int \theta^2 \theta^{a-1} e^{-\theta b}\, d\theta$$

$$\frac{b^a}{\Gamma(a)} \int \theta^{a+2-1} e^{-\theta b}\, d\theta$$

$$E(\theta^2) = \frac{b^a}{\Gamma(a)} \frac{\Gamma(a+2)}{b^{a+2}} = \frac{(a+1)a}{b^2}$$

$$E(\theta^2) - E(\theta)^2$$

$$\frac{a^2+a}{b^2} - \frac{a^2}{b^2} = \frac{a}{b^2}$$

$$\prod_{i=1}^{n} \theta e^{-\theta x_i}$$

$$\theta^n e^{-\theta \sum_{i=1}^{n} x_i}$$

$$\theta^n e^{-\theta T}$$

$$\prod_{i=1}^{n} f(y_i|\theta) = g(T|\theta) h(y)$$

$$\prod_{i=1}^{n} f(y_i|\theta) p(\theta) = g(T|\theta) p(\theta)$$

$$h(x) c(\theta) e^{\sum_{i=1}^{k} w_i(\theta) t_i(x)} \qquad h(x) e^{w(\theta)^T t(x) - (-\log c(\theta))}$$

$$\lambda e^{-\lambda x}$$

$$e^{\log(\alpha^x (1-\alpha)^{1-x})}$$

$$c(\theta) = \lambda$$

$$e^{x \log \alpha + (1-x) \log(1-\alpha)}$$

$$f(x) = x$$

$$w(\lambda) = -\lambda$$

$$x \log(\alpha) + \log(1-\alpha) - x \log(1-\alpha)$$

$$x \left(\log \frac{\alpha}{1-\alpha}\right) + \log(1-\alpha)$$

e

$e^{x\theta + A(\theta)}$

$A(\log \frac{\alpha}{1-\alpha}) = \log(1-\alpha)$

$\theta = \log \frac{\alpha}{1-\alpha}$

$\log(1 + e^{\log \frac{\alpha}{1-\alpha}})$

$\lambda e^{-\lambda x}$

$\log(1 + \frac{\alpha}{1-\alpha})$

$e^{-\lambda x - (-\log \lambda)}$

$\log(\frac{1-\alpha+\alpha}{1-\alpha})$

$T(x) = x$

$\eta(\lambda) = -\lambda$

$A(\eta) = -\log \lambda$

$-\log(-\eta) = -\log(\lambda)$

$\theta^n \frac{1}{(\prod_{i=1}^n x_i)^2} I(\theta \leq \min x_i)$

$E(X) = \int_\theta^\infty \frac{\theta}{x} dx$