MIS 381: Assignment #1

Due on Thursday, January 25, 2018

Kumar 11: 00am

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Problem 1

We will first show that $(AB)^T = B^T A^T$

Proof. $((AB)^T)_{ij} = (AB)_{ji} = \sum_{k=1}^n a_{jk} b_{ki}$
$(B^T A^T)_{ij} = \sum_{k=1}^n (B^T)_{ik} (A^T)_{kj} = \sum_{k=1}^n b_{ki} a_{jk}$
Thus, $(AB)^T = B^T A^T$ □

Now, we will show that $(A^{-1})^T = (A^T)^{-1}$

Proof. $I^T = I$
$(AA^{-1})^T = I$
$(A^{-1})^T (A)^T = I$ since $(AB)^T = B^T A^T$
$(A^{-1})^T (A)^T (A^T)^{-1} = (A^T)^{-1}$
$(A^{-1})^T = (A^T)^{-1}$ □

Problem 2

Denote
x_1 =Amount invested in the first mortgage
x_2 =Amount invested in the second mortage
x_3 =Amount invested in home improvement
x_4 =Amount invested in personal overdraft

$x_1 + x_2 + x_3 + x_4 = 250$
$0.25x_1 - 0.75x_2 + 0.25x_3 + 0.25x_4 = 0$
$-0.45x_1 + 0.55x_2 = 0$
$0.14x_1 + 0.2x_2 + 0.2x_3 + 0.1x_4 = 250 * 0.15 = 37.5$

We will write this system of linear equations in matrix form.

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.25 & -0.75 & 0.25 & 0.25 \\ -0.45 & 0.55 & 0 & 0 \\ 0.14 & 0.2 & 0.2 & 0.1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0.3055 & -1.2222 & -2.2222 & 0 \\ 0.25 & -1 & 0 & 0 \\ -1.372 & 1.4888 & 0.8888 & 10 \\ 1.816 & 0.7333 & 1.333 & -10 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$b = \begin{bmatrix} 250 \\ 0 \\ 0 \\ 37.5 \end{bmatrix}$$

R-code:
$A = matrix(c(1, 0.25, -0.45, 0.14, 1, -0.75, 0.55, 0.2, 1, 0.25, 0, 0.2, 1, 0.25, 0, 0.1), nrow = 4)$
$b = matrix(c(250, 0, 0, 37.5), nrow = 4)$
$A\_inv = solve(A)$
$x = A\_inv\% * \%b$
$Ax = b$
$x = A^{-1}b$

$$x = \begin{bmatrix} 76.3889 \\ 62.5 \\ 31.9444 \\ 79.1667 \end{bmatrix}$$

Problem 3

Denote
$x_i = \#$ of units produced for variant i $\forall i \in \{1,2,3,4\}$
maximize:
$f = 1.5x_1 + 2.5x_2 + 3x_3 + 4.5x_4$
subject to:
Assembly Constraint - $2x_1 + 4x_2 + 3x_3 + 7x_4 \leq 100000$
Polish Constraint - $3x_1 + 2x_2 + 3x_3 + 4x_4 \leq 50000$
Pack Constraint - $2x_1 + 3x_2 + 2x_3 + 5x_4 \leq 60000$
Non-Negativity Constraint - $-x_i \leq 0 \ \forall i \in \{1,2,3,4\}$

Let

$$A = \begin{bmatrix} 2 & 4 & 3 & 7 \\ 3 & 2 & 3 & 4 \\ 2 & 3 & 2 & 5 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$b = \begin{bmatrix} 100000 \\ 50000 \\ 60000 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

R-code:
$z = c(1.5, 2.5, 3, 4.5)$
$A\_temp = matrix(c(2,3,2,4,2,3,3,3,2,7,4,5), nrow = 3)$
$I = -diag(4)$
$A = rbind(A\_temp, I)$
$b = c(100000, 50000, 60000, 0, 0, 0, 0)$
$signs = rep("<=", 7)$
$ans = lp("max", z, A, signs, b)$

$$x = \begin{bmatrix} 0 \\ 16000 \\ 6000 \\ 0 \end{bmatrix}$$

The objective function is 58000. Thus, we would produce 16000 units of variant 2 and 6000 of variant 3 and should obtain a profit of 58000.

Problem 4

Denote

$r_i =$ rating of team i $\forall i \in \{1, 2, 3, 4, 5\}$

Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix}$$

$$b = \begin{bmatrix} -45 \\ -3 \\ -31 \\ -45 \\ 18 \\ 8 \\ 20 \\ 2 \\ -27 \\ -38 \\ 0 \end{bmatrix}$$

We will solve $Ar = b$. However, b is not in the column space of A so we will find $\hat{r}$ by projecting b into column space of A.

$A\hat{r} = b$
$A^T A\hat{r} = A^T b$
$(A^T A)^{-1}(A^T A)\hat{r} = (A^T A)^{-1}A^T b$
$\hat{r} = (A^T A)^{-1}A^T b$

R-code:
$A = matrix(rep(0, 50), nrow = 10)$
$i = 1$
$k = 4$

$l = k$
$j = 1$
$p = 1$
$for(rin1 : 10)\{$
$A[i, j] = 1$
$A[i, j + p] = -1$
$l = l - 1$
$i = i + 1$
$p = p + 1$
$if(l == 0)\{$
$j = j + 1$
$l = k - 1$
$k = k - 1$
$p = 1$
$\}$
$\}$
$A = rbind(A, rep(1, 5))$
$b = matrix(c(7 - 52, 21 - 24, 7 - 38, 0 - 45, 34 - 16, 25 - 17, 27 - 7, 7 - 5, 3 - 30, 14 - 52, 0), nrow = 11)$
$A.T = t(A)$
$A.T\_A = A.T\%\%A$
$x = (solve(A.T\_A)\%\%A.T)\%\%b$

$$\hat{r} = \begin{bmatrix} -24.8 \\ 18.2 \\ -8.0 \\ -3.4 \\ 18.0 \end{bmatrix}$$

Thus, $r_1 = -24.8, r_2 = 18.2, r_3 = -8.0, r_4 = -3.4, r_5 = 18.0$