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Exercise 29. (X_n) is not a Markov Chain. Indeed, consider X_0, X_1 and X_2. Suppose that $X_1 = 2$. This necessarily implies $Y_0 = 1$ and $Y_1 = 1$. Suppose *also* $X_2 = 1$. This necessarily implies $Y_2 = 0$. It follows that under both these assumption, $X_3 \neq 2$, i.e.

$$\mathbb{P}[X_3 = 2 | X_2 = 1, X_1 = 2] = 0.$$

But

$$\mathbb{P}[X_3 = 2 | X_2 = 1] = \frac{\mathbb{P}[X_3 = 2, X_2 = 1]}{\mathbb{P}[X_2 = 1]} = \\ \frac{\mathbb{P}[Y_3 = 1, Y_2 = 1, Y_1 = 0]}{\mathbb{P}[(Y_2 = 1 \wedge Y_1 = 0) \vee (Y_2 = 0 \wedge Y_1 = 1)]} = \frac{1/8}{1/2} = \frac{1}{4} \neq 0.$$

Exercise 30. We have

- $\mathbb{P}[X_{n+1} = x - 1 | X_n = x] = \left(\frac{x}{5}\right)^2$ for all $x = 1, ..., 5$.
- $\mathbb{P}[X_{n+1} = x | X_n = x] = \left(\frac{5-x}{5}\right) \cdot \left(\frac{x}{5}\right)$ for all $x = 0, ..., 5$.
- $\mathbb{P}[X_{n+1} = x + 1 | X_n = x] = \left(\frac{5-x}{5}\right)^2$ for all $x = 0, ..., 4$.
- The transition probability is 0 in any other case.

Exercise 31.

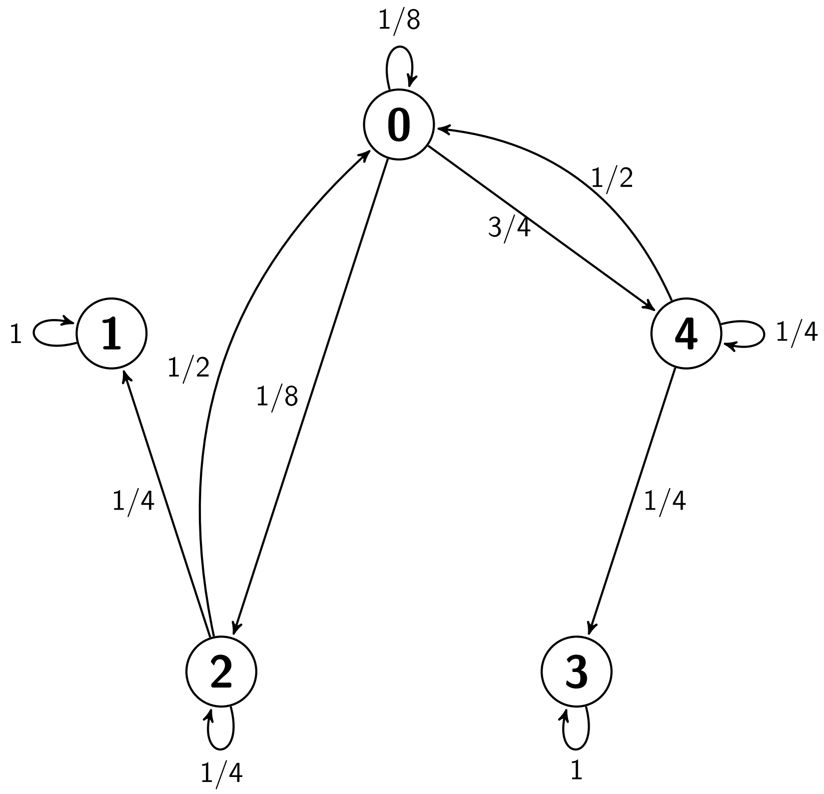
(a) By the given transition probabilities we can find the missing ones:

- $p(4, 0) = 2p(4, 3) = 2/3 \cdot (1 - 1/4) = 1/2$.
- $p(0, 4) = 1 - 1/8 - 1/8 = 3/4$.
- $p(2, 0) = 1 - 1/4 - 1/4 = 1/2$.

Therefore the transition matrix $P = (p(i, j))_{i,j}$ is

$$P = \begin{pmatrix} 1/8 & 0 & 1/8 & 0 & 3/4 \\ 0 & 1 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1/4 & 1/4 \end{pmatrix}$$

So, the transition graph is the following:



(b) The Markov Chain is not irreducible because, for example, $p(1,1) = 1$, i.e. $p(1,x) = 0$ for any $x \neq 1$.

(c) For every state x, $p(x,x) > 0$. This means that there are no transient states, i.e. every state is recurrent.

(d) The absorbing states are the ones such that $p(x,x) = 1$. So $S = \{1,3\}$.

(e) We trivially have $\mathbb{P}[X_n = 3 \text{ for } n \geq 1 | X_0 = 2] \leq \mathbb{P}[X_1 = 3 | X_0 = 2] = p(2,3) = 0$.

(f) For all $n \geq 0$, define $Y_n := \mathbb{1}_{[X_n \notin S | X_0 = 0]}$. Its expectation is

$$\begin{aligned}
\mathbb{E}[Y_n] &= \mathbb{P}[X_n \notin S | X_0 = 0] \\
&= \mathbb{P}[X_n = 0 | X_0 = 0] + \mathbb{P}[X_n = 2 | X_0 = 0] + \mathbb{P}[X_n = 4 | X_0 = 0] \\
&= p^{(n)}(0,0) + p^{(n)}(0,2) + p^{(n)}(0,4).
\end{aligned}$$

Now observe that

$$T_0 = \sum_{n=0}^{\infty} Y_n$$

almost surely, because $\mathbb{P}[X_{n+k} \notin S, X_n \in S] = 0$ for all $k \geq 0$, since S is the set of absorbing states.
Thus

$$\mathbb{E}[T_0] = \sum_{n=0}^{\infty} \mathbb{E}[Y_n] = \sum_{n=0}^{\infty} \Big(p^{(n)}(0,0) + p^{(n)}(0,2) + p^{(n)}(0,4) \Big).$$