7.3 - Operational Properties I

7.3.1 - Translation on S-axis

First Translation theorem:
$\mathscr{L}\{f(t)\} = F(s)$ and a is a real number, then $\mathscr{L}\{e^{at}f(t)\} = F(s)|_{s\to s-a} = F(s-a)$

Examples: Find

1) $\mathscr{L}\left\{9e + 3t - 4te^{3t} + 10e^{3t}\sin\frac{t}{2}\right\}$
$= \mathscr{L}\{9e^{3t}\} - \mathscr{L}\{4te^{3t}\} + \mathscr{L}\left\{10e^{3t}\sin\frac{t}{2}\right\}$
$= 9\mathscr{L}\{e^{3t}\} - 4\mathscr{L}\{te^{3t}\} + 10\mathscr{L}\left\{e^{3t}\sin\frac{t}{2}\right\}$
$= 9\mathscr{L}\{1\}|_{s\to s-3} - 4\mathscr{L}\{t\}|_{s\to s-3} + 10\mathscr{L}\left\{\sin\frac{t}{2}\right\}|_{s\to s-3}$
$= 9\frac{1}{s-3} - 4\frac{1}{(s-3)^2} + 10\frac{1/2}{(s-3)^2+1/4}$

2) $\mathscr{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\}$
$= \mathscr{L}^{-1}\left\{\frac{2s+5}{(s+3)^2+25}\right\} = \mathscr{L}^{-1}\left\{2\frac{s+3}{s^2+25} - \frac{1}{s^2+25}\right\}$
$= \mathscr{L}^{-1}\left\{2\frac{s+3}{s^2+25}\right\} - \frac{1}{5}\mathscr{L}^{-1}\left\{\frac{51}{s^2+25}\right\}$
$= 2e^{-3t}\cos 5t - \frac{1}{5}e^{-3t}\sin 5t$

Proof: $\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s)|_{s\to s-a}$

Inverse form of the first translation thm:

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t)$$

Ex: solve $y'' - y' = e^t\cos t, y(0) = 0, y'(0) = 0$

$(s^2Y(s) - sy(0) - y'(0)) - (sY(s) - y(0)) = \frac{s-1}{(s-1)^2+1}$

$(s^2 - s)Y(s) = \frac{s-1}{(s-1)^2+1}$

$Y(s) = \frac{s-1}{s(s-1)((s-1)^2+1)} = \frac{1}{s((s-1)^2+1)}$

$Y(s) = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s-1}{(s-1)^2+1} + \frac{1}{2}\cdot\frac{1}{(s-1)^2+1}$

$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{1}{2} - \frac{1}{2}e^t \cos t + \frac{1}{2}e^t \sin t$

Definition: (Unit step function)
The unit step function, a.k.a. the Heaveside function, is denoted $U(t-a)$ and is defined by $\mathscr{U}(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$

Ex: $g(t) = (3t-4)\mathscr{U}(t-1)$ may be written as $g(t) = \begin{cases} 0 & 0 \le t < 1 \\ 3t-4 & t \ge 1 \end{cases}$

Second Translation Thm:
If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$ then $\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s)$

ex: $\mathscr{L}\{\cos 2t\mathscr{U}(t-\pi)\}$

$= \mathscr{L}\{\cos 2(t-\pi)\mathscr{U}(t-\pi)\}$

$= e^{-\pi s}\frac{s}{s^2+4}$