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Exercise 48.

1. We have to check that I is closed under addition and multiplication by elements of R. The closure under addition is trivial since I is an ideal. The closure under multiplication by elements of R follows by the following:

$$\lambda \in R, x \in I \Rightarrow \lambda x = \lambda(1_A x) = (\lambda 1_A)x$$

where the last item is in I since $\lambda 1_A$ is an element of A and thus $(\lambda 1_A)x \in I$ since I is an ideal of A.

2. A is a noetherian R-module because it is surjective image of $\oplus_{i=1}^n R$ for some $n \in \mathbb{N}$ (trivial), which is noetherian since R is noetherian, and by theory we know that finite direct sum of noetherian modules is noetherian and that surjective image of a noetherian module is noetherian. We want now to prove that A is a noetherian ring, that is, A is a noetherian A-module. The A-submodules of A are the ideals of A. But thanks to the first point, these are also R-modules, thus any ascending chain of them must stabilize.

Exercise 49. $\mathbb{Z}$ is a reduced ring (every integral domain is trivially a reduced ring).

1. Let $[x] \in R/I$ such that $[x]^n = 0$ for some $n \in \mathbb{N}$. This means $x^n \in I$, but since I is radical we have $x \in I$, that is $[x] = 0$.

2. Thanks to the first point, it suffices to prove that $\mathcal{I}(V)$ is radical. But this has already been proven in the proof of Hilbert's Nullstellenstatz (proof: If $f \in K[\mathbf{X}]$ with $f^n \in \mathcal{I}(V)$, then $f^n(\underline{a}) = 0$ for all $\underline{a} \in V$, and thus $f(\underline{a}) = 0$ for all $\underline{a} \in V$, since L is a field and thus a domain).

Exercise 50.

1. We know by Proposition 3.23 that

$$I = \sqrt{I} = \bigcap_{\mathfrak{p} \in \nu(I)} \mathfrak{p},$$

where $\nu(I) := \{\mathfrak{p} \in \mathrm{Spec}(I) \mid \mathfrak{p} \supseteq I\}$. But by Theorem 3.18 (whose proof seems to work even getting rid of the "domain" hypothesis) we know that every noetherian ring has only a finite number of prime ideals. Therefore $\nu(I)$ is finite, and we are done.

2. Of course we have that
$$\mathfrak{p}_1 \cap ... \cap \mathfrak{p}_r \subseteq \mathfrak{q}_k$$
for all $k = 1..s$. By Theorem 3.4, this means that $\mathfrak{p}_i \subseteq \mathfrak{q}_k$ for some $i = 1..r$. Repeating the same argument, we obtain that for all $i = 1..r$ there exists $k = 1..s$ such that $\mathfrak{q}_k \subseteq \mathfrak{p}_i$. We must have that the equality $\mathfrak{p}_i = \mathfrak{q}_k$ holds. In fact, suppose to the contrary that $\mathfrak{p}_i \subseteq \mathfrak{q}_k$ and $\mathfrak{q}_{k'} \subseteq \mathfrak{p}_i$. Then $\mathfrak{q}_{k'} \subseteq \mathfrak{q}_k$, which contradicts the hypothesis. So we are done.

Exercise 51.
(a)$\Rightarrow$(b): By contraposition, suppose there exist $f, g \notin \mathcal{J}(V)$ with $fg \in \mathcal{J}(V)$. Consider $\mathcal{J}(V) + (f) \supsetneq \mathcal{J}(V)$ and $\mathcal{J}(V) + (g) \supsetneq \mathcal{J}(V)$, and define $V_1 := \mathcal{V}_L(\mathcal{J}(V) + (f))$, $V_2 := \mathcal{V}_L(\mathcal{J}(V) + (g))$. Of course $V_1, V_2 \subsetneq V$. But $V = V_1 \cup V_2$, because (since $fg \in \mathcal{J}(V)$) we have $x \in V \Rightarrow fg(x) = 0 \Rightarrow f(x)g(x) = 0 \Rightarrow f(x) = 0$ or $g(x) = 0 \Rightarrow x \in V_1$ or $x \in V_2$.
(b)$\Rightarrow$(a): By contraposition, suppose $V = V_1 \cup V_2$ with $V_1, V_2 \neq V$ varieties. Then of course[1] $\mathcal{J}(V) \subsetneq \mathcal{J}(V_1)$ and $\mathcal{J}(V) \subsetneq \mathcal{J}(V_2)$, that is, there exist $f \in \mathcal{J}(V_1) \setminus \mathcal{J}(V)$ and $g \in \mathcal{J}(V_2) \setminus \mathcal{J}(V)$. But fg vanishes on $V = V_1 \cup V_2$, so $fg \in \mathcal{J}(V)$, i.e. $\mathcal{J}(V)$ is not prime.

Exercise 52.

1. TODO

2. We have $\mathcal{J}(V) = \mathcal{J}(V_1 \cup ... \cup V_s = \mathcal{J}(V_1) \cap ... \cap \mathcal{J}(V_s)$. $\mathcal{J}(V_i)$ is a prime ideal for all $i = 1..s$ thanks to Exercise 51, and thus $\mathcal{J}(V_i) = \mathfrak{p}_i$ (after renumbering if necessary) thanks to Exercise 50.2. Therefore $V_i = \mathcal{V}_L(\mathcal{J}(V_i)) = \mathcal{V}_L(\mathfrak{p}_i)$.

[1] If $\mathcal{J}(V) = \mathcal{J}(V_1)$, then $V = \mathcal{V}_L(\mathcal{J}(V)) = \mathcal{V}_L(\mathcal{J}(V_1)) = V_1$.