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Exercise 27.
(b)$\Rightarrow$(a) Suppose $M = M_1 \oplus M_2$. Then $e_1 \in \operatorname{End}_R(M)$ given by $e_1(x_1, x_2) = (x_1, 0)$ is a non trivial idempotent endomorphism.
(a)$\Rightarrow$(b) Let $f \in \operatorname{End}_R(M)$ be a non-trivial idempotent endomorphism of M. Consider the exact sequence

$$0 \to \ker f \hookrightarrow M \xrightarrow{f} f[M] \to 0.$$

Consider the inclusion $i : f[M] \to M$. Since f is idempotent, we have that $f \circ i(f(m)) = f(f(m)) = f(m)$, that is $f \circ i = \mathrm{id}_{f[M]}$. By splitting lemma, we obtain $M \simeq f[M] \oplus \ker f$.
Now we want to show that $f \neq 0, 1 \Rightarrow f[M] \neq \{0\} \wedge \ker f \neq \{0\}$. If $f[M] = \{0\}$, of course $f = 0$. If $\ker f = \{0\}$, then f is injective. But since $f(f(m)) = f(m)$ for all $m \in M$, this means that $f(m) = m$ for all $m \in M$, thus $f = 1$. $\square$

Exercise 28. Straightforward by the hint. $\square$

Exercise 29. Consider the series of the exercise. If we prove that the composition factors are $(\mathbb{Z}/p_1\mathbb{Z}, ..., \mathbb{Z}/p_r\mathbb{Z})$, then the series is a composition series, since every $\mathbb{Z}/p_1\mathbb{Z}$ is simple because every submodule is also a subgroup, and the only subgroups of $\mathbb{Z}/p\mathbb{Z}$ with p prime are trivial (since the order of any subgroup must divide p).

Claim. For all i, $p_1...p_i\mathbb{Z}/n\mathbb{Z} \simeq (p_1...p_i\mathbb{Z})/n\mathbb{Z}$, and there exists an isomorphism φ such that $\varphi[p_1...p_ip_{i+1}\mathbb{Z}/n\mathbb{Z}] = (p_1...p_ip_{i+1}\mathbb{Z})/n\mathbb{Z}$.
Proof of the claim. Consider the function $\varphi : p_1...p_i\mathbb{Z}/n\mathbb{Z} \to (p_1...p_i\mathbb{Z})/n\mathbb{Z}$ given by $p_1...p_i[x] \mapsto [p_1...p_ix]$. This is trivially a well-defined isomorphism, since it's precisely the definition of scalar multiplication in the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$. $\square$

So now we have that, for all $i = \{1, ..., r\}$,

$$\begin{aligned} p_1...p_i\mathbb{Z}/n\mathbb{Z}/p_1...p_ip_{i+1}\mathbb{Z}/n\mathbb{Z} &\simeq (p_1...p_i\mathbb{Z})/n\mathbb{Z}/(p_1...p_ip_{i+1}\mathbb{Z})/n\mathbb{Z} \\ &\simeq p_1...p_i\mathbb{Z}/p_1...p_ip_{i+1}\mathbb{Z} \simeq \mathbb{Z}/p_{i+1}\mathbb{Z}, \end{aligned}$$

where the first "equality" of the second line holds thanks to the third isomorphism theorem, and the last "equality" holds thanks to the following isomorphism:

$$\varphi : p_1...p_i\mathbb{Z}/p_1...p_ip_{i+1}\mathbb{Z} \to \mathbb{Z}/p_{i+1}\mathbb{Z},\ [p_1...p_ik] \mapsto [p_1...p_ik].$$

φ is well-defined, since $[p_1...p_ik] = [p_1...p_il]$ implies $p_1...p_ik - p_1...p_il = p_1...p_ip_{i+1}m$ for some $m \in \mathbb{Z}$, which means $k - l = p_{i+1}m$, i.e. $[k]_{\mathbb{Z}/p_{i+1}\mathbb{Z}} = [l]_{\mathbb{Z}/p_{i+1}\mathbb{Z}}$. The surjectivity is totally trivial by the way φ is defined, and for the injectivity just observe that $p_1...p_ik + p_{i+1}\mathbb{Z} = 0 \Rightarrow p_{i+1}|k \Rightarrow p_1...p_ip_{i+1}|p_1...p_ik$. $\square$

Observe that this give a further proof of the Fundamental Theorem of Aritmetic in $\mathbb{Z}$. In fact, suppose $n = p_1...p_r = q_1...q_t$. Then of course $\mathbb{Z}_{p_1...p_r} = \mathbb{Z}_{q_1...q_t}$. So, by the first part of the exercise, we can write two composition series as the one given in the text. But thanks to Jordan-Hölder theorem, we know that they must have the same length (i.e. $r = t$) and that the factors of the series must be unique up to a permutation of the indeces in $\{1, ..., r\}$. But we know from the first part of the exercise that the factors are of the form $\mathbb{Z}_{p_i}$ and $\mathbb{Z}_{q_j}$ with $i, j \in \{1, ..., r\}$, and so by the theorem $\mathbb{Z}_{p_i} \simeq \mathbb{Z}_{q_j}$, which trivially implies $p_i = q_j$. (note: to be perfectly formal we should use a permutation...for the moment I just want to go to bed). $\square$

Exercise 30. Consider the following series:

$$\mathbb{Z}_{20} \oplus \mathbb{Z}_{27} \supset 2\,\mathbb{Z}_{20} \oplus \mathbb{Z}_{27} \supset 2 \cdot 2\,\mathbb{Z}_{20} \oplus \mathbb{Z}_{27} \supset 2 \cdot 2 \cdot 5\,\mathbb{Z}_{20} \oplus \mathbb{Z}_{27} \supset 2 \cdot 2 \cdot 5\,\mathbb{Z}_{20} \oplus 3\,\mathbb{Z}_{27}$$
$$\supset 2 \cdot 2 \cdot 5\,\mathbb{Z}_{20} \oplus 3 \cdot 3\,\mathbb{Z}_{27} \supset 2 \cdot 2 \cdot 5\,\mathbb{Z}_{20} \oplus 3 \cdot 3 \cdot 3\,\mathbb{Z}_{27} = \{0\}.$$

We want to prove that this is a composition series. To this aim, we just prove that the first factor is $\mathbb{Z}_2$, i.e. simple, since the argument can be repeated in the very same way for the other factors as well (observe that switching from a sum to the following one, one component always stays the same). Observe that, by (a very trivial application of) the second isomorphism theorem, we have:

$$(\mathbb{Z}_{20} \oplus \mathbb{Z}_{27})/\,\mathbb{Z}_{27} \simeq \mathbb{Z}_{20} \quad \text{and} \quad (2\,\mathbb{Z}_{20} \oplus \mathbb{Z}_{27})/\,\mathbb{Z}_{27} \simeq 2\,\mathbb{Z}_{20}\,.$$

This implies

$$\frac{\mathbb{Z}_{20} \oplus \mathbb{Z}_{27}}{2\,\mathbb{Z}_{20} \oplus \mathbb{Z}_{27}} \simeq \frac{(\mathbb{Z}_{20} \oplus \mathbb{Z}_{27})/\,\mathbb{Z}_{27}}{(2\,\mathbb{Z}_{20} \oplus \mathbb{Z}_{27})/\,\mathbb{Z}_{27}} \simeq \frac{\mathbb{Z}_{20}}{2\,\mathbb{Z}_{20}} \simeq \mathbb{Z}_2,$$

where the first equality holds thanks to the third isomorphism theorem, and the last one holds because it's precisely what we proved in the second part of the previous exercise (which covers also the other cases). $\square$

Exercise 31. M is finitely generated by hypothesis, that is $M = Rx_1 + ... + Rx_k$ for some $n \in \mathbb{N}$ and some $x_1, ..., x_k \in M$. Consider now the morphism $\mathrm{Hom}_R(M, N) \to N^k$ given by

$$f \mapsto \big(f(x_1), ..., f(x_n)\big).$$

This is clearly a morphism. It is also injective since $\ker f = \{0\}$, because $\big(f(x_1), ..., f(x_n)\big) = (0, ..., 0)$ clearly implies $f(x_i) = 0$ for all i, which means that $f = 0$. Thus

$\mathrm{Hom}_R(M,N)$ is isomorphic to a submodule of N^k. But N is noetherian, so N^k is noetherian as well, thus finally $\mathrm{Hom}_R(M,N)$ is noetherian. $\square$

If we drop the hypothesis "M finitely generated", then the statement is not necessarily true even for vector spaces. Consider the following example: let $R = \mathbb{Z}_2$ (which is a field), $M = (\mathbb{Z}_2)^{\mathbb{N}}$ and $N = \mathbb{Z}_2$ (which are vector spaces on $\mathbb{Z}_2$). Of course N is noetherian, since it has no non-trivial submodules. Now take $\mathrm{Hom}_R(M,N)$. Suppose towards a contradiction that $\mathrm{Hom}_R(M,N)$ is noetherian. Then it is a finitely generated vector space on $\mathbb{Z}_2$. Since $\mathbb{Z}_2$ is a finite set, every vector space with a finite basis must be a finite set. Thus $\mathrm{Hom}_R(M,N)$ is a finite set. But this is obviously not true, since $\mathrm{Hom}_R(M,N)$ must contain at least the projections $e_j : (\mathbb{Z}_2)^{\mathbb{N}} \to \mathbb{Z}_2$, which are countably many. $\square$