Inviscid Burgers' Equation

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1 Nonlinear Wave Equation

We first consider the nonlinear one-way wave equation of the form:

$$\frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = 0 \tag{1}$$

with the initial wave profile

$$u(x,0) = F(x) \tag{2}$$

where $c(u)$ is the wave speed. We define the characteristic curves of (1) by the differential equation

$$\frac{dx}{dt} = c(u) \tag{3}$$

Then, along a particular such curve $x = x(t)$ we have

$$\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = 0$$

Therefore u is constant along the characteristics, and the characteristics are straight lines since

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dc(u)}{dt} = c'\frac{du}{dt} = 0$$

In the nonlinear case, however, the speed of the characteristics as defined by (3) depends on the value u of the solution at a given point. To find the equation of the characteristic Γ through (x,t) we note that its speed is

$$\frac{dx}{dt} = c(u(x,t)) = c(u(\xi,0)) = c(F(\xi)) \tag{4}$$

This results from applying (3) at $(\xi,0)$. Equation (4) shows that the characteristics are straight lines emanating from $(\xi,0)$ with speed $c(F(\xi))$. Direct integration of (4) gives the equation of characteristic curve Γ

$$x = c(F(\xi))t + \xi \tag{5}$$

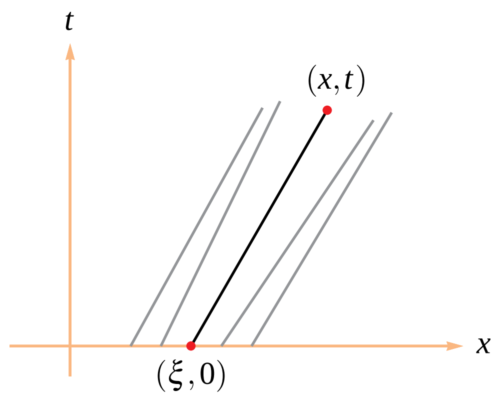

Figure 1: Characteristic curves for nonlinear wave equation.

where ξ is the x-intercept of the characteristic curve. This represents straight line whose slope is not a constant, but depends on ξ (see Fig. 1). Equation (5) defines $\xi = \xi(x,t)$ implicitly as a function of x and t. The solution $u(x,t)$ of the initial value problem (1) and (2) is given by

$$u(x,t) = u(\xi,0) = F(\xi) \tag{6}$$

where ξ is implicitly defined by (5).

We next obtain the necessary condition that (6) represents the solution of (1). Putting $G(\xi) = c(F(\xi))$, equation (5) can be rewritten as

$$x = G(\xi)t + \xi$$

Differentiating with respect to x and t, we obtain

$$1 = \left[1 + tG'(\xi)\right]\frac{\partial \xi}{\partial x} \qquad \text{and} \qquad 0 = G(\xi) + \left[1 + tG'(\xi)\right]\frac{\partial \xi}{\partial t}$$

Again, differentiating (6) with respect to x and t, we obtain

$$\frac{\partial u}{\partial x} = F'(\xi)\frac{\partial \xi}{\partial x} \qquad \text{and} \qquad \frac{\partial u}{\partial t} = F'(\xi)\frac{\partial \xi}{\partial t}$$

Eliminating ξ_x and ξ_t from the above equations gives

$$\frac{\partial u}{\partial x} = \frac{F'(\xi)}{1 + tG'(\xi)} \qquad \text{and} \qquad \frac{\partial u}{\partial t} = -\frac{F'(\xi)G(\xi)}{1 + tG'(\xi)} \tag{7}$$

Clearly, equation (1) is satisfied only if $1 + tG'(\xi) \neq 0$. The solution (6) also satisfies the initial condition at $t = 0$, since $\xi = x$, and the solution (6) is unique.

In summary, we have the following statement: The nonlinear initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= 0, \qquad \infty < x < \infty, \quad t > 0, \\ u(x,0) &= F(x), \qquad \infty < x < \infty \end{aligned}$$

has a unique solution provided $1+tG'(\xi)\neq 0$, F and c are C^1 functions where $G(\xi)=c(F(\xi))$. The solution is given in the parametric form:

$$\begin{aligned} u(x,t) &= F(\xi), \\ x &= c(F(\xi))t+\xi. \end{aligned}$$

Consider the solution of the nonlinear wave equation

$$u(x,t) = u(G(\xi)t+\xi,t) = f(\xi).$$

It is easy to see that the point $(\xi, f(\xi))$ moves parallel to the x-axis in the positive direction through a distance $G(\xi)t = ct$, and the distance moved $(x = \xi + ct)$ depends on ξ. This is a typical nonlinear phenomenon. In the linear case, the curve moves parallel to the x-axis with constant velocity c, and the solution represents waves travelling without change of shape. Thus, there is a striking difference between the linear and the nonlinear solution.

The solution of the linear wave equation can be obtained as a special case of the nonlinear wave equation (1). When $c(u) =$ constant, the characteristic curves are $x = ct+\xi$ and the solution u is given by

$$u(x,t) = F(\xi) = F(x-ct).$$

1.1 Breaking time

We have seen that the solution (a differentiable function $u(x,t)$) of the nonlinear initial value problem exists provided

$$1+tG'(\xi)\neq 0. \tag{8}$$

However, for smooth initial data, this condition is always satisfied for sufficiently small time t. It follows from results (1) that both u_x and u_t tend to infinity as $1+tG'(\xi)\to 0$. This means that the solution develops a singularity (discontinuity) when $1+tG'(\xi)=0$. We consider a point $(x,t)=(\xi,0)$ so that this condition is satisfied on the characteristics through the point $(\xi,0)$ at a time t such that

$$t = -\frac{1}{G'(\xi)} \tag{9}$$

which is positive provided $G'(\xi)=c'(F)F'(\xi)<0$. If we assume $c'(F)>0$, the above inequality implies that $F'(\xi)<0$. Hence, the solution ceases to exist for all time if the initial data is such that $F'(\xi)<0$ for some value of ξ. The time t^* at which this happens for the first time is called the breaking time. We will see more about the breaking time with regard to the inviscid Burgers' equation which is discussed in the next section.

2 The Inviscid Burgers' Equation

Inviscid Burgers' equation is a special case of nonlinear wave equation where wave speed $c(u) = u$. The inial value problem in this case can be posed as

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= 0 \\ u(x,0) &= F(x) \end{aligned} \tag{10}$$

The characteristic curves are defined by the differential equation

$$\frac{dx}{dt} = u \tag{11}$$

Since u is constant along the characteristics, the equation of the characteristic Γ through (x,t) can be found:

$$\frac{dx}{dt} = u(x,t) = u(\xi,0) = F(\xi) \tag{12}$$

The slope of characteristics curves in (x,t) plane are constant and is given by $1/F(\xi)$. Direct integration of (12) gives the equation of characteristic curve Γ

$$x = F(\xi)t + \xi \tag{13}$$

where ξ is the x-intercept of the characteristic curve. Equation (13) defines $\xi = \xi(x,t)$ implicitly as a function of x and t.

Figure 2 shows a typical initial waveform for the inviscid Burgers' equation and the corresponding characteristic curves. It can be seen from (12) that the characteristics are straight lines emanating from $(\xi,0)$ with speed $c = F(\xi)$. Observe that, the larger the $F(\xi)$ is, the flatter the characteristic line, and faster that part of the wave travels. Also, since the slope of the characteristics depend on $F(\xi)$, the slope can change from characteristics to characteristics; this leads to the possibility of intersecting the characteristics as shown in Fig. 2.

The solution $u(x,t)$ of the initial value problem (10) is given by

$$u(x,t) = u(\xi,0) = F(\xi) = F(x-ut) \tag{14a}$$

which may also be written as

$$\left.\begin{aligned} u(x,t) &= F(\xi) \\ \xi &= x - F(\xi)t \end{aligned}\right\} \tag{14b}$$

Consider, for example, the initial profile given by

$$u(x,0) = F(x) = \alpha x + \beta \tag{15}$$

where α and β are constants. This initial profile is a straight line with slope α and u-intercept β. Thus, the solution (14) becomes

$$u(x,t) = \alpha(x-ut) + \beta \tag{16}$$

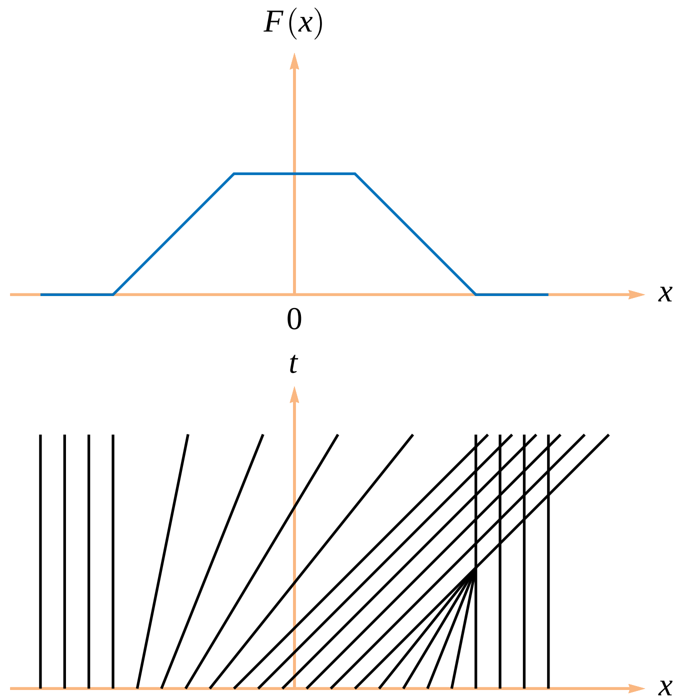

Figure 2: Initial profile and the characteristic curves for inviscid Burgers' equation.

This can be solved explicitly to yield the solution

$$u(x,t) = \frac{\alpha x + \beta}{\alpha t + 1} \tag{17}$$

It can be seen from equation (17) that, for each fixed time t, the solution represents a straight line with slope $\alpha/(1+\alpha t)$. If $\alpha > 0$, the slope of the straight line decreases as time increase and thus the solution flattens out with time. On the other hand, if $\alpha < 0$, the straight line rapidly steepens to vertical as t approaches to critical time called *breaking time* $t^* = -1/\alpha$ at which point the solution ceases to exist.

*

The case when $\alpha = -1$ and $\beta = 0$ The initial condition in this case is

$$u(x,0) = F(x) = -x$$

Now (15) gives the implicit form of the solution as

$$u(x,t) = F(x-ut) = -(x-ut)$$

from which it follows that

$$u(x,t) = \frac{x}{t-1}$$

The solution shows that as t increases, the initial waveform executes a clockwise rotation around the origin in the xu-plane. Since $u = 0$ at $x = 0$, the origin remains stationary (Fig. 2). Also,

$|u|$ increases linearly with $|x|$, and points x farther away from the origin have linearly increasing velocity. At $t=1$ the solution blows up as the waveform $u(x,t)$ coincide with the u-axis and thus becomes infinitely multivalued.

The breakdown of the solution and its multivaluedness at $t=1$ may also be determined by considering the characteristic curves. The equation for characteristics can be directly obtained from (13) and is given by

$$t = -\frac{x}{\xi} + 1$$

The slopes of the characteristic lines are given by $-1/\xi$. Hence the slopes are negative for $\xi > 0$ and it is positive for $\xi < 0$. The magnitude slopes of the characteristic lines decreases with distance from the origin. Since the t-intercepts of all the characteristic lines are 1, all of them converge at $t=1$ and thus the solution becomes multivalued, see Fig. 15.

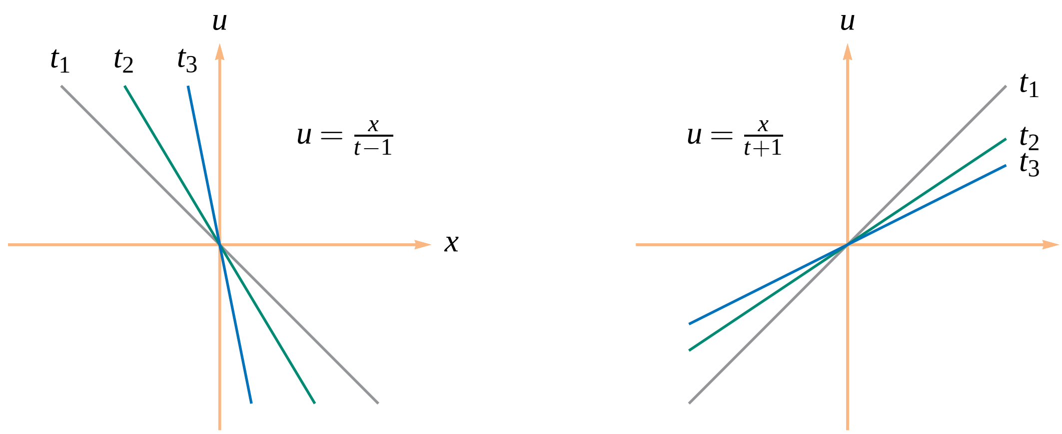

Figure 3: Solution of inviscid Burgers' equation at time $t=0,0.4,0.8$ for initial conditions $F(x)=-x$ and x.

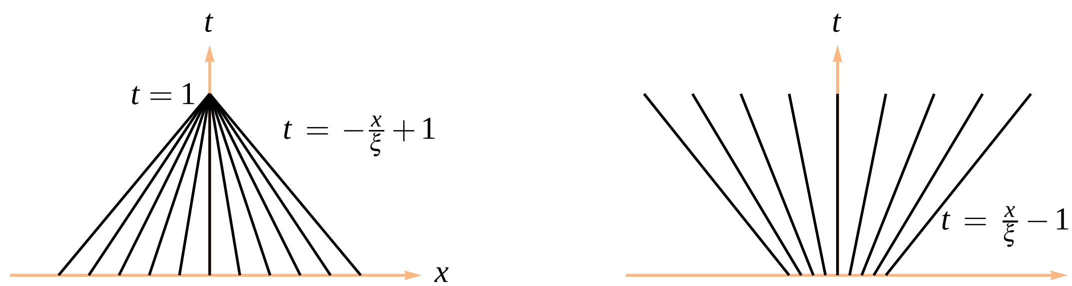

Figure 4: Characteristic curves of Burgers' equation for initial conditions $F(x)=-x$ and x.

The case when $\alpha = 1$ and $\beta = 0$

If the initial condition is

$$u(x,0) = F(x) = x$$

the solution becomes

$$u(x,t) = \frac{x}{t+1}$$

This solution is defined for all $t > 0$ and hence never breaks. It shows that as t increases, the initial waveform executes a clockwise rotation around the origin in the xu-plane. Since $u = 0$ at $x = 0$, the origin remains stationary (Fig. 14). Also, $|u|$ increases linearly with $|x|$, and points x farther away from the origin have linearly increasing velocity. The equation for characteristics in this case is given by

$$t = \frac{x}{\xi} - 1$$

The slopes of the characteristic lines are given by $1/\xi$. Hence the slopes are positive for $\xi > 0$ and it is negative for $\xi < 0$. The magnitude slopes of the characteristic lines decreases with distance from the origin. Since the t-intercepts of all the characteristic lines are -1, none of them converges for $t > 0$ and hence they diverges from each other as shown in Fig. 15. No two characteristics intersect for $t > 0$.

It is instructive to compare the solution (14) of the quasilinear PDE in (10) with the solution $u(x,t) = F(x - ct)$ of the linear equation subject to the same initial condition $u(x,0) = F(x)$. In the case of linear equation the slope of the characteristic is $1/c =$ constant that the solution represents a steady translation of the initial wave profile along the x axis with speed c, and without change of shape or scale. In the (x,t) plane, where the solution represents a propagating wave, the function $u(x,t)$ is said to define the wave profile at time t. On the other hand, in the quasilinear case (inviscid Burgers' equation) the speed of translation of the wave depends on u, so different part of the wave will move with different speeds, causing it to distort as it propagates. It is this distortion that can lead to the nonuniqueness of solution in the quasilinear case. A physical example of this phenomena is found in the theory of shallow water waves, where the speed of propagation of a surface element of water is proportional to the square root of the depth. This has the effect that in shallow water the crest of the wave moves faster than the trough, leading to wave breaking close to the shore line.

2.1 Wave distortion

To illustrate the phenomenon of distortion of nonlinear waves as it propagates and the formation of the envelope of characteristics, let us consider the initial value problem governed by the quasilinear PDE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

subject to the Cauchy data $u(x,0) = F(x) = \sin x$. The characteristic equations of the PDE are given by

$$\frac{dx}{dt} = u(x,t) \qquad \text{and} \qquad \frac{du}{dt} = 0$$

Thus, the slope of the characteristic curve is

$$\frac{dx}{dt} = u(x,t) = u(\xi,0) = F(\xi) = \sin(\xi)$$

which leads to

$$x = \sin(\xi)t + \xi = ut + \xi$$

The solution $u(x,t)$ of the initial value problem is given by (13)

$$u(x,t) = u(\xi,0) = F(\xi) = \sin(\xi) = \sin(x-ut)$$

Points on the wave with larger values of u propagates faster and consequently overtakes parts of the wave propagating with smaller values of u. At time $t=0$, the wave profile is pure sinusoid. At $t=1$ the wave profile has steepened to the point where the tangent to the wave profile has become vertical at the point where the solution crosses the x axis, while at $t=1.5$, shows that the wave is no longer single valued in periodic intervals of x. This illustrates how, as the time progresses, the wave profile distorts as it propagates eventually, after time $t=1$, becoming a multiple-valued function with respect to x, thereby leading to the breakdown of the differentiability of the solution.

The characteristic through the arbitrary point $(\xi,0)$ on the x axis has the equation

$$x = ut + \xi = t\sin\xi + \xi$$

By defining

$$\psi(x,t,\xi) = x - t\sin\xi - \xi$$

the equation of this characteristic can be written in the parametric form $\psi(x,t,\xi) = 0$, with the ξ serving as the parameter. From elementary calculus it is known that, when it exists, the envelope formed by a family of curves $\psi(x,t,\xi) = 0$ with ξ as a parameter is found by eliminating ξ between the equations $\psi(x,t,\xi) = 0$ and $\partial\psi/\partial\xi = 0$. A simple calculation shows that in this case the envelope has the equation

$$x = \sqrt{t^2-1} + \pi - \cos^{-1}(1/t)$$

As the term $\sqrt{t^2-1}$ is real-valued for $t > 1$, this result confirms the uniqueness of the solution for $0 < t < 1$, because no envelope can form during this time interval. Defining the critical time as $t^* = 1$ it follows that a unique solution exists for $0 < t < t^*$, while the solution become nonunique for $t > t^*$.

3 Shock formation

Let us look at an example in which the solution develops discontinuity even if the initial waveform is continuous. The basic idea is simple: An initial value of u that is greater on the left side of a particular x than on the right side of the same x will create waves that travel faster on the left side of x than on the right side. The fast waves will overtake the slow waves, causing a discontinuity. Let us consider the following initial waveform:

$$u(x,0) = F(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_l - \alpha x & \text{if } 0 \le x \le x_r \\ u_r & \text{if } x > x_r \end{cases} \tag{18}$$

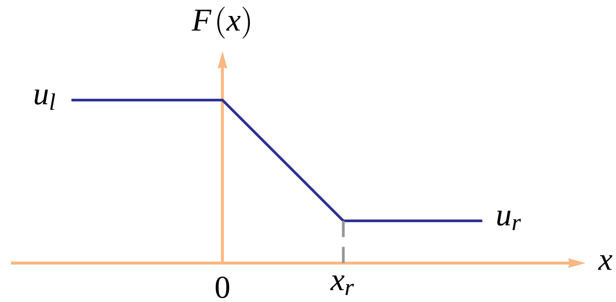

Figure 5: Initial profile with a ramp function.

where u_l and u_r are *positive* constants. This profile represents a ramp function with a negative slope of magnitude $\alpha = (u_l - u_r)/x_r$ as shown in Fig. 5. Using the initial condition (18), the equation of the characteristics become

$$x = \begin{cases} u_l t + \xi & \text{if } \xi < 0 \\ (u_l - \alpha\xi)t + \xi & \text{if } 0 \le \xi \le x_r \\ u_r t + \xi & \text{if } \xi > x_r \end{cases} \tag{19}$$

where ξ is the x-intercept of the characteristics. We can also express the characteristic lines to give t as a function of x:

$$t = \begin{cases} \frac{1}{u_l}(x - \xi) & \text{if } \xi < 0 \\ \dfrac{x - \xi}{u_l - \alpha\xi} & \text{if } 0 \le \xi \le x_r \\ \frac{1}{u_r}(x - \xi) & \text{if } \xi > x_r \end{cases} \tag{20}$$

The plot of the equation for characteristic lines for the initial condition (18) is sketched in Fig. 6. The characteristics originating from region where $\xi < 0$ have slopes equal to $1/u_l$ and those originating from region where $\xi > x_r$ have slopes equal to $1/u_r$. Since $u_r < u_l$, the right characteristics are steeper than left characteristics. In the region where $0 < \xi < x_r$, the slope of characteristics increases continuously with ξ from $1/u_l$ at $\xi = 0$ to $1/u_r$ at $\xi = x_r$. In this region, a characteristics that passes through $\xi = x_r/2$ is also shown.

From (14), the solution of the Burgers' equation is given by

$$u(x,t) = F(x - ut) = \begin{cases} u_l & \text{if } x - ut < 0 \\ u_l - \alpha(x - ut) & \text{if } 0 \le x - ut \le x_r \\ u_r & \text{if } x - ut > x_r \end{cases}$$

which can also be written as

$$u(x,t) = \begin{cases} u_l & \text{if } x < u_l t \\ \dfrac{u_l - \alpha x}{1 - \alpha t} & \text{if } u_l t \le x \le x_r + u_r t \\ u_r & \text{if } x > x_r + u_r t \end{cases} \tag{21}$$

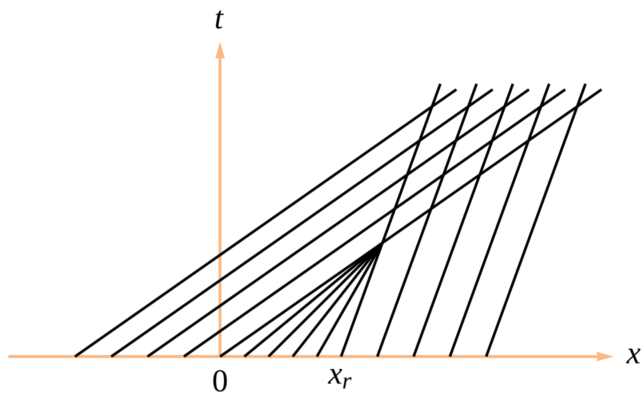

Figure 6: The characteristic curves for inviscid Burgers' equation.

We will now see what happens to characteristics with advancement of time. Looking at Fig. 5 we see that, initially the characteristics do not cross, and the solution remains a well defined, single-valued function. However, since the characteristic lines are not parallel to each other they must cross each other in a finite time. The intersection points lie on two distinct characteristics with different slopes, and thus the solution u is no longer uniquely determined. The solution at intersection point becomes multivalued since the point can be traced back along either of the characteristics to an initial value of either u_l or u_r, given by the initial condition (18). This phenomena is called *shock* and the time t^* at which this happens for the first time is called the *breaking time*. Note that, if u represent a physical quantity, the multivalued solution associated with shock is not acceptable since physical quantity should have unique value at each point. The mathematical model has broken down, and fails to agree with the physical reality.

To fully appreciate what is going on, let us look at the solution (21). It is clear that the waves that issue from the region $\xi < 0$ move faster than the waves that issues from the region $\xi > x_r$. Because of this the ramp rapidly steepens to vertical as $(\alpha t - 1) \to 0$ or $t \to 1/\alpha$ and thus the solution becomes *discontinuous* at a finite time $t = 1/\alpha$.

3.1 Breaking time

A general expression for breaking time can be found as follows. As the time approaches breaking time t^*, the solution becomes vertical at $\xi = x^*$. Thus we have,

$$\frac{\partial u}{\partial x}(x^*, t) \to \infty \qquad \text{as} \qquad t \to t^*$$

The breaking time can be determined from the implicit solution formula (14a). Differentiating (14a) with respect to x yields

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} F(x - ut) = F'(\xi)\left(1 - \frac{\partial u}{\partial x}\right)$$

where $\xi = x - ut$ is the characteristic variable, which is constant along the characteristic lines. Rearranging the above equation to obtain

$$\frac{\partial u}{\partial x} = \frac{F'(\xi)}{1 + tF'(\xi)} \tag{22}$$

This shows that if $F' > 0$, a *classical solution*, where the differential equation is satisfied at every point, exists for all time. On the other hand, if $F' < 0$ the classical solution exists only for a small period of time as the solution blows up when $\partial u/\partial x \to 0$. This happens when the denominator of equation (22) tends to zero. In other words

$$t \to -\frac{1}{F'(\xi)}$$

Hence if the initial profile has negative slope at position x, then the solution along the characteristic line issuing from the point $(x, 0)$ will break down at the time $-1/F'(x)$. As a consequence, the earliest breaking time is

$$t^* = \min\left\{-\frac{1}{F'(x)}\right\} \qquad \text{when} \qquad F'(x) < 0 \tag{23}$$

In the present problem, $F'(x) = 0$ if $x \leq 0$ and $x \geq x_r$. However, in $0 \leq x \leq x_r$, the value of the derivative $F'(x) = -\alpha$. Therefore the breaking time $t^* = 1/\alpha$.

Breaking time from characteristics

Expression for breaking time can also determined from the characteristics. Consider characteristics that emanate from the points ξ_1 and $\xi_2 = \xi_1 + \Delta\xi$. Since the breaking time corresponds to the first intersection of the characteristics, we have from equation (13)

$$\xi_1 + F(\xi_1)t = \xi_2 + F(\xi_2)t$$

Solving for t to obtain

$$t = -\frac{\xi_2 - \xi_1}{F(\xi_2) - F(\xi_1)} = -\frac{\Delta x}{F(\xi_1 + \Delta\xi) - F(\xi_1)}$$

When $\Delta x \to 0$, the expression for the earliest breaking time becomes

$$t^* = \min\left\{\lim_{\Delta\xi \to 0} -\frac{\Delta\xi}{F(\xi_1 + \Delta\xi) - F(\xi_1)}\right\} = \min\left\{-\frac{1}{F'(x)}\right\}$$

which is same as the expression (23).

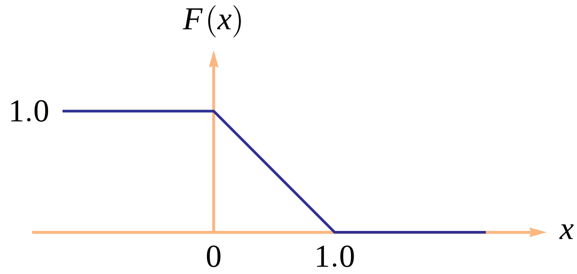

Figure 7: Initial profile with a ramp function.

The special case of $u_l = 1$, $u_r = 0$, $x_r = 1$

The value of $\alpha = 1$. The initial condition is

$$u(x,0) = F(x) = \begin{cases} 1 & \text{if } x < 0 \\ 1 - x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases} \tag{24}$$

which is sketched in Fig. 7. The equation of the characteristics becomes

$$x = \begin{cases} t + \xi & \text{if } \xi < 0 \\ (1-\xi)t + \xi & \text{if } 0 \leq \xi \leq 1 \\ \xi & \text{if } \xi > 1 \end{cases}$$

We can also express the characteristic lines to give t as a function of x:

$$t = \begin{cases} x - \xi & \text{if } \xi < 0 \\ \dfrac{x-\xi}{1-\xi} & \text{if } 0 \leq \xi \leq 1 \end{cases} \tag{25}$$

$$x = \xi \qquad \text{if } \xi > 1$$

The characteristics are plotted in Fig. 8. The solution of the Burgers' equation for the initial condition (24) is given by

$$u(x,t) = \begin{cases} 1 & \text{if } x - ut < 0 \\ 1 - (x - ut) & \text{if } 0 \leq x - ut \leq 1 \\ 0 & \text{if } x - ut > 1 \end{cases}$$

which can also be written as

$$u(x,t) = \begin{cases} 1 & \text{if } x < t \\ \dfrac{1-x}{1-t} & \text{if } t \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases} \tag{26}$$

The solution surface is sketched in Fig. 8. For $x < 0$ the lines have speed 1; for $x > 1$ the lines have speed 0; for $0 < x < 1$ the lines have speed $1 - x$. Since the breaking time $t^* = 1/\alpha$, for the present case we find that $t^* = 1$. So the solution cannot exist for $t > 1$, since the characteristics cross beyond that time and they carry different constant values of u. Figure 9 shows several wave profiles that indicate the steepening that is occurring. At $t = 1$ breaking of the wave occurs, which is the first instant when the solution becomes multiple valued. The characteristics first intersect at $(1, 1)$. In general the nonlinear initial value problem (1) and (2) may have a solution only up to a finite breaking time t^*.

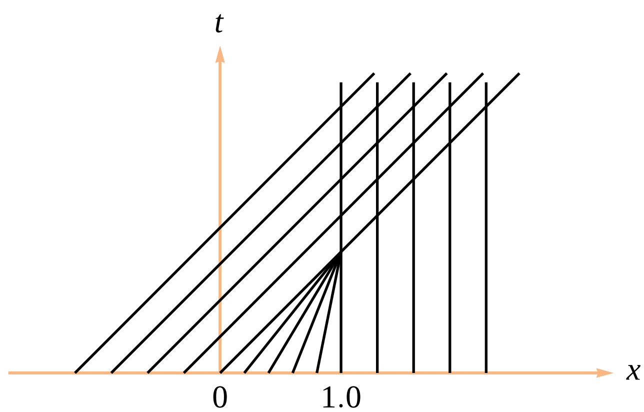

Figure 8: The characteristic curves for inviscid Burgers' equation.

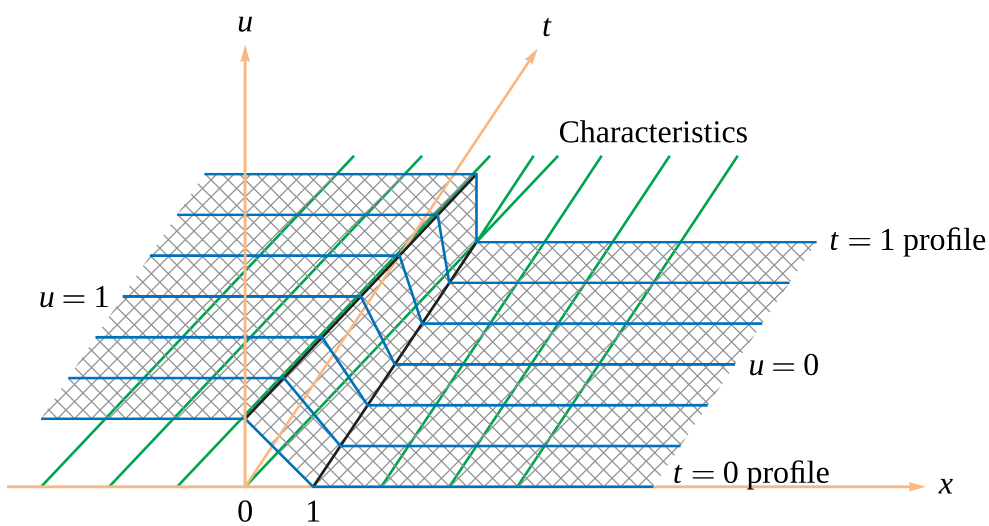

Figure 9: Solution surface of inviscid Burgers' equation.

Example

Consider the initial value problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

subject to the Cauchy data

$$u(x,0) = F(x) = \begin{cases} 2 & \text{if } x < 0 \\ 2 - x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Obtain and sketch the characteristic equation and the solution.

Solution Since the PDE is the inviscid Burgers' equation, the characteristics are straight lines emanating from $(\xi, 0)$ with speed $c = F(\xi)$. The equation of the characteristics becomes

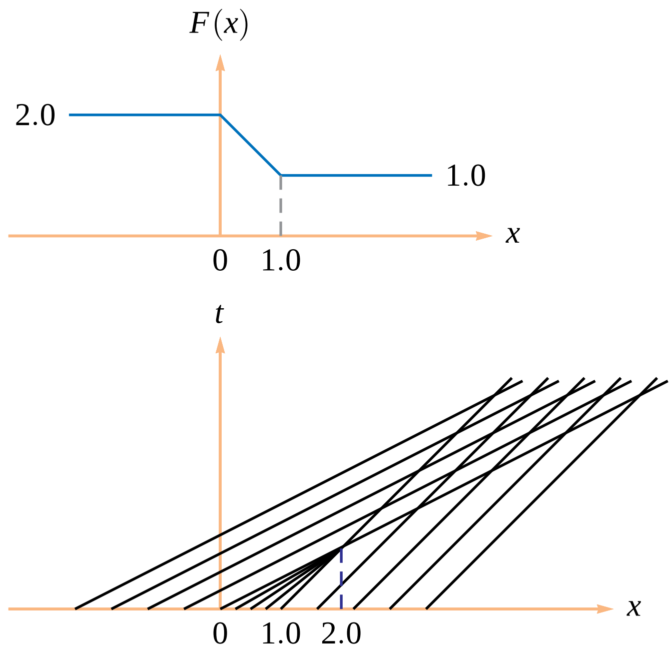

Figure 10: Initial profile and the characteristic curves for inviscid Burgers' equation.

$$x = \begin{cases} 2t + \xi & \text{if } \xi < 0 \\ (2-\xi)t + \xi & \text{if } 0 \le \xi \le 1 \\ t + \xi & \text{if } \xi > 1 \end{cases}$$

We can also express the characteristic lines to give t as a function of x:

$$t = \begin{cases} (x-\xi)/2 & \text{if } \xi < 0 \\ \dfrac{x-\xi}{2-\xi} & \text{if } 0 \le \xi \le 1 \\ x - \xi & \text{if } \xi > 1 \end{cases}$$

The characteristics are plotted in Fig. 10. The solution of the Burgers' equation for the given initial condition becomes

$$u(x,t) = \begin{cases} 2 & \text{if } x - ut < 0 \\ 2 - (x - ut) & \text{if } 0 \le x - ut \le 1 \\ 1 & \text{if } x - ut > 1 \end{cases}$$

which can also be written as

$$u(x,t) = \begin{cases} 2 & \text{if } x < 2t \\ \dfrac{2-x}{1-t} & \text{if } 2t \le x \le 1+t \\ 1 & \text{if } x > 1+t \end{cases} \tag{27}$$

The solution surface is sketched in Fig. 11. For $x < 0$ the lines have speed 2; for $x > 1$ the lines have speed 1; for $0 < x < 1$ the lines have speed $2 - x$. Since the breaking time $t^* = 1/\alpha$, for the present case we find that $t^* = 1$. So the solution cannot exist for $t > 1$, since the characteristics cross beyond that time and they carry different constant values of u. Figure 11 shows several wave profiles that indicate the steepening that is occurring. At $t = 1$ breaking of the wave occurs, which is the first instant when the solution becomes multiple valued. The characteristics first intersect at $(2, 1)$.

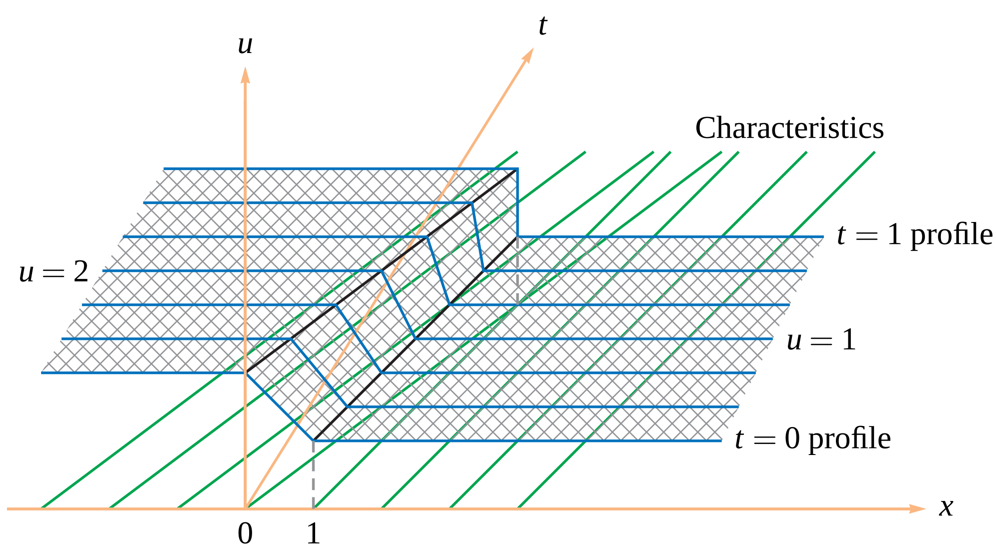

Figure 11: Solution surface of inviscid Burgers' equation.

The special case of jump discontinuity as initial waveform

An extreme case of wave breaking arises when the initial waveform has a jump discontinuity with the value of u behind the discontinuity greater than that ahead. If we have the initial waveform

$$F(x) = u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

with the condition that $u_l > u_r$, then breaking occurs immediately. This is illustrated in Fig. 12. The multivalued region starts right at the origin and is bounded by the characteristics $x = u_1 t$ and $x = u_2 t$; the boundary is no longer a cusped envelope since $F(x)$ and its derivatives are not continuous. Nevertheless, the result may be considered as the limit of a series of smoothed-out steps, and the breaking point moves closer to the origin as the initial profile approaches the discontinuous step.

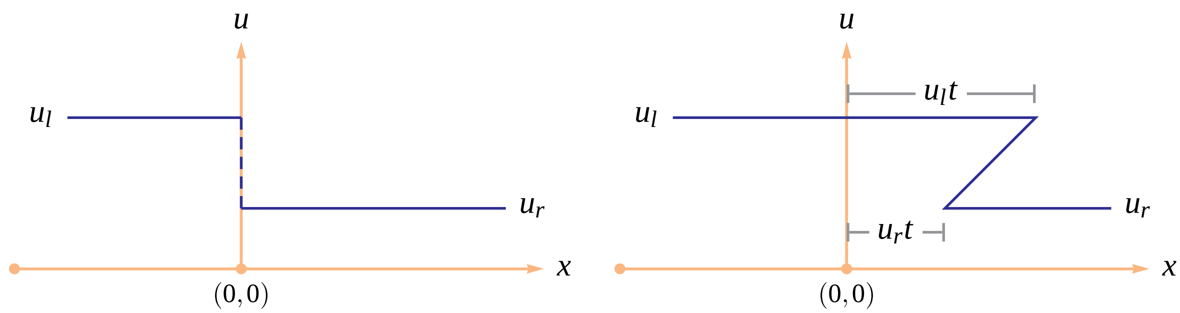

Figure 12: Initial waveform (jump discontinuity) and the solution at time t.

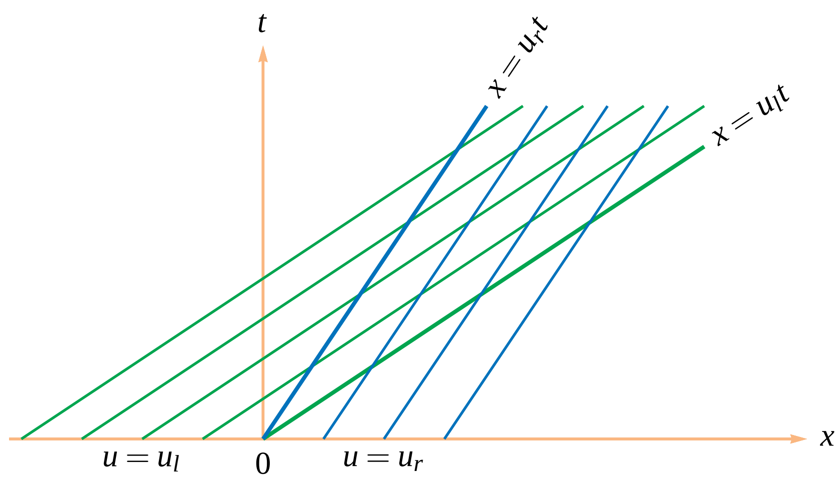

Figure 13: Characteristics of nonlinear wave equation with jump discontinuity as initial waveform.

In most physical problems where (10) represents the mathematical model of a physical problem, (x,t) is just the property of some medium and is inherently single-valued. Therefore when breaking occurs (10) must cease to be valid as a description of the physical problem. Thus the situation is that some assumption or approximate relation in the formulation leading to (10) is no longer valid and we need to look for the methods to extend the solution beyond the breaking point. In this case, it can be done through the concept of a "weak solution" which is beyond the scope of this short notes.

4 Boundary conditions

Until now all of the wave equations we have examined were considered on an infinite domain $-\infty < x < \infty$. To find the solution to (1) in a practical problem, we need to specify the boundary conditions. To illustrate the basic points it is sufficient to study the simple wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

with initial conditions

$$u(x,0) = F(x)$$

We have already seen that on the interval $(-\infty, \infty)$ the solution to the problem (for $c > 0$) was a unidirectional right travelling wave

$$u(x,t) = F(x - ct)$$

On a semi-infinite interval, $0 < x < \infty$ with $c > 0$, we must specify a boundary condition at $x = 0$, say

$$u(0,t) = v(t) \qquad \text{for} \quad t > 0$$

in addition to the specified initial condition. The solution is then

$$u(x,t) = \begin{cases} v(t - x/c) & \text{if} \quad x \le ct \\ F(x - ct) & \text{if} \quad x > ct \end{cases} \tag{28a}$$

which is illustrated in Fig. 14. The solution in part of the domain, namely points that can be reached by characteristics from the positive x-axis, is given by the initial data; the solution in the remainder of the domain is given by tracing back along characteristics to the boundary data on the positive t-axis.

If the boundary value v is independent of time, equation (28a) becomes

$$u(x,t) = \begin{cases} v & \text{if} \quad x \le ct \\ F(x - ct) & \text{if} \quad x > ct \end{cases} \tag{28b}$$

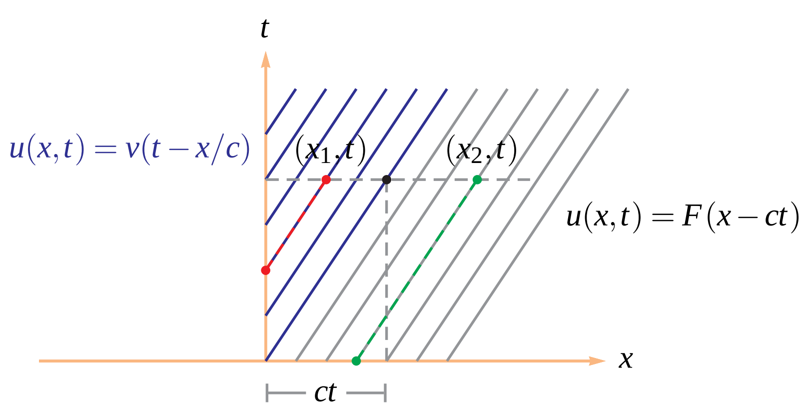

Figure 14: Characteristics of linear wave equation on a half-line.

Note that, since the wave travels from left to right (information travels from left to right), the boundary condition is needed at the left boundary and no boundary condition required on the right boundary, which is at infinity.

Now we consider a bounded domain $0 < x < L$ with $c > 0$. We make reference to Fig. 15. Since u is given by the initial condition $F(x)$ along the initial line $t = 0$, $0 < x < L$, data cannot be prescribed arbitrarily on the segment 'A' along the boundary $x = L$. This is because the characteristics $x - ct =$ constant carry the initial data to the segment 'A'. Boundary data can be imposed along the line $x = 0$, since those data would be carried along the forward-going characteristics to the segment 'B' along $x = L$. Then boundary conditions along 'B' cannot be prescribed arbitrarily.

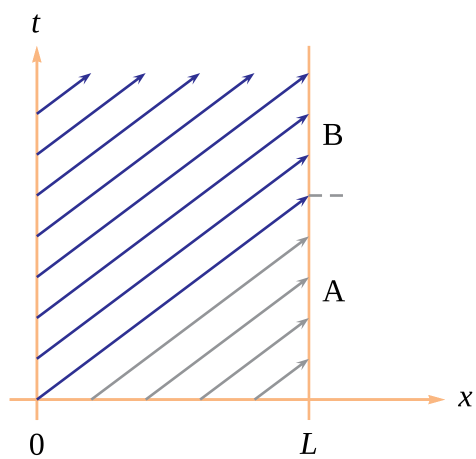

Figure 15: Characteristic diagram of linear wave equation on a finite domain.

Thus it is clear that the problem

$$\begin{aligned} \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= 0, && 0 < x < L, \quad c > 0 \\ u(x,0) &= F(x), && 0 \leq x \leq L \\ u(0,t) &= v(t), && t > 0 \end{aligned}$$

is properly posed in the sense that there is a unique solution. There are no backward-going characteristics in this problem so there are no left traveling waves. Thus waves are not reflected from the boundary $x = L$. In summary, care must be taken to properly formulate boundary value problems for unidirectional wave equation. In fact, the solution would be impossible to determine had the boundary conditions been given on the right boundary $x = L$ as the problem would then be ill-posed for lack of proper boundary conditions. We shall see that the situation is much different for second order hyperbolic partial differential equations like $u_{tt} - u_{xx} = 0$. In this case both forward- and backward-going characteristics exist, and so left traveling waves are also possible as well as a mechanism for reflections from boundaries.

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