Normalizations and changes in notation

I set $L_{nt} = 1$ for all country and time. This is just a normalization, as the level of producitivity can pick up the slack.

I drop T_n. There is no need to carry a time invariant productivity around.

I use A_{njt} rather than Z_{njt}. Because $L_{nt} = 1$, the difference is only the exponent.

We used to have y_{njt}/ψ_{njt} in the equation for productivity. This is only needed to express wages, so I just write out w_{njt} explicitly in the formulas below.

1 Model solution

Introduce a new variable for the factory-gate price of intermediate goods,

$$\varrho_{njt} = \xi B_j T_{nj}^{-1/\theta} A_{njt}^{-1} w_{njt}^{\beta_j} \prod_{k=1}^{J} P_{nkt}^{\gamma_{kj}}. \tag{1}$$

With this variable, we can write the price of the final good as

$$P_{mjt} = \left[\sum_{n=1}^{N} (\varrho_{njt}/\kappa_{mnjt})^{-\theta} \right]^{-1/\theta}. \tag{2}$$

This is a standard CES price index. We can write import shares as

$$d_{mnjt} = \left[\frac{\varrho_{njt}/\kappa_{mnjt}}{P_{mjt}} \right]^{-\theta}. \tag{3}$$

Wages can be expressed by inverting (1),

$$w_{njt} = (\xi B_j)^{-1/\beta_j} \varrho_{njt}^{1/\beta_j} T_{nj}^{1/(\beta_j \theta)} A_{njt}^{1/\beta_j} \prod_{k=1}^{J} P_{nkt}^{-\gamma_{kj}/\beta_j}. \tag{4}$$

Taking the market clearing condition,

$$R_{njt} = \sum_{m=1}^{N} d_{mnjt} \left[\alpha_{jt} \left(\sum_{k=1}^{J} \beta_k R_{mkt} - S_{mt} \right) + \sum_{k=1}^{J} \gamma_{jk} R_{mkt} \right], \tag{5}$$

$$R_{njt}\varrho_{njt}^{\theta} = \sum_{m=1}^{N}(\kappa_{mnjt}P_{mjt})^{\theta}\left[\alpha_{jt}\left(\sum_{k=1}^{J}\beta_k R_{mkt} - S_{mt}\right) + \sum_{k=1}^{J}\gamma_{jk}R_{mkt}\right],$$

$$\varrho_{njt} = R_{njt}^{-1/\theta}\left\{\sum_{m=1}^{N}(\kappa_{mnjt}P_{mjt})^{\theta}\left[\alpha_{jt}\left(\sum_{k=1}^{J}\beta_k R_{mkt} - S_{mt}\right) + \sum_{k=1}^{J}\gamma_{jk}R_{mkt}\right]\right\}^{1/\theta}. \tag{6}$$

Here both R_{njt} and ϱ_{njt} depend on equilibrium prices. But starting from a guess for R (say, the free-trade equilibrium), we can easily compute a shadow price ϱ supporting that allocation.

1. Set $R_{njt}^{(0)}$ and $P_{njt}^{(0)}$ to the autarky equilibrium value or other suitable starting value.
2. Given $R_{njt}^{(k)}$ and $P_{njt}^{(k)}$, solve for $\varrho_{njt}^{(k+1)}$ using equation (6).
3. Solve for $w_{njt}^{(k+1)}$ and $P_{njt}^{(k+1)}$ using (4) and (2).
4. Calculate $R_{njt}^{(k+1)} = w_{njt}^{(k+1)}L_{njt}/\beta_j$. (This step sounds the most divergent, nothing ensures that relative sectoral revenue shares will remain close to equilibrium.)
5. Go back to Step 2 and repeat until convergence.

Under CES final demand, α_{jt} should be replaced with

$$\alpha_{mjt}(\mathbf{P}_t) = \nu_{jt}\left(\frac{P_{mjt}}{P_{mt}}\right)^{1-\sigma},$$

and the consumer price index becomes

$$P_{mt} = \left(\sum_{j=1}^{J}\nu_{jt}P_{mjt}^{1-\sigma}\right)^{1/(1-\sigma)}$$

1.1 Dual loop

What are the sectoral expenditures across countries?

$$E_{mjt} = \alpha_{mjt}(\mathbf{P}_t)\left(\sum_{k=1}^{J}\beta_k R_{mkt} - S_{mt}\right) + \sum_{k=1}^{J}\gamma_{jk}R_{mkt}$$

The sectoral expenditure share is

$$e_{mjt} = \frac{\sum_{k=1}^{J} (\alpha_{mjt}(\mathbf{P}_t)\beta_k + \gamma_{jk}) R_{mkt} - S_{mt}}{\sum_{k=1}^{J} R_{mkt} - S_{mt}}. \tag{7}$$

Even if $S_{mt} = 0$, this is not constant, because it depends on the revenue distribution across sectors. If a country has a comparative advantage in car production, it will have high revenue (not expenditure) in cars. But then it will have high expenditure in car inputs.

For the purposes of finding the equilibrium, fix sectoral expenditure shares in each country. (We can start from free-trade expenditure shares, which are exogenous and are the same across countries.) Rewriting the equilibrium condition,

$$\varrho_{njt} = R_{njt}^{-1/\theta} \left[\sum_{m=1}^{N} (\kappa_{mnjt} P_{mjt})^{\theta} e_{mjt} \left(\sum_{k=1}^{J} R_{mkt} - S_{mt} \right) \right]^{1/\theta}.$$

Inner loop: solve for ϱ and R, holding e fixed. This does not allow for divergence across sectors off the equilibrium path, as the sectoral composition is held fixed.

Middle loop: given the solved Rs, calculate the new expenditure shares from equation (7).

1.2 Suitable starting values

Suppose that trade is free, $\kappa_{nmjt} \equiv 1$. Then $P_{njt} \equiv P_{jt}$ and

$$\varrho_{njt}^{(0)} = P_{jt} R_{njt}^{-1/\theta} \left\{ \sum_{m=1}^{N} \left[\alpha_{jt}(\mathbf{P}_t) \left(\sum_{k=1}^{J} \beta_k R_{mkt} - S_{mt} \right) + \sum_{k=1}^{J} \gamma_{jk} R_{mkt} \right] \right\}^{1/\theta}.$$

Let R_{wjt} denote the world revenue in sector j.

$$\varrho_{njt}^{(0)} = P_{jt} R_{njt}^{-1/\theta} \left[\sum_{k=1}^{J} (\alpha_{jt}\beta_k + \gamma_{jk}) R_{wkt} \right]^{1/\theta}.$$

This only determines ϱ_{njt}/P_{jt}, what determines P_{jt} relative to other sectors?

$$\varrho_{njt}/P_{jt} = \beta_j^{1/\theta} (w_{njt} L_{njt})^{-1/\theta} \left[\sum_{k=1}^{J} (\alpha_{jt} + \gamma_{jk}/\beta_k) \sum_{m=1}^{N} w_{mkt} L_{mkt} \right]^{1/\theta}.$$

Sum across ns,

$$(\varrho_{njt}/P_{jt})^{-\theta} = \beta_j^{-1}(w_{njt}L_{njt})\left[\sum_{k=1}^{J}(\alpha_{jt}+\gamma_{jk}/\beta_k)\sum_{m=1}^{N}w_{mkt}L_{mkt}\right]^{-1}.$$

$$w_{wjt}L_{wjt} = \beta_j\sum_{k=1}^{J}(\alpha_{jt}+\gamma_{jk}/\beta_k)w_{wkt}L_{wkt}.$$

$$R_{wjt} = \sum_{k=1}^{J}(\alpha_{jt}\beta_k+\gamma_{jk})R_{wkt}.$$

We can solve this for eigenvalue equation for R_{wjt} without knowing anything about equilibrium prices, conditional a global numeraire.

Write the log of equation (1) in matrix notation, and recognizing that under free trade, P does not depend on n,

$$\ln\varrho_{nt} = \ln\xi + \ln\mathbf{B} - \ln\mathbf{A}_{nt} + \mathrm{diag}(\beta)\ln\mathbf{w}_{nt} + \Gamma\ln\mathbf{P}_t$$

$$\ln\varrho_{nt} - \ln\mathbf{P}_t = \frac{1}{\theta}\ln\beta - \frac{1}{\theta}(\ln\mathbf{w}_{nt}+\ln\mathbf{L}_{nt}) + \frac{1}{\theta}\ln\mathbf{E}_{wt}$$

$$\frac{1}{\theta}\ln\beta - \frac{1}{\theta}(\ln\mathbf{w}_{nt}+\ln\mathbf{L}_{nt}) + \frac{1}{\theta}\ln\mathbf{E}_{wt} = \ln\xi + \ln\mathbf{B} - \ln\mathbf{A}_{nt} + \mathrm{diag}(\beta)\ln\mathbf{w}_{nt} + (\Gamma-\mathbf{I})\ln\mathbf{P}_t$$

$$\ln\beta - \ln\mathbf{L}_{nt} + \ln\mathbf{E}_{wt} + \theta\ln\mathbf{A}_{nt} - \theta\ln\xi - \theta\ln\mathbf{B} + \theta(\mathbf{I}-\Gamma)\ln\mathbf{P}_t = \mathrm{diag}(1+\beta\theta)\ln\mathbf{w}_{nt}$$

Guess a P. Solve above for w. Then solve for next P.

1.2.1 General problem for free trade

Pricing equation:

$$\ln\varrho_{nt} - \ln\mathbf{P}_t = \ln\xi + \ln\mathbf{B} - \ln\mathbf{A}_{nt} + \mathrm{diag}(\beta)(\ln\mathbf{w}_{nt} - \ln\mathbf{P}_t) + \mathbf{C}\ln\mathbf{P}_t$$

The last matrix $\mathbf{C} = \Gamma + \mathrm{diag}(\beta) - \mathbf{I}$ has the property $\mathbf{C1} = \mathbf{0}$ so this equation is hom(0) in prices.

Import share equation:

$$\ln\varrho_{nt} - \ln\mathbf{P}_t = +\frac{1}{\theta}(\ln\beta + \ln\mathbf{E}_{wt} - \ln\mathbf{w}_{nt} - \ln\mathbf{L}_{nt})$$

This is also hom(0) in prices. World expenditure E_{wt} can be precomputed up to a choice of numeraire.

CES price index:

$$\ln \mathbf{P}_t = -\frac{1}{\theta} \ln \left[\sum_{n=1}^{N} \exp(-\theta \ln \varrho_{nt}) \right]$$

$$\mathbf{1} = \sum_{n=1}^{N} \exp[-\theta(\ln \varrho_{nt} - \ln \mathbf{P}_t)]$$

Again, hom(0) in prices. Substituting in the import share equation,

$$\mathbf{1} = \sum_{n=1}^{N} \exp(\ln \mathbf{w}_{nt} - \ln \mathbf{E}_{wt} - \theta \mathbf{b}_2).$$

This just says that the import shares add up to 1. This suggests that we can limit the search to import shares on the simplex. Let $\mathbf{d}_{nt}$ denote the import share vector.

$$\ln \varrho_{nt} - \ln \mathbf{P}_t = -\frac{1}{\theta} \ln \mathbf{d}_{nt}$$

Wages are

$$\ln \mathbf{w}_{nt} = \ln \beta + \ln \mathbf{d}_{nt} + \ln \mathbf{E}_{wt} - \ln \mathbf{L}_{nt}$$

so that

$$-\frac{1}{\theta} \ln \mathbf{d}_{nt} = \ln \xi + \ln \mathbf{B} - \ln \mathbf{A}_{nt} + \mathrm{diag}(\beta)(\ln \beta + \ln \mathbf{d}_{nt} - \ln \mathbf{L}_{nt} + \ln \mathbf{E}_{wt}) - (\mathbf{I} - \Gamma) \ln \mathbf{P}_t$$

$$\mathrm{diag}(\beta + 1/\theta) \ln \mathbf{d}_{nt} = -\ln \xi - \ln \mathbf{B} + \ln \mathbf{A}_{nt} + \mathrm{diag}(\beta)(\ln \mathbf{L}_{nt} - \ln \mathbf{E}_{wt}) + (\mathbf{I} - \Gamma) \ln \mathbf{P}_t$$

This holds for all n and t. In particular, it also holds for country 0, which we can subtract:

$$\mathrm{diag}(\beta + 1/\theta)(\ln \mathbf{d}_{nt} - \ln \mathbf{d}_{0t}) = \ln \mathbf{A}_{nt} - \ln \mathbf{A}_{0t} + \mathrm{diag}(\beta)(\ln \mathbf{L}_{nt} - \ln \mathbf{L}_{0t})$$

$$(\ln \mathbf{d}_{nt} - \ln \mathbf{d}_{0t}) = \mathrm{diag}(\theta/(1+\beta\theta))(\ln \mathbf{A}_{nt} - \ln \mathbf{A}_{0t}) + \mathrm{diag}(\beta\theta/(1+\beta\theta))(\ln \mathbf{L}_{nt} - \ln \mathbf{L}_{0t})$$

This suggests the following solution for import shares,

$$d_{njt} = c_{jt} A_{njt}^{\theta/(1+\beta_j\theta)} L_{njt}^{\beta_j\theta/(1+\beta_j\theta)},$$

where the c_{jt} constant is such that import shares add up to 1 for all j and all t.

Given d_{njt} and the precomputed E_{wjt}, we can calculate wages in all sectors.

$$-\frac{1}{\theta}\ln\mathbf{d}_{nt} = \ln\xi + \ln\mathbf{B} - \ln\mathbf{A}_{nt} + \text{diag}(\beta)(\ln\mathbf{w}_{nt} - \ln\mathbf{P}_t) + \mathbf{C}\ln\mathbf{P}_t$$

$$[\text{diag}(\beta) - \mathbf{C}]\ln\mathbf{P}_t = \frac{1}{\theta}\ln\mathbf{d}_{nt} + \ln\xi + \ln\mathbf{B} - \ln\mathbf{A}_{nt} + \text{diag}(\beta)\ln\mathbf{w}_{nt}$$

Prices can be calculated from

$$\ln\mathbf{P}_t = (\mathbf{I} - \Gamma)^1[\frac{1}{\theta}\ln\mathbf{d}_{nt} + \ln\xi + \ln\mathbf{B} - \ln\mathbf{A}_{nt} + \text{diag}(\beta)\ln\mathbf{w}_{nt}]$$

Note: This gamma matrix is the transpose of gammajk in the code.

1.3 Expectations and outer loop

An equilibrium condition states

$$\frac{L_{njt}}{L_{nt}} = E_{t-1}\frac{w_{njt}L_{njt}}{w_{nt}L_{nt}}. \tag{8}$$

We calculate the expected wage share as follows. Given past productivities $A_{nj,t-1}$, draw S realizations for current productivities, A_{njs}. Because we are looking for a share bounded between 0 and 1, we don't need a sophisticated sampling to calculate the numerical integral.

Start from a candidate labor allocation $L^{(0)}_{njt}$. Calculate expected wage share as the simple average of wage shares across the S realizations. Set $L^{(n+1)}_{njt} = L_{nt}\eta^{(n)}_{njt}$, where $\eta^{(n)}_{njt}$ is the wage share in iteration n. This may be dampened, but in practice, it convergence in a handful of steps.

1.3.1 Time series

Given the autoregressive nature of productivity, the vector of A_{jnt}s is a state variable. The problem is drastically simplified by the fact that no decision variable affects the state. The representative agent solves

$$\max_{\{L_{njt}, L^*_{njt}\}} E\sum_{t=0}^{\infty}\delta^t\left[\ln\frac{\sum_j w_{njt}(\mathbf{A}_t, \mathbf{L}_t)L_{njt}}{P_{nt}(\mathbf{A}_t, \mathbf{L}_t)} - \frac{\varrho}{2}\sum_{j=1}^{J}\left(\frac{L_{njt}}{L_{nt}} - \frac{L^*_{njt}}{L_{nt}}\right)^2\right]$$

subject to $\sum_j L_{njt} = \sum_j L^*_{njt} = L_{nt}$ and w and P being determined in equilibrium. The starred labor allocation is determined before shock realizations are known.

The corresponding Bellman equation is

$$V_n^N(\mathbf{A}_{t-1}) = \max_{L^*_{njt}} E_A V_n^D(\mathbf{A}_t, \mathbf{L}^*_t) \tag{9}$$

for the "night" period before productivity shocks have been realized. For the "day" period,

$$V_n^D(\mathbf{A}_t, \mathbf{L}^*_t) = \max_{\{L_{njt}\}} \ln \frac{\sum_j w_{njt}(\mathbf{A}_t, \mathbf{L}_t) L_{njt}}{P_{nt}(\mathbf{A}_t, \mathbf{L}_t)} - \frac{\varrho}{2} \sum_{j=1}^{J} \left(\frac{L_{njt}}{L_{nt}} - \frac{L^*_{njt}}{L_{nt}} \right)^2 + \delta E V_n^N(\mathbf{A}_t). \tag{10}$$

The day value depends on both time-t productivity and on labor allocations decided during the night period, before A_t was known.

Note that choice L_{njt} does not affect future state A_t. Also, wages and prices do not depend on L^* directly, only through L.

The first-order-condition for the night Bellman is

$$\frac{\partial E_A V_n^D(\mathbf{A}_t, \mathbf{L}^*_t)}{\partial L^*_{njt}} = 0$$

for all j. To determine the partial derivative of the day value, we can use the envelope theorem,

$$\frac{\partial V_n^D(\mathbf{A}_t, \mathbf{L}^*_t)}{\partial L^*_{njt}} = \frac{\varrho}{L_{nt}} \left(\frac{L_{njt}}{L_{nt}} - \frac{L^*_{njt}}{L_{nt}} \right).$$

Equating this with zero in expectation,

$$L^*_{njt} = E_A L_{njt}.$$

Or, in other words,

$$L_{njt} = L^*_{njt} + \varepsilon_{njt}(\mathbf{A}_t),$$

where ε_{njt} is the equilibrium deviation from labor, with zero mean conditional on past productivity.

Using the first-order-condition for the day period,

$$\frac{w_{njt}(\mathbf{A}_t, \mathbf{L}^*_t + \varepsilon_t)}{\sum_k w_{nkt}(\mathbf{A}_t, \mathbf{L}^*_t + \varepsilon_t) L_{nkt}/L_{nt}} \equiv \frac{w_{njt}(\mathbf{A}_t, \mathbf{L}^*_t + \varepsilon_t)}{w_{nt}(\mathbf{A}_t, \mathbf{L}^*_t + \varepsilon_t)} = \lambda L_{nt} + \varrho \frac{\varepsilon_{njt}(\mathbf{A}_t)}{L_{nt}}.$$

In sectors where the wage rate is higher than the weighted-average wage of the economy, the optimal allocation of labor is higher than the ex-ante expected allocation.

The challenge is that both sides depend on ε_t, the labor response to productivity innovations.

Because in expectation the last term on the right-hand-side of this equation is zero, the L^* has to be such that

$$E_A \frac{w_{njt}(\mathbf{A}_t, \mathbf{L}_t^* + \varepsilon_t)}{w_{nt}(\mathbf{A}_t, \mathbf{L}_t^* + \varepsilon_t)} = \lambda L_{nt}$$

constant. There can be no expected deviations in the (ratio) relative wage across sectors.

$$\frac{\varepsilon_{njt}}{L_{nt}} = \frac{1}{\varrho} \left[\frac{w_{njt}(\mathbf{A}_t, \mathbf{L}_t^* + \varepsilon_t)}{w_{nt}(\mathbf{A}_t, \mathbf{L}_t^* + \varepsilon_t)} - \lambda L_{nt} \right], \tag{11}$$

and λ is such that $\sum_j \varepsilon_{njt} = 0$ because of the resource constraint. That is,

$$\lambda L_{nt} = \frac{1}{J} \sum_{j=1}^{J} \frac{w_{njt}}{w_n},$$

the average wage ratio across sectors (which may be different from one due to Jensen inequality terms).

This can be solved for ε implicitly, as the RHS is (likely) decreasing in ε_{njt}. We then have $\varepsilon_{njt}(\mathbf{A}_t, \mathbf{L}_t^*)$ and $\tilde{w}_{njt}(\mathbf{A}_t, \mathbf{L}_t^*)$. Note that $\tilde{w}()$ is a different function from $w()$, because it already captures the endogenous labor response. They are only identical when $\varrho = \infty$. Otherwise, the distribution of $\tilde{w}$ depends on adjustment costs.

The night labor share is a fixed point of

$$E_A \frac{\tilde{w}_{njt}(\mathbf{A}_t, \mathbf{L}_t^*)}{\tilde{w}_{nt}(\mathbf{A}_t, \mathbf{L}_t^*)} = \lambda L_{nt}.$$

1.4 Calculating expectations

Agents have rational expectations. The DGP of producitivities is as follows

$$\ln A_{njt} = a_{njt} + \lambda_{jt} + \mu_{nt} + \varepsilon_{njt}. \tag{12}$$

The first term a_{njt} is a deterministic trend, perfectly foreseen by agents. The second term λ_{jt} is a global sectoral shock as in Koren and Tenreyro (2007). The third term μ_{nt} is a country shock, and ε_{njt} is the idiosyncratic shock.

These shocks have the following DGP

$$\lambda_{jt} = \rho_j \lambda_{jt-1} + \sigma_j u_{jt} \tag{13}$$

$$\mu_{nt} = \rho_n \mu_{nt-1} + \sigma_n u_{nt} \tag{14}$$

$$\varepsilon_{njt} = \rho_{nj} \varepsilon_{njt-1} + \sigma_{nj} u_{njt} \tag{15}$$

where the us are serially and cross-sectionally independent standard normal variables. That is, each shock component follows an independent AR(1) process.

Expectations are taken conditional on $(a_{njt}, \lambda_{jt-1}, \mu_{nt-1})$, that is, over realizations of $(u_{jt}, u_{nt}, u_{njt})$. In practice we draw $S = 100$ realizations of the innovations to compute the expected wage share.

Numerically integrating the expected wage shares over us converges fast, and hence relatively low S is sufficient, because wage shares are bounded between 0 and 1

2 Calibration

2.1 Calibrating the CES model*

To calibrate the CES demand shifters ν, we need data on how final expenditure is split across sectors in each country. We will recover this from the split of revenue, subtracting intermediate expenditure.

Let

$$\Delta_{jt} = [d_{mnjt}]$$

denote the matrix of import shares (including the domestic market). We know that $\Delta_{jt}\mathbf{1} = 1$. We write equation (5) in matrix form as

$$\tilde{\mathbf{R}}_{jt} = \tilde{\mathbf{E}}_{jt} \Delta_{jt}, \tag{16}$$

which we can invert

$$\tilde{\mathbf{E}}_{jt} = \tilde{\mathbf{R}}_{jt} \Delta_{jt}^{-1}, \tag{17}$$

and we can use $\tilde{\mathbf{Y}}_{jt}/\beta_j = \tilde{\mathbf{R}}_{jt}$. Once we have E_{mjt}, we subtract $\sum_k \gamma_{jk} R_{mkt}$ intermediate expenditure to get the final expenditure. Note that either of these expenditures may be negative in the data, in which case they are replaced by numerical zero.

Given a set of expenditure shares, we proceed as follows. First, we recover demand shifters and aggregate price index for the baseline country (US), using the observed sectoral prices there.

$$\nu_{0jt} = e_{0jt} P_{0jt}^{\sigma-1} P_{0t}^{1-\sigma},$$

where P_{0t} is selected such that $\sum_j \nu_{0jt} = 1$ for all t.

Second, for the tradable sectors, we recover sectoral prices P_{njt} using the trade share formula. This expresses prices relative to the baseline country, but we already have the baseline prices.

Third, we recover tradable demand shifters using the price of the sector relative to the consumption aggregate obtained from PWT.

$$\nu_{njt} = e_{njt} P_{njt}^{\sigma-1} (P_{0t} PWT_{nt})^{1-\sigma},$$

for all $j < J$.

Fourth, for services, we calculate the demand shifter as

$$\nu_{nJt} = 1 - \sum_{j \neq J} \nu_{njt},$$

and construct its prices as

$$P_{nJt} = (\nu_{nJt}/e_{nJt})^{1/(\sigma-1)} P_{0t} PWT_{nt}.$$

This is the only step that has to be done separately for $\sigma = 1$, although approximations may be valid. CHECK.

Computing the price index for US, 1972,

$$\left[\sum_{j=1}^{J} \nu_{US,jt} 1^{1-\sigma}\right]^{1/(1-\sigma)} = \left[\sum_{j=1}^{J} \nu_{US,jt}\right]^{1/(1-\sigma)} = 1, \tag{18}$$

so we are measuring all nominal quantities in 1972 US dollars. We then average across countries to get ν_{jt}.

However, calibrating nontradable prices this way relies on our estimates of final expenditure shares. These tend to be particularly noisy as they involve a matrix inversion and the subtraction of intermediate expenditure, which, in reality, may not be drive by the same technology matrix across countries. Hence our price estimates and productivity estimates are very noisy. This leads to GDP volatility that is much higher than in the data (Figure 1).

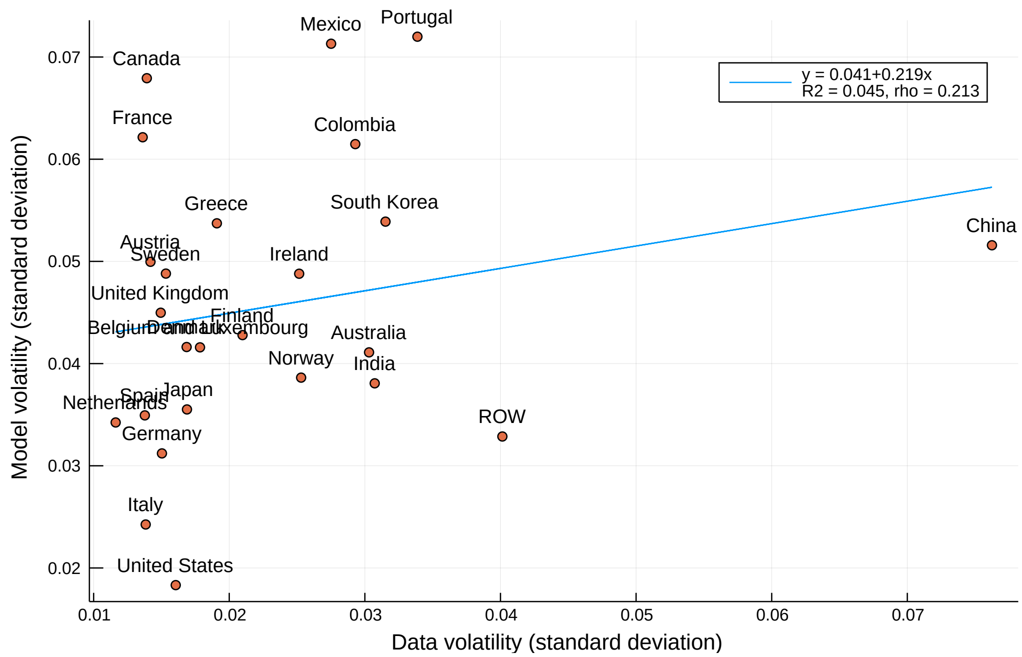

Figure 1: GDP volatility in the data and in the model calibrated to $\sigma = 2$

As an alternative calibration, we use the price index formula for the Cobb-Douglas specification,

$$P_{nJt} = (P_{0t} PWT_{nt} / P_{n.t})^{1/\nu_{Jt}}$$

with

$$P_{n.t} \equiv \prod_{j=1}^{J-1} P_{njt}^{\nu_{jt}}.$$

The benefit of this formula is that it does not depend on the noisily measured country-specific expenditure shares. This leads to the same calibrated productivity process as under the Cobb-Douglas scenario ($\sigma = 1$). This provides a much better fit with the data (Figure 2).

2.2 Calibrating US productivity

Import shares are only informative about relative productivities. For a benchmark country, the US, we calibrate productivity to match observed value

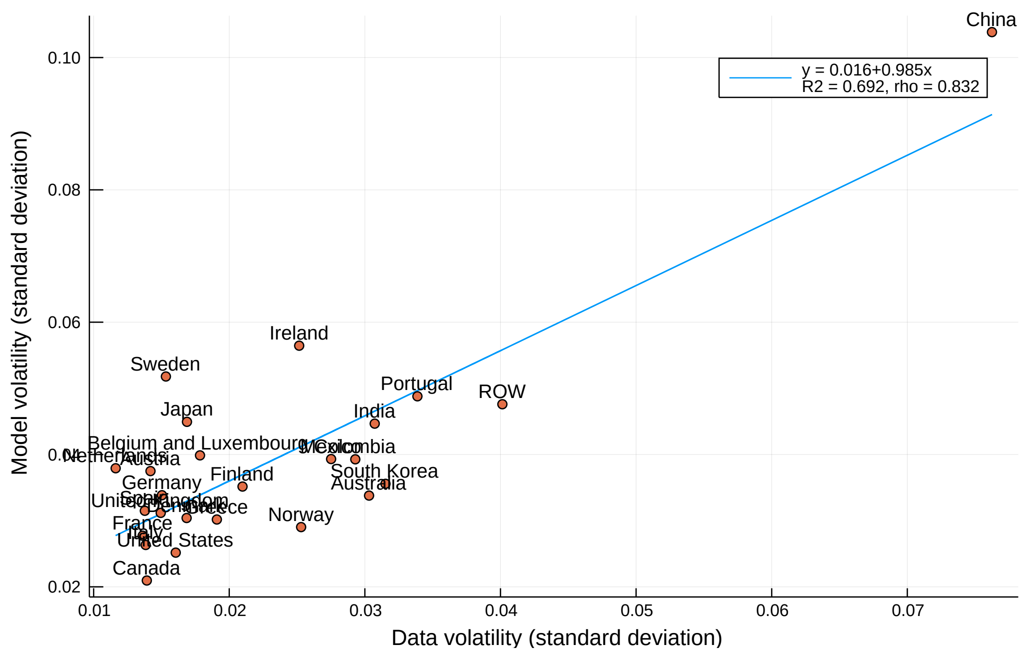

Figure 2: GDP volatility in the data and in the model with $\sigma = 2$ but productivity calibrated to Cobb-Douglas case

added data. Using (4), we can express productivity as a function of sectoral wages,

$$A_{njt} = \xi B_j w_{njt}^{\beta_j} \varrho_{njt}^{-1} \prod_{k=1}^{J} P_{nkt}^{\gamma_{kj}},$$

and

$$d_{nnjt} = (\varrho_{njt}/P_{njt})^{-\theta}$$

can be used to express

$$\varrho_{njt} = P_{njt} d_{nnjt}^{-1/\theta}.$$

$$A_{njt} = \xi B_j d_{nnjt}^{1/\theta} (w_{njt}/P_{njt})^{\beta_j} \prod_{k=1}^{J} (P_{nkt}/P_{njt})^{\gamma_{kj}}. \tag{19}$$

Productivity is high when domestic market share is high, when real wages in sector are high, or when real input costs are high.

Using (11) to express sectoral wages in the US,

$$w_{njt} = w_{nt} \left[\varrho(L_{njt} - L^*_{njt}) + \lambda_{nt} \right],$$

with the normalization that $L_{nt} = 1$. How do we calibrate this without observing L_{njt}? Multiply by L_{njt},

$$L_{njt}^2 + (\lambda_{nt}/\varrho - L^*_{njt}) L_{njt} - \frac{1}{\varrho} \frac{w_{njt} L_{njt}}{w_n t} = 0.$$

Solving this quadratic equation,

$$L_{njt} = \frac{1}{2}(L^*_{njt} - \lambda_{nt}/\varrho) + \sqrt{\frac{1}{4}(L^*_{njt} - \lambda_{nt}/\varrho)^2 + \frac{1}{\varrho} V_{njt}},$$

where V_{njt} is the value added share in sector j. Recall that λ_{nt} equals the unweighted average wage ratio across sectors, and is approximately one, ignoring Jensen inequality terms. Also, in equilibrium L^*_{njt} is the expected value added share, which can be approximated by the long-run trend.

This helps us calibrate the sectoral labor share as

$$L_{njt} \approx (L^*_{njt} - 1/\varrho) \left[\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{1}{\varrho} \frac{V_{njt}}{L^*_{njt} - 1/\varrho}} \right].$$

Given a set of labor shares, we can calculate wages as $w_{njt} = Y_{njt}/L_{njt}$.

2.3 Decomposing productivity shocks

Given a series of productivities calibrated to the data, we decompose productivity into these four components:

$$\ln A_{njt} = a_{njt} + \lambda_{jt} + \mu_{nt} + \varepsilon_{njt}. \tag{20}$$

The first term is a Baxter-King trend. The second is a global sector shock, calculated as the (j, t) average of productivity deviations across countries. The third term is calculated as a weighted average of remaining shocks within a country-year. (Weighting is necessary because otherwise idiosyncratic shock of services, the larges sector, would dominate fluctuations in all countries.) The last term is a residual.

The parameter vector ρ and σ is estimated using simple OLS. Shocks u are drawn for iid standard normal distribution.