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Exercise 13 (to be simplified). As for (a), we first prove that

$$\sum_{n=1}^{\infty} \mathbb{P}[X > n] = \sum_{k=1}^{\infty} k\mathbb{P}[k < X \leq k+1].$$

Observe that

$$\{X > n\} = \biguplus_{k \geq n} \{k < X \leq k+1\}$$

and thus

$$\mathbb{P}[X > n] = \sum_{k=n}^{\infty} \mathbb{P}[k < X \leq k+1].$$

Therefore

$$\sum_{n=1}^{\infty} \mathbb{P}[X > n] = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \mathbb{P}[k < X \leq k+1] = \sum_{k=1}^{\infty} \sum_{n=1}^{k} \mathbb{P}[k < X \leq k+1] =$$
$$\sum_{k=1}^{\infty} \mathbb{P}[k < X \leq k+1] \sum_{n=1}^{k} 1 = \sum_{k=1}^{\infty} \mathbb{P}[k < X \leq k+1] \cdot k.$$

So the two sums are equal. Now we want to prove that

$$\mathbb{E}[X] < \infty \Leftrightarrow \sum_{k=1}^{\infty} k\mathbb{P}[k < X \leq k+1] < \infty.$$

($\Longrightarrow$)

$$\infty > \int_{\Omega} X d\mathbb{P} > \int_{\Omega} \sum_{k=1}^{\infty} \left(k \cdot \mathbb{1}_{\{\omega : k < X(\omega) \leq k+1\}}\right) d\mathbb{P} = \sum_{k=1}^{\infty} \int_{\Omega} k \cdot \mathbb{1}_{\{\omega : k < X(\omega) \leq k+1\}} d\mathbb{P} =$$
$$\sum_{k=1}^{\infty} k \int_{\Omega} \mathbb{1}_{\{\omega : k < X(\omega) \leq k+1\}} d\mathbb{P} = \sum_{k=1}^{\infty} k\mathbb{P}[k < X \leq k+1].$$

($\Longleftarrow$)

$$\int_{\Omega} X d\mathbb{P} \leq \int_{\Omega} \sum_{k=0}^{\infty} \left((k+1) \cdot \mathbb{1}_{\{\omega : k < X(\omega) \leq k+1\}}\right) d\mathbb{P} = \sum_{k=0}^{\infty} \int_{\Omega} (k+1) \cdot \mathbb{1}_{\{\omega : k < X(\omega) \leq k+1\}} d\mathbb{P} =$$
$$\sum_{k=0}^{\infty} (k+1) \int_{\Omega} \mathbb{1}_{\{\omega : k < X(\omega) \leq k+1\}} d\mathbb{P} = \sum_{k=0}^{\infty} (k+1)\mathbb{P}[k < X \leq k+1] < \infty.$$

And we are done.
As for (b), consider

$$X_n := \underbrace{\sum_{x_k<n} x_k \mathbb{1}_{[X=x_k]}}_{=:Y_n} + n\mathbb{1}_{[X\geq n]}.$$

Of course $X_n \nearrow X$, and so by monotone convergence

$$\mathbb{E}[X] = \mathbb{E}[\lim_{n\to\infty} X_n] = \lim_{n\to\infty} \mathbb{E}[X_n] = \lim_{n\to\infty} \sum_{x_k<n} x_k \mathbb{P}[X = x_k] + n\mathbb{P}[X \geq n].$$

Now observe that $\mathbb{P}[X = +\infty] > 0$ implies $\mathbb{E}[X] = +\infty$ because $\mathbb{P}[X \geq n] \searrow \mathbb{P}[X = +\infty]$ by continuity. Since $\mathbb{E}[X < +\infty]$ by hypothesis, we trivially get $\mathbb{P}[X = +\infty] = 0$. This means

$$X \cdot \mathbb{1}_{[X<+\infty]} = X \text{ almost surely} \tag{$*$}$$

Now, first observe that $Y_n \leq Y_{n+1}$ for all $n \in \mathbb{N}$. If $X(\omega) < \infty$, then $\exists n_0 \in \mathbb{N}$ s.t. $X(\omega) \leq n_0$. So $n\mathbb{1}_{[X\geq n]}(\omega) = 0$ for all $n > n_0$, i.e. $X(\omega) = \lim_n Y_n(\omega)$. Thus $Y_n \nearrow X \cdot \mathbb{1}_{[X<+\infty]}$, i.e. $Y_n \nearrow X$ almost surely by ($*$).
Hence by monotone convergence we obtain $\mathbb{E}[Y_n] \to \mathbb{E}[X]$ and $\mathbb{E}[X_n] \to \mathbb{E}[X]$. Since $\mathbb{E}[X] < +\infty$, we can finally conclude that $\mathbb{E}[X_n] - \mathbb{E}[Y_n] \to 0$, i.e. $n\mathbb{P}[X \geq n] \to 0$.

Exercise 14. Observe that

$$|Z - Y| = |X_n - Y - X_n + Z| \leq |X_n - Y| + |Z - X_n|$$

by the triangular inequality. Thus

$$\begin{aligned}
\mathbb{P}[|Z - Y| \geq 2\varepsilon] &\leq \mathbb{P}[|X_n - Y| + |Z - X_n| \geq 2\varepsilon] \\
&\leq \mathbb{P}[|X_n - Y| \geq \varepsilon \text{ or } |X_n - Z| \geq \varepsilon] \\
&\leq \mathbb{P}[|X_n - Y| \geq \varepsilon] + \mathbb{P}[|X_n - Z| \geq \varepsilon] \to_{n\to\infty} 0 + 0 = 0
\end{aligned}$$

for all $\varepsilon > 0$, i.e. $\mathbb{P}[|Z - Y| \geq 2\varepsilon] = 0$ for all $\varepsilon > 0$. Finally, observing that

$$[Y \neq Z] = \bigcup_{k=1}^{\infty} \left[|Y - Z| \geq \frac{1}{k} \right]$$

we immediately conclude $\mathbb{P}[Y \neq Z] = 0$ by subadditivity.

Exercise 15. Observe that $\mathrm{Var}[S_n/n^p] = \frac{1}{n^{2p}} nC = \frac{1}{n^{2p-1}} C$. So, by Chebyshev's inequality, we get

$$\mathbb{P}[|S_n/n^p| \geq a] \leq \frac{C}{n^{2p-1}a^2} \to 0$$

because $2p - 1 > 0$.

Exercise 16.
Solution 1. This solution is way easier, and it follows immediately by the following Corollary we've seen in class (I didn't remember that at first):

If X_n are independent with finite variances and $\sum \frac{\mathrm{Var}[X]}{n^2} < \infty$, then $\sum \frac{X_n - \mathbb{E}[X_n]}{n}$ converges almost surely to a finite random variable.

Solution 2. Define $Y := \sum_{n=1}^{\infty} X_n/n$. First observe that $\mathbb{E}[X_n] = 0$ and $\mathrm{Var}[X_n] = 1$. Define $Y_m := \sum_{n=1}^{m} X_n/n$. We immediately have $\mathbb{E}[Y_m] = 0$. Furthermore by independence

$$\mathrm{Var}[Y_m] = \sum_{n=1}^{m} \frac{1}{n^2} \mathrm{Var}[X_n] = \sum_{n=1}^{m} \frac{1}{n^2}.$$

So by Chebyshev's inequality

$$0 \le \mathbb{P}[|Y_m - \mathbb{E}[Y_m]| \ge k] \le \frac{\mathrm{Var}[Y_m]}{k^2} = \frac{\sum_{n=1}^{m} \frac{1}{n^2}}{k^2}$$

for all $a > 0$. So

$$0 \le \lim_{m\to\infty} \mathbb{P}[|Y_m| \ge k] \le \lim_{m\to\infty} \frac{\sum_{n=1}^{m} \frac{1}{n^2}}{k^2} = \frac{1}{k^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{k^2} \frac{\pi}{6}.$$

So $\lim_{m\to\infty} \mathbb{P}[|Y_m| \ge k] \le \frac{1}{k^2}\frac{\pi}{6}$, which implies (TO BE PROVEN) $\mathbb{P}[\lim_{m\to\infty} |Y_m| \ge k] \le \frac{1}{k^2}\frac{\pi}{6}$. But $\lim_{m\to\infty} |Y_m| = |Y|$ (we are allowed to assume that the limit exists, see my question in the newsgroup), so $\mathbb{P}[|Y| \ge k] \le \frac{1}{k^2}\frac{\pi}{6}$. Therefore

$$\sum_{k=1}^{\infty} \mathbb{P}[|Y| > k] \le \sum_{k=1}^{\infty} \frac{1}{k^2}\frac{\pi}{6} = \frac{\pi}{6} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{36} < \infty.$$

By Exercise 16(a) this implies $\mathbb{E}[|Y|] < \infty$, which implies $\mathbb{P}[|Y| = +\infty] = 0$ (see solution of Exercise 13). So Y is finite a.s., i.e. Y converges a.s.

Exercise 17. Let's show that $\mathbb{E}[N_n/n] \to e^{-c}$ (the "almost surely" here makes no sense. $\mathbb{E}[N_n/n]$ is a number). It's trivial to check that $\mathbb{E}[N_n] = n(1 - 1/n)^r$. So $\mathbb{E}[N_n/n] = (1 - 1/n)^r$. Now, since $r/n \to c$,

$$\lim_{n\to+\infty} \left(1 - \frac{1}{n}\right)^r = \lim_{n\to+\infty} \left(1 - \frac{1}{n}\right)^{nc} = \lim_{n\to+\infty} \left(1 + \frac{1}{-n}\right)^{nc} = \lim_{n'\to-\infty} \left(1 + \frac{1}{n'}\right)^{-n'c} =$$
$$\lim_{n'\to-\infty} \left(\left(1 + \frac{1}{n'}\right)^{n'}\right)^{-c} = \left(\lim_{n'\to-\infty} \left(1 + \frac{1}{n'}\right)^{n'}\right)^{-c} = e^{-c}.$$

Now we are asked to show that $N_n/n \to e^{-c}$ in probability. We follow the hint. We will soon need the following:
Lemma. The following hold:

1. $\mathrm{Var}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathrm{Var}[X_i] + 2\sum_{i<j} \mathrm{Cov}[X_i, X_j]$, where $\mathrm{Cov}[X,Y] := \mathbb{E}\left[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])\right]$.

2. $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}[A]$, $\mathrm{Var}[\mathbb{1}_A] = \mathbb{P}[A](1-\mathbb{P}[A])$, $\mathrm{Cov}[\mathbb{1}_A, \mathbb{1}_B] = \mathbb{P}[A\cap B] - \mathbb{P}[A]\mathbb{P}[B]$.

3. $\displaystyle\sum_{\substack{i<j\\ j\le n}} 1 = \sum_{j=1}^n \sum_{i=1}^{j-1} 1 = \frac{1}{2}(n^2-n).$

The proof of the lemma is very easy, therefore we skip it. Define $A_i :=$ [i-th box is empty]. Observe that $\mathbb{P}[A_i \cap A_j] = \mathbb{P}[A_j|A_i]\mathbb{P}[A_i] = \left(1-\frac{1}{n-1}\right)^r \mathbb{P}[A_i] \to e^{-2c}$. Observe also that $\mathbb{P}[A_i] = \mathbb{P}[A_j]$ for all $i,j \le n$ and $\mathbb{P}[A_i|A_j] = \mathbb{P}[A_h|A_k]$ for all $i \neq j$, $h \neq k$. Putting everything together we get

$$\begin{aligned}
\mathrm{Var}[N_n] &= \mathrm{Var}\left[\sum_{i=1}^n \mathbb{1}_{A_i}\right] \\
&= \sum_{i=1}^n \mathrm{Var}[\mathbb{1}_{A_i}] + 2\sum_{i<j}\mathrm{Cov}[\mathbb{1}_{A_i}, \mathbb{1}_{A_j}] \\
&= \sum_{i=1}^n \mathbb{P}[A_i](1-\mathbb{P}[A_i]) + 2\sum_{i<j}(\mathbb{P}[A_i\cap A_j] - \mathbb{P}[A_i]\mathbb{P}[A_j]) \\
&= \sum_{i=1}^n \mathbb{P}[A_i](1-\mathbb{P}[A_i]) + 2\sum_{i<j}(\mathbb{P}[A_j|A_i]\mathbb{P}[A_i] - \mathbb{P}[A_i]\mathbb{P}[A_j]) \\
&= \mathbb{P}[A_i](1-\mathbb{P}[A_i])\sum_{i=1}^n 1 + 2(\mathbb{P}[A_j|A_i]\mathbb{P}[A_i] - \mathbb{P}[A_i]\mathbb{P}[A_j])\sum_{i<j} 1 \\
&= n\mathbb{P}[A_i](1-\mathbb{P}[A_i]) + (n^2-n)(\mathbb{P}[A_j|A_i]\mathbb{P}[A_i] - \mathbb{P}[A_i]\mathbb{P}[A_j]).
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{Var}[N_n/n] &= \frac{1}{n^2}\mathrm{Var}[N_n] \\
&= \frac{1}{n}\mathbb{P}[A_i](1-\mathbb{P}[A_i] - \mathbb{P}[A_j|A_i] + \mathbb{P}[A_j]) + (\mathbb{P}[A_j|A_i]\mathbb{P}[A_i] - \mathbb{P}[A_i]\mathbb{P}[A_j]) \\
&\to \underbrace{\frac{1}{n}e^{-c}(1-e^{-c}-e^{-c}+e^{-c})}_{\to 0} + \underbrace{(e^{-2c}-e^{-2c})}_{=0} \\
&\to 0.
\end{aligned}$$

Finally, observe that by Chebyshev's inequality

$$\mathbb{P}[|N_n/n - \mathbb{E}[N_n/n]| \ge a] \le \frac{\mathrm{Var}[N_n/n]}{a^2} \to 0$$

for all $a > 0$, i.e. $N_n/n \to \lim_{n\to\infty} \mathbb{E}[N_n/n] = e^{-c}$ in probability.