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Exercise 44.
(a)⇒(b): Let $\mathfrak{p}$ be the unique prime ideal of R. Since every ring has a maximal ideal, and every maximal ideal is prime, $\mathfrak{p}$ must be that maximal ideal. Thus $R/\mathfrak{p}$ is a field. So, for any $x \in R \setminus \mathfrak{p}$, we must have $[x][y] = [1]$ for some $[y] \in R/\mathfrak{p}$, which means that $xy \in 1 + \mathfrak{p}$. Furthermore, $\mathfrak{p} = \sqrt{0}$, since $\sqrt{0}$ is the intersection of all the prime ideals, and $\mathfrak{p}$ is the only one. This means that $xy = 1 + z$ where z is nilpotent. By the fourth remark after the definition of $\mathcal{J}(R)$ we know that

$$\mathcal{J}(R) = \{x \in R \mid 1 + Rx \subseteq R^{\times}\}.$$

But we know (immediate by definitions) that $\sqrt{0} \subseteq \mathcal{J}(R)$, so it must be true that $1 + Rz \subseteq R^{\times}$, thus $1 + z \in R^{\times}$. Therefore xy is invertible, which implies that x is invertible ($y(y^{-1}x^{-1})$ is the inverse of x).
(b)⇒(a): Suppose that every element is either a unit or nilpotent. Consider $\sqrt{0}$. $\sqrt{0}$ is maximal, since every element in $R \setminus \sqrt{0}$ is a unit (and thus generates the whole ring R). Hence $\sqrt{0}$ is prime as well. But $\sqrt{0}$ is the intersection of all the prime ideals of R, and since it is maximal it must be the unique prime ideal of R.

Exercise 45.

1. We know by the second remark after the definition of artinian rings that in artinian rings every prime ideal is maximal. Thus $\mathcal{J}(R) = \sqrt{0}$ trivially by definition.

2. Let $\Omega := \{\mathfrak{m}_1 \cap ... \cap \mathfrak{m}_n \mid n \in \mathbb{N}, \mathfrak{m}_i \in \max(R)\}$. $\Omega \neq \emptyset$, since every ring has a maximal ideal. But we know by theory that a ring is artinian iff every non-empty family of ideals contains a minimal element[1]. Hence there exists $\mathfrak{m}_1 \cap ... \cap \mathfrak{m}_n \in \Omega$ which is minimal. Now take any maximal ideal $\mathfrak{m}$. By minimality we must have $\mathfrak{m} \cap \mathfrak{m}_1 \cap ... \cap \mathfrak{m}_n = \mathfrak{m}_1 \cap ... \cap \mathfrak{m}_n$, and therefore $\mathfrak{m}_1 \cap ... \cap \mathfrak{m}_n \subseteq \mathfrak{m}$. Since $\mathfrak{m}$ is prime as well, by Theorem 3.4 we obtain $\mathfrak{m}_i \subseteq \mathfrak{m}$ for some $i = 1..n$. Since $\mathfrak{m}_i$ is maximal, this means $\mathfrak{m}_i = \mathfrak{m}$. Hence the $\mathfrak{m}_i$'s are the only maximal ideals of R.

[1] Proof: immediate by Zorn's lemma, ordering the family by reverse inclusion.