Exercise 6.

Proof. Suppose $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. We want to show $1 - \mathbb{P}(\limsup_{n\to\infty} A_n) = 0$. First of all note that for all $N \in \mathbb{N}$ we have $\sum_{n=N}^{\infty} \mathbb{P}(A_n) = \infty$. Now observe that

$$
\begin{aligned}
1 - \mathbb{P}(\limsup_{n\to\infty} A_n) = 1 - \mathbb{P}\left(\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} A_n\right)\right) = \mathbb{P}\left(\left(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} A_n\right)^c\right) = \\
= \mathbb{P}\left(\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty} A_n^c\right) = \mathbb{P}\left(\liminf_{n\to\infty} A_n^c\right) = \lim_{N\to\infty} \mathbb{P}\left(\bigcap_{n=N}^{\infty} A_n^c\right).
\end{aligned}
$$

So it is enough to show that $\mathbb{P}\left(\bigcap_{n=N}^{\infty} A_n^c\right) = 0$ for all $N \in \mathbb{N}$. Since the $(A_n)_{n=1}^{\infty}$ are independent and $1 - x \le e^{-x}$ for all $x \in \mathbb{R}^+$:

$$
\begin{aligned}
\mathbb{P}\left(\bigcap_{n=N}^{\infty} A_n^c\right) &= \prod_{n=N}^{\infty} \mathbb{P}\left(A_n^c\right) \\
&= \prod_{n=N}^{\infty} \left(1 - \mathbb{P}\left(A_n\right)\right) \\
&\le \prod_{n=N}^{\infty} e^{-\mathbb{P}(A_n)} \\
&= e^{-\sum_{n=N}^{\infty} \mathbb{P}(A_n)} \\
&= e^{-\infty} \\
&= 0,
\end{aligned}
$$

and we are done.
The first $(\Longrightarrow)$ is the Borel-Cantelli lemma (already proven). So we proved both the $(\Longrightarrow)$'s implications. Now it's just a matter of elementary logic to see that the $(\Longleftarrow)$'s hold as well (observe before that the sums must exist since they are sums of non-negative numbers). □

Exercise 7.

Proof. **General remark.** Given two random variables X, Y (which can both assume only finitely-many values) defined on $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ respectively, the *sum of* X *and* Y, written $X + Y$, is the random variable defined on the product probability space $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$ and given by

$$Z(\omega_1, \omega_2) := X(\omega_1) + Y(\omega_2).$$

Back to the exercise. Without loss of generality (because what really matters about a random variable is its distribution, not the probability space on which it's defined), we can assume that X is defined on the probability space $(\{1,2\}, \mathcal{P}(\{1,2\}), \mathbb{P}_1)$ and is the identity function, and Y is defined on the probability space $(\{0,1,2\}, \mathcal{P}(\{0,1,2\}), \mathbb{P}_2)$, where $\mathbb{P}_1[\omega] := \frac{1}{2}$ for all $\omega \in \{1,2\}$ and $\mathbb{P}_2$ is defined as in the assignment. Now, the sample space on which Z is defined is by definition

$$\{1,2\} \times \{0,1,2\}.$$

By the way the the product σ-algebra is defined (which I won't discuss here), it is immediate to see that it must contain every singleton. Thus, since the sample space is clearly finite, this means that the σ-algebra of Z is $\mathcal{P}(\{1,2\} \times \{0,1,2\})$.
By the way the the product probability measure $\mathbb{P} := \mathbb{P}_1 \otimes P_2$ is defined (which I won't discuss here), we must have that

$$\mathbb{P}[(\omega_1, \omega_2)] = \mathbb{P}_1[\omega_1]\mathbb{P}_2[\omega_2]$$

for all $\omega_1 \in \{1,2\}$ and $\omega_2 \in \{0,1,2\}$. Since the probability space is clearly finite, this completely determines $\mathbb{P}$ on the whole σ-algebra (by the additive property).
It is left to find the distribution of Z. Observe that

$$\begin{aligned}
\mathbb{P}[Z = z] &= \mathbb{P}[X + Y = z] \\
&= \mathbb{P}[Y = z - X] \\
&= \mathbb{P}\Big[[X = 1 \text{ and } Y = z - 1] \uplus [X = 2 \text{ and } Y = z - 2]\Big] \\
&= \mathbb{P}[X = 1 \text{ and } Y = z - 1] + \mathbb{P}[X = 2 \text{ and } Y = z - 2] \\
&= \mathbb{P}[X = 1]\mathbb{P}[Y = z - 1] + \mathbb{P}[X = 2]\mathbb{P}[Y = z - 2] \quad \text{(by independence)} \\
&= \frac{1}{2}\mathbb{P}[Y = z - 1] + \frac{1}{2}\mathbb{P}[Y = z - 2]. \qquad (*)
\end{aligned}$$

Since Y can take values only among $\{0,1,2\}$, $(*)$ can be different from 0 only if $z = 1, 2, 3, 4$ (which *of course* makes sense, doesn't it?). Now the last computations (which I am not gonna do):

- $\mathbb{P}[Z = 1] = ...$
- $\mathbb{P}[Z = 2] = ...$
- $\mathbb{P}[Z = 3] = ...$
- $\mathbb{P}[Z = 4] = ...$

$\square$

Exercise 8.

$\mathbb{P}[X=0] = 0.03 + 0.16 + 0.12 = 0.31$.
$\mathbb{P}[X=1] = ... = 0.69$.
$\mathbb{P}[Y=-1] = 0.03 + 0.07 = 0.10$.
$\mathbb{P}[Y=2] = ... = 0.51$.
$\mathbb{P}[Y=3] = ... = 0.39$.
X and Y are not independent, because e.g. $\mathbb{P}[(X,Y) = (0,-1)] = 0.03 \neq 0,031 = 0.31 \cdot 0.10 = \mathbb{P}[X=0] \cdot \mathbb{P}[Y=-1]$.

Exercise 9.

Proof. Let $\mu := \mathbb{E}[X]$. Since X is discrete, it can take at most countable-many values $\{x_n \mid n \in \mathbb{N}\}$. Recall that $\mathrm{Var}[X] := \mathbb{E}[(X-\mu)^2] = \sum_{n\in\mathbb{N}} (x_n - \mu)^2 \cdot \mathbb{P}[X = x_n]$. Now observe that the terms in the sum are all ≥ 0. If $\mathrm{Var}[X] = 0$, it clearly means that they are all 0. So

$$(x_n - \mu)^2 \mathbb{P}[X = x_n] = 0 \text{ for all } n \in \mathbb{N}\,.$$

If $\mathbb{P}[X = x_n] = 0$, we "store" x_n in the set of all the values with 0 probability. This whole set clearly has probability 0, and this is where the "almost surely" in the statement comes from.
If $\mathbb{P}[X = x_n] > 0$, it means that $(x_n - \mu)^2 = 0$, i.e. $x_n = \mu$.
So $X(\omega) = \mu$ except for a set of probability 0, which is exactly what we must prove. $\square$

Exercise 10. Observe that

$$\begin{aligned}
E\left[\frac{1}{X+1}\right] &= \sum_{k=0}^{n} \frac{1}{k+1}\mathbb{P}[X=k] \\
&= \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} p^k(1-p)^{n-k} \\
&= \sum_{k=0}^{n} \frac{1}{k+1}\frac{n!}{k!(n-k)!} p^k(1-p)^{n-k} \\
&= \sum_{k=0}^{n} \frac{n!}{(k+1)!(n-k)!} p^k(1-p)^{n-k} \\
&= \frac{1}{(n+1)p}\sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!(n-k)!} p^{k+1}(1-p)^{n-k} \\
&= \frac{1}{(n+1)p}\sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!(n-k)!} p^{k+1}(1-p)^{n-k} \\
&= \frac{1}{(n+1)p}\sum_{k=0}^{n} \binom{n+1}{k+1} p^{k+1}(1-p)^{n-k} \\
&= \frac{1}{(n+1)p}\left(\sum_{k=-1}^{n} \binom{n+1}{k+1} p^{k+1}(1-p)^{n-k}\right) - \frac{1}{(n+1)p}\binom{n+1}{0}p^0(1-p)^{n+1} \\
&= \frac{1}{(n+1)p}\left(\sum_{k'=0}^{n+1} \binom{n+1}{k'} p^{k'}(1-p)^{n-(k'-1)}\right) - \frac{1}{(n+1)p}(1-p)^{n+1} \\
&= \frac{1}{(n+1)p}\left(\sum_{k'=0}^{n+1} \underbrace{\binom{n+1}{k'} p^{k'}(1-p)^{n+1-k'}}_{=:\varphi(k)}\right) - \frac{1}{(n+1)p}(1-p)^{n+1} \\
&= \frac{1}{(n+1)p} - \frac{1}{(n+1)p}(1-p)^{n+1} \qquad (*) \\
&= \frac{1-(1-p)^{n+1}}{(n+1)p},
\end{aligned}$$

where $(*)$ holds because $\varphi(k)$ is the distribution function of $\mathrm{Bin}(n+1, \frac{1}{2})$, and so its sum from 0 to $n+1$ is 1.

Exercise 11. Note: I am pretty sure that we also need the hypothesis $X \geq 0$, i.e. $x_i \geq 0$ for all $i \in \mathbb{N}$, like in the continuous case seen in class. Moreover, we also need the hypothesis $\mathbb{E}[X] \neq 0$, otherwise the statement is clearly false (this is because our statement is slightly different from the standard one, but the professor forgot to add this hypothesis). However:

Proof. Call $\mu := \mathbb{E}[X]$.

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{i\in\mathbb{N}} x_i \mathbb{P}[X = x_i] \\
&= \sum_{x_i < a\mu} x_i \mathbb{P}[X = x_i] + \sum_{x_i \geq a\mu} x_i \mathbb{P}[X = x_i] \\
&\geq \sum_{x_i \geq a\mu} x_i \mathbb{P}[X = x_i] && \text{(because } X > 0\text{)} \\
&\geq \sum_{x_i \geq a\mu} a\mu \mathbb{P}[X = x_i] \\
&= a\mu \sum_{x_i \geq a\mu} \mathbb{P}[X = x_i] \\
&= a\mu \mathbb{P}[X \geq a\mu].
\end{aligned}$$

□