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Exercise 31. Let $Y = \mathbb{E}[X \mid \mathcal{F}]$. We know that Y is constant over every C_j, i.e. $Y_{|C_j} = y_j \in \mathbb{R}$. Furthermore, we know

$$y_j = \frac{\mathbb{E}[X \cdot \mathbb{1}_{C_j}]}{\mathbb{P}[C_j]}.$$

Now observe that $X_{|C_j} = -1$ for all $j \geq 3$. So $\mathbb{E}[X \cdot \mathbb{1}_{C_j}] = -1$. Since $\mathbb{P}[C_j] = 2^{-j+1} - 2^{-j} = 2^{-j}$, we obtain

$$y_j = -2^{-j} \quad \forall j \geq 3.$$

We shall compute y_1 and y_2. We have $C_1 = (1/2, 1]$ and $C_2 = (1/4, 1/2]$. So

$$\mathbb{E}[X \cdot \mathbb{1}_{C_1}] = 0 \cdot \left(\frac{2}{3} - \frac{1}{2}\right) + 1 \cdot \left(1 - \frac{2}{3}\right) = \frac{1}{3}$$

and

$$\mathbb{E}[X \cdot \mathbb{1}_{C_2}] = -1 \cdot \left(\frac{1}{4} - 0\right) + 0 \cdot \left(\frac{1}{3} - \frac{1}{4}\right) = -\frac{1}{4}.$$

Finally:

$$y_1 = \frac{1}{3} \cdot 2^1 = \frac{2}{3} \quad \text{and} \quad y_2 = -\frac{1}{4} \cdot 2^2 = -1.$$

□

Exercise 32. Observe that, for every r.v. X, we have

$$X = \frac{X(\omega) + X(-\omega)}{2} + \frac{X(\omega) - X(-\omega)}{2}.$$

Now define $X_1(\omega) := \frac{X(\omega)+X(-\omega)}{2}$ and $X_2 := \frac{X(\omega)-X(-\omega)}{2}$. By linearity of conditional expectation, we get $\mathbb{E}[X \mid \mathcal{F}] = \mathbb{E}[X_1 \mid \mathcal{F}] + \mathbb{E}[X_2 \mid \mathcal{F}]$.
Now let $A \in \mathcal{F}$. We have

$$\int_A X_2 \, d\mathbb{P} = \frac{1}{2} \int_A X_2 \, d\lambda = 0,$$

since X_2 is an odd function and A is symmetric. So (by uniqueness) it follows that $\mathbb{E}[X_2 \mid \mathcal{F}] \equiv 0$.
Now observe that X_1 is an even function, and it is immediate to see that every even function is $\mathcal{F}$-measurable. Thus, by the theory (but actually it's immediate by the definition of conditional expectation), $\mathbb{E}[X_1 \mid \mathcal{F}] = X$, whereby $\mathbb{E}[X \mid \mathcal{F}] = X$. □

Exercise 33. First, observe that $\mathcal{F}_{n-1} := \sigma(Y_1, ..., Y_{n-1})$ is atomic. In fact, calling $\mathbf{Y} := (Y_1, ..., Y_{n-1})$ the r.v. $\Omega \to \mathcal{Y}^{n-1}$, we have $\mathcal{F}_{n-1} = \sigma(\mathbf{Y}) = \{\mathbf{Y}^{-1}(S) \mid S \subseteq \mathcal{Y}^{n-1}\}$. Observe that

$$\mathbf{Y}^{-1}(S) = \bigcup_{s \in S} \mathbf{Y}^{-1}(\{s\}).$$

Since $\mathcal{Y}$ is countable, $\mathcal{Y}^{n-1}$ is countable as well, and so is every subset $S \subseteq \mathcal{Y}^{n-1}$. This implies that the union above is actually a countable union. It follows that

$$\sigma(\mathbf{Y}) = \sigma\big(\{\mathbf{Y}^{-1}(\{s\}) \mid s \in \mathcal{Y}^{n-1}\}\big).$$

We have trivially

$$\biguplus_{s \in \mathcal{Y}^{n-1}} \mathbf{Y}^{-1}(\{s\}) = \Omega.$$

Finally, since $\mathcal{Y}^{n-1}$ is countable, the union above is countable. So $\mathcal{F}_{n-1}$ is atomic, where the atoms are the sets of the form $\mathbf{Y}^{-1}(\{s\})$.
Now we can proceed with the exercise.

1. By the theory, we immediately have

$$\mathbb{E}[X \mid Y_1, \dots, Y_{n-1}] \restriction \mathbf{Y}^{-1}(\{s\}) = \frac{\mathbb{E}[X \cdot \mathbb{1}_{\mathbf{Y}^{-1}(\{s\})}]}{\mathbb{P}[\mathbf{Y}^{-1}(\{s\})]} = \frac{\mathbb{E}[X \cdot \mathbb{1}_{\mathbf{Y}^{-1}(\{s\})}]}{p_{n-1}(s_1, \dots, s_{n-1})}$$

   where $s = (s_1, ..., s_{n-1})$.

2. If $X_n = g_n(Y_1, ..., Y_n)$, then $X_n \cdot \mathbb{1}_{\mathbf{Y}^{-1}(\{s\})} = g_n(s_1, ..., s_{n-1}, Y_n)$. So

$$\mathbb{E}[X_n \mid Y_1, \dots, Y_{n-1}] \restriction \mathbf{Y}^{-1}(\{s\}) = \frac{\mathbb{E}[g_n(s_1, \dots, s_{n-1}, Y_n)]}{p_{n-1}(s_1, \dots, s_{n-1})}.$$

3.
   - X_n is $\mathcal{F}_n$-measurable. By the way X_n is defined, this is equivalent to say that g_n is $\mathcal{F}_n$-measurable.
   - $\mathbb{E}[|X_n|] < \infty$ for all $n \in \mathbb{N}$.
   - $\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] = X_{n-1}$, i.e.

$$\frac{\mathbb{E}[g_n(s_1, \dots, s_{n-1}, Y_n)]}{p_{n-1}(s_1, \dots, s_{n-1})} = X_{n-1} \restriction \mathbf{Y}^{-1}(\{s\}), \quad \forall s \in \mathcal{Y}^{n-1}.$$