Exercise 21.

Proof. Let $M = \langle x_1, ..., x_n \rangle$. Consider the morphism $\varphi : R \to M^n$ given by $r \mapsto (rx_1, ..., rx_n)$. It is immediate to check that $\ker \varphi = \operatorname{Ann}_R M$. So $\varphi(R) \simeq R/\operatorname{Ann}_R M$. But $\varphi(R)$ is a submodule of M^k, which is noetherian by theory. Thus $\varphi(R)$ is noetherian as well, and we are done. $\square$

Exercise 22.

Proof.

1. Let $M_{\text{tor}} = \{x \in M \mid \operatorname{Ann}_R(x) \neq 0\}$. Let $x, y \in M_{\text{tor}}$. Then there exist $\lambda, \mu \in R$ s.t. $\lambda x = \mu x = 0$. Thus $\lambda\mu(x+y) = 0$, i.e. $x + y \in M_{\text{tor}}$. The closure under scalar multiplication is trivial.

2. Suppose $M = \langle x_1, ..., x_n \rangle$. By hypothesis, there exist $\lambda_1, ..., \lambda_n \in R$ s.t. $\lambda_i x_i = 0$ for all i. Thus, for every $x \in M$, $(\lambda_1 \cdot ... \cdot \lambda_n)x = \lambda_1 \cdot ... \cdot \lambda_n(a_1x_1 + ... + a_nx_n) = 0$.

3. Let $(x_1, ..., x_n)$ be a basis for M. Let $0 \neq x = a_1x_1 + ... + a_nx_n$. Suppose $0 = \lambda x = \lambda(a_1x_1 + ... + a_nx_n)$. Since $(x_1, ..., x_n)$ is a basis, this means $\lambda a_i = 0$ for all a_i. By hypothesis, $x \neq 0$, so at least one a_k is not 0. Since R is a domain, we have $\lambda = 0 \vee a_k = 0$, thus $\lambda = 0$. MANCA IL CONTROESEMPIO!!!

$\square$

Exercise 23.

Proof. First of all, we prove that $\{a \in S \mid aS \subseteq R\} = \operatorname{Ann}_R(S/R)$. The inclusion $\subseteq$ is trivial consequence of the definition of the factor ring. For the inclusion $\supseteq$, observe that $a(x + R) = 0 \Rightarrow ax + R = 0 \Rightarrow ax \in R$.
It is immediate to show that $\mathfrak{f}_{S/R}$ is an ideal of S and R. Consider now an ideal I of S which is also an ideal of R. Since every ideal is closed under multiplication of elements of the ring, we have that, for all $a \in I$ and $x \in S$, $ax \in I \subseteq R$, thus $aS \subseteq R$. That is, $I \subseteq \mathfrak{f}_{S/R}$. $\square$

Exercise 24.

Proof. Suppose $S = \langle x_1, ..., x_n \rangle$. Since $S \subseteq q(R)$, we must have $x_i = a_i b_i^{-1}$ for some $a, b \in R$. Consider an arbitrary $x \in S$, that is $x = \lambda_1 a_1 b_1^{-1} + ... + \lambda_n a_n b_n^{-1}$, with $\lambda_1, ..., \lambda_n \in R$. Then obviously $(b_1 \cdot ... \cdot b_n)x \in R$. Thus $b_1 \cdot ... \cdot b_n \in \mathfrak{f}_{S/R}$. $\square$

Exercise 26.

Proof. Follows trivially by Corollary 2.36. $\square$