7.2 - Inverse Transforms and Transforms of derivatives

Lapace inverse is linear

Examples:

1) $\mathscr{L}^{-1}\left\{\frac{1}{s^6}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{5!}\cdot\frac{5!}{s^{5+1}}\right\} = \frac{1}{120}t^5$

2) $\mathscr{L}^{-1}\left\{\frac{1}{s^2+5}\right\} = \frac{1}{\sqrt{5}}\mathscr{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+5}\right\} = \frac{1}{\sqrt{5}}\sin\sqrt{5}t$

3) $\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} = -2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = -2\cos 2t + 3\sin 2t$

4) $\mathscr{L}^{-1}\left\{\frac{6x^2+50}{(s+3)(s^2+4)}\right\} = \mathscr{L}^{-1}\left\{\frac{8}{s+3}\right\} + \mathscr{L}^{-1}\left\{\frac{-2}{s^2+4}\right\} + \mathscr{L}^{-1}\left\{\frac{6}{s^2+4}\right\}$
$= 8e^{-3t} - 2cos2t + 3sin2t$

We have used the fact that the Laplace transform is a linear operator

$$\mathscr{L}^{-1}\left\{cf(t) + g(t)\right\} = c\mathscr{L}^{-1}\left\{f(t)\right\} + \mathscr{L}^{-1}\left\{g(t)\right\}$$

Transforms of derivatives given $\mathscr{L}\left\{f(t)\right\} = F(t)$

$$\mathscr{L}\left\{f'(t)\right\} = sF(s) - f(0)$$

$$\mathscr{L}\left\{f''(t)\right\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}\left\{f'''(t)\right\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

Proof of $L(f'(t))$

$$
\begin{aligned}
\mathscr{L}\left\{f'(t)\right\} &= \int_0^\infty e^{-st} f'(t)\, dt \\
&= e^{-st} f(t)|_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt \\
&= 0 - f(0) + sF(s)
\end{aligned}
$$

Ex2

$$
y'' - 3y' + 2y = e^{-4t}, y(0) = 1, y'(0) = 5
$$

$$
\begin{aligned}
\mathscr{L}\left\{y''\right\} - 3\mathscr{L}\left\{y'\right\} + 2\mathscr{L}\left\{y\right\} &= \mathscr{L}\left\{e^{-4t}\right\} \\
= (s^2Y(s) - sy(0) - y'(0)) - 3(sY(s) - y(0)) + 2(Y(s)) &= \frac{1}{s+4}
\end{aligned}
$$

$$
\begin{aligned}
Y(s)(s^2 - 3s + 2) &= s + 2 + \frac{1}{s+4} \\
Y(s) &= \frac{s^2 + 6s + 9}{(s^2 - 3s + 2)(s+4)} \\
Y(s) &= \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} \\
&= \frac{-{}^{16}\!/_{5}}{s-1} + \frac{{}^{25}\!/_{6}}{s-2} + \frac{{}^{1}\!/_{30}}{s+4} \\
\mathscr{L}^{-1}\left\{Y(s)\right\} &= -\frac{16}{4}e^t + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}
\end{aligned}
$$

Solve for $Y(s)$, then find $y(t) = \mathscr{L}^{-1}\left\{Y(s)\right\}$