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Exercise 27. First, we want show that

$$m_k := \mathbb{E}[X^k] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^k e^{-\frac{x^2}{2}} dx = \begin{cases} (k-1)!! & \text{if } k \text{ is even;} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Let's define

$$I(k) := \int_{\mathbb{R}} x^k e^{-\frac{x^2}{2}} dx.$$

If we can prove the following claim, we are done.

Claim: For all $p \in \mathbb{N}$, we have

1. $I(2p) = (2p-1)!!\sqrt{2\pi}$;
2. $I(2p+1) = 0$.

Proof: Both the proofs proceed by induction on p.

1. Suppose $p = 0$. Then

$$I(2p) = I(0) = \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} = (2 \cdot 0 - 1)!!\sqrt{2\pi}$$

   Suppose now $p > 0$. We have

$$\begin{aligned} I(2p) &= \int_{\mathbb{R}} x^{2p} e^{-\frac{x^2}{2}} dx \\ &= \int_{\mathbb{R}} -x^{2p-1} \left[-x e^{-\frac{x^2}{2}} \right] dx \\ &= -x^{2p-1} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{+\infty} + (2p-1) \int_{\mathbb{R}} x^{2p-2} e^{-\frac{x^2}{2}} dx \\ &= 0 + (2p-1)[(2p-3)!!\sqrt{2\pi}] \\ &= (2p-1)!!\sqrt{2\pi}. \end{aligned}$$

2. For $p = 0$ we obtain

$$I(2p+1) = I(1) = \int_{\mathbb{R}} x e^{-\frac{x^2}{2}} dx = \mathbb{E}[X] = 0.$$

   For $p > 0$, proceeding similarly to point (1), we obtain

$$I(2p+1) = \int_{\mathbb{R}} x^{2p+1} e^{-\frac{x^2}{2}} dx = -x^{2p} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{+\infty} + 2p \int_{\mathbb{R}} x^{2p-1} e^{-\frac{x^2}{2}} dx = 0 + 0 = 0.$$

$\blacksquare$

Now we want to show that

$$|m|_k := \mathbb{E}[|X|^k] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x|^k e^{-\frac{x^2}{2}} dx = \begin{cases} (k-1)!! & \text{if } k \text{ is even;} \\ (k-1)!!\sqrt{\frac{2}{\pi}} & \text{if } k \text{ is odd.} \end{cases}$$

If k is even, the statement follows trivially by $m_k = |m|_k$. Suppose then p odd. If we can prove the following claim, we are done.
Claim: For all $p \in \mathbb{N}$, we have

$$\int_{\mathbb{R}} |x|^{2p+1} e^{-\frac{x^2}{2}} dx = (2p)!! \cdot 2.$$

Proof: The proof proceeds by induction on p. Suppose $p = 0$. Then

$$\int_{\mathbb{R}} |x|^{2p+1} e^{-\frac{x^2}{2}} dx = \int_{\mathbb{R}} |x| e^{-\frac{x^2}{2}} dx = 2\int_0^{+\infty} |x| e^{-\frac{x^2}{2}} dx = -2e^{-\frac{x^2}{2}} \Big|_0^{+\infty} = 0+2 = (2\cdot 0)!!\cdot 2.$$

Suppose now $p > 0$. We have

$$\begin{aligned} I(2p+1) &= \int_{\mathbb{R}} |x|^{2p+1} e^{-\frac{x^2}{2}} dx \\ &= \int_{\mathbb{R}} -x^{2p} \left[-|x| e^{-\frac{x^2}{2}} \right] dx \\ &= 2\int_0^{+\infty} -x^{2p} \left[-x e^{-\frac{x^2}{2}} \right] dx \\ &= -2x^{2p} e^{-\frac{x^2}{2}} \Big|_0^{+\infty} + 4p \int_0^{+\infty} x^{2p-1} e^{-\frac{x^2}{2}} dx \\ &= 0 + 2p \int_{\mathbb{R}} |x|^{2p-1} e^{-\frac{x^2}{2}} dx \\ &= 2p[(2p-2)!! \cdot 2] = (2p)!! \cdot 2 \end{aligned}$$

$\blacksquare$

In order to find the characteristic function, we will soon need the following:

Lemma 0.0.1. For every $p \in \mathbb{N}_{\geq 1}$ we have

$$\log 1 \log 2 + ... + \log p \geq \int_1^p \log x dx.$$

Proof.

$$\begin{aligned}
\log 1 + \log 2 + ... + \log p &= 0 + \log 2 \int_1^2 1dx + \log 3 \int_2^3 1dx + ... + \log p \int_{p-1}^p 1dx \\
&= \int_1^2 \log 2dx + \int_2^3 \log 3dx + ... + \int_{p-1}^p \log pdx \\
&\geq \int_1^2 \log xdx + \int_2^3 \log xdx + ... + \int_{p-1}^p \log xdx \\
&= \int_1^p \log xdx,
\end{aligned}$$

where the inequality holds since log is an increasing function, thus $\log x \leq \log(i+1)$ for all $x \in [i, i+1]$. $\square$

To lighten the notation, suppose WLOG that $k = 2p$ for some $p \in \mathbb{N}$. Observe now that

$$\begin{aligned}
\lim_{k\to+\infty} \left(\frac{|m|_k}{k!}\right)^{\frac{1}{k}} &\leq \lim_{k\to+\infty} \left(\frac{(k-1)!!}{k!}\right)^{\frac{1}{k}} \\
&= \lim_{k\to+\infty} \left(\frac{1}{k!!}\right)^{\frac{1}{k}} && \text{(since } k!! \geq (k/2)!\text{)} \\
&\leq \lim_{k\to+\infty} \left(\frac{1}{(k/2)!}\right)^{\frac{1}{k}} \\
&= \lim_{p\to+\infty} \left(\frac{1}{p!}\right)^{\frac{2}{p}} \\
&= \lim_{p\to+\infty} e^{\frac{2}{p}\log\left(\frac{1}{p!}\right)} \\
&= \lim_{p\to+\infty} e^{-\frac{2}{p}(\log p!)} \\
&= \lim_{p\to+\infty} e^{-\frac{2}{p}(\log 1 + \log 2 + ... + \log p)} \\
&\leq \lim_{p\to+\infty} e^{-\frac{2}{p}\int_1^p \log xdx} && \text{(thanks to the lemma)} \\
&= \lim_{p\to+\infty} e^{-\frac{2}{p}(p\log p - p + 1)} \\
&= \lim_{p\to+\infty} e^{-2\log p + 2 - \frac{2}{p}} = e^{-\infty} = 0.
\end{aligned}$$

This means that we can apply the corollary we saw during the lecture, i.e.

$$\varphi(t) = \sum_{k=0}^{\infty} m_k \frac{(it)^k}{k!}.$$

Since $m_k = 0$ if k is odd, we obtain

$$\sum_{k=0}^{\infty} m_k \frac{(it)^k}{k!} = \sum_{h=0}^{\infty} m_{2h} \frac{(it)^{2h}}{(2h)!} = \sum_{h=0}^{\infty} (2h-1)!!(-1)^h \frac{t^{2h}}{(2h)!} = \sum_{h=0}^{\infty} (-1)^h \frac{t^{2h}}{(2h)!!}.$$

It is very easy to check that $(2h)!! = 2^h h!$, for all $h \in \mathbb{N}$. Therefore

$$\sum_{h=0}^{\infty} (-1)^h \frac{t^{2h}}{(2h)!!} = \sum_{h=0}^{\infty} (-1)^h \frac{t^{2h}}{2^h h!} = \sum_{h=0}^{\infty} \frac{\left(-\frac{t^2}{2}\right)^h}{h!} = e^{-\frac{t^2}{2}}.$$

That is, $\varphi(t) = e^{-\frac{t^2}{2}}$.

Exercise 28.

$$\varphi_{aX+b}(t) = \mathbb{E}\left[e^{it(aX+b)}\right] = \mathbb{E}\left[e^{itaX}e^{itb}\right] = \mathbb{E}\left[e^{itaX}\right]\mathbb{E}\left[e^{itb}\right] = \varphi_X(at)e^{itb}.$$

Thus, since $X \sim N(\mu, \sigma^2) \Rightarrow X = \sigma Y + \mu$ with $Y \sim N(0,1)$, we have

$$\varphi_X(t) = \varphi_{\sigma Y+\mu}(t) = e^{-\frac{\sigma^2 t^2}{2}} e^{it\mu} = e^{it\mu - \frac{1}{2}\sigma^2 t^2}.$$

Exercise 29. We easily have

$$\varphi_{-X}(t) = \mathbb{E}\left[e^{i(-t)X}\right] = \varphi_X(-t)$$

and thus

$$\varphi_{X-Y}(t) = \mathbb{E}\left[e^{it(X+(-Y))}\right] = \varphi_X(t)\varphi_X(-t).$$

Now:

a) Consider $f(t) := e^{2(\cos t - 1)}$. We want to find explicity a random variable Z such that $\varphi_Z(t) = f(t)$. Let Y be a Bernoulli r.v. with $\mathbb{P}[0] = \mathbb{P}[1] = \frac{1}{2}$. Then

$$\varphi_Y(t) = \frac{1}{2}e^{-it} + \frac{1}{2}e^{it} = \cos t.$$

Now let N be distributed like a Poisson distribution of parameter λ, and let (Y_j) be i.i.d. random variables, independent with N. Define

$$S = \sum_{j=1}^{N} Y_j.$$

We have

$$\begin{aligned}
\varphi_S(t) = \mathbb{E}\left[e^{itS}\right] &= \mathbb{E}\left[e^{it\sum\limits_{j=1}^{N} Y_j}\right] \\
&= \mathbb{E}\left[\sum_{n\geq 0} e^{it\sum\limits_{j=1}^{n} Y_j} \mathbb{1}_{\{N=n\}}\right] \\
&= \sum_{n\geq 0} \mathbb{E}\left[e^{it\sum\limits_{j=1}^{n} Y_j}\right] \cdot \mathbb{E}\left[\mathbb{1}_{\{N=n\}}\right] \qquad (N \text{ and } (Y_j) \text{ indep.}) \\
&= \sum_{n\geq 0} \mathbb{E}\left[e^{it\sum\limits_{j=1}^{n} Y_j}\right] \cdot \mathbb{P}[N=n] \\
&= \sum_{n\geq 0} \underbrace{\varphi_{Y_1}(t)^n}_{\mathbb{E}e^{it\sum_{j=1}^{n} Y_j}} \cdot \underbrace{e^{-\lambda}\frac{\lambda^n}{n!}}_{\mathbb{P}[N=n]} = e^{\lambda(\varphi_{Y_1}(t)-1)}.
\end{aligned}$$

Therefore, if we take

$$Z := \sum_{j=1}^{N} Y_j$$

with (Y_j) i.i.d. Bernoulli symmetric r.v. with range $\{1, -1\}$ and $N \sim \text{Pois}(2)$ independent with (Y_j), we obtain

$$\varphi_Z(t) = e^{2(\cos t - 1)},$$

as wanted.

- Consider $f(t) = e^{-|t|^3}$. We have

$$f'(t) = -3|t|te^{-|t|^3}$$

and

$$f''(t) = -3\left(|t|e^{-|t|^3} + |t|e^{-|t|^3} - 3|t|te^{-|t|^3}|t|t\right).$$

So $f'(0) = f''(0) = 0$. Let X be a random variable. We now that $\varphi_X'(0) = i\,\mathbb{E}[X]$. So, if $f(t) = \varphi_X(t)$, we would have $\mathbb{E}[X] = 0$. But since $\varphi_X''(0) = -\mathbb{E}[X^2] = \mathbb{E}[(X-0)^2] = \text{Var}[X]$, we would have $\text{Var}[X] = 0$. Therefore X must be the constant function in 0. But the characteristic function of $X \equiv 0$ is trivially $e^{it0} = 1 \neq e^{-|t|^3}$. Thus $e^{-|t|^3}$ is not the characteristic function of any random variable.