Exercise 1. $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$

Dimostrazione. First of all, observe that any probability measure is monotone, since $B \supseteq A \Rightarrow \mathbb{P}(B) = \mathbb{P}(B \setminus A \uplus A) = \mathbb{P}(B \setminus A) + \mathbb{P}(A) \geq \mathbb{P}(A)$.
Define $A'_0 := A_0$ and $A'_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i$. It's immediate to check that $\bigcup_{n=1}^{\infty} A'_n = \bigcup_{n=1}^{\infty} A_n$. Of course, $i \neq j \Rightarrow A'_i \cap A'_j \neq \emptyset$ and $A'_n \subseteq A_n$ for all n. Therefore

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} A'_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A'_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

□

Exercise 2. TO-DO $P(\liminf A_n) = \mathbb{P}(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m)$. Define $B_n := \bigcap_{m=n}^{\infty} A_m$ and observe that $B_n \subseteq B_{n+1}$, thus by continuity $\mathbb{P}(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m) = \mathbb{P}(\lim_{n\to\infty} \bigcup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mathbb{P}(\bigcup_{n=1}^{\infty} B_n) = \liminf \mathbb{P}(\bigcup_{n=1}^{\infty} B_n)$

Exercise 3. Prove the Borel-Cantelli lemma:

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}(\limsup A_n) = 0.$$

Dimostrazione. Since $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, we have necessarily that

$$\lim_{m\to\infty} \sum_{n=m}^{\infty} \mathbb{P}(A_n) = 0.$$

Then, by monotony

$$\mathbb{P}(\limsup A_n) = \mathbb{P}\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n\right) \leq \mathbb{P}\left(\bigcup_{n=m}^{\infty} A_n\right), \quad \forall m \in \mathbb{N}$$

But, by countable subadditivity

$$\mathbb{P}\left(\bigcup_{n=m}^{\infty} A_n\right) \leq \sum_{n=m}^{\infty} \mathbb{P}(A_n) \overset{n\to\infty}{\longrightarrow} 0$$

and we are done. □

Exercise 4.

Dimostrazione.

$$\sum_{i\in I}\mathbb{P}(A \mid B_i)\mathbb{P}(B_i) = \sum_{\{i\in I:\mathbb{P}(B_i)\neq 0\}} \frac{\mathbb{P}(A\cap B_i)}{\mathbb{P}(B_i)}\mathbb{P}(B_i) = \sum_{i\in I}\mathbb{P}(A\cap B_i) =$$

$$= \mathbb{P}\left(\biguplus_{i\in I} A\cap B_i\right) = \mathbb{P}\left(A\cap \biguplus_{i\in I} B_i\right) = \mathbb{P}(A).$$

□

Exercise 5. $\mathbb{P}(A \mid B) = \mathbb{P}(B \mid A)\cdot\kappa$. Determine κ.

Dimostrazione. Since $\mathbb{P}(A) = 0 \vee \mathbb{P}(B) = 0 \Rightarrow \mathbb{P}(A \mid B) = \mathbb{P}(B \mid A) = 0$, in that case we can choose an arbitrary κ, for example $\kappa = 0$.
If $\mathbb{P}(A) \neq 0 \neq \mathbb{P}(B)$, then $\mathbb{P}(B \mid A) = 0 \Rightarrow \mathbb{P}(B\cap A) = 0 \Rightarrow \mathbb{P}(A \mid B) = 0$, and we can choose an arbitrary κ again. If we suppose $P(A \mid B)$ too, then

$$\frac{\mathbb{P}(A\mid B)}{\mathbb{P}(B\mid A)} = \frac{\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}}{\frac{\mathbb{P}(B\cap A)}{\mathbb{P}(A)}} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)}.$$

So $\kappa = \frac{\mathbb{P}(A)}{\mathbb{P}(B)}$. □

Exercise 6.

- Let D_i^C be the event "the ball drawn from i-th urn is of colour C". Of course $D_1^R \uplus D_1^W = \Omega$. So, by Exercise 4, we obtain

  $$\mathbb{P}(D_2^R) = \mathbb{P}(D_2^R \mid D_1^R)\cdot\mathbb{P}(D_1^R) + \mathbb{P}(D_2^R \mid D_1^W)\cdot\mathbb{P}(D_1^W) = 7/10\cdot 3/10 + 6/10\cdot 7/10 = 63/100.$$

- By Exercise 5 (Bayes' formula) we obtain

  $$\mathbb{P}(D_1^W \mid D_2^W) = \mathbb{P}(D_2^W \mid D_1^W)\frac{\mathbb{P}(D_1^W)}{\mathbb{P}(D_2^W)} = \mathbb{P}(D_2^W \mid D_1^W)\frac{\mathbb{P}(D_1^W)}{1-\mathbb{P}(D_2^R)} = 4/10\cdot\frac{7/10}{37/100} = 28/37.$$

Exercise 7.

Dimostrazione. We want to prove that for any $n \in \mathbb{N}$ and any set $\{A_1, ..., A_n\} \in \mathcal{P}(\Omega)$ the following holds:

$$\forall\{i_1, ..., i_k\} \subseteq \{1, ..., n\} \quad \left[\mathbb{P}(A_{i_1}\cap A_{i_2}\cap ...\cap A_{i_k}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2})...\mathbb{P}(A_{i_k})\right]$$

$$\Updownarrow$$

$$\forall\{\epsilon_1, ..., \epsilon_n\} \subseteq \{-1, 1\} \quad \left[\mathbb{P}(A_1^{\epsilon_1}\cap ...\cap A_n^{\epsilon_n}) = \mathbb{P}(A_1^{\epsilon_1})...\mathbb{P}(A_n^{\epsilon_n})\right]$$

($\Longrightarrow$) We proceed by induction on $N := |\{\epsilon_i : \epsilon_i = -1\}|$. For the case $n = 1$, observe that
$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c),$$
so
$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A)(1 - \mathbb{P}(B)) = \mathbb{P}(A)\mathbb{P}(B^c)$$
($\Longleftarrow$) Fix $n \in \mathbb{N}$. We proceed by induction on $d = n - k$. If $d = 0$ there is nothing to prove. Now suppose that the statement holds for $n - k = d$, i.e. for $k = n - d$. We want to show that it holds for $d = n - k + 1$, i.e. for $n - d = k - 1$. Observe that

$$\begin{gathered}\mathbb{P}(A_{i_1} \cap ... \cap A_{i_{k-1}}) = \mathbb{P}(A_{i_1} \cap ... \cap A_{i_{k-1}} \cap B) + \mathbb{P}(A_{i_1} \cap ... \cap A_{i_{k-1}} \cap B^c) = \\ \mathbb{P}(A_{i_1})...\mathbb{P}(A_{i_{k-1}})\mathbb{P}(B) + \mathbb{P}(A_{i_1})...\mathbb{P}(A_{i_{k-1}})\mathbb{P}(B^c) = \\ \mathbb{P}(A_{i_1})...\mathbb{P}(A_{i_{k-1}})\mathbb{P}(B) + \mathbb{P}(A_{i_1})...\mathbb{P}(A_{i_{k-1}})(1 - \mathbb{P}(B)) = \mathbb{P}(A_{i_1})...\mathbb{P}(A_{i_{k-1}})\end{gathered}$$

where the second equality holds thanks to the inductive hypothesis.
Since n is arbitrary, the proof above holds for any set of events $\{A_1, ..., A_n\}$.

□

Exercise 8.

Dimostrazione. $i \neq j \Rightarrow \mathbb{P}(A_i \cap A_j) = 1/4 = 2/4 \cdot 2/4 = \mathbb{P}(A_i)\mathbb{P}(A_j)$.
But $\mathbb{P}(A_1 \cap A_2 \cap A_3) = 1/4 \neq 2/4 \cdot 2/4 \cdot 2/4 = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$. □

Lezione 14/10
Remark: d'ora in poi consideriamo le RV *uguali* se $\mathbb{P}[X = X'] = 1$.

Definizione 0.0.1. The *expectation* (or *expected value, mean*) of X is
$$\mathbb{E}(X) := \int_\Omega X d\mathbb{P}$$
if the integral exists.

Reminder: Construction of Lebesgue integral:

1. Simple RV:
$$X = \sum_{i=1}^{n} c_i \mathbb{1}$$
with $c_i \in \mathbb{R}$, $A_i \in \mathcal{A}$. Then
$$\mathbb{E}(X) := \sum_{i=1}^{n} c_i \mathbb{P}(A_i)$$

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