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Exercise 53. Let R be a UFD. Consider $x = c^{-1}b \in \mathfrak{q}(R)$. Suppose that there exists a polynomial of $R[X]$ which has x as a solution. This means

$$x^n + a_{n-1}x^{n-1} + ... + a_0 = 0$$

for some $a_0, ..., a_{n-1} \in R$. Since R is a UFD, we can suppose that b, c have no non-unit common divisor. Multiply the above equation by c^n to get

$$b^n + a_{n-1}cb^{n-1} + ... + a_0b^n = 0$$

i.e.

$$b^n = -a_{n-1}cb^{n-1} - ... - a_0b^n$$

Let now d be an irreducible divisor of c. Then d is prime since R is a UFD. Now, $d|b^n$ since it divides the member on the right and thus (since d is prime) $d|b$. But b, c have no non-unit common divisors, so d must be a unit. Thus c is a unit as well and hence $x \in R$, as wanted.

Exercise 55. We follow the hint: let $R_1 = K[X]$. So $K[X, Y] = R_1[Y]$. Observe that, since K is a field, $K[X]$ is a PID, and thus a UFD. Thanks to Exercise 53, we have that $R_1 = K[X]$ is integrally closed. Thus we satisfy the hypothesis of Exercise 54, and so there exist $p, q \in R_1[Y]$ s.t. $pf + qg = a \in R_1 = K[X]$. Now suppose $(\alpha, \beta) \in \mathcal{V}_{\overline{K}}(f) \cap \mathcal{V}_{\overline{K}}(g)$. This implies

$$0 = p(\alpha)f(\alpha) + q(\alpha)g(\alpha) = a(\alpha),$$

i.e. α must be a zero of a. By the Foundamental Theorem of Algebra, this means that we have only finitely many choices for α.
The same argument works defining $R_2 = K[Y]$ and considering $K[X, Y] = R_2[X]$. Hence there are only finitely many choices for β as well, whereby there are only finitely many choices for (α, β).

Exercise 56. First, observe that we can see $R/(Q \cap R)$ as a subring of S/Q because

$$\varphi : R \to S/Q, \quad r \mapsto r + Q$$

is a homomorphism and $\ker \varphi = Q \cap R$. Now, let $x + Q \in S/Q$. Since $x \in S$, by hypothesis there exist $a_0, ..., a_{n-1} \in R$ s.t.

$$x^n + a_{n-1}x^{n-1} + ... + a_0 = 0.$$

We trivially have (in S/Q)

$$(x+Q)^n + (a_{n-1}+Q)(x+Q)^{n-1} + ... + (a_0+Q) = 0+Q,$$

so

$$X^n + (a_{n-1}+Q)X^{n-1} + ... + (a_0+Q)$$

is a polinomial in $S/Q[X]$ which has x as a zero. But since $a_0, ..., a_{n-1} \in R$, we have that $a_0+Q, ..., a_{n-1}+Q \in \varphi[R]$, i.e. we can see them as elements of $R/(Q \cap R)$, which means that the polynomial is actually in $R/(Q \cap R)[X]$.