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Exercise 1.

(a) $\mathcal{A}$ must contain $\emptyset$, Ω, $\{1,3,5\}$ and $\{2,4,6\}$. Such a collection is already closed under union, intersection and complement. So it's precisely $\mathcal{A}$.

(b) $\mathcal{A}$ must contain $\emptyset$, Ω, $\{1,2\}$, $\{3,4\}$ and $\{5,6\}$, thus (using the union) also $\{1,2,3,4\}$, $\{1,2,5,6\}$ and $\{3,4,5,6\}$. It's immediate to check that such a collection is already closed under intersection and complement. So it's precisely $\mathcal{A}$.

(c) $\mathcal{A}$ must contain every singleton, and thus $\mathcal{A} = \mathcal{P}(\Omega)$, because Ω is finite and thus every subset can be written as the *countable* (finite) union of the singletons of its elements.

Exercise 2. Let $\Omega := \{r, b, g, w\}$. We can construct the σ-algebra formed by $\emptyset, \Omega$ and:

1. nothing else.
2. One singleton, and thus also its complement. These are 4 different algebras (one for every singleton).
3. One set of cardinality 2, and thus also its complement. These are $6/2 = 3$ different algebras (there are 6 choices for sets of cardinality 2, but every such a set determines also the other one).
4. *One set of cardinality 3: these algebras are exactly the same of the second point.*
5. *One set of cardinality 4: same of first point.*
6. Two singletons, and thus also their complements, their union and the complement of their union. These are 6 different algebras (one for every choice of two singletons).
7. Two sets of cardinality 2. *If the two sets are disjoint, the algebras are the same of third point.* If the two sets are not disjoint, then their intersection is a singleton. Now it's easy to check that we can construct every singleton, and thus these algebras are all $\mathcal{P}(\Omega)$. Thus we have found just 1 more algebra.
8. *Two sets of cardinality 3: same of sixth point.*

9. *Three singletons: it's immediate to construct the other singleton, and thus we obtain $\mathcal{P}(\Omega)$.*

10. *Three sets of cardinality 2: it's easy to check that in this case we can always construct every singleton, and thus we get $\mathcal{P}(\Omega)$.*

11. *Three sets of cardinality 3: it's easy to check that in this case we can always construct every singleton, and thus we get $\mathcal{P}(\Omega)$.*

12. *Four singletons: this is trivially $\mathcal{P}(\Omega)$.*

So the different σ-algebras are 15. The non-isomorphic ones are as many as the non-italic points, i.e. 5.

Exercise 3.

(a) $\mathbb{P}[A \cap B^c] = \mathbb{P}[A] - \mathbb{P}[A \cap B] = \mathbb{P}[A] - \mathbb{P}[A]\mathbb{P}[B] = \mathbb{P}[A](1 - \mathbb{P}[B]) = \mathbb{P}[A]\mathbb{P}[B^c]$.

(b) $\mathbb{P}[A^c \cap B^c] = \mathbb{P}[B^c] - \mathbb{P}[A \cap B^c] = \mathbb{P}[B^c] - \mathbb{P}[A]\mathbb{P}[B^c] = \mathbb{P}[B^c](1 - \mathbb{P}[A]) = \mathbb{P}[B^c]\mathbb{P}[A^c]$.

Exercise 4.
Note: the following solution might look too complex/heavy/elaborated. I am pretty sure there aren't much easier ways to solve the exercise, but I could be wrong. Nevertheless, one thing is for sure: **answers of the type "if the firstborn is a girl, then the chance that the second is a boy is 1/2, because the events are *clearly* independent" MAKE. NO. SENSE.** The exercise is given *precisely* to force us to set up a theoretical environment which models the real problem, but where we can *formally prove* the statements. And indeed such independence can be *proved* in the following model. Of course, one can argue that my probability space doesn't faithfully model the real problem, but that's a different story.

Solution. We define $\Omega := \{0,1\} \times \{0,1\} \times \{0,1\}$. By 0 we mean boy and by 1 we mean "girl".

(a) We take $\mathcal{A} := \mathcal{P}(\Omega)$ and $\mathbb{P}[\omega] := \frac{1}{2^3}$ for all $\omega \in \Omega$. Since Ω is finite, this completely determines $\mathbb{P}$ on $\mathcal{A}$ (by the additive property).

(b) Recall that $\mathbb{P}[A|B] = \mathbb{P}[A \cap B]/\mathbb{P}[B]$. Let $A := \{(x,y,z) \in \Omega \mid y = 0\}$ and $B := \{(x,y,z) \in \Omega \mid x = 1\}$. Then the exercise asks us to to compute $\mathbb{P}[A|B]$, and it's very easy to check that $\mathbb{P}[A \cap B] = \frac{1}{2^2}$ and $\mathbb{P}[B] = \frac{1}{2}$, thus the result is $\frac{1}{2}$.

(c) Define $A := \{(x, y, z) \in \Omega \mid x = y = z = 1\} = \{(1, 1, 1)\}$ and $B := \{(x, y, z) \in \Omega \mid x = 1 \vee y = 1 \vee z = 1\}$. We shall compute $\mathbb{P}[A|B] = \mathbb{P}[A \cap B]/\mathbb{P}[B]$. Observe that $\mathbb{P}[A] = \frac{1}{2^3}$. Furthermore, trivially $A \subseteq B$, thus $\mathbb{P}[A \cap B] = \mathbb{P}[A] = \frac{1}{2^3}$. Now observe that $\mathbb{P}[B] = 1 - \mathbb{P}[B^c]$, and trivially $B^c = \{(0, 0, 0)\}$. Thus $\mathbb{P}[B] = 1 - \frac{1}{2^3}$. So

$$\mathbb{P}[A|B] = \frac{1}{2^3} \cdot \frac{2^3}{2^3 - 1} = \frac{1}{7}.$$

Exercise 5.

(a) Let A denote the event "$X = 4$" and B denote the event "all coins show head". We are asked to find $\mathbb{P}[A|B]$. By Bayes theorem we know that

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[B|A]\mathbb{P}[B]}{\mathbb{P}[A]}.$$

Now observe that (after setting up an appropriate abstract model for this problem we could prove that) $\mathbb{P}[A] = \frac{8}{110}$ and $\mathbb{P}[B|A] = \frac{1}{2^4}$. It is left to compute $\mathbb{P}[B]$. Let C_k be the event "$X = k$". Of course such events are pairwise disjoint and they cover Ω. So by the rule of total probability we have

$$\mathbb{P}[B] = \sum_{k=1}^{10} \mathbb{P}[B|C_k]\mathbb{P}[C_k] = \sum_{k=1}^{10} \frac{1}{2^k} \frac{2k}{110} = \frac{1}{110} \sum_{k=1}^{10} k \left(\frac{1}{2}\right)^{k-1} = \frac{1}{110} \frac{509}{128}.$$

So

$$\mathbb{P}[A|B] = \frac{1}{2^4} \frac{1}{110} \frac{509}{128} \frac{110}{8} = \frac{509}{2^{14}}.$$

(b) Let D be the event "X is even". We are asked to check whether

$$\mathbb{P}[D \cap B] = \mathbb{P}[D]\mathbb{P}[B].$$

We already know $\mathbb{P}[B]$. Let's compute $\mathbb{P}[D \cap B]$. Observe that

$$\begin{aligned}
\mathbb{P}[D \cap B] &= \mathbb{P}[B|D]\mathbb{P}[D] \\
&= \mathbb{P}[B|C_2 \uplus C_4 \uplus C_6]\mathbb{P}[D] \\
&= \Big(\mathbb{P}[B|C_2] + \mathbb{P}[B|C_4] + \mathbb{P}[B|C_6]\Big)\mathbb{P}[D] \qquad \text{(immediate to check)} \\
&= \left(\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6}\right)\mathbb{P}[D] \\
&= \frac{25}{64}\mathbb{P}[D].
\end{aligned}$$

But $\mathbb{P}[B] \neq \frac{25}{64}$, and thus D and B are not independent.