Consistency results concerning ω_1-trees

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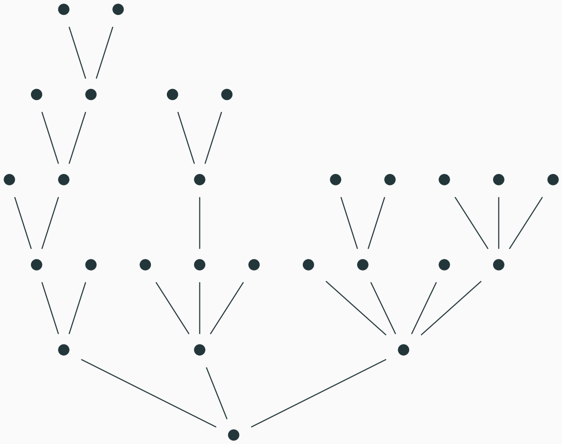
“Short” trees

Definitions of tree

How can we define trees?



Definitions of tree

Definition 1

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"When set theorists say large, they **mean** *large."*

(Scott Aaronson)

Definitions of tree

We will see three definitions in total:

- Definition 1. Easy, not general.
- Definition 2. Quite easy, not general.
- Definition 3. Less easy, very general.

Ordered sets

Definition

Let A be a set. A **partial order** "$<$" on A is a binary relation between elements of A which is:

(i) irreflexive, i.e. $x \not< x$;

(ii) transitive, i.e. $x < y$ and $y < z$ imply $x < z$.

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Example

Let S be the set of all the finite binary strings. Define the relation "$<$" as follows:

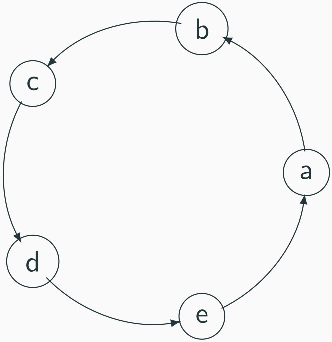
$$s < t \text{ if and only if } s \text{ is shorter than } t.$$

Then $<$ is a partial order on S. Observe that if two strings have the same length, then they are not comparable.

Ordered sets

Observation

There are no "cycles" in partially ordered sets!



If $a < e$ and $e < a$, then $a < a$. Impossible!

Ordered sets

Definition

A **linear order** (or **total order**) “$<$” on a set A is a partial order such that every two elements of A are comparable, i.e.

$$\text{for all } x, y \in A: \quad x < y \text{ or } y < x \text{ or } x = y.$$

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Example

The canonical order on $\mathbb{N}$ and the canonical order on $\mathbb{R}$.

Trees of height $\leq \omega$

We can now give the simplified definition of tree.

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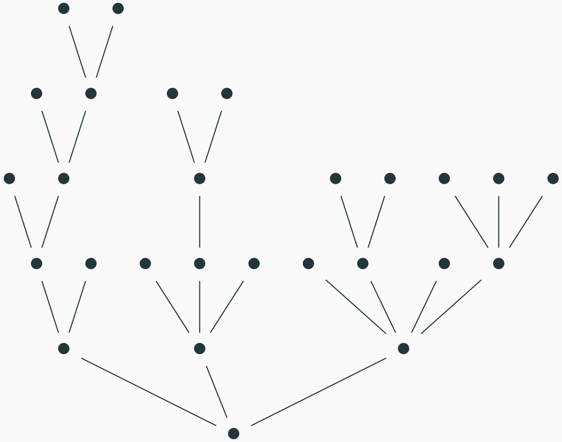
Definition 2

A **tree** is a partially ordered set $(T, <)$ such that, for each $x \in T$, the set

$$\downarrow x := \{y \in T : y < x\}$$

is finite and linearly ordered by $<$.

Trees of height $\leq \omega$



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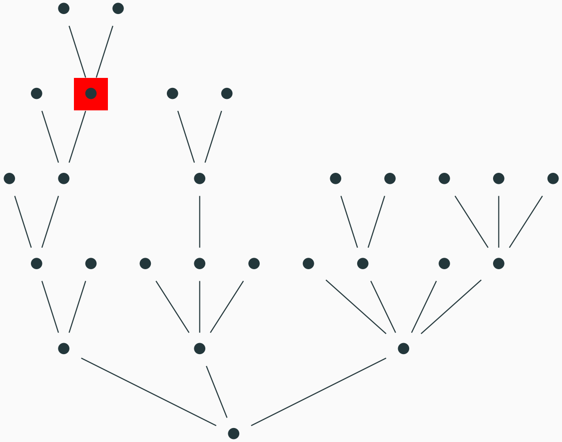
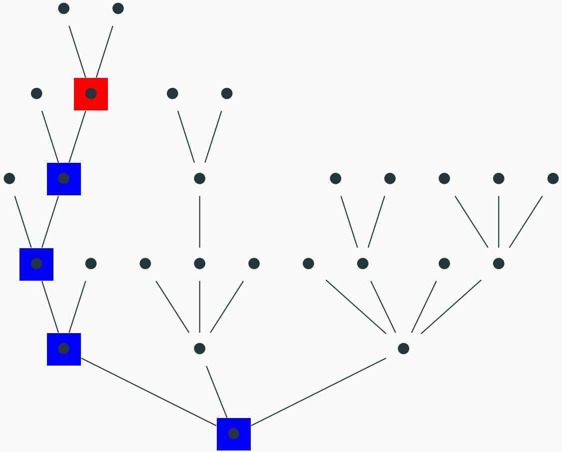


$\times$

Trees of height $\leq \omega$



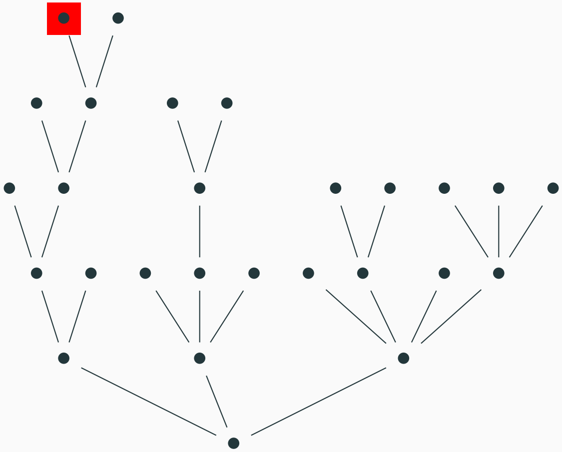
x, $\downarrow x$

Trees of height $\leq \omega$

How tall can a tree be, according to Definition 2?

Let T be a tree. We can define the **height** of a node in T and the n-**th level** of T in the obvious way. Moreover, if the maximum height of every node in T is n, then we say that T has height $n+1$.

Trees of height $\leq \omega$



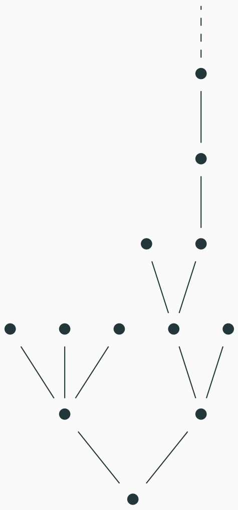
x is a highest node, and height$(x) = 5$. So the tree has height 6.

Trees of height $\leq \omega$

But the nodes of T could also be arbitrarily large, meaning that their heights are not bounded by any fixed amount. So T has **infinite** height. More precisely, we say that T has height $\mathbb{N}$. Actually, in this situation we prefer to indicate $\mathbb{N}$ by ω. So we say that T has height ω.

If T has finite height, then we say that T has height $< \omega$.

A simple tree of height ω



The tree FST

Recall that ω is just $\mathbb{N}$.

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The set of **finite binary strings** is

$$2^{<\omega} := \{s \mid s\colon \{0, \ldots, n-1\} \to \{0,1\} \text{ for some } n < \omega\}.$$

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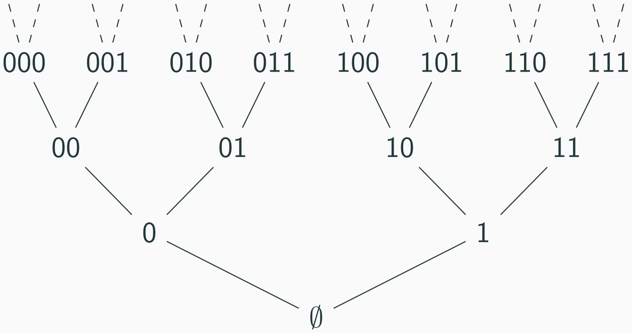
Then we obtain a binary tree FST (*finite strings tree*).



Branches

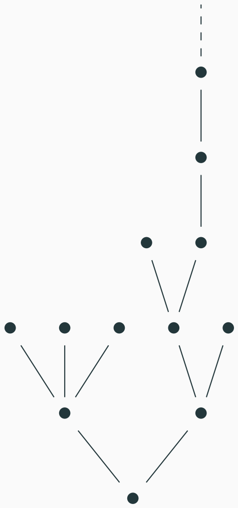
The "classical" definition of branch is: a path from the root to a leaf.

In our new context, the idea is the same, but the formal definition is different.

Definition

A **branch** in a tree T is a *maximal linearly ordered subset* of T. (i.e. a subset of T which is linearly ordered and which is not properly contained in any other linearly ordered subset)

Branches: examples



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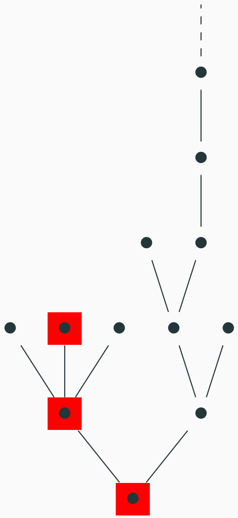
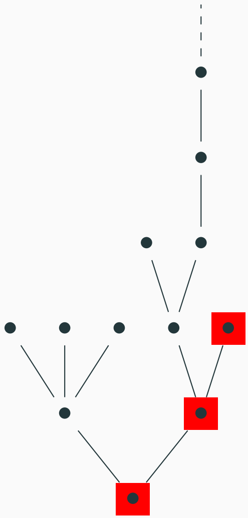


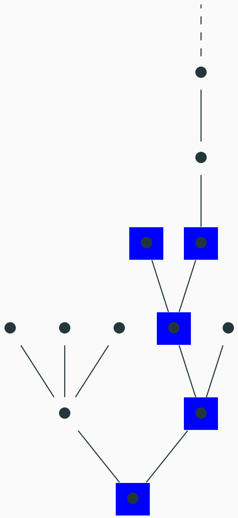
A branch.

Branches: examples



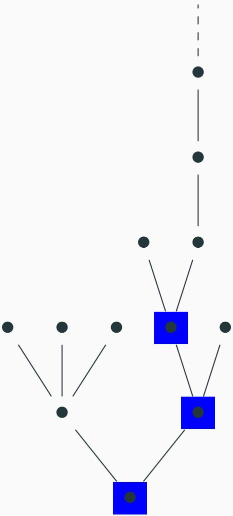
Another branch.

Branches: examples



Not a branch (not linearly ordered).

Branches: examples



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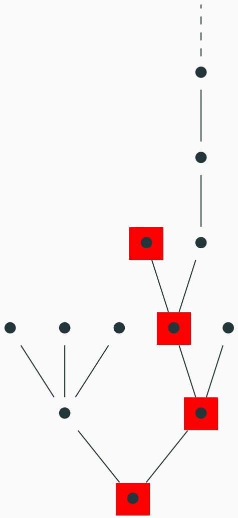
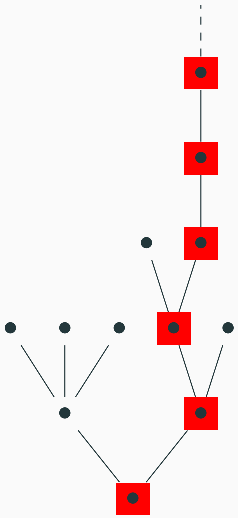
Branches: examples



Not a branch (too small). Contained in this.

Branches: examples



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Cofinal branches

It is immediate to define the height of a branch, in the same way we did for trees.

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We are interested in "special" branches.

Definition

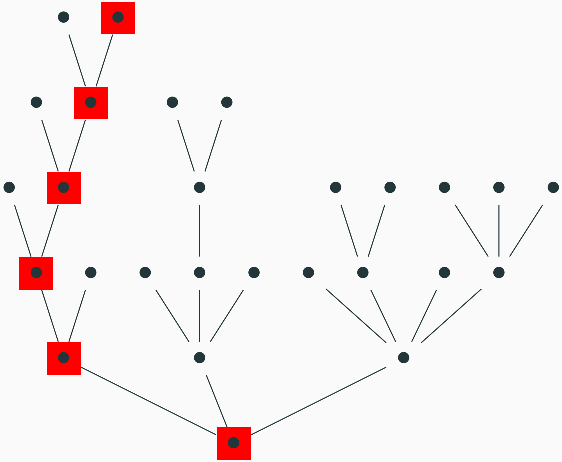
A **cofinal** branch of a tree T is a branch which has the same height of T.

Cofinal branches

In nature, every tree has a "cofinal" branch.

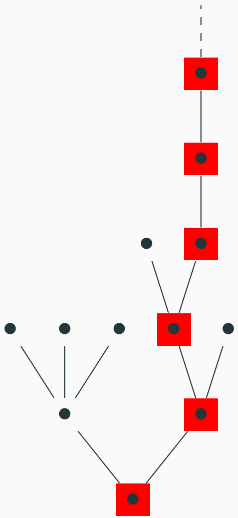
Cofinal branches

This is clearly true also for trees of finite height. If a tree has height (e.g.) 6, then there is a node of height 5, i.e. a branch of height 6.



Cofinal branches

Some trees of height ω have cofinal branches.



Cofinal branches

Does *every* tree have a cofinal branch?

The answer is:

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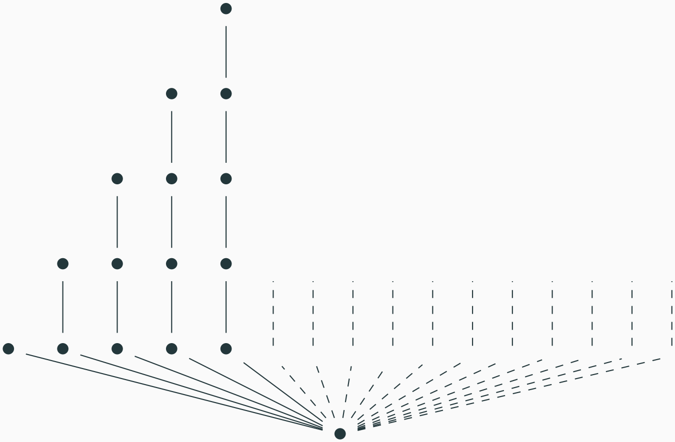


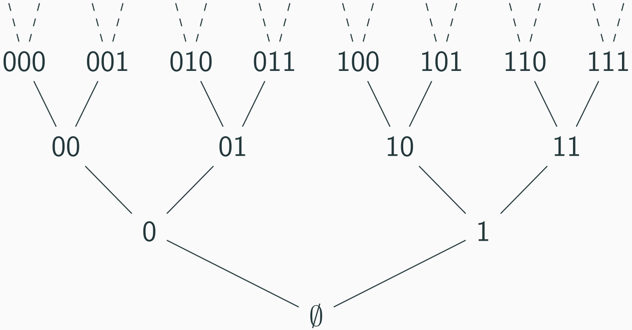
Cofinal branches

Question

Is there any sufficient condition for a tree of height ω to have a cofinal branch?

Back to FST

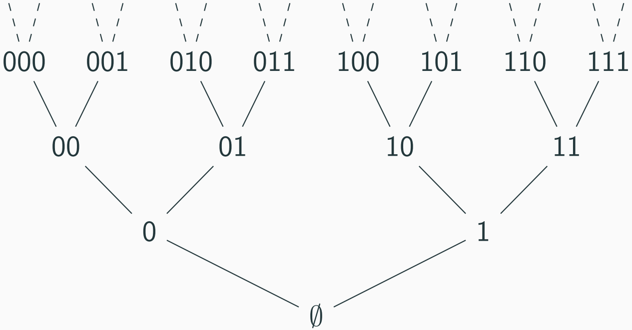
In FST, the height of each node corresponds to its length as string. Hence the n-th level contains exactly the strings in FST of length n.



Back to FST

So:

- FST has height ω;
- the levels of FST are all finite sets;
- $\{0^n : n < \omega\}$ is a cofinal branch (where 0^n means $\overbrace{0000\ldots}^{n}$).
  $\{1^n : n < \omega\}$ as well.



König's lemma

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Question

Is it possible to generalize König's lemma to cardinals greater than ω?

Cardinals

Cardinalities

Let A be a set. The **cardinality** of A is *the number of elements which belong to* A.

But this definition works only for finite sets. For the general case, we say that:

Definition

Two sets A and B have the same cardinality if and only if there exists a bijection between A and B (i.e. a one-to-one correspondence between the elements of A and B).

A set is called **countable** if it has the same cardinality of $\mathbb{N}$. The cardinality of $\mathbb{N}$ is indicated by ω, and it is the *smallest infinite cardinal*. A set A is called **uncountable** if its cardinality is greater than ω (i.e. there exists no function from $\mathbb{N}$ **onto** A).
Example: $\mathbb{R}$ is uncountable. (*Cantor's diagonal argument*)

Cardinalities

For every cardinal κ, there is a smallest cardinal greater than κ. We indicate such cardinal with κ^+.

So ω^+ is the smallest uncountable cardinal. The cardinal ω^+ is usually called ω_1.

BIG trees

Well-ordered sets

To deal with trees of arbitrary height, we first need to generalize Definition 2. In order to do this, we need to define well-ordered sets.

Definition

A partial order $(P, <)$ is a **well-order** if it is linear and every non-empty subset of P has a minimum.

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Definition

A partial order $(P, <)$ is a **well-order** if it is linear and every non-empty subset of P has a minimum.

Examples

- The canonical order on $\mathbb{N}$ is a well-order.
- The canonical order on $\mathbb{Z}$ is *not* a well-order.
- The canonical order on $\mathbb{R}$ is *not* a well-order.

Definition 3

A **tree** is a partially ordered set $(T, <)$ such that, for each $x \in T$, the set

$$\downarrow x := \{y \in T : y < x\}$$

is ~~finite and linearly ordered~~ well-ordered by $<$.

Trees

Definition 3

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The definitions of level and height of a node/tree/branch can be easily generalized to work also with Definition 3.

Attempt of generalization

Lemma (König, 1927)

Let T be a tree of height ω. If the levels of T are finite sets, then T has a cofinal branch.

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Conjecture

Let T be a tree of height ω_1. If the levels of T are countable sets, then T has a cofinal branch.

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The generalization of König's lemma for ω_1 would be:

Conjecture

Let T be a tree of height ω_1. If the levels of T are countable sets, then T has a cofinal branch.

Question

Is the conjecture true?

A tree of height ω_1

Recall that ω_1 is the smallest cardinal greater than ω.

A tree of height ω_1

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Consider the tree $(\mathsf{CST}, <)$ of **countable binary strings**, i.e.

$\mathsf{CST} := \{s \mid s\colon \alpha \to \{0,1\}$ with $\alpha < \omega_1$ and s has finitely-many 1's$\}$

and $s < t$ iff s is a prefix of t.

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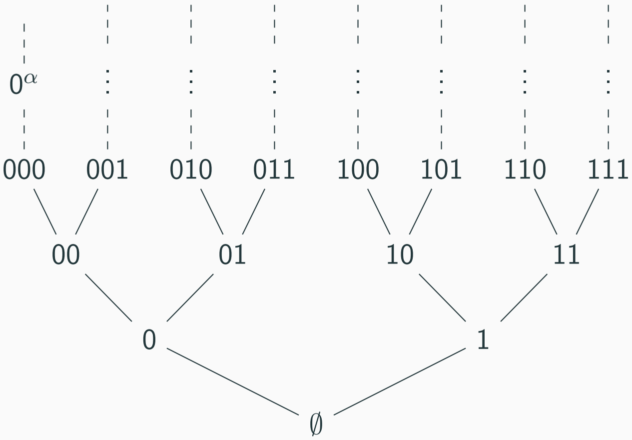
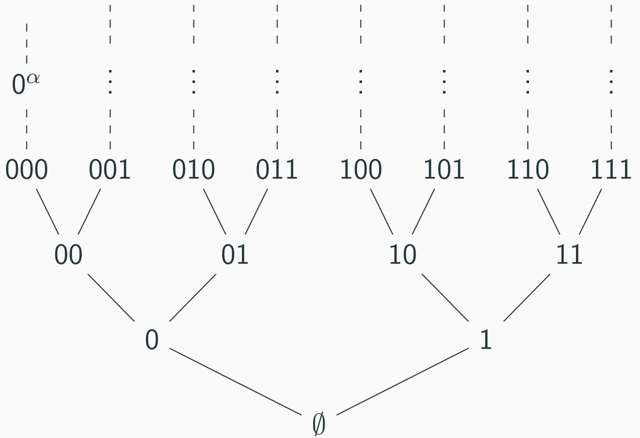


A tree of height ω_1

Observe that:

- CST has height ω_1;
- the levels of CST are all countable;
- $\{0^\alpha : \alpha{<}\omega_1\}$ is a (cofinal) branch in CST. $\{1^\alpha : \alpha{<}\omega_1\}$ is <u>not</u>.



Aronszajn trees

Conjecture

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Aronszajn trees

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Let T be a tree of height ω_1. If the levels of T are countable sets, then T has a cofinal branch.

The conjecture is **false**.

Theorem (Aronszajn, 1934)

There exists an Aronszajn tree, i.e. a tree of height ω_1 whose levels are countable, which has no cofinal branch.

Suslin's hypothesis

Antichains

Let's try to strengthen the assumptions in our conjecture to see if we get at least something...

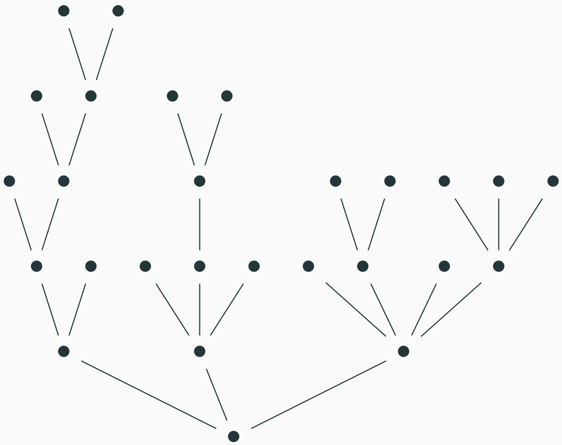
Definition

A subset A of a tree is an **antichain** if every two nodes in A are incomparable, i.e.

$$\text{for all } x, y \in A: \quad x \not< y \text{ and } y \not< x.$$

Every level is trivially an antichain.

Antichains



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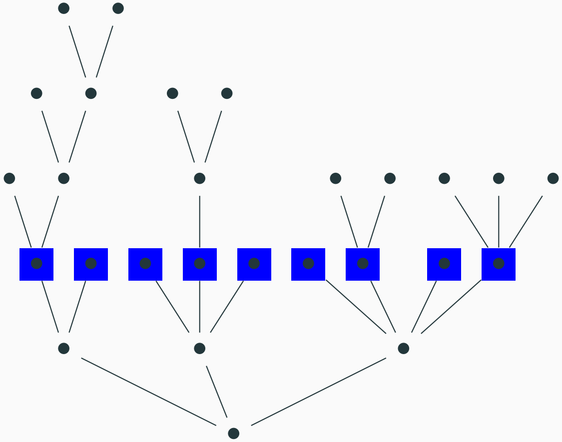

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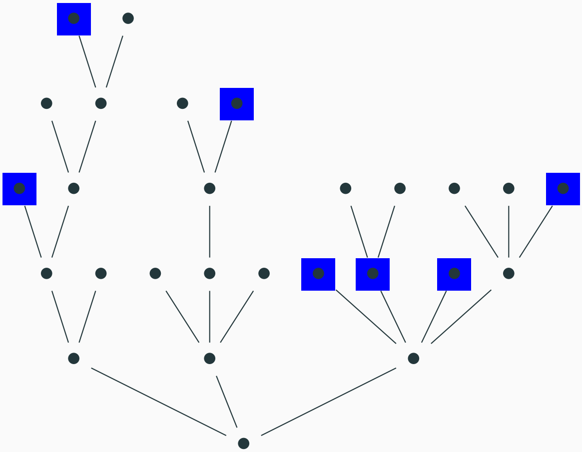

Suslin trees

Conjecture (FALSE)

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Suslin trees

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Suslin's hypothesis

Let T be a tree of height ω_1. If the ~~levels~~ antichains of T are countable sets, then T has a cofinal branch.

Definition

A **Suslin tree** is a tree of height ω_1 whose antichains are countable, but which has no cofinal branch.

Suslin trees

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Suslin's hypothesis

Let T be a tree of height ω_1. If the ~~levels~~ antichains of T are countable sets, then T has a cofinal branch.

Definition

A **Suslin tree** is a tree of height ω_1 whose antichains are countable, but which has no cofinal branch.

So Suslin's hypothesis is true if and only if there exists **no** Suslin tree.

Suslin trees

Question

Is there any Suslin tree?

The answer is:

Suslin trees

Question

Is there any Suslin tree?

The answer is: *There is no answer.*

Suslin trees

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Suslin trees

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The answer is: *There is no answer.* And we can prove that there is no answer.

In other words, the existence of Suslin trees is not provable nor refutable in ZFC, the system of *standard axioms* for mathematics. We say that it is **independent** of ZFC. More precisely:

Theorem (Tennenbaum, 1963)

There is a "universe" (i.e. a model of ZFC*) in which there exists a Suslin tree.*

Theorem (Solovay-Tennenbaum, 1971)

There is a "universe" (i.e. a model of ZFC*) in which there exists no Suslin tree.*

Forcing

The proofs of both theorems are based on **forcing**. Forcing is an extremely advanced technique, which can be used to produce universes which satisfy theorems “at will”.

It was invented by Paul Cohen in 1963 to show the independence of the **Continuum Hypothesis**.

Forcing is still the main tool in Set Theory. It is used and improved by set theorists every day to prove the independence of new statements (many of them about trees).

Thank you for your attention!

Slides and thesis (with LaTeX sources): `github.com/korg91`.