Exercise 16. Suppose F strictly increasing. This means that F is bijective, i.e. there exists F^{-1} inverse function of F. Note also that, given a r.v. Y, $F^{-1}(Y)$ is still a r.v., since F^{-1} is still continuous (should be proved formally), hence measurable. Observe now that

$$\mathbb{P}[F(X) \leq x] = \mathbb{P}[F^{-1}(F(X)) \leq F^{-1}(x)] = \mathbb{P}[X \leq F^{-1}(x)] = F(F^{-1}(x)) = x.$$

That is, $F(X) \sim U$, the uniform distribution on $F[\operatorname{Im} X]$.

We want to prove that the same holds even if F is not strictly monotone. Suppose then F increasing, but not strictly. That is, there exist intervals where F is constant, and other intervals where F is injective (restricted on those intervals). We may assume that those intervals are maximal, i.e. disjoint (otherwise we can just sustitute them with the union).
Claim: such intervals are at most countably many.

Proof (euristic). Thanks to the disjoint hypothesis, we have that on every interval F increases of some factor $a_\alpha > 0$ compared to the previous ones. Let $\{a_\alpha : \alpha \in \mathcal{A}\}$ be the set of increases. Since $\lim_\infty F(x) = 1$, we have that

$$\sum_{\alpha \in \mathcal{A}} a_\alpha = 1 < +\infty.$$

A well known analysis lemma assures us that, since the sum is finite, $\mathcal{A}$ must be at most countable. $\square$

Therefore we can write F as

$$F(x) = F_0(x)\mathbb{1}_{A_0}(x) + \sum_{i=1}^{\infty} c_i \mathbb{1}_{A_i}(x),$$

where A_0 is a set, $F_0(x)$ is strictly increasing (on A_0), c_i are constant ≥ 0, A_i is a closed interval $[a_i, b_i]$ for any $i \geq 1$, and all the A_i, $i \in \mathbb{N}$, are pairwise disjoint and their union is the whole domain of F.

We can provide a right inverse F^{-1} of F, e.g. the one defined by

$$F^{-1}(x) := F_0^{-1}(x)\mathbb{1}_{F[A_0]}(x) + \sum_{i=1}^{\infty} a_i \mathbb{1}_{F[A_i]}(x).$$

It is easy to check that F^{-1} is (strictly) increasing (try with a simple drawing). Furthermore, it is measurable, since it can be written as a limit of (sums of) measurable functions. We have:

$$
\begin{aligned}
\mathbb{P}[F(X) \leq x] &= \mathbb{P}\left[F^{-1}(F(X)) \leq F^{-1}(x)\right] \\
&= \mathbb{P}\left[F_0^{-1}(F(X))\mathbb{1}_{F[A_0]}(F(X)) + \sum_{i=1}^{\infty} a_i \mathbb{1}_{F[A_i]}(F(X)) \leq F^{-1}(x)\right] \\
&= \mathbb{P}\left[X\,\mathbb{1}_{A_0}(X) + \sum_{i=1}^{\infty} a_i \mathbb{1}_{A_i}(X) \leq F^{-1}(x)\right] \\
&= \mathbb{P}\left[X\,\mathbb{1}_{A_0}(X) + X\mathbb{1}_{\cup_{i=1}^{\infty}A_i}(X) \leq F^{-1}(x)\right] \\
&= \mathbb{P}\left[X \leq F^{-1}(x)\right] = F(F^{-1}(x)) = x,
\end{aligned}
$$

where the last equality holds because F^{-1} is a right inverse of F, and the fourth equality holds thanks to the following observation: $X\,\mathbb{1}_{A_0}(X) + \sum_{i=1}^{\infty} a_i \mathbb{1}_{A_i}(X)$ and $X\,\mathbb{1}_{A_0}(X) + X\mathbb{1}_{\cup_{i=1}^{\infty}A_i}(X)$ are equal almost everywhere (thus, our equality holds). In order to prove that, it is sufficient to prove that $\mathbb{1}_{\cup_{i=1}^{\infty}A_i}(X)$ is null almost everywhere. Observe that, for every $i \geq 1$,

$$\mathbb{1}_{A_i}(X) \neq 0 \iff X \in A_i \iff a_i \leq X \leq b_i.$$

Hence $P(X \in A_i) = F(b_i) - F(a_i)$. But by construction of the A_i's, we have $0 = F(b_i) - F(a_i) = \mathbb{P}(X \in A_i)$. Thus every A_i has zero measure, and therefore also $\bigcup_{i=1}^{\infty} A_i$ has zero measure, since it is the countable union of zero-measure sets. □

Exercise 17. We already consider the general case where F is increasing, but not necessarily strictly. Let F^{-1} be the right inverse of F defined like in the previous exercise (which is measurable, as explained above). We want $\mathbb{P}[G(U) \leq x] = F(x)$, thus $\mathbb{P}[G(U) \leq F^{-1}(x)] = F(F^{-1}(x))$, thus $\mathbb{P}[F(G(U)) \leq F(F^{-1}(x))] = x$, thus $\mathbb{P}[F \circ G(U) \leq x] = x$, thus if $F \circ G = \mathrm{id}$ it works, i.e. $G = F^{-1}$. It is straightforward to check that defining $G := F^{-1}$ we are done. □

Exercise 18. We want to find $\lambda(\{\omega \in [0,1) \mid \lfloor 2^n\omega \rfloor \text{ is even}\})$. Observe that, given $\omega \in \mathbb{R}$, $\lfloor 2^n\omega \rfloor$ is even iff $2^n\omega = 2m + \mu$ for some $m \in \mathbb{N}$ and $\mu \in [0,1)$. Thus $\omega = \frac{2m}{2^n} + \frac{\mu}{2^n}$. Let's now fix m. We obviously have that

$$\lambda\left(\left\{\omega \in \mathbb{R} \,\middle|\, \omega = \frac{2m}{2^n} + \frac{\mu}{2^n},\ \mu \in [0,1)\right\}\right) = \frac{1}{2^n}.$$

Observe now that changing m, we obtain a different disjoint set:

$$\frac{2m}{2^n} + \frac{\mu}{2^n} = \frac{2m'}{2^n} + \frac{\mu'}{2^n} \Rightarrow \mu = 2(m' - m) + \mu' \Rightarrow m = m' \text{ since } \mu \in [0,1).$$

Finally, we must find the maximum possible value for m (since $0 \leq \omega \leq 1$):

$$\frac{2m+\mu}{2^n} < 1 \Rightarrow m < \frac{2^n - \mu}{2} = 2^{n-1} - \mu/2 \Rightarrow m \leq 2^{n-1} - 1.$$

So we obtain

$$\lambda(\{\omega \in [0,1) \mid \lfloor 2^n\omega \rfloor \text{ is even}\}) = \lambda\left(\biguplus_{m=0}^{2^{n-1}-1}\left\{\omega \in \mathbb{R} \middle| \omega = \frac{2m}{2^n} + \frac{\mu}{2^n},\ \mu \in [0,1)\right\}\right)$$
$$= \sum_{m=0}^{2^{n-1}-1} \lambda\left(\left\{\omega \in \mathbb{R} \middle| \omega = \frac{2m}{2^n} + \frac{\mu}{2^n},\ \mu \in [0,1)\right\}\right) = \sum_{m=0}^{2^{n-1}-1} \frac{1}{2^n} = 2^{n-1} \cdot \frac{1}{2^n} = \frac{1}{2}.$$

So $\mathbb{P}[X_n = 0] = \frac{1}{2}$, and of course $\mathbb{P}[X_n = 1] = 1 - \frac{1}{2} = \frac{1}{2}$. Thus X_n is distributed like a bernullian $B(1, \frac{1}{2})$, for every n.
We now want to prove that the X_n's are independent. First, observe that for all $\omega \in (0,1]$, ω can be written in binary as $0.x_1x_2x_3x_4...$, where $x_n = 0, 1$. Observe that this means

$$\omega = \sum_{n=1}^{\infty} 2^{-n} x_n.$$

Now consider $\lfloor 2^n\omega \rfloor = 2^{n-1}x_1 + 2^{n-2}x_2 + ... + 2x_{n-1} + x_n$. It is thus clear that $\lfloor 2^n\omega \rfloor$ is even iff $x_n = 0$. I.e., $X_n(\omega) = x_n$.
So consider $\mathbb{P}[X_1 = i_1, X_2 = i_2, ..., X_n = i_n]$ for any arbitrary $n \in \mathbb{N}$. For what we just said, we are calculating the probability that the first n digits of the binary representation of ω are $i_1, ..., i_n$. That is, we want to calculate the Lebesgue measure of the set of the ω's such that $\omega \in [0.i_1i_2...i_n0000...; 0.i_1i_2...i_n1111...]$, which by invariance under translations equals the Lebesgue measure of $[0; 0.000..00011111....]$. The Lebesgue measure of that interval is of course

$$\sum_{i=n}^{\infty} 2^{-i}.$$

Since this holds for any $n \in \mathbb{N}$, this means that $\mathbb{P}[X_1 = i_1, X_2 = i_2, ..., X_n = i_n] = \sum_{i=n}^{\infty} 2^{-i} = \frac{1}{2} \cdot \sum_{i=n}^{\infty} 2^{-i+1} = \frac{1}{2} \cdot \sum_{i=n-1}^{\infty} 2^{-i} = \mathbb{P}[X_n = i_n] \cdot \mathbb{P}[X_1 = i_1, ..., X_{n-1} = i_{n-1}]$.
The last step is to prove independence in the general case. So, consider $\{X_{k_1}, ..., X_{k_n}\}$. We can suppose WLOG that $k_1 < k_2 < ... < k_n$. Furthermore, in order to lighten the notation, we suppose that $n > 2$, $k_1 = 2$, $k_2 = 4$ and $k_{i+1} = k_i + 1$ for all $i \geq 2$.

Observe that, thanks to the previous less general case, we have

$$\begin{aligned}
\mathbb{P}[X_{k_1} = i_1, ..., X_{k_n} = i_n] &= \mathbb{P}[X_1 \in \mathbb{R}, X_{k_1} = i_1, X_2 \in \mathbb{R}, X_{k_2} = i_2, ..., X_{k_n} = i_n] \\
&= \mathbb{P}[X_1 = 0, X_{k_1} = i_1, X_3 = 0, X_{k_2} = i_2, ..., X_{k_n} = i_n] \\
&+ \mathbb{P}[X_1 = 1, X_{k_1} = i_1, X_3 = 0, X_{k_2} = i_2, ..., X_{k_n} = i_n] \\
&+ \mathbb{P}[X_1 = 0, X_{k_1} = i_1, X_3 = 1, X_{k_2} = i_2, ..., X_{k_n} = i_n] \\
&+ \mathbb{P}[X_1 = 1, X_{k_1} = i_1, X_3 = 1, X_{k_2} = i_2, ..., X_{k_n} = i_n] \\
&= \text{... we split into products thanks to previous point ...} \\
&= \mathbb{P}[X_{k_1} = i_1] \cdot ... \cdot \mathbb{P}[X_{k_n} = i_n]\big(\mathbb{P}[X_1 = 0]\mathbb{P}[X_3 = 0] + \mathbb{P}[X_1 = 1]\mathbb{P}[X_3 = 0] \\
&+ \mathbb{P}[X_1 = 0]\mathbb{P}[X_3 = 1] + \mathbb{P}[X_1 = 1]\mathbb{P}[X_3 = 1]\big) \\
&= \mathbb{P}[X_{k_1} = i_1] \cdot ... \cdot \mathbb{P}[X_{k_n} = i_n] \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) \\
&= \mathbb{P}[X_{k_1} = i_1] \cdot ... \cdot \mathbb{P}[X_{k_n} = i_n],
\end{aligned}$$

and we are done (since the most general argument proceeds by induction, for example on the number of "holes"). □

Exercise 19. Of course $0 \leq \sum_{n=1}^{\infty} 2^{-n} X_n \leq 1$. We want to calculate

$$\mathbb{P}\left[\sum_{n=1}^{\infty} 2^{-n} X_n \leq x\right].$$

We can write x in base 2 as $x = \sum_{n=1}^{\infty} 2^{-n} x_n$ with $x_n = 0, 1$. Observe now that

$$\sum_{n=1}^{\infty} 2^{-n} X_n \leq \sum_{n=1}^{\infty} 2^{-n} x_n$$

if and only if

$$(X_1 = x_1 \wedge X_2 = x_2 \wedge ...) \vee (X_1 < x_1) \vee (X_1 = x_1 \wedge X_2 < x_2) \vee (X_1 = x_1 \wedge X_2 = x_2 \wedge X_3 < x_3) \vee ...$$

Observe now that $\mathbb{P}[X_i < x_i] \neq 0 \iff x_i = 1 \wedge X_i = 0$. Since $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{2}$, we have

$$\begin{aligned}
\mathbb{P}\left[\sum_{n=1}^{\infty} 2^{-n} X_n \leq x\right] &= \mathbb{P}[X_1 = x_1 \wedge X_2 = x_2 \wedge ...] + \sum_{i=1}^{\infty} \mathbb{P}\left[X_i < x_i \wedge \bigwedge_{j=1}^{i-1} (X_j = x_j)\right] \\
&= 0 + \sum_{i=1}^{\infty} \left(\mathbb{P}[X_i < x_i] \prod_{j=1}^{i-1} \mathbb{P}[X_j = x_j]\right) \\
&= \sum_{i=1}^{\infty} \mathbb{P}[X_i < x_i] \cdot \frac{1}{2^{i-1}} \\
&= \sum_{\{i | x_i = 1\}} \mathbb{P}[X_i < x_i] \cdot \frac{1}{2^{i-1}} \\
&= \sum_{\{i | x_i = 1\}} \mathbb{P}[X_i = 0] \cdot \frac{1}{2^{i-1}} = \sum_{\{i | x_i = 1\}} \frac{1}{2} \cdot \frac{1}{2^{i-1}} = \sum_{\{i | x_i = 1\}} \frac{1}{2^i} = x,
\end{aligned}$$

where the second equality holds because of the independence of the X_i's.
Therefore the distribution $\sum_{n=1}^{\infty} 2^{-n} X_n$ is distributed like the uniform distribution on $[0, 1]$. □

Exercise 20. Take the sequence $(X_n)_{n \geq 1}$ of Bernoulli r.v. of Exercise 18. Let $\varphi : \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 1} \times \mathbb{N}_{\geq 1}$ be a bijection (for example the standard diagonal enumeration). We can extract countably many *disjoint* sequences of r.v. this way: for any $i \geq 1$, define

$$(X_{i,j})_{j \geq 1} := (X_{\varphi(i,j)})_{j \geq 1}.$$

By the previous exercise, it is clear that, for every $i \geq 1$, $U_i := \sum_{j=1}^{\infty} 2^{-j} X_{i,j}$ is a uniform distribution on $[0, 1]$.
We want to prove that the U_i's are independent. In order to lighten the notation, we will consider just two random variables (the general case being similar) and we

will write them as $U = \sum_{i=1}^{\infty} 2^{-i} X_i$ and $V = \sum_{i=1}^{\infty} 2^{-i} Y_i$. Observe that:

$$\begin{aligned}
\mathbb{P}[U \leq t, V \leq z] &= \mathbb{P}\left[\sum_{n=1}^{\infty} 2^{-n} X_n \leq \sum_{n=1}^{\infty} 2^{-n} t_n \ \wedge\ \sum_{n=1}^{\infty} 2^{-n} Y_n \leq \sum_{n=1}^{\infty} 2^{-n} z_n\right] \\
&= \mathbb{P}\big[\big((X_1 = t_1 \wedge X_2 = t_2 \wedge ...) \veebar (X_1 < t_1) \veebar (X_1 = t_1 \wedge X_2 < t_2) \veebar ...\big) \\
&\wedge \big((Y_1 = z_1 \wedge Y_2 = z_2 \wedge ...) \veebar (Y_1 < z_1) \veebar (Y_1 = z_1 \wedge Y_2 < z_2) \veebar ...\big)\big] \\
&= \mathbb{P}[\text{ ... ``convolution product'' of logical connectives ... }] \\
&= \text{ ... convolution product of the two sums (thanks to independence) ...} \\
&= \left(\sum_{\{i | t_i = 1\}} \frac{1}{2^i}\right) \cdot \left(\sum_{\{i | z_i = 1\}} \frac{1}{2^i}\right) = t \cdot z = \mathbb{P}[U \leq t] \cdot \mathbb{P}[V \leq z],
\end{aligned}$$

and we are done. □

Exercise 21. Take the sequence $(U_n)_{n \geq 1}$ of independent uniform r.v. of the previous exercise. Define $X_n := F^{-1}(U_n)$ like in our solution of Exercise 17. Thanks to Exercise 17, we know that every X_n's cumulative distribution is $F(x)$.
We shall prove that the X_n's are independent. Observe that

$$\begin{aligned}
\mathbb{P}[X_n \leq t, X_m \leq z] &= \mathbb{P}[F^{-1}(U_i) \leq t, F^{-1}(U_j) \leq z] \\
&= \mathbb{P}[F(F^{-1}(U_i)) \leq F(t), F(F^{-1}(U_j)) \leq F(z)] \\
&= \mathbb{P}[U_i \leq F(t), U_j \leq F(z)] \\
&= \mathbb{P}[U_i \leq F(t)] \cdot \mathbb{P}[U_j \leq F(z)] \\
&= F(t) \cdot F(z) = \mathbb{P}[X_n \leq t] \cdot \mathbb{P}[X_m \leq z],
\end{aligned}$$

where the third equality holds because F^{-1} is a right inverse of F and the fourth equality holds thanks to the independence of the U_n's. □