Exercise 10.

Proof. To prove that the two sums are equal, first observe that

$$\{|X| > n\} = \biguplus_{k \geq n} \{k < |X| \leq k+1\}$$

and thus

$$\mathbb{P}[|X| > n] = \sum_{k=n}^{\infty} \mathbb{P}[k < |X| \leq k+1].$$

Therefore

$$\sum_{n=1}^{\infty} \mathbb{P}[|X| > n] = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \mathbb{P}[k < |X| \leq k+1] = \sum_{k=1}^{\infty} \sum_{n=1}^{k} \mathbb{P}[k < |X| \leq k+1] =$$
$$\sum_{k=1}^{\infty} \mathbb{P}[k < |X| \leq k+1] \sum_{n=1}^{k} 1 = \sum_{k=1}^{\infty} \mathbb{P}[k < |X| \leq k+1] \cdot k.$$

So the two sums are equal. Now we want to prove that

$$\int_{\Omega} |X| d\mathbb{P} < \infty \iff \sum_{k=1}^{\infty} k\mathbb{P}[k < |X| \leq k+1] < \infty.$$

$(\Longrightarrow)$

$$\infty > \int_{\Omega} |X| d\mathbb{P} > \int_{\Omega} \sum_{k=1}^{\infty} \left(k \cdot \mathbb{1}_{\{\omega : k < |X(\omega)| \leq k+1\}}\right) d\mathbb{P} = \sum_{k=1}^{\infty} \int_{\Omega} k \cdot \mathbb{1}_{\{\omega : k < |X(\omega)| \leq k+1\}} d\mathbb{P} =$$
$$\sum_{k=1}^{\infty} k \int_{\Omega} \mathbb{1}_{\{\omega : k < |X(\omega)| \leq k+1\}} d\mathbb{P} = \sum_{k=1}^{\infty} k\mathbb{P}[k < |X| \leq k+1].$$

$(\Longleftarrow)$

$$\int_{\Omega} |X| d\mathbb{P} \leq \int_{\Omega} \sum_{k=0}^{\infty} \left((k+1) \cdot \mathbb{1}_{\{\omega : k < |X(\omega)| \leq k+1\}}\right) d\mathbb{P} = \sum_{k=0}^{\infty} \int_{\Omega} (k+1) \cdot \mathbb{1}_{\{\omega : k < |X(\omega)| \leq k+1\}} d\mathbb{P} =$$
$$\sum_{k=0}^{\infty} (k+1) \int_{\Omega} \mathbb{1}_{\{\omega : k < |X(\omega)| \leq k+1\}} d\mathbb{P} = \sum_{k=0}^{\infty} (k+1)\mathbb{P}[k < |X| \leq k+1] < \infty.$$

□

Exercise 11.

Proof. Suppose $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. We want to show $1 - \mathbb{P}(\limsup_{n\to\infty} A_n) = 0$. First of all note that for all $N \in \mathbb{N}$ we have $\sum_{n=N}^{\infty} \mathbb{P}(A_n) = \infty$. Now observe that

$$1 - \mathbb{P}(\limsup_{n\to\infty} A_n) = 1 - \mathbb{P}\left(\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} A_n\right)\right) = \mathbb{P}\left(\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} A_n\right)^c\right) =$$
$$= \mathbb{P}\left(\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty} A_n^c\right) = \mathbb{P}\left(\liminf_{n\to\infty} A_n^c\right) = \lim_{N\to\infty} \mathbb{P}\left(\bigcap_{n=N}^{\infty} A_n^c\right).$$

So it is enough to show that $\mathbb{P}\left(\bigcap_{n=N}^{\infty} A_n^c\right) = 0$ for all $N \in \mathbb{N}$. Since the $(A_n)_{n=1}^{\infty}$ are independent and $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}^+$:

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{n=N}^{\infty} A_n^c\right) &= \prod_{n=N}^{\infty} \mathbb{P}\left(A_n^c\right) \\
&= \prod_{n=N}^{\infty} \left(1 - \mathbb{P}\left(A_n\right)\right) \\
&\leq \prod_{n=N}^{\infty} e^{-\mathbb{P}(A_n)} \\
&= e^{-\sum_{n=N}^{\infty} \mathbb{P}(A_n)} \\
&= e^{-\infty} \\
&= 0,
\end{aligned}$$

and we are done.
The first $(\Longrightarrow)$ is the Borel-Cantelli lemma (already proven). So we proved both the $(\Longrightarrow)$'s implications. Now it's just a matter of elementary logic to see that the $(\Longleftarrow)$'s hold as well (observe before that the sums must exist since they are sums of non-negative numbers). $\square$

Exercise 12. Let $\mathcal{S}$ be a semiring over X and let $\mathcal{R}(\mathcal{S})$ the ring generated by $\mathcal{S}$. Then

$$\mathcal{R}(\mathcal{S}) = \left\{ A \subseteq X \mid A = \biguplus_{i=1}^{n} S_i \text{ for some disjoint elements of } \mathcal{S} \right\} =: \mathcal{A}$$

Proof. The inclusion $\supseteq$ is trivial.
We will show now that $\mathcal{A}$ is a ring. Take any $A, B \in \mathcal{A}$, i.e.

$$A = \bigcup_{i=1}^{m} S_i \text{ and } B = \bigcup_{j=1}^{n} T_j$$

for some disjoint collections $\{S_i\}$ and $\{T_j\}$ in $\mathcal{S}$. Then

$$\begin{aligned}
A \setminus B &= \left(\bigcup_{i=1}^{m} S_i\right) \cap \left(\bigcup_{j=1}^{n} T_j\right)^c \\
&= \left(\bigcup_{i=1}^{m} S_i\right) \cap \left(\bigcap_{j=1}^{n} T_j^c\right) \\
&= \bigcup_{i=1}^{m} \left(S_i \cap \left(\bigcap_{j=1}^{n} T_j^c\right)\right) \\
&= \bigcup_{i=1}^{m} \left\{\bigcap_{j=1}^{n} (S_i \setminus T_j)\right\}. \qquad \text{(a)}
\end{aligned}$$

Since $\mathcal{S}$ is a semiring, we have

$$S_i \setminus T_j = \biguplus_{l=1}^{L_{ij}} H_{ijl}$$

for some disjoint $\{H_{ijl}\}$ of $\mathcal{S}$. Thus

$$\begin{aligned}
\bigcap_{j=1}^{n} (S_i \setminus T_j) &= \bigcap_{j=1}^{n} \biguplus_{l=1}^{L_{ij}} H_{ijl} \\
&= \left(\biguplus_{l_1=1}^{L_{i1}} H_{i1l_1}\right) \cap \left(\biguplus_{l_2=1}^{L_{i2}} H_{i2l_2}\right) \cap \ldots \cap \left(\biguplus_{l_n=1}^{L_{in}} H_{inl_n}\right) \\
&= \bigcup_{l_1=1}^{L_{i1}} \bigcup_{l_2=1}^{L_{i2}} \ldots \bigcup_{l_n=1}^{L_{in}} (H_{i1l_1} \cap H_{i2l_2} \cap \ldots \cap H_{inl_n}), \qquad \text{(b)}
\end{aligned}$$

which is a disjoint union. Furthermore, since $\mathcal{S}$ is a semiring, we have

$$H_{i1l_1} \cap H_{i2l_2} \cap \ldots \cap H_{inl_n} \in \mathcal{S}.$$

Therefore, combining (a) and (b), we see that $A \setminus B$ is a disjoint union of sets of $\mathcal{S}$, i.e. $A \setminus B \in \mathcal{A}$.
A similar argument shows that $S_i \cup T_j \in \mathcal{S}$ as well. So

$$A \cap B = \left(\bigcup_{i=1}^{m} S_i\right) \cap \left(\bigcup_{j=1}^{n} T_j\right) = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (S_i \cap T_j) \in \mathcal{A}\,.$$

Then $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$ is a disjoint union of sets that we have seen above are in $\mathcal{A}$, so $A \cup B \in \mathcal{A}$.
We have therefore proven that $\mathcal{A}$ is a ring. Since $\mathcal{S} \subseteq \mathcal{A}$, this implies $\mathcal{R}(\mathcal{S}) \subseteq \mathcal{A}$. $\square$

Let now $\mu : \mathcal{S} \to [0, +\infty)$ be a finitely additive (finite) measure on $\mathcal{S}$. Then $\mu^* : \mathcal{R}(\mathcal{S}) \to [0, +\infty)$ given by

$$\mu^*(A) = \mu^* \left(\biguplus_{i=1}^{n} S_i \right) = \sum_{i=1}^{n} \mu(S_i)$$

is the unique extension of μ to a finitely additive measure on $\mathcal{R}(\mathcal{S})$.

Proof. The proof that μ^* is a measure on $\mathcal{R}(\mathcal{S})$ is immediate.
It is also trivial that it is unique, since

$$\mu'(A) \neq \mu^*(A) \Rightarrow \mu' \left(\biguplus_{i=1}^{n} S_i \right) \neq \mu^* \left(\biguplus_{i=1}^{n} S_i \right) \Rightarrow \sum_{i=1}^{n} \mu'(S_i) \neq \sum_{i=1}^{n} \mu(S_i)$$

which means that μ' does not extend μ. $\square$

Exercise 13.

Proof. First observe that μ^* is monotone:

$$A \subseteq B \Rightarrow \mu^*(B) = \mu^*(A \uplus (B \setminus A)) = \mu^*(A) + \mu^*(B \setminus A) \geq \mu^*(A).$$

Thus $\mu^*(\biguplus_{i=1}^{\infty} A_i) \geq \mu^*(\biguplus_{i=1}^{N} A_i)$ for all $N \in \mathbb{N}$ (note that a priori μ is not defined on the partial union). So we obtain

$$\mu \left(\biguplus_{i=1}^{\infty} A_i \right) = \mu^* \left(\biguplus_{i=1}^{\infty} A_i \right) \geq \sup_{N \in \mathbb{N}} \left\{ \mu^* \left(\biguplus_{i=1}^{N} A_i \right) \right\} =$$

$$\sup_{N \in \mathbb{N}} \left\{ \sum_{i=1}^{N} \mu^*(A_i) \right\} = \sum_{i=1}^{\infty} \mu^*(A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

$\square$

Exercise 14.

Proof. Immediate. $\square$

Exercise 15.

Proof. First of all, observe that every open and closed interval is an element of $\mathcal{R}(\mathcal{S})$, since $\mathcal{R}(\mathcal{S})$ is closed under complement (and intersection).
Choose an arbitrary $\varepsilon > 0$. Since F is right continuous, we can find $d > a$ such that $F(a) \leq F(d) \leq F(a) + \varepsilon$. This means that

$$\mu((d,b]) - \mu((a,b]) = F(b) - F(d) - (F(b) - F(a)) = F(a) - F(d) \geq -\varepsilon,$$

i.e. $\mu((d,b]) \geq \mu((a,b]) - \varepsilon$.

Thus, consider the closed interval $[d,b] \subseteq (a,b]$. We trivially obtain

$$\mu^*([d,b]) \geq \mu((d,b]) \geq \mu((a,b]) - \varepsilon.$$

Claim. For all $n \in \mathbb{N}$ we can find $(a_n, d_n) \supseteq (a_n, b_n]$ s.t.

$$\sum_{n=1}^{\infty} \mu^*((a_n, d_n)) \leq \sum_{n=1}^{\infty} \mu^*((a_n, b_n]) + \varepsilon.$$

We will prove the claim later. Now observe that $\{(a_n, d_n)\}_{n\in\mathbb{N}}$ is an open cover of $[d,b]$, which is a compact set. Thus there exists a finite open subcover $\{(a_n, d_n)\}_{n\in F}$ of $[d,b]$. Therefore

$$\mu^*([d,b]) \leq \mu^*\left(\bigcup_{n\in F}(a_n, d_n)\right) \leq \sum_{n\in F} \mu^*((a_n, d_n)) \leq \sum_{n=1}^{\infty} \mu^*((a_n, d_n)),$$

where the second inequality holds because every additive measure is finitely subadditive (to show this, just repeat the proof of the Exercise 1. in the finite case). Therefore

$$\mu((a,b]) - \varepsilon \leq \mu^*([d,b]) \leq \sum_{n=1}^{\infty} \mu^*((a_n, d_n)) \leq \sum_{n=1}^{\infty} \mu^*((a_n, b_n]) + \varepsilon.$$

that is

$$\mu((a,b]) \leq \sum_{n=1}^{\infty} \mu((a_n, b_n]) + 2\varepsilon.$$

Thanks to the arbitrary choice of ε, we are done. □

Proof of the Claim. Take any b_n. Since F is right continuous, we can find $d_n > b_n$ such that $F(b_n) \leq F(d_n) \leq F(b_n) + \frac{\varepsilon}{2^n}$. This means that

$$\mu((a_n, b_n]) - \mu((a_n, d_n]) = F(b_n) - F(a_n) - (F(d_n) - F(a_n)) = F(b_n) - F(d_n) \geq -\frac{\varepsilon}{2^n},$$

i.e. $\mu^*((a_n, d_n)) \leq \mu((a_n, d_n]) \leq \mu((a_n, b_n]) + \frac{\varepsilon}{2^n}$.
Thus, we obtain

$$\sum_{n=1}^{\infty} \mu^*((a_n, d_n)) \leq \sum_{n=1}^{\infty} \left(\mu^*((a_n, b_n]) + \frac{\varepsilon}{2^n} \right) = \sum_{n=1}^{\infty} \mu^*((a_n, b_n]) + \varepsilon,$$

which is what we wanted to prove. $\square$