Preliminaries

Unless otherwise stated, small greek letters always refer to ordinals.
$\operatorname{type}(X, <)$ is the order type of the well-order $(X, <)$.
Let X be a set and R a binary relation on X. If $x \in X$ then $\downarrow x := \{y \in X \mid R(y, x)\}$ and similarly $\uparrow x := \{y \in X \mid R(x, y)\}$.
Let $(X, \lhd)$ be a linearly ordered set. The *lexicographic order* on ${}^{\omega}X$ is defined by

$$f <_{\mathrm{lex}} g \Leftrightarrow \exists n \in \omega[f(n) \lhd g(n) \text{ and } \forall i < n(f(i) = g(i))].$$

Let α be a limit ordinal. We say that a sequence $\langle \alpha_\xi \mid \xi < \beta \rangle$, with β limit ordinal, is *cofinal in* α if it's strictly increasing and $\sup_{\xi<\beta} \alpha_\xi = \alpha$.

Chapter 1

Trees

The main aim of this chapter is to introduce the basic concepts about trees and to present several results discussed in Jech's paper [6], while trying to fill every proof with details.

Definition 1.1. A *tree* is a partially ordered set $(T, <)$ such that for all $x \in T$ the set $\downarrow x$ is well-ordered by $<$. We define the following basic notions related to trees:

- If $x \in T$, the *order* of x is $o(x) := \text{type}(\downarrow x)$.
- The *αth level of* T is $U_\alpha := \{x \in T \mid o(x) = \alpha\}$.
- The *height of* T is the least ordinal such that every $x \in T$ has smaller order type, i.e. $\text{height}(T) := \sup\{o(x) + 1 \mid x \in T\}$.
- A *branch in* T is a maximal linearly ordered subset of T. If b is a branch in T, of course we can define $\text{height}(b) := \text{type}(b)$.
- An α-tree is a tree of height α, and similarly for an α-branch.
- $T|\alpha$ is the subset of T which contains every element of order strictly less than α, i.e. $T|\alpha := \cup_{\xi<\alpha} U_\xi$. Obviously $T|\alpha$ has height α if $\alpha \leq \text{height}(T)$.
- We say that a tree $(T_2, <_2)$ is an *extension* of $(T_1, <_1)$ if $<_1 \, = \, <_2 \cap (T_1 \times T_1)$, and *end-extension* if $T_1 = T_2|\alpha$ for some α.

Example 1.2. We consider the family of trees given by all T which satisfy the following properties: for some $\alpha < \omega_1$,

(i) every element $t \in T$ is a function $t\colon \beta \to \omega$, with $\beta < \alpha$;

(ii) T is closed under initial segments, i.e. if $t \in T$ then $t \restriction \beta$ is in T as well, for any β;

(iii) if $t\colon \beta \to \omega$ is in T and $\beta + 1 < \alpha$, then $t^\frown n \in T$ for all $n \in \omega$;

(iv) if $t\colon \beta \to \omega$ is in T and $\beta \leq \gamma < \alpha$, then there exists $s\colon \gamma \to \omega$ such that $t \subseteq s$;

(v) $T \cap {}^{\beta}\omega$ is at most countable for all $\beta < \alpha$.

Observe that T is a countable set and the βth level consists precisely of the function whose length is β.

Definition 1.3. Let $\alpha \leq \omega_1$. An α-tree T is *normal* if:.

(i) T has a unique least point (which we call *root*);

(ii) every level of T is at most countable;

(iii) if x is not maximal in T, then are infinitely many y at level $o(x) + 1$ (we call these *immediate successors of* x);

(iv) if $x \in T$ then there is $y > x$ at each higher level less than α;

(v) the order $<$ is extensional within each level U_γ such that $\gamma < \alpha$ is a limit ordinal, that is: for all $x, y \in U_\gamma$, if $\downarrow x = \downarrow y$ then $x = y$.

It is very easy to check that the trees of last example are normal. We shall use them as forcing conditions later because of these nice properties they enjoy.

1.1 The tree property

We start with an easy and well-known fact:

1.4. König's lemma. If T is an ω-tree whose levels are all finite, then T has an ω-branch.

Proof. Define $T' := \{x \in T \mid \uparrow x \text{ is infinite}\}$. It is immediate to construct an ω-branch in T' by recursion. Such branch is trivially an ω-branch in T. □

Does König's lemma hold for cardinals greater than ω? More precisely, we say that a cardinal κ has *the tree property*, in symbols $\mathsf{TP}(\kappa)$, if the following statement is true:

If T is a κ-tree and if every level has cardinality $< \kappa$, then T has a κ-branch.

Of course $\mathsf{TP}(\kappa)$ is false if κ is singular: if $\langle \alpha_\xi \mid \xi < \lambda \rangle$ is a cofinal sequence in κ with $\lambda < \kappa$, then take the tree given by the disjoint union of branches of length α_ξ for all $\xi < \lambda$, where elements of two different branches are incomparable.
We will show now that the tree property fails already at ω_1. This is a classical result due to Aronszajn.

Definition 1.5. Let κ be a cardinal. An *Aronszajn κ-tree* is a κ-tree whose levels are of power less than κ but has no κ-branch.

Thus, there exists an Aronszajn κ-tree if and only if $\mathsf{TP}(\kappa)$ is false.

Theorem 1.6. There is an Aronszajn ω_1-tree.

Proof. We will construct the tree T in such a way that

- every $x \in T$ is a bounded and strictly increasing sequence of rational numbers;
- the order on T is defined by: $x \leq y$ iff y extends x, i.e. $x \subseteq y$;
- T is closed under inital segments.

By last condition, the αth level will consist precisely of the sequences of length α of T. Of course such a tree can't have an uncountable branch, since its union would yield a strictly increasing (and thus injective) sequence of length ω_1 into $\mathbb{Q}$, which is countable. Note that T must be constructed carefully: if we let any sequence be in T, then the ωth level would be uncountable already.
We construct T by induction on levels. To make sure that everything works, we will need to preserve the following properties (inductive hypotheses) at each level $\alpha < \omega_1$:

(1.1) $|U_\alpha| \leq \aleph_0$;
(1.2) For all $\beta < \alpha$, $x \in U_\beta$ and $q > \sup x$, there is $y \in U_\alpha$ such that $x \subseteq y$ and $q \geq \sup y$.

Define $U_0 := \{\emptyset\}$. For the successor step, suppose that we have already constructed level U_α. Then we define

$$U_{\alpha+1} := \{x^\frown r \mid x \in U_\alpha, r \in \mathbb{Q} \text{ with } r > \sup x\}.$$

It's easy to check that also $U_{\alpha+1}$ satisfies (1.1) and (1.2) w.r.t. $\alpha + 1$ (but note that one needs that $\mathbb{Q}$ is dense).
For the limit step, let α be a limit ordinal and suppose we have already defined U_β for all $\beta < \alpha$.
Claim: For each $x \in T|\alpha$ and each $q > \sup x$ there exists a strictly increasing α-sequence of rationals y such that y extends x, $q \geq \sup y$ and $y \restriction \beta \in T|\alpha$ for all $\beta < \alpha$.
Proof: Since $\alpha < \omega_1$, its cofinality is ω. Let $\langle \alpha_n \mid n \in \omega \rangle$ be cofinal in α and such that $x \in U_{\alpha_0}$. Now let $\langle q_n \mid n \in \omega \rangle$ be a strictly increasing sequence of rationals such that $q_0 = \sup x$ and $\lim_n q_n \leq q$. Using the inductive hypothesis (1.2) at each step, we can recursively find for each $n \geq 1$ a sequence $y_n \in U_{\alpha_n}$ which extends y_{n-1}

and such that $\sup y_n \leq q_n$. By defining $y := \cup_n y_n$ we are done. ■

For all $x \in T|\alpha$ and all $q > \sup x$ we choose an y as provided by the claim, and we define U_α as the set of all such y's. It's clear that (1.2) holds for U_α. Because $\mathbb{Q}$ and $T|\alpha = \bigcup_{\beta<\alpha} U_\beta$ are countable, also (1.1) is preserved.
Of course T is an Aronszajn ω_1-tree by construction. □

In the proof of the claim we exploited the fact that all limit ordinals smaller than ω_1 have countable cofinality. Actually, we could use the fact that for every $\alpha < \omega_1$ there is an order-embedding of α into any interval of $\mathbb{Q}$. This makes the proof more involved, but it will be the strategy to prove the following generalization [7]:

Theorem 1.7. Let κ be an infinite cardinal. If $\kappa^{<\kappa} = \kappa$, then there exists an Aronszajn k^+-tree.

First we need some lemmas.

Lemma 1.8. Let α be a limit ordinal. There exists a sequence $\langle \alpha_\xi \mid \xi < \mathrm{cf}(\alpha)\rangle$ cofinal in α which is also *continuous*, i.e. $\alpha_\gamma = \sup_{\xi<\gamma} \alpha_\xi$ for all $\gamma < \mathrm{cf}(\alpha)$ limit.

Proof. Let $\langle \beta_\xi \mid \xi < \mathrm{cf}(\alpha)\rangle$ be cofinal in α. Define $\langle \alpha_\xi \mid \xi < \mathrm{cf}(\alpha)\rangle$ by

$$\alpha_\xi := \begin{cases} \beta_\xi, & \text{if } \xi \text{ successor} \\ \cup_{\eta<\xi}\beta_\eta, & \text{if } \xi \text{ limit.} \end{cases}$$

Of course this sequence is continuous and still cofinal in α. □

Lemma 1.9. Let κ be an infinite cardinal. Let $\mathcal{Q} := \{f \in \kappa^\omega \mid f(n) = 1$ for finitely-many $n \in \omega\}$. Then every $\alpha < \kappa^+$ embeds in $\mathcal{Q}$, ordered lexicographically.

Proof. We proceed by induction on α. Suppose $\varphi\colon \alpha \to \mathcal{Q}$ is an order-embedding. Then $\varphi^+\colon \alpha + 1 \to \mathcal{Q}$ defined by

$$\varphi^+(\xi) := \begin{cases} 0^\frown\varphi(\xi), & \text{if } \xi \in \alpha \\ 1^\frown 0^\omega, & \text{if } \xi = \alpha \end{cases}$$

is an order-embedding of $\alpha+1$. Now suppose that α is a limit ordinal and that each $\beta < \alpha$ can be order-embedded in $\mathcal{Q}$. Let $\lambda := \mathrm{cf}(\alpha) \leq \kappa$ and let $\langle \alpha_\xi \mid \xi < \lambda\rangle$ be cofinal in α and such that $\alpha_0 = 0$. For $\xi < \lambda$ consider the interval $I_\xi = [\alpha_\xi, \alpha_{\xi+1})$; clearly $\mathrm{type}(I_\xi) \leq \alpha_{\xi+1} < \alpha$, so there is an order-embedding φ_ξ of I_ξ into $\mathcal{Q}$. For $\eta \in \alpha$ let $\xi(\eta) < \lambda$ be such that $\eta \in I_{\xi(\eta)}$. Now define $\varphi\colon \alpha \to \mathcal{Q}$ by

$$\varphi(\eta) := \xi(\eta)^\frown\varphi_{\xi(\eta)}(\eta).$$

Clearly φ order-embeds α in $\mathcal{Q}$. □

Corollary 1.10. Every $\alpha < \kappa^+$ embeds in any non-trivial open interval of $\mathcal{Q}$.

Proof. Let $f, g \in \mathcal{Q}$ be sequences with $f < g$. Let n be the least such that $f(n) < g(n)$ and let $m > n$ be such that $f(m) = 0$. It's immediate to check that $\mathcal{Q}' := \{h \in \mathcal{Q} \mid h(i) = f(i) \text{ for all } i < m \text{ and } h(m) = 1\}$ is order-isomorphic to $\mathcal{Q}$. By last lemma every $\alpha < \kappa^+$ embeds in $\mathcal{Q}'$, and since $\mathcal{Q}' \subseteq (f, g)$ open interval we are done. $\square$

We can finally proceed with the

Proof of Theorem 1.7. We will adapt the proof of Theorem 1.6. Instead of $\mathbb{Q}$, we shall use $\mathcal{Q}$ of Lemma 1.9. The only properties of $\mathcal{Q}$ we will need are that $|\mathcal{Q}| = \kappa$, a well-known fact, and the statement of Corollary 1.10. Every $x \in T$ will be a bounded and strictly increasing sequence of elements of $\mathcal{Q}$ such that $\mathrm{length}(x) = \alpha$ for $\alpha < \kappa^+$. As before, T will be such that $o(x) = \mathrm{length}(x)$ for all $x \in T$.
Again, we construct T by induction on levels, preserving for every $\alpha < \kappa^+$ conditions (1.1) (of course now we require $|U_\alpha| \leq \kappa$) and (1.2) [1], plus the following additional condition:

(1.3) If α is limit with $\mathrm{cf}(\alpha) < \kappa$ and $\mathfrak{b}$ is a branch in $T|\alpha$, then $\bigcup \mathfrak{b} \in U_\alpha$.

$U_0 := \{\emptyset\}$ and the successor step are just as before:

$$U_{\alpha+1} := \{x^\frown q \mid x \in U_\alpha, q \in \mathcal{Q} \text{ with } q > \sup x\},$$

which satisfies (1.1) and (1.2).
For U_α with α limit, we have again the claim:
Claim: For each $x \in T|\alpha$ and each $q > \sup x$ there is a strictly increasing α-sequence y in $\mathcal{Q}$ such that y extends x, $q \geq \sup y$ and $y \upharpoonright \beta \in T|\alpha$ for all $\beta < \alpha$.
Proof: Let $\lambda := \mathrm{cf}(\alpha) \leq \kappa$. By Corollary 1.10 there exists $\langle q_\xi \mid 1 \leq \xi < \lambda\rangle$ strictly increasing and contained in the interval $(\sup x, q)$ of $\mathcal{Q}$ [2]. By Lemma 1.8, let $\langle \alpha_\xi \mid \xi < \lambda\rangle$ cofinal in α, continuous and such that $x \in U_{\alpha_0}$.
As before, we want to recursively define $\langle y_\xi \mid \xi < \lambda\rangle$ such that for all $\xi < \lambda$ the following hold:

(i) $y_\xi \in U_{\alpha_\xi}$;

(ii) if $\eta < \xi$ then $y_\eta \subseteq y_\xi$;

(iii) $\sup y_\xi \leq q_\xi$ (for $\xi \geq 1$).

[1] Formally, the supremum here lives in the Dedekind completion of $\mathcal{Q}$.

[2] Actually, $\sup x$ might not be in $\mathcal{Q}$, but in that case we can simply take $q' \in \mathcal{Q}$ such that $\sup x < q' < q$ and consider the interval (q', q).

Let $y_0 := x$. Suppose we have already defined y_ξ. Then there exists $y_{\xi+1}$ which satisfies our requests by the inductive hypothesis (1.2), just as in Theorem 1.6. The limit case is where we finally use the additional condition. Suppose $\xi < \lambda$ is a limit ordinal. First observe that $\mathrm{cf}(\alpha_\xi) \le \xi < \lambda \le \kappa$ because we assumed $\langle \alpha_\xi \rangle_{\xi<\lambda}$ continuous. Now suppose we have already defined y_γ for every $\gamma < \xi$. Then it's clear that $y := \bigcup_{\gamma<\xi} y_\gamma$ satisfies (ii) and (iii). Condition (i) is also true, because $\sup_{\gamma<\xi} \alpha_\gamma = \alpha_\xi$ by continuity again, and thus $\langle y_\gamma \rangle_{\gamma<\xi}$ induces a branch in $T|\alpha_\xi$. So $y_\xi \in U_{\alpha_\xi}$ by hypothesis (1.3).

By defining $y := \bigcup_{\xi<\lambda} y_\xi$ we are done. ■

Now, suppose α is limit and $\mathrm{cf}(\alpha) = \kappa$. For all $x \in T|\alpha$ and all $q > \sup x$ we choose an y as provided by the claim, and we define U_α as the set of all such y's. It's clear that (1.1) and (1.2) hold for U_α.
The only case left is α limit with $\mathrm{cf}(\alpha) < \kappa$. Then we define $U_\alpha := \{\bigcup \mathfrak{b} \mid \mathfrak{b}$ is a branch in $T|\alpha\}$, so that condition (1.3) certainly holds. By the claim, also (1.2) is true, since "$y \restriction \beta \in T|\alpha$ for all $\beta < \alpha$" means precisely that $\mathfrak{b} := \{y \restriction \beta : \beta < \alpha\}$ is a branch in $T|\alpha$, so $y = \bigcup \mathfrak{b} \in U_\alpha$ by definition. Finally, observe that $T|\alpha = \bigcup_{\beta<\alpha} U_\alpha$, so $|T|\alpha| \le \kappa$. Hence

$$|U_\alpha| \le |\{\mathfrak{b} : \mathfrak{b} \text{ is a branch in } T|\alpha\}| \le |{}^\alpha\kappa|.$$

But of course every branch in $T|\alpha$ is completely determined by $\mathrm{cf}(\alpha)$-many entries, therefore $|U_\alpha| \le \kappa^{\mathrm{cf}(\alpha)}$. Since $\mathrm{cf}(\alpha) < \kappa$ and by hypothesis $\kappa^{<\kappa} = \kappa$, we obtain that $|U_\alpha| \le \kappa$, i.e. also condition (1.3) is satisfied.

Clearly T is an Aronszajn κ^+-tree by construction. □

Last theorem is totally useless for κ singular, since in that case the hypothesis is always false: if $\mathrm{cf}(\kappa) < \kappa$ then $\kappa^{<\kappa} = \sup\{\kappa^\lambda \mid \lambda < \kappa, \lambda \text{ cardinal}\} \ge \kappa^{\mathrm{cf}(\kappa)}$. But $\mathrm{cf}(\kappa^{\mathrm{cf}(\kappa)}) > \mathrm{cf}(\kappa)$ by König's theorem, so $\kappa^{\mathrm{cf}(\kappa)} > \kappa$ and hence $\kappa^{<\kappa} > \kappa$. Nonetheless:

Proposition 1.11. Let κ be a regular cardinal. Suppose that GCH holds. Then $\kappa^{<\kappa} = \kappa$.

Proof. The following is a well-known fact under GCH (see [1]):

Let $\kappa, \lambda \ge 1$ be cardinals with $\max(\kappa, \lambda)$ infinite. Then $\kappa^\lambda = \kappa$ if $\lambda < \mathrm{cf}(\kappa)$.

So $\kappa^{<\kappa} = \sup\{\kappa^\lambda \mid \lambda < \kappa = \mathrm{cf}(\kappa), \lambda \text{ cardinal}\} = \kappa$. □

Hence, if we assume GCH we have that for every κ regular there exists an Aronszajn κ^+-tree.

Note for Professor Friedman. While checking everything before sending you this first part, I noticed that in the proof of Theorem 1.7 I actually never need that every $\alpha < \kappa^+$ embeds into every interval of $\mathcal{Q}$, but only that $(*)$ every $\alpha \leq \kappa$ does. This is because if $\alpha < \kappa^+$, then $\mathrm{cf}(\alpha) \leq \kappa$, and thus $(*)$ is sufficient to provide the sequence $\langle q_\xi \mid 1 \leq \xi < \mathrm{cf}(\alpha) \rangle$. If my observations are correct, this is the only step where I need Corollary 1.10. So I could avoid stating Lemma 1.9 and Corollary 1.10 and just state $(*)$ instead, whose proof is immediate. Am I right?

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