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Exercise 12.

Note: the notation here is probably heavier than anywhere else. I'm sorry for that, but personally this is the only way I can really explain (and even understand) what's really happening. (I had already read something about conditional expectations, and the notation was definitely lighter, but it sometimes lead to some paradoxical conclusions like $\mathbb{P}[X = X] \neq 1$)

(a) Let $\psi(x) := \mathbb{E}[Y|X = x] := \sum_{y\in A} y\mathbb{P}[Y = y|X = x]$. Recall that $\mathbb{E}[Y|X]$ is defined as

$$\mathbb{E}[Y|X] := \psi(X) = \psi(X(\omega)) = \sum_{y\in A} y \underbrace{\mathbb{P}[Y = y|X = X(\omega)]}_{=:\varphi(X(\omega))}$$

So

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[Y|X]] &= \mathbb{E}\left[\sum_{y\in A} y\varphi(X)\right] \\
&= \sum_{y\in A} y\,\mathbb{E}\left[\varphi(X)\right] \\
&= \sum_{y\in A} y\left[\sum_{\omega\in\Omega} \varphi(X(\omega))\mathbb{P}[X = X(\omega)]\right] \quad \text{(law unconsc. statistician)} \\
&= \sum_{y\in A} y\left[\sum_{\omega\in\Omega} \mathbb{P}[Y = y|X = X(\omega)]\ \mathbb{P}[X = X(\omega)]\right] \\
&= \sum_{y\in A} y\mathbb{P}[Y = y] \quad \text{(rule of tot. prob.)} \\
&= \mathbb{E}[Y].
\end{aligned}$$

(b) Let $K := \mathrm{Im}(g)$. Observe that for all $\omega \in \Omega$ we have

$$\begin{aligned}
\mathbb{E}[g(X)Y|X](\omega) &= \sum_{k\in K, y\in A} k\, y\; \mathbb{P}[g(X) = k\, Y = y \mid X = X(\omega)] \\
&= \sum_{y\in A} y \sum_{k\in K} k\; \underbrace{\mathbb{P}[g(X) = k,\, Y = y \mid X = X(\omega)]}_{=0 \text{ except when } k=g(X(\omega))} \\
&= \sum_{y\in A} y\; g(X(\omega))\, \mathbb{P}[g(X) = g(X(\omega)),\, Y = y \mid X = X(\omega)] \\
&= \sum_{y\in A} y\; g(X(\omega))\, \mathbb{P}[Y = y \mid X = X(\omega)] \\
&\qquad\qquad\qquad \text{(immediate using def. of cond. prob.)} \\
&= g(X(\omega)) \sum_{y\in A} y\; \mathbb{P}[Y = y \mid X = X(\omega)] \\
&= g(X(\omega))\, \mathbb{E}[Y|X](\omega),
\end{aligned}$$

i.e. $\mathbb{E}[g(X)Y|X] = g(X)\,\mathbb{E}[Y|X]$.

(c) Suppose X and Y are independent. Then

$$\mathbb{E}[Y|X] = \sum_{y\in A} y\mathbb{P}[Y = y|X = X(\omega)] = \sum_{y\in A} y\mathbb{P}[Y = y] = \mathbb{E}[Y].$$

Putting this together with (b), we are done.

Exercise 13 (to be simplified). As for (a), we first prove that

$$\sum_{n=1}^{\infty} \mathbb{P}[X > n] = \sum_{k=1}^{\infty} k\mathbb{P}[k < X \le k+1].$$

Observe that

$$\{X > n\} = \biguplus_{k\ge n} \{k < X \le k+1\}$$

and thus

$$\mathbb{P}[X > n] = \sum_{k=n}^{\infty} \mathbb{P}[k < X \le k+1].$$

Therefore

$$\sum_{n=1}^{\infty} \mathbb{P}[X > n] = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \mathbb{P}[k < X \le k+1] = \sum_{k=1}^{\infty}\sum_{n=1}^{k} \mathbb{P}[k < X \le k+1] =$$

$$\sum_{k=1}^{\infty} \mathbb{P}[k < X \le k+1] \sum_{n=1}^{k} 1 = \sum_{k=1}^{\infty} \mathbb{P}[k < X \le k+1] \cdot k.$$

So the two sums are equal. Now we want to prove that

$$\mathbb{E}[X] < \infty \Leftrightarrow \sum_{k=1}^{\infty} k\mathbb{P}[k < X \leq k+1] < \infty.$$

$(\Longrightarrow)$

$$\infty > \int_\Omega X d\mathbb{P} > \int_\Omega \sum_{k=1}^{\infty} \left(k \cdot \mathbb{1}_{\{\omega : k < X(\omega) \leq k+1\}}\right) d\mathbb{P} = \sum_{k=1}^{\infty} \int_\Omega k \cdot \mathbb{1}_{\{\omega : k < X(\omega) \leq k+1\}} d\mathbb{P} =$$
$$\sum_{k=1}^{\infty} k \int_\Omega \mathbb{1}_{\{\omega : k < X(\omega) \leq k+1\}} d\mathbb{P} = \sum_{k=1}^{\infty} k\mathbb{P}[k < X \leq k+1].$$

$(\Longleftarrow)$

$$\int_\Omega X d\mathbb{P} \leq \int_\Omega \sum_{k=0}^{\infty} \left((k+1) \cdot \mathbb{1}_{\{\omega : k < X(\omega) \leq k+1\}}\right) d\mathbb{P} = \sum_{k=0}^{\infty} \int_\Omega (k+1) \cdot \mathbb{1}_{\{\omega : k < X(\omega) \leq k+1\}} d\mathbb{P} =$$
$$\sum_{k=0}^{\infty} (k+1) \int_\Omega \mathbb{1}_{\{\omega : k < X(\omega) \leq k+1\}} d\mathbb{P} = \sum_{k=0}^{\infty} (k+1)\mathbb{P}[k < X \leq k+1] < \infty.$$

And we are done.
As for (b), consider

$$X_n := \underbrace{\sum_{x_k < n} x_k \mathbb{1}_{[X = x_k]}}_{=:Y_n} + n\mathbb{1}_{[X \geq n]}.$$

Of course $X_n \nearrow X$, and so by monotone convergence

$$\mathbb{E}[X] = \mathbb{E}[\lim_{n\to\infty} X_n] = \lim_{n\to\infty} \mathbb{E}[X_n] = \lim_{n\to\infty} \sum_{x_k < n} x_k \mathbb{P}[X = x_k] + n\mathbb{P}[X \geq n].$$

Now observe that $\mathbb{P}[X = +\infty] > 0$ implies $\mathbb{E}[X] = +\infty$ because $\mathbb{P}[X \geq n] \searrow \mathbb{P}[X = +\infty]$ by continuity. Since $\mathbb{E}[X < +\infty]$ by hypothesis, we trivially get $\mathbb{P}[X = +\infty] = 0$. This means

$$X \cdot \mathbb{1}_{[X < +\infty]} = X \text{ almost surely} \tag{$*$}$$

Now, first observe that $Y_n \leq Y_{n+1}$ for all $n \in \mathbb{N}$. If $X(\omega) < \infty$, then $\exists n_0 \in \mathbb{N}$ s.t. $X(\omega) \leq n_0$. So $n\mathbb{1}_{[X \geq n]}(\omega) = 0$ for all $n > n_0$, i.e. $X(\omega) = \lim_n Y_n(\omega)$. Thus $Y_n \nearrow X \cdot \mathbb{1}_{[X < +\infty]}$, i.e. $Y_n \nearrow X$ almost surely by $(*)$.
Hence by monotone convergence we obtain $\mathbb{E}[Y_n] \to \mathbb{E}[X]$ and $\mathbb{E}[X_n] \to \mathbb{E}[X]$. Since $\mathbb{E}[X] < +\infty$, we can finally conclude that $\mathbb{E}[X_n] - \mathbb{E}[Y_n] \to 0$, i.e. $n\mathbb{P}[X \geq n] \to 0$.

Exercise 14. Observe that

$$|Z - Y| = |X_n - Y - X_n + Z| \leq |X_n - Y| + |Z - X_n|$$

by the triangular inequality. Thus

$$\begin{aligned}\mathbb{P}[|Z - Y| \geq 2\varepsilon] &\leq \mathbb{P}[|X_n - Y| + |Z - X_n| \geq 2\varepsilon] \\ &\leq \mathbb{P}[|X_n - Y| \geq \varepsilon \text{ or } |X_n - Z| \geq \varepsilon] \\ &\leq \mathbb{P}[|X_n - Y| \geq \varepsilon] + \mathbb{P}[|X_n - Z| \geq \varepsilon] \to_{n\to\infty} 0 + 0 = 0\end{aligned}$$

for all $\varepsilon > 0$, i.e. $\mathbb{P}[|Z - Y| \geq 2\varepsilon] = 0$ for all $\varepsilon > 0$. Finally, observing that

$$[Y \neq Z] = \bigcup_{k=1}^{\infty} \left[|Y - Z| \geq \frac{1}{k}\right]$$

we immediately conclude $\mathbb{P}[Y \neq Z] = 0$ by subadditivity.

Exercise 15. Observe that $\mathrm{Var}[S_n/n^p] = \frac{1}{n^{2p}} nC = \frac{1}{n^{2p-1}} C$. So, by Chebyshev's inequality, we get

$$\mathbb{P}[|S_n/n^p| \geq a] \leq \frac{C}{n^{2p-1}a^2} \to 0$$

because $2p - 1 > 0$.

Exercise 16.
Solution 1. This solution is way easier, and it follows immediately by the following Corollary we've seen in class (I didn't remember that at first):

If X_n are independent with finite variances and $\sum \frac{\mathrm{Var}[X]}{n^2} < \infty$, then $\sum \frac{X_n - \mathbb{E}[X_n]}{n}$ converges almost surely to a finite random variable.

Solution 2. Define $Y := \sum_{n=1}^{\infty} X_n/n$. First observe that $\mathbb{E}[X_n] = 0$ and $\mathrm{Var}[X_n] = 1$. Define $Y_m := \sum_{n=1}^{m} X_n/n$. We immediately have $\mathbb{E}[Y_m] = 0$. Furthermore by independence

$$\mathrm{Var}[Y_m] = \sum_{n=1}^{m} \frac{1}{n^2} \mathrm{Var}[X_n] = \sum_{n=1}^{m} \frac{1}{n^2}.$$

So by Chebyshev's inequality

$$0 \leq \mathbb{P}[|Y_m - \mathbb{E}[Y_m]| \geq k] \leq \frac{\mathrm{Var}[Y_m]}{k^2} = \frac{\sum_{n=1}^{m} \frac{1}{n^2}}{k^2}$$

for all $a > 0$. So

$$0 \leq \lim_{m\to\infty} \mathbb{P}[|Y_m| \geq k] \leq \lim_{m\to\infty} \frac{\sum_{n=1}^{m} \frac{1}{n^2}}{k^2} = \frac{1}{k^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{k^2} \frac{\pi}{6}.$$

So $\lim_{m\to\infty} \mathbb{P}[|Y_m| \geq k] \leq \frac{1}{k^2}\frac{\pi}{6}$, which implies (TO BE PROVEN) $\mathbb{P}[\lim_{m\to\infty} |Y_m| \geq k] \leq \frac{1}{k^2}\frac{\pi}{6}$. But $\lim_{m\to\infty} |Y_m| = |Y|$ (we are allowed to assume that the limit exists, see my question in the newsgroup), so $\mathbb{P}[|Y| \geq k] \leq \frac{1}{k^2}\frac{\pi}{6}$. Therefore

$$\sum_{k=1}^{\infty} \mathbb{P}[|Y| > k] \leq \sum_{k=1}^{\infty} \frac{1}{k^2}\frac{\pi}{6} = \frac{\pi}{6}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{36} < \infty.$$

By Exercise 16(a) this implies $\mathbb{E}[|Y|] < \infty$, which implies $\mathbb{P}[|Y| = +\infty] = 0$ (see solution of Exercise 13). So Y is finite a.s., i.e. Y converges a.s.

Exercise 17. Let's show that $\mathbb{E}[N_n/n] \to e^{-c}$ (the "almost surely" here makes no sense. $\mathbb{E}[N_n/n]$ is a number). It's trivial to check that $\mathbb{E}[N_n] = n(1-1/n)^r$. So $\mathbb{E}[N_n/n] = (1-1/n)^r$. Now, since $r/n \to c$,

$$\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^r = \lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{nc} = \lim_{n\to+\infty}\left(1+\frac{1}{-n}\right)^{nc} = \lim_{n'\to-\infty}\left(1+\frac{1}{n'}\right)^{-n'c} =$$
$$\lim_{n'\to-\infty}\left(\left(1+\frac{1}{n'}\right)^{n'}\right)^{-c} = \left(\lim_{n'\to-\infty}\left(1+\frac{1}{n'}\right)^{n'}\right)^{-c} = e^{-c}.$$

Now we are asked to show that $N_n/n \to e^{-c}$ in probability. We follow the hint. We will soon need the following:
Lemma. The following hold:

1. $\mathrm{Var}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathrm{Var}[X_i] + 2\sum_{i<j}\mathrm{Cov}[X_i, X_j]$, where $\mathrm{Cov}[X, Y] := \mathbb{E}\left[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])\right]$.
2. $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}[A]$, $\mathrm{Var}[\mathbb{1}_A] = \mathbb{P}[A](1-\mathbb{P}[A])$, $\mathrm{Cov}[\mathbb{1}_A, \mathbb{1}_B] = \mathbb{P}[A\cap B] - \mathbb{P}[A]\mathbb{P}[B]$.
3. $\displaystyle\sum_{\substack{i<j\\ j\leq n}} 1 = \sum_{j=1}^{n}\sum_{i=1}^{j-1} 1 = \frac{1}{2}(n^2-n).$

The proof of the lemma is very easy, therefore we skip it. Define $A_i := [i\text{-th box is empty}]$. Observe that $\mathbb{P}[A_i \cap A_j] = \mathbb{P}[A_j|A_i]\mathbb{P}[A_i] = \left(1-\frac{1}{n-1}\right)^r\mathbb{P}[A_i] \to e^{-2c}$. Observe also that $\mathbb{P}[A_i] = \mathbb{P}[A_j]$ for all $i, j \leq n$ and $\mathbb{P}[A_i|A_j] = \mathbb{P}[A_h|A_k]$ for all $i \neq j$, $h \neq k$.

Putting everything together we get

$$\begin{aligned}
\mathrm{Var}[N_n] &= \mathrm{Var}\left[\sum_{i=1}^n \mathbb{1}_{A_i}\right] \\
&= \sum_{i=1}^n \mathrm{Var}[\mathbb{1}_{A_i}] + 2\sum_{i<j} \mathrm{Cov}[\mathbb{1}_{A_i}, \mathbb{1}_{A_j}] \\
&= \sum_{i=1}^n \mathbb{P}[A_i](1-\mathbb{P}[A_i]) + 2\sum_{i<j}(\mathbb{P}[A_i \cap A_j] - \mathbb{P}[A_i]\mathbb{P}[A_j]) \\
&= \sum_{i=1}^n \mathbb{P}[A_i](1-\mathbb{P}[A_i]) + 2\sum_{i<j}(\mathbb{P}[A_j|A_i]\mathbb{P}[A_i] - \mathbb{P}[A_i]\mathbb{P}[A_j]) \\
&= \mathbb{P}[A_i](1-\mathbb{P}[A_i])\sum_{i=1}^n 1 + 2(\mathbb{P}[A_j|A_i]\mathbb{P}[A_i] - \mathbb{P}[A_i]\mathbb{P}[A_j])\sum_{i<j} 1 \\
&= n\mathbb{P}[A_i](1-\mathbb{P}[A_i]) + (n^2-n)(\mathbb{P}[A_j|A_i]\mathbb{P}[A_i] - \mathbb{P}[A_i]\mathbb{P}[A_j]).
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{Var}[N_n/n] &= \frac{1}{n^2}\mathrm{Var}[N_n] \\
&= \frac{1}{n}\mathbb{P}[A_i](1-\mathbb{P}[A_i] - \mathbb{P}[A_j|A_i] + \mathbb{P}[A_j]) + (\mathbb{P}[A_j|A_i]\mathbb{P}[A_i] - \mathbb{P}[A_i]\mathbb{P}[A_j]) \\
&\to \underbrace{\frac{1}{n}e^{-c}(1-e^{-c}-e^{-c}+e^{-c})}_{\to 0} + \underbrace{(e^{-2c}-e^{-2c})}_{=0} \\
&\to 0.
\end{aligned}$$

Finally, observe that by Chebyshev's inequality

$$\mathbb{P}[|N_n/n - \mathbb{E}[N_n/n]| \geq a] \leq \frac{\mathrm{Var}[N_n/n]}{a^2} \to 0$$

for all $a > 0$, i.e. $N_n/n \to \lim_{n\to\infty} \mathbb{E}[N_n/n] = e^{-c}$ in probability.