Exercise 27.

Proof.
(b)$\Rightarrow$(a) Suppose $M = M_1 \oplus M_2$. Then $e_1 \in \mathrm{End}_R(M)$ given by $e_1(x_1, x_2) = (x_1, 0)$ is a non trivial idempotent endomorphism.
(a)$\Rightarrow$(b) Let $f \in \mathrm{End}_R(M)$ be a non-trivial idempotent endomorphism of M. Consider the exact sequence

$$0 \to \ker f \hookrightarrow M \xrightarrow{f} f[M] \to 0.$$

Consider the inclusion $i : f[M] \to M$. Since f is idempotent, we have that $f \circ i(f(m)) = f(f(m)) = f(m)$, that is $f \circ i = \mathrm{id}_{f[M]}$. By splitting lemma, we obtain $M \simeq f[M] \oplus \ker f$.
Now we want to show that $f \neq 0, 1 \Rightarrow f[M] \neq \{0\} \wedge \ker f \neq \{0\}$. If $f[M] = \{0\}$, of course $f = 0$. If $\ker f = \{0\}$, then f is injective. But since $f(f(m)) = f(m)$ for all $m \in M$, this means that $f(m) = m$ for all $m \in M$, thus $f = 1$. □

Exercise 29. Consider the series of the exercise. If we prove that the composition factors are $(\mathbb{Z}/p_1\mathbb{Z}, ..., \mathbb{Z}/p_r\mathbb{Z})$, then the series is a composition series, since every $\mathbb{Z}/p_1\mathbb{Z}$ is simple because every submodule is also a subgroup, and the only subgroups of $\mathbb{Z}/p\mathbb{Z}$ with p prime are trivial (since the order of any subgroup must divide p).

Claim. For all i, $p_1...p_i\mathbb{Z}/n\mathbb{Z} \simeq (p_1...p_i\mathbb{Z})/n\mathbb{Z}$, and there exists an isomorphism φ such that $\varphi[p_1...p_ip_{i+1}\mathbb{Z}/n\mathbb{Z}] = (p_1...p_ip_{i+1}\mathbb{Z})/n\mathbb{Z}$.
Proof of the claim. Consider the function $\varphi : p_1...p_i\mathbb{Z}/n\mathbb{Z} \to (p_1...p_i\mathbb{Z})/n\mathbb{Z}$ given by $p_1...p_i[x] \mapsto [p_1...p_ix]$. This is trivially a well-defined isomorphism, since it's precisely the definition of scalar multiplication in the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$. □

So now we have that, for all $i = \{1, ..., r\}$,

$$\begin{aligned} p_1...p_i\mathbb{Z}/n\mathbb{Z}/p_1...p_ip_{i+1}\mathbb{Z}/n\mathbb{Z} &\simeq (p_1...p_i\mathbb{Z})/n\mathbb{Z}/(p_1...p_ip_{i+1}\mathbb{Z})/n\mathbb{Z} \\ &\simeq p_1...p_i\mathbb{Z}/p_1...p_ip_{i+1}\mathbb{Z} \simeq \mathbb{Z}/p_{i+1}\mathbb{Z}, \end{aligned}$$

where the first "equality" of the second line holds thanks to the third isomorphism theorem, and the last "equality" holds thanks to the following isomorphism:

$$\varphi : p_1...p_i\mathbb{Z}/p_1...p_ip_{i+1}\mathbb{Z} \to \mathbb{Z}/p_{i+1}\mathbb{Z},\ [p_1...p_ik] \mapsto [p_1...p_ik].$$

φ is well-defined, since $[p_1...p_ik] = [p_1...p_il]$ implies $p_1...p_ik - p_1...p_il = p_1...p_ip_{i+1}m$ for some $m \in \mathbb{Z}$, which means $k - l = p_{i+1}m$, i.e. $[k]_{\mathbb{Z}/p_{i+1}\mathbb{Z}} = [l]_{\mathbb{Z}/p_{i+1}\mathbb{Z}}$. The surjectivity is totally trivial by the way φ is defined, and for the injectivity just observe that $p_1...p_ik + p_{i+1}\mathbb{Z} = 0 \Rightarrow p_{i+1}|k \Rightarrow p_1...p_ip_{i+1}|p_1...p_ik$.