Chapter 7 - The Laplace Transforms

D.E.(IVP) to Algebraic Equation to(inverse Laplace Transform) solution

Definition: Given $f(t)$, $t > 0$, its Leplace Transform is denoted $F(x) = \mathscr{L}\{f(t)\}$ and is given by $F(x) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

We say that the transform converges if the limit on the RHS(the imporper integral) converges, and diverges if the limit diverges

1) $f(t) = 1, t \geq 0$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)\, dt$$
$$= \lim_{A\to\infty} \left(\frac{e^{-st}}{-s} \bigg|_0^A \right) = \lim_{A\to\infty} \left(\frac{e^{-sA}}{-s} - \frac{e^{-s0}}{-s} \right) = \frac{1}{-s}$$

2) $f(t) = e^{at}$ where a is a const, and $t \geq 0$

$$F(x) = \int_0^\infty e^{-st} e^{at}\, dt = \lim_{A\to\infty} \int_0^A e^{-(s-a)t}\, dt$$
$$= \lim_{A\to\infty} \left(\frac{e^{-(s-a)A}}{-(s-a)} - \frac{e^{-(s-a)0}}{-(s-a)} \right) = \frac{1}{s-a}$$

3) $f(t) = t^n, t > 0$, where n is a positive integer

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} t^n\, dt = \lim_{A\to\infty} \int_0^A e^{-st} t^n\, dt$$
$$= \lim_{A\to\infty} \left(-\frac{1}{s} e^{-st} t^n \bigg|_0^A + \frac{n}{s} \int_0^A e^{-st} t^{n-1}\, dt \right)$$

$$= \lim_{A\to\infty} \left(\frac{n}{s} \int_0^A e^{-st} t^{n-1}\, dt \right) = \frac{n!}{s^{n+1}}$$

4) $f(t) = e^{iat}$

$$e^{iat} = \cos at + i \sin at$$

$$\mathscr{L}\left\{e^{iat}\right\} = \frac{1}{s-ia} * \frac{s+ia}{s+ia}$$
$$= \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

$$\mathscr{L}\left\{e^{iat}\right\} = \mathscr{L}\left\{\cos at\right\} + i\mathscr{L}\left\{\sin at\right\}$$

$$\mathscr{L}\left\{\cos at\right\} = \frac{s}{s^2+a^2}$$

$$\mathscr{L}\left\{\sin at\right\} = \frac{a}{s^2+a^2}$$

Linearity of the laplace tansform

$$\mathscr{L}\left\{cf(t) + g(t)\right\} = c\mathscr{L}\left\{f(t)\right\} + \mathscr{L}\left\{g(t)\right\}$$

Note: from here on $\int_o^\infty f(x)\, dx = \lim_{A\to\infty} \int_0^A f(x)\, dx$

Using the definition of the laplace transform find $\mathscr{L}\left\{f(t)\right\}$ for $f(t) = \begin{cases} 1 & 0 \leq t < 2 \\ t-2 & t \geq 2 \end{cases}$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st}(1)\,dt + \int_2^\infty e^{-st}(t-2)\,dt$$

' finish integration '

$$= -\frac{1}{s}e^{-st}\Big|_0^2$$

$$= -\frac{1}{s}(e^{-2s}-1) + \frac{s}{s^2}e^{-2s}$$

$$\begin{aligned}
\mathscr{L}\left\{1\right\} &= \frac{1}{s} \\
\mathscr{L}\left\{t^n\right\} &= \frac{n!}{s^{n+1}} \\
\mathscr{L}\left\{e^{at}\right\} &= \frac{1}{s-a} \\
\mathscr{L}\left\{\sin kt\right\} &= \frac{k}{s^2+k^2} \\
\mathscr{L}\left\{\cos kt\right\} &= \frac{s}{s^2+k^2} \\
\mathscr{L}\left\{\sinh kt\right\} &= \frac{k}{s^2-k^2} \\
\mathscr{L}\left\{\cosh kt\right\} &= \frac{s}{s^2-k^2}
\end{aligned}$$