Exercise 10. Let G be an abelian simple group. Then G is cyclic and has prime order.

Dimostrazione. G is abelian, therefore every subgroup is normal. Then, since G is simple by hypothesis, $\{1\}$ and G are its only subgroups. G is cyclic because otherwise there would exist an element $x \in G$ such that $\langle x \rangle \neq \{1\}$ is a proper subgroup of G. Let's consider a generator g. Furthermore, the order must be finite, since otherwise $\langle g^2 \rangle$ would be a proper subgroup of $\langle g \rangle = G$. Finally, suppose towards a contradiction that G has order $n = pn$ with p prime. Then $\langle g^p \rangle$ would be a proper subgroup of $\langle g \rangle$. □

Exercise 11. Let M be an R-module and $R_M = \{\lambda \cdot \mathrm{id}_M \mid \lambda \in R\} \subseteq \mathrm{End}_R(M)$. Then $R_M \cong R/\operatorname{Ann}_R(M)$.

Dimostrazione. Consider $\varphi : R_M \to R/\operatorname{Ann}_R(M)$, $\lambda \cdot \mathrm{id}_M \mapsto [\lambda]$. φ is trivially an epimorphism and it's also injective since

$$[\lambda_1] = [\lambda_2] \Rightarrow \lambda_1 = \lambda_2 + r \Rightarrow \lambda_1 \cdot \mathrm{id}_M = (\lambda_2 + r) \cdot \mathrm{id}_M = \lambda_2 \cdot \mathrm{id}_M + r \cdot \mathrm{id}_M = \lambda_2 \cdot \mathrm{id}_M$$

□

Exercise 12. Let M be an R-module.

1. M is simple iff $M = Rx$ for all $0 \neq x \in M$. (**Observation:** this means that every simple module is cyclic.)
2. Give an example of a cyclic module which is not simple.
3. If M is simple, then $\mathrm{End}_R(M)$ is a division ring.

Dimostrazione.

1. Trivial.
2. $M = \mathbb{Z}$ is a cyclic $\mathbb{Z}$-module but $2\,\mathbb{Z}$ is a proper submodule.
3. Thanks to the first point, if $0 \neq x \in M$ then $Rx = M$. Therefore, if $0 \not\equiv f \in \mathrm{End}_R(M)$ we have $Rf(x) = M$, i.e.

   $$\forall x \in M \exists \lambda_x \in R[\lambda_x f(x) = x].$$

   Defining $g(x) := \lambda_x \,\mathrm{id}_M$ we obtain $g \circ f = f \circ g = \mathrm{id}_M$.

□

Exercise 13. Let $n \in \mathbb{N}$.

1. $\operatorname{End}_R(R^n) \cong \mathcal{M}_n(R)$.
2. $p \in \mathbb{N}$ prime $\Rightarrow \operatorname{End}_{\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^n) \cong \mathcal{M}_n(\mathbb{Z}/p\mathbb{Z})$.
3. $\operatorname{End}_R(M^n) \cong \mathcal{M}_n(\operatorname{End}_R(M))$.

Dimostrazione.

1. Consider $\varphi : \operatorname{End}_R(R^n) \to \mathcal{M}_n(R)$, $f \mapsto (a_{i,j})_{i,j}$ where $a_{i,j} = (\pi_i \circ f)(e_j)$. It is immediate to check that φ is a morphism. Consider now $\psi : \mathcal{M}_n(R) \to \operatorname{End}_R(R^n)$, $(a_{i,j})_{i,j} \mapsto f$ where $f(\lambda_1 e_1 + ... + \lambda_n e_n)_i = \lambda_1 a_{i,1} + ... + \lambda_n a_{i,n}$. It's easy to check that $\psi = \varphi^{-1}$.

2. Follows immediatly from the last point and the following trivial observation:
$$\operatorname{End}_{\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^n) \cong \operatorname{End}_{\mathbb{Z}/p\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^n)$$

3. Consider $\varphi : \operatorname{End}_R(M^n) \to \mathcal{M}_n(\operatorname{End}_R(M))$, $f \mapsto (f_{i,j})_{i,j}$ where $f_{i,j} : M \to M$, $x \mapsto \pi_i(f(0e_0 + 0e_1, ..., +xe_j, ..., 0e_n))$.

$\square$

Exercise 14. Let $I_1, ..., I_n \lhd R$ be ideals of R and let M be an R-module. TFAE:

(a) $M \cong R/I_1 \oplus ... \oplus R/I_n$.

(b) There exist cyclic submodules $C_1, ...C_n \subseteq M$ s.t. $\operatorname{Ann}_R(C_j) = I_j$ and $M = \oplus_{j=1}^n C_j$.

Dimostrazione.
(a) $\Rightarrow$ (b): R/I_j is a submodule of M. We want to show that $\operatorname{Ann}(R/I_j) = I_j$. The inclusion $\supseteq$ is immediate. For the other inclusion, observe that
$$\forall [x](\lambda[x] = [0]) \Rightarrow \lambda[1] = [0] \Rightarrow [\lambda] = [0] \Rightarrow \lambda \in I_j.$$
Finally, $R/I_j = \{[\lambda] \mid \lambda \in R\} = \{\lambda[1] \mid \lambda \in R\} = \langle [1] \rangle$, i.e. is cyclic. Therefore defining $C_j := R/I_j$ we are done.
(b) $\Rightarrow$ (a): Let $g_j \in C_j$ be a generator of C_j. Consider $\varphi : C_j \to R$, $\lambda g_j \mapsto \lambda$. φ is clearly an epimorphism. If we can prove $\ker \varphi = \operatorname{Ann}_R(C_j) = I_j$, we obtain $C_j \cong R/I_j$ and so we are done. Thus observe
$$\lambda \in \operatorname{Ann}_R(C_j) \Rightarrow \lambda g_j$$
and viceversa
$$\lambda g_j = 0 \wedge c_j \in C_j \Rightarrow \lambda g_j = 0 \wedge c_j = \mu g_j \Rightarrow \lambda c_j = \lambda(\mu g_j) = \mu(\lambda g_j) = 0 \Rightarrow \lambda \in \operatorname{Ann}_R(C_j).$$

$\square$

Exercise 15. Let M be an R-module, $\sim$ a congruence relation on M and $N = [0]_\sim$. Then there is a uniquely determined R-modulo structure on M/N such that $\pi : M \to M/N$ is a R-epimorphism. We have that the scalar multiplication $R \times M/N \to M/N$ is given by $(\lambda, [a]) \mapsto [\lambda a]$.

Dimostrazione. Let $\alpha, \beta \in M/N$. We can write them as $\alpha = [a]$ and $\beta = [b]$ for some $a, b \in M$. By definition we have that $\pi(a + b) = [a + b]$ and by hypothesis we need to have $\pi(a + b) = \pi(a) + \pi(b) = [a] + [b]$. So $[a] + [b] = [a + b]$. Similarly, by definition $\pi(\lambda a) = [\lambda a]$, and by hypothesis $\pi(\lambda a) = \lambda\pi(a) = \lambda[a]$, so $\lambda[a] = [\lambda a]$. $\square$