TRAFFIC FLOW MODEL

A PROJECT REPORT

submitted by

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under the guidance of

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THESIS CERTIFICATE

This is to certify that the Dissertation titled **Traffic Flow Model** submitted by **Sushmita Rose John** to the **Department of Mathematics, Indian institute of Technology Madras** in partial fulfillment of requirements for the award of degree of **Master of Science in Mathematics** is a bonafide record of the work done by her under my supervision. The contents of this dissertation, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Prof. S. Sundar

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Sushmita Rose John

Abstract

In this dissertation, we use the LWR traffic flow model with linear velocity-density relation to study traffic flow properties. We examine the behavior of traffic near a traffic signal. We study how the traffic flows when the signal is red and how the flow changes when the signal turns green. We also analyze the traffic flow changes when a speed breaker is placed near a signal. The visualizations of all the above scenarios were implemented using HTML and Java script.

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Introduction

1.1 Traffic Flow Model

The first mathematical model to study traffic flow is the LWR traffic flow model, named after the authors M.J.Lighthill & G.B.Whitham (1955)[**6**] and P.I.Richards (1956) [**8**]. In this model the individual vehicle properties are ignored and the traffic stream as a whole is treated as a 1-D compressible fluid. The traffic state is described in terms of continuum variables traffic flow, density and mean speed.

Consider the flow of traffic on a single lane highway (i.e. overtaking is impossible). Let ρ(x,t) denote the density of cars and $v(x,t)$ denote the velocity of the cars in $x \in \mathbb{R}$ at time t$\geqslant$ 0. Then the number of cars which pass through x at time t (in unit length) is $\rho(x,t)v(x,t)$. The number of cars which are in the interval (x_1,x_2) at time t is

$$\int_{x_1}^{x_2} \rho(x,t)dx.$$

The number of cars in the interval (x_1,x_2) changes according to the number of cars which enter or leave this interval (assuming that no cars are created or destroyed within this interval), i.e.

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = \rho(x_2,t)v(x_2,t) - \rho(x_1,t)v(x_1,t). \tag{1.1}$$

This equation is the **integral conservation law**. It represents the fact that change in number of cars is due to the flow at the boundaries.
Integration of this equation with respect to time and assuming that ρ and v are regular functions

yields

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{d}{dt}\rho(x,t)dxdt = \int_{t_1}^{t_2}(\rho(x_2,t)v(x_2,t)-\rho(x_1,t)v(x_1,t))dxdt \tag{1.2}$$

$$= -\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{d}{dx}(\rho(x,t)v(x,t))dxdt. \tag{1.3}$$

Since $x_1,\ x_2 \in \mathbb{R}$ and $t_1,\ t_2 > 0$ was arbitrary, we conclude

$$\rho_t + (\rho v)_x = 0 \tag{1.4}$$

This equation is the **1-dimensional conservation law**.
To complete the model we also need an equation for the velocity v. We assume that v depends only on ρ.

- If the highway is empty, i.e., $\rho = 0$ then $v = v_{max}$, the maximum velocity;
- in heavy traffic when the cars are bumper to bumper, i.e., $\rho = \rho_{max}$, the maximum density, then v=0;
- velocity decreases as density increases i.e. $\dfrac{d\rho}{dv} \leqslant 0$.

The simplest model that satisfy these conditions was given by Greenshield(1934)[**7**] which is the linear relation

$$v(\rho) = v_{max}(1-\frac{\rho}{\rho_{max}}), \quad 0 \leqslant \rho \leqslant \rho_{max}. \tag{1.5}$$

Hence, our model becomes

$$\rho_t + (\rho v_{max}(1-\frac{\rho}{\rho_{max}}))_x = 0, \quad x \in \mathbb{R},\ \ t \geqslant 0. \tag{1.6}$$

with initial conditions

$$\rho(x,0) = \rho_0(x), \quad x \in \mathbb{R}. \tag{1.7}$$

This is the traffic flow model we will be using in this project.

1.2 Burgers Equation

The equation (1.6) can be simplified by bringing it to a dimensionless form. Let L and T be typical length and time, respectively, such that $\dfrac{L}{T} = v_{max}$. Introducing $x_L = \dfrac{x}{L}$, $\ t_T =$

$\frac{t}{T}$ *and* $u = 1 - \frac{2\rho}{\rho max}$, we obtain

$$\begin{aligned}\rho_t &= \frac{1}{T}\frac{d}{dt_T}(\frac{\rho max}{2}(1-u)) \\ &= -\frac{\rho max}{2T}u_{t_T} \qquad (1.8)\end{aligned}$$

Similarly, we get

$$\begin{aligned}(\rho v max(1-\frac{\rho}{\rho max}))_x &= \frac{1}{L}\frac{d}{dx_L}(v max\frac{\rho max}{2}(1-u)\frac{1}{2}(1+u)) \\ &= -\frac{\rho max}{2T}(\frac{u^2}{2})_{x_L} \qquad (1.9)\end{aligned}$$

Substituting (1.8)-(1.9) in the model we can write (1.6)-(1.7) as (with (x,t) instead of (x_L, t_T))

$$u_t + (\frac{u^2}{2})_x = 0 \qquad (1.10)$$

$$u(x,0) = u_0(x) \qquad (1.11)$$

where

$$u_0(x) = \frac{1-2\rho_0}{\rho max}. \qquad (1.12)$$

This is the inviscid Burgers Equation. We will be working mostly with this form of the traffic flow model.

1.3 Analytic Solution

In this section, we study the problem

$$u_t + (f(u))_x = 0, \quad x \in \mathbb{R},\ t > 0 \qquad (1.13)$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R} \qquad (1.14)$$

for some function $f : \mathbb{R} \to \mathbb{R}$ which is the general form of the traffic flow model we are studying. This problem can be solved using the **method of characteristics**.

1.3.1 Method of Characteristics

Let $u = u(x,t)$ be a smooth solution of equation. Let $x = x(t)$ define a smooth curve C in the $x-t$ plane. Then the total derivative of u along C is given by

$$\frac{du(x(t),t)}{dt} = u_t(x(t),t) + u_x(x(t),t)\frac{dx}{dt} \tag{1.15}$$

Comparing (1.15) with (1.13) we see that

$$\frac{du}{dt} = 0 \text{ along the curves defined by } \frac{dx}{dt} = f'(u) \tag{1.16}$$

i.e,

$$u(x,t) \text{ is constant along the curves defined by} \frac{dx}{dt} = f'(u). \tag{1.17}$$

These curves defined by $\frac{dx}{dt} = f'(u)$ where u is a constant are called the **characteristic curves** for equation.
Now,

$$\frac{d^2x}{dt^2} = \frac{d}{dt}(f'(u)) = f''(u)\frac{du}{dt} = 0. \tag{1.18}$$

Hence, we see that the characteristic curves are always straight lines. Therefore, in short, characteristic curves are straight lines emanating from the boundaries of the $x-t$ plane such that u is constant along these curves.
What will happen if two characteristic curves will intersect (i.e. two different u values at a particular point)? This situation represents mathematical discontinuity (abrupt changes to u). Therefore, we need a solution concept including discontinuous functions.

1.3.2 Weak solutions

Definition 1.3.1. A function $u : \mathbb{R} \times (0,T) \to \mathbb{R}$ is called a **weak solution**[**2**] of (1.13) if for all $\phi \in C_0^1(\mathbb{R}^2)$,

$$\int_0^\infty \int_{\mathbb{R}} (u\phi_t + f(u)\phi_x)dxdt = -\int_{\mathbb{R}} u_0(x)\phi(x,0)dx. \tag{1.19}$$

We can now understand what happens when two characteristic lines intersect. We study this by considering special discontinuous initial data.

The problem (1.13) with initial data

$$u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x \geqslant 0 \end{cases}$$

where $u_l, u_r \in \mathbb{R}$ is called a **Reimann problem[2]**.
We consider the following three cases :

1. CASE 1: $u_l = u_r$. In this case $u(x,t) = u_l = u_r$ for all $x \in \mathbb{R},\ t > 0$.

2. CASE 2: $u_l > u_r$. In this case the characteristics intersect leading to a shock line, i.e, a discontinuity curve $x = \psi(t)$ (See Figure 1.1). The speed of discontinuity is given by the **Rankine-Hugoniot condition**[**2**], i.e.,

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r}. \tag{1.20}$$

For a general initial data, this condition generalizes to

$$s = \frac{f(u_l(t)) - f(u_r(t))}{u_l(t) - u_r(t)} \tag{1.21}$$

where $u_l(t) = \lim_{x \to \psi(t)^-} u(x,t)$ and $u_r(t) = \lim_{x \to \psi(t)^+} u(x,t)$

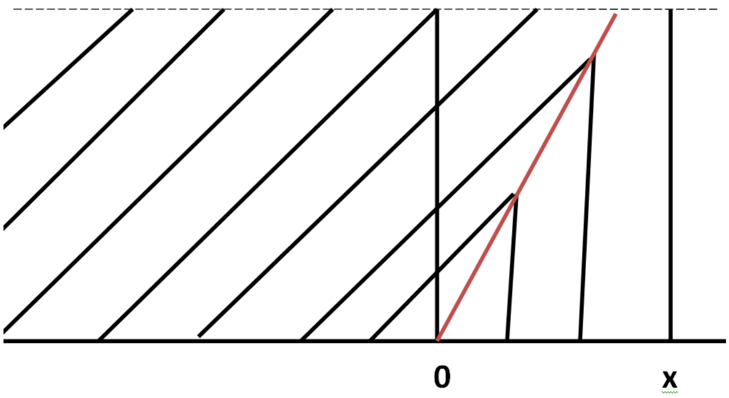

Figure 1.1: The characteristic lines when $u_l = 1 > 0 = u_r$. They intersect to form a shock wave which is shown in red colour.

3. CASE 3: $u_l > u_r$. In this case, the characteristics separate out and a void is formed. The problem, in this case, has more than one weak solution. Figure 1.2 shows two possible solutions. In this case we need to select the physically relevant solution. This is done with the help of the entropy condition.

A function $u : \mathbb{R} \times (0,T) \to \mathbb{R}$ is said to satisfies the **entropy condition of Oleinik**[**2**] if and only if along each discontinuity curve $x = \psi(t)$,

$$\frac{f(u_l(t) - f(v))}{u_l(t) - v} \geqslant \psi'(t) \geqslant \frac{f(u_r(t) - f(v))}{u_r(t) - v} \tag{1.22}$$

for all $t \in (0,T)$ and for all v between $u_l(t)$ and $u_r(t)$.
The second solution in Figure 1.2 satisfies the entropy condition and is called a rarefaction wave.

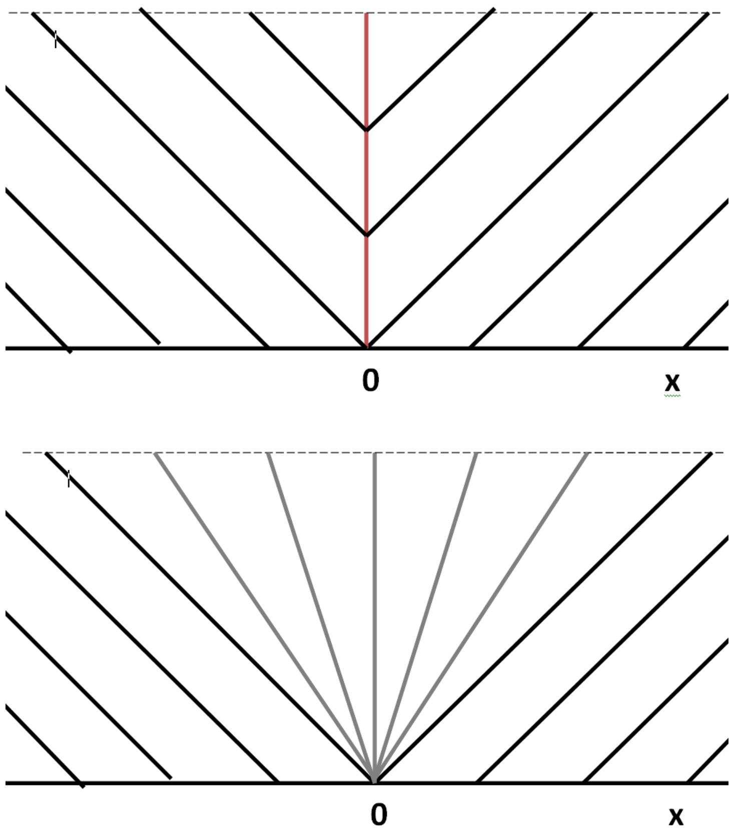

Figure 1.2: These are two different weak solutions when $u_l = -1 < 1 = u_r$. In the first figure, the characteristic curves intersect leading to a shock wave, whereas in the second figure, a rarefaction is formed.

1.4 Numerical Schemes

A non-linear hyperbolic partial differential equation can be written in two forms:

1. Conservative form : $u_t + (f(u))_x = 0$

2. Quasi-linear form : $u_t + f'(u)u_x = 0$

Both these forms are equivalent for smooth solutions but not for shocks and fans. For example, consider the two hyperbolic p. d. e.'s

$$u_t + (\frac{u^2}{2})_x = 0$$

and

$$(u^2)_t + (\frac{2u^3}{3})_x = 0.$$

They have the same quasi-linear form

$$u_t + uu_x = 0$$

but they have different weak solutions. Hence, numerical schemes for quasi-linear form may not give us the correct solution (in case of weak solutions). Therefore, we study conservative numerical methods.

1.4.1 Conservative Methods

Definition 1.4.1. A numerical scheme is said to be in **conservative form**[**1**] if it is of the form

$$u_i^{n+1} = u_i^n - \frac{k}{h}[F(u_{i-p}^n, ...u_{i+q}^n) - F(u_{i-1-p}^n, ...u_{i-1+q}^n)]$$

for some function F of $p+q+1$ arguments. Function F is called the numerical flux function.

Definition 1.4.2. A conservative scheme is called **consistent**[**1**] if F is locally Lipschitz continuous and

$$F(u, ...u) = f(u)$$

for all $u \in \mathbb{R}$.

Definition 1.4.3. The **total variation TV(v)**[**1**] of a function $v : \mathbb{R} \to \mathbb{R}^n$ is defined by

$$TV(v) = sup \sum_{i=1}^{N} |v(\psi_i) - v(\psi_{i-1})|,$$

where the supremum is taken over all subdivisions $-\infty = \psi_0 < \psi_1 .... \psi_n = \infty$ of the real line.

Theorem 1. *(Lax-Wendroff Theorem) Let $u_j(x,t)$ be a numerical solution computed with a consistent and conservative method on a mesh size h_j and time step k_j, with $h_j, k_j \to 0$ as $j \to 0$. Assume that there exists a function u(x,t) such that*

1. *for all a,b* $\in \mathbb{R}, T > 0$

$$\int_0^T \int_a^b |u_j(x,t) - u(x,t)| dxdt \to 0 \ \ as \ j \to \infty,$$

2. *for all* $T > 0$ *there is a number* $K > 0$ *such that*

$$TV(u_j(.,t)) \leqslant K$$

for all $0 \leqslant t \leqslant T, j \in \mathbb{N}$*. Then u(x,t) is a weak solution of (1.13) [**1**].*

Lax and Wendroff proved that we can correctly approximate discontinuous weak solutions to the conservation law by using a conservative method, this is, if the numerical solution using a conservative method converges to some function $u(x,t)$ as the grid is refined, then this function will in fact be a weak solution of the conservation law. However the theorem does not guarantee that the solution will converge.

Some examples of conservative schemes [**1**] :

1. **Upwind Method**:
   If $f'(u) > 0$
   $$u_i^{n+1} = u_i^n - \frac{k}{h}(f(u_i^n) - f(u_{i-1}^n))$$
   If $f'(u) < 0$
   $$u_i^{n+1} = u_i^n - \frac{k}{h}(f(u_{i+1}^n) - f(u_i^n))$$

2. **Lax-Friedrichs scheme**:
   $$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{k}{2h}(f(u_{i+1}^n) - f(u_{i-1}^n))$$

3. **Richtmyer two-step Lax-Wendroff scheme** :
   $$u_i^* = \frac{1}{2}(u_i^n + u_{i+1}^n) - \frac{k}{h}(f(u_{i+1}^n) - f(u_i^n))$$
   $$u_i^{n+1} = u_i^n - \frac{k}{h}(f(u_i^*) - f(u_{i-1}^*))$$

4. **Mac-Cornack scheme** :
   $$u_i^* = u_i^n - \frac{k}{h}(f(u_{i+1}^n) - f(u_i^n))$$
   $$u_i^{n+1} = \frac{1}{2}(u_i^n + u_i^*) - \frac{k}{h}(f(u_i^*) - f(u_{i-1}^*))$$

All these schemes give numerical solutions which if it converges, converges to a weak solution of the problem. But when there are more than one weak solution, it may not converge to the entropy satisfying solution.
One scheme which will give us the correct entropy satisfying solution is the Gudonov Method.

Gudonov Method:
For all i,n do :

- if $f'(u) \geqslant 0$ and $f'(u^n_{i+1}) \geqslant 0$ then $u^*_i = u^n_i$;
- if $f'(u) < 0$ and $f'(u^n_{i+1}) < 0$ then $u^*_i = u^n_{i+1}$;
- if $f'(u) \geqslant 0$ and $f'(u^n_{i+1}) < 0$ then $u^*_i = u^n_i$ if $(s \geqslant 0)$ or $u^*_i = u^n_{i+1}$ if $(s < 0)$ where $s = \dfrac{f(u^n_{i+1}) - f(u^n_i)}{u^n_{i+1} - u^n_i}$;
- if $f'(u) < 0$ and $f'(u^n_{n+1}) \geqslant 0$ then u^*_i is the unique solution of $f'(u^*_i) = 0$;
- Set
$$u^{n+1}_i = u^n_i - \frac{k}{h}(f(u^*_i) - f(u^*_{i-1}))$$

1.4.2 Shock Waves

In this section, we study the formation of shock using an example. We then solve the problem using the different conservative numerical methods described in section 1.4. Consider the Burgers equation

$$u_t + (\frac{u^2}{2})_x = 0, \quad x \in \mathbb{R}, \ t \geqslant 0$$

with initial density

$$u_0(x) = \begin{cases} 1, & x \leqslant 0 \\ 1 - x, & 0 < x < 1 \\ 0 & x \geqslant 0 \end{cases}$$

The characteristic curves for this equation is

$$\frac{dx}{dt} = u$$

Solving this problem using the method of characteristics (see Figure 1.3) we see that the characteristics intersect leading to a shock wave.

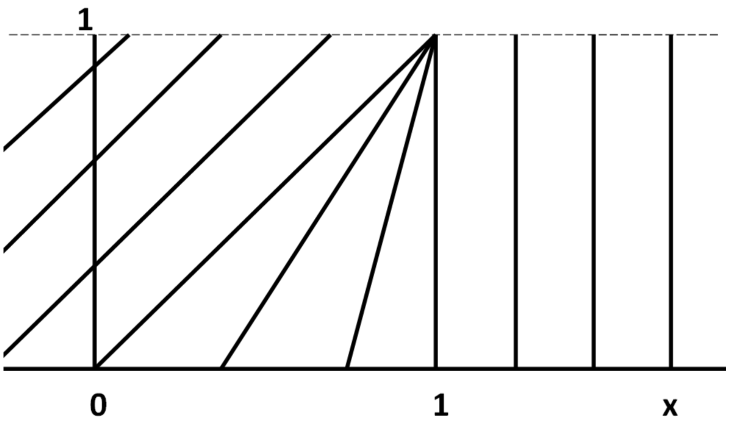

Figure 1.3: Characteristic curves intersect a $t = 1$ to producing a shock.

We now try to solve this problem using the different numerical schemes discussed in section 1.4 to see which of them gives us the solution we are looking for.
Let us model the problem for x in -0.4 to 0.3. Taking h=0.14 and k=0.005 we got the following results :

1. The Upwind scheme and Mac-Cornack scheme gave us reasonable results (Figures 1.4 and 1.5).

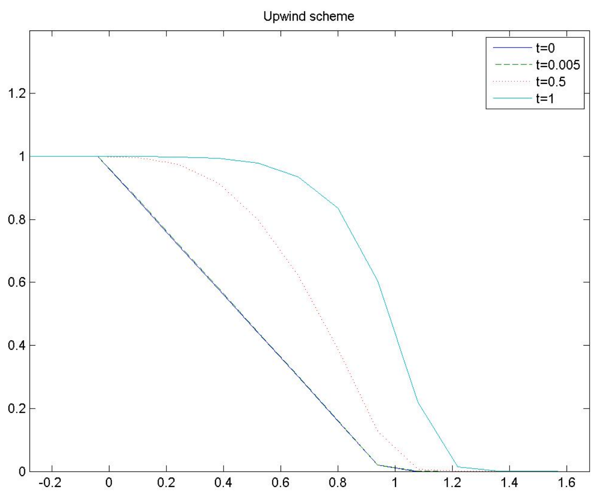

Figure 1.4: Matlab plot of the solution using Upwind scheme for $t = 0$, $t = 0.005$, $t = 0.5$ and $t = 1$.

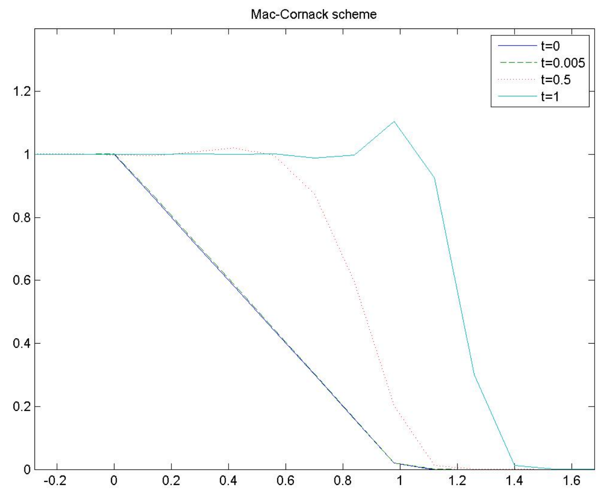

Figure 1.5: Matlab plot of the solution using Mac-Cornack scheme for $t = 0$, $t = 0.005$, $t = 0.5$ and $t = 1$.

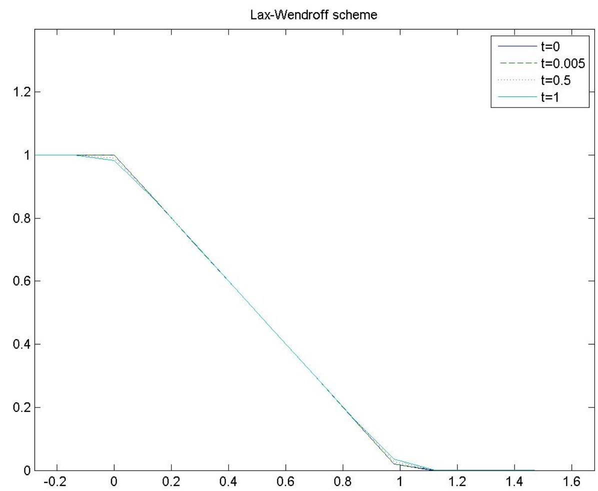

Figure 1.6: Matlab plot of the solution using Lax-Wendroff scheme for $t = 0$, $t = 0.005$, $t = 0.5$ and $t = 1$.

2. Richtmyer two-step Lax-Wendroff scheme converged to $u_0(x)$ which is not the weak solution we are looking for (Figure 1.6).

3. Lax-Friedrichs scheme did not converge.

4. The Gudonov method gave the same results as that of Upwind scheme

These results are consistent with what we know from section 1.4.

1.4.3 Rarefaction waves

In this section, we study the formation of rarefaction (or fan) using an example. We then solve the problem using the different conservative numerical methods described in section 1.4. Consider the burgers equation

$$u_t + (\frac{u^2}{2})_x = 0, \quad x \in \mathbb{R}, \; t \geqslant 0$$

with initial density

$$u_0(x) = \begin{cases} -1, & x \leqslant 0 \\ 1, & x > 0 \end{cases}$$

In this case the characteristics separate leading to a void. Both the figures in Figure 1.7 are characteristic curves corresponding to a weak solution of the problem. But the solution we want is the solution satisfying the entropy condition. This is the solution characterized by the second figure in Figure 1.7.

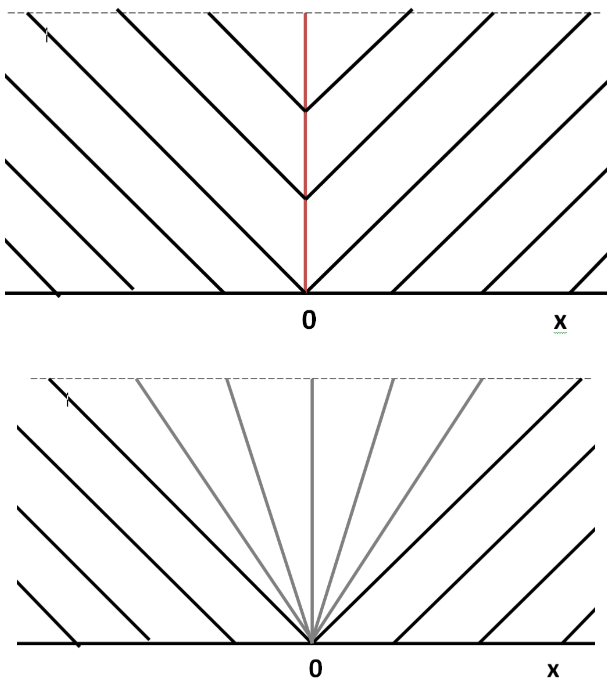

Figure 1.7: Characteristic curves showing the two weak solutions of the burgers equation with u_l=-1 and u_r=1.

Solving this problem using the different numerical schemes, we observe the following results :

1. The Gudonov Method gave us a approximation to the entropy satisfying solution (See Figure 1.8).

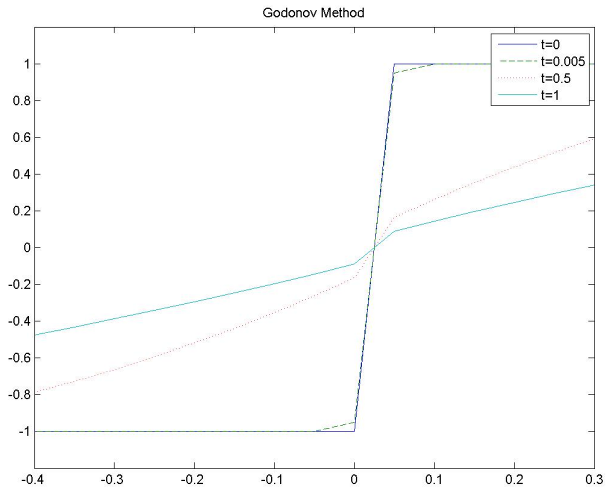

Figure 1.8: Matlab Plot for solution using Gudonov Method for $t = 0,\ t = 0.005,\ t = 0.5$ and $t = 1$.

2. Upwind Scheme and Mac-Cornack scheme converged to $u_0(x)$.

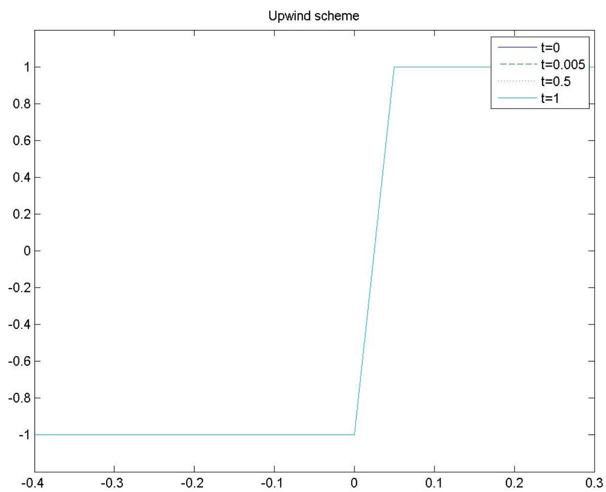

Figure 1.9: Matlab Plot for solution using Upwind Scheme for $t = 0,\ t = 0.005,\ t = 0.5$ and $t = 1$.

3. Richtmyer two-step Lax-Wendroff scheme and Lax-Friedrichs scheme did not converge.

These results are consistent with what we know from section 1.4.

Chapter 2

Traffic Signal

In this chapter, we will analyze the traffic flow near a signal. We will look into the following three cases:

1. how the traffic moves when the signal is red.
2. how the traffic flow changes when signal turns from red to green.
3. how the traffic flow changes when signal turns from green to red.

Suppose that the highway is initially filled with cars with uniform density ρ_0 in the interval $(-\infty, \bar{x})$, that there is a traffic signal at $x = \bar{x}$, and that the highway is empty in $(\bar{x}, \infty)$ for some $\bar{x} \in \mathbb{R}$. We want to model the traffic flow when the signal is red. We will work this problem with Burgers equation and the uniform density $u_0 = 1 - \dfrac{\rho_0}{\rho_{max}}$. The characteristic curves are given by

$$\frac{dx}{dt} = u.$$

2.1 Red Light

From practical experience, we know that when the signal is red there will be a tailback in front of the traffic light. Our aim is to obtain this solution from the model.

Let $\rho_{max} = 1$. When the signal at $x = \bar{x}$ is red, the traffic flow is stopped at $x = \bar{x}$ and the cars before the signal are not allowed to pass through it. Therefore, the signal acts like a boundary with $\rho(\bar{x}, t) = 1$ for all t, that is,

$$u(\bar{x}, t) = -1 \text{ for all t.}$$

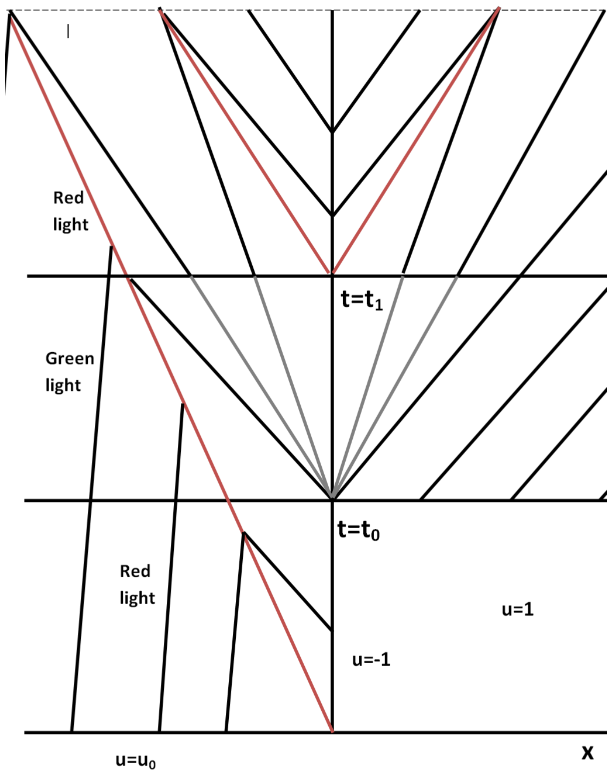

Figure 2.1: Characteristic curves for the traffic signal problem.

Figure 2.1 shows the characteristic curves corresponding to this problem. We can see from Figure 2.1 that a shock wave is formed and as time passes the cars pile up before the signal.

For doing the numerical computation, let us discretize the road from x_1 to x_{15} into 14 equal intervals with h=0.05. Let k=0.005. Also suppose the signal is between x_6 and x_7. Using the upwind scheme with u_0=-0.1 and the above mentioned boundary condition, the following results were obtained (See Table 2.1). It is clear from the data obtained that as time increases

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
0	0.55	0.55	0.55	0.55	0.55	0.554	0	0	0	0	0	0	0	0	0
0.1	0.55	0.55001	0.550409	0.561946	0.677687	0.904947	0	0	0	0	0	0	0	0	0
0.2	0.550059	0.551451	0.576059	0.717113	0.908046	0.98727	0	0	0	0	0	0	0	0	0
0.3	0.552749	0.589143	0.744626	0.915865	0.984046	0.998433	0	0	0	0	0	0	0	0	0
0.4	0.602143	0.767777	0.924342	0.983805	0.997602	0.999809	0	0	0	0	0	0	0	0	0
0.5	0.788663	0.932553	0.984693	0.997289	0.999656	0.999977	0	0	0	0	0	0	0	0	0

Table 2.1: Data obtained using Upwind scheme when the signal is red.

there will be tailback in front of the signal. Hence the data agrees with practical experience.

2.1.1 Visualization

The visualization is done using HTML and Java script. Here the road being modeled is divided into 14 equal intervals. Cars are placed in the intervals corresponding to the solution obtained from using the upwind scheme. The following are the images for $t = 0$ and $t = 0.5$ (See Figures 2.2 and 2.3).

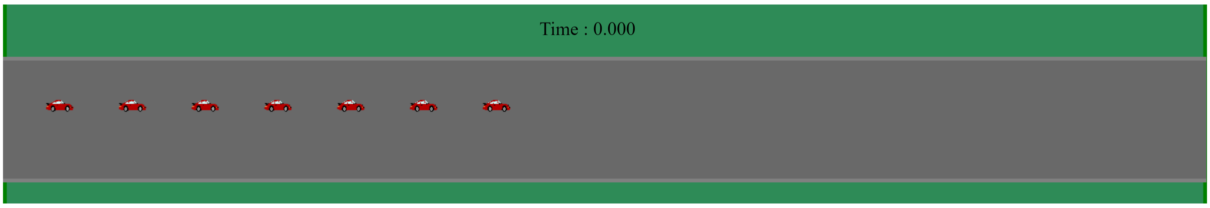

Figure 2.2: Initial traffic density at $t = 0$.

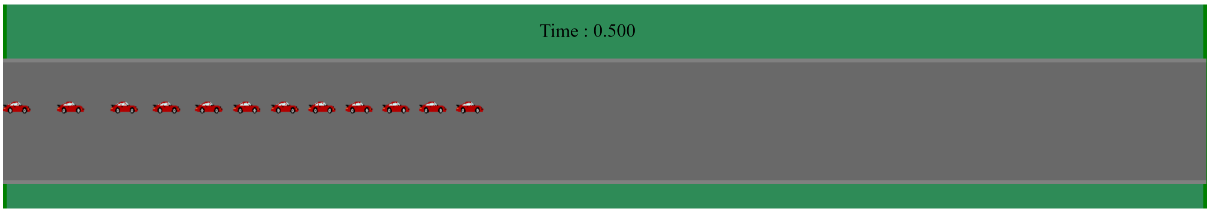

Figure 2.3: Traffic density pattern at $t = 0.5$ when the signal is red.

2.2 Green Light

Now at time $t = t_0$, suppose, the traffic light changes from red to green, and the cars starts moving from the left to the right. We want to know what happens with the tailback at time $t > t_0$.

For this problem we consider the burgers equation with $u_0(x) = u(x,t_0)$ without the boundary at x=$\bar{x}$. See the Figure 2.1 to see how the characteristics change. In this case a rarefaction is developed.

Using the Gudonov method with h=0.05 and k=0.005, the following results were obtained (Table 2.2).

It can be seen from the data obtained that the traffic density starts spreading out to the right of the signal when the signal turns green. But since traffic keeps entering from left at constant rate, the density in the left increases in the beginning.

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
0.5	0.78866	0.93255	0.98469	0.99728	0.99965	0.99997	0	0	0	0	0	0	0	0	0
0.6	0.93783	0.97706	0.97021	0.93252	0.86033	0.74554	0.25444	0.13960	0.067017	0.02700	0.0089124	0.0023857	0.00051888	$9.2e-5$	$1.3e-5$
0.7	0.94991	0.92946	0.88726	0.82856	0.75460	0.6635	0.33642	0.24537	0.17131	0.11213	0.067604	0.036933	0.018035	$7.7e-3$	$2.9e-3$
0.8	0.89889	0.86061	0.81284	0.75745	0.69474	0.62281	0.37718	0.30524	0.24250	0.18696	0.13860	0.097884	0.06519	0.04054	0.02333
0.9	0.84302	0.80310	0.75858	0.71002	0.6572	0.59837	0.4016	0.34274	0.28995	0.24134	0.19663	0.15599	0.11982	0.08855	0.06254
1	0.79631	0.75903	0.71917	0.67687	0.63174	0.58208	0.41791	0.36825	0.32311	0.28078	0.24085	0.20331	0.16833	0.13619	0.10722

Table 2.2: Data obtained using Gudonov Method when the traffic light turns green at $t = 0.5$.

We can also model the flow of traffic already present on road, i.e, when there are no new cars entering the road when the signal turns green. This can be done by taking

$$u_0(x) = \begin{cases} u(x, t_0), & x \geqslant x_1 \\ 1, & x < x_1 \end{cases}$$

In this case, using Gudonov method we get the results in Table 2.3.

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
0.5	0.78866	0.93255	0.98469	0.99728	0.99965	0.99997	0	0	0	0	0	0	0	0	0
0.6	0.71714	0.97706	0.97021	0.9325	0.8603	0.7455	0.25444	0.13960	0.067017	0.02706	0.00891	0.00238	0.00051	$9.2e-005$	$1.3e-005$
0.7	0.63935	0.92946	0.88726	0.82856	0.75460	0.66357	0.33642	0.24537	0.17131	0.11213	0.06760	0.03693	0.01803	0.00779	0.00297
0.75	0.56063	0.89441	0.84738	0.78927	0.72085	0.64027	0.35972	0.27913	0.21066	0.15229	0.10413	0.06649	0.03916	0.02103	0.01022

Table 2.3: Data obtained using Gudonov Method for the density changes in the existing traffic when the signal turned green at $t = 0.5$.

Here we can see that, unlike the previous case, when the signal turns green, the density starts decreasing from the left as well since there are no new incoming cars.

2.2.1 Visualization

The image for the traffic density at $t = 1$ is the following (See Figures 2.4 and 2.5):

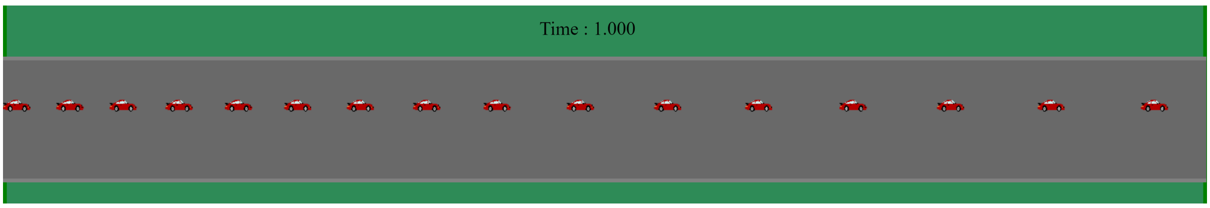

Figure 2.4: Traffic density pattern at $t = 1$ when the signal is green.

The image for the traffic density at $t = 0.75$ when there are no incoming traffic is the following:

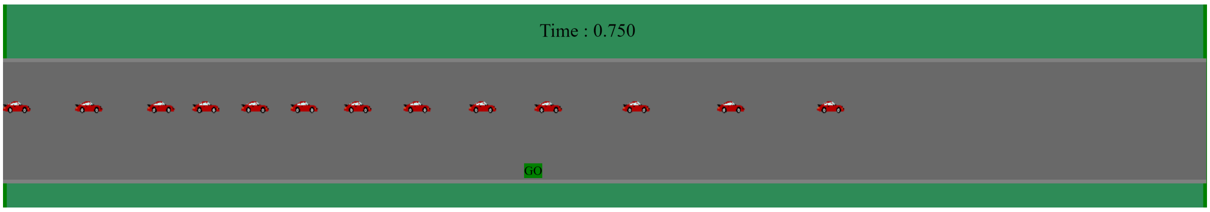

Figure 2.5: Traffic density pattern at $t = 0.75$ when there is no incoming cars.

2.3 Red Light Again

Now suppose at $t = t_1$ the signal changes to red again.
We can model this by dividing the road into two sections at the signal. For the section before the signal we use the boundary condition $u(\bar{x}, t) = -1$ which models the red light. For the section of the road after the signal we assume the boundary condition $u(\bar{x}, t) = 1$ (since cars wont cross the signal and the incoming traffic density for this section is 0). The initial density will be $u(x, t_1)$.
Once the signal changes to red, the cars before the signal will pile up near the signal and the cars beyond the signal will move ahead. This is reflected in the data obtained by using Gudonov method with the above boundary and initial conditions (See Table 2.4).

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
1	0.79631	0.75903	0.71917	0.67687	0.63174	0.58208	0.41791	0.36825	0.32311	0.28078	0.24085	0.20331	0.16833	0.13619	0.10722
1.1	0.75918	0.72552	0.69219	0.67403	0.74356	0.91597	0.084022	0.25643	0.32596	0.30778	0.27442	0.24065	0.20827	0.17758	0.14881
1.2	0.73178	0.71025	0.72062	0.80895	0.92967	0.98887	0.011122	0.070324	0.19103	0.27936	0.28971	0.26813	0.24024	0.21204	0.18478
1.3	0.73233	0.76617	0.8561	0.94535	0.98765	0.99863	0.00136	0.01234	0.05464	0.14381	0.23380	0.26761	0.26014	0.23896	0.21476
1.4	0.80972	0.89272	0.95883	0.98911	0.99810	0.99983	0.000166	0.00189	0.01089	0.04116	0.10726	0.19024	0.24091	0.2491	0.23611
1.5	0.92101	0.9695	0.99116	0.99809	0.99972	0.99998	$2.0e-005$	0.000275	0.00190	0.00883	0.03042	0.07897	0.15032	0.21013	0.23389

Table 2.4: Data obtained using Gudonov scheme when the traffic signal turned red at $t = 1$.

2.3.1 Visualization

The images for the traffic density at $t = 1.5$ is the following (See Figure 2.6):

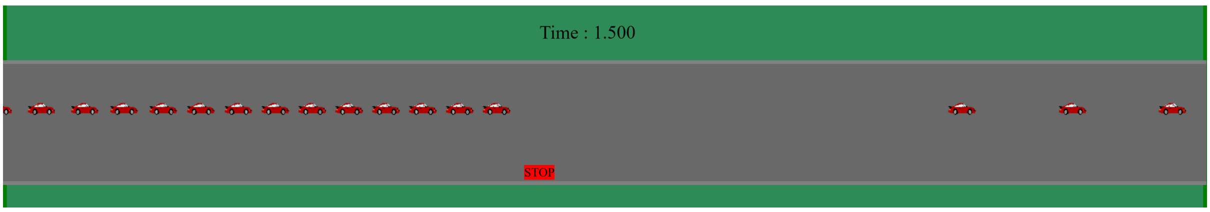

Figure 2.6: Traffic density pattern at $t = 1.5$ when the signal is red.

Chapter 3

Speed Breaker

A speed breaker or a road bump is a like a slowdown section on the road, where the vehicles passing through it are forced to reduce their velocity. This leads to reduction in the local road capacity and will lead to different traffic properties up to a certain time. In this chapter, we analyze the effect of a speed breaker in two particular cases :

1. when the speed breaker is placed after a traffic signal.

2. when the speed breaker is placed before a traffic signal.

We model the speed drop section by locally reducing the maximum speed parameter v_{max}. Consider the vehicular traffic flowing on the single-lane highway with a section of slowdown. When the vehicles enter into a section of slowdown, we assume that the vehicles are forced to decelerate their speeds. On the slowdown section, the velocities of vehicles must be less than the limit value, while they move with their expected speeds on normal road.

For this problem we will be working with the model

$$\rho_t + (\rho v_{max}(1 - \frac{\rho}{\rho_{max}}))_x = 0$$

with different values of v_{max} in the different sections of the road. The characteristic curves for this equation is

$$\frac{dx}{dt} = v_{max}(1 - 2\frac{\rho}{\rho_{max}}).$$

Let the maximum velocity in the normal section of the road be $v_m = v_{max}$ and let the maximum velocity in the slow down section of the road be v_s.

3.1 After Traffic Signal

When the speed breaker is placed after the signal and the signal turns green, we would expect the cars to start flowing at a slow pace. But once it crosses the speed breaker it should move with its full velocity.
See the characteristics for this problem in Figure 3.1 .

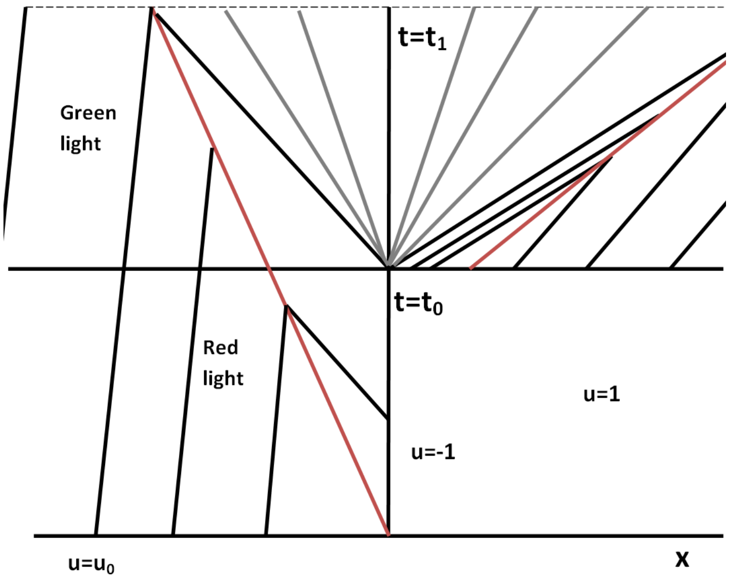

Figure 3.1: Characteristic curves for the speed breaker problem when the speed breaker is placed after a traffic signal.

For the numerical computation let us take $\rho_{max} = 1, v_m = 1$ and $v_s = 0.1$. Let us model the road from x_1 to x_{15} discretized into 14 equal intervals with h=0.05. Let k=0.005. Also suppose the signal is placed at $\bar{x}$ between x_6 and x_7. Using Gudonov method gave us the result in Table 3.1.

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
0.1	0.99761	0.99108	0.97299	0.93298	0.86039	0.74555	0.25444	0.13960	0.06701	0.02700	0.00891	0.00238	0.00051	$9.26e-5$	$1.3e-5$
0.2	0.96306	0.9323	0.8878	0.8286	0.75462	0.66357	0.33642	0.2453	0.1713	0.1121	0.06760	0.0369	0.01803	0.00779	0.00297
0.5	0.79668	0.7591	0.7192	0.6768	0.63174	0.58208	0.41791	0.36825	0.32311	0.28078	0.24085	0.20331	0.16833	0.13619	0.10722

Table 3.1: Data obtained using Gudonov method when there is no speed breaker on the road.

Now suppose that x_7 and x_8 are slowdown sections on the road. Using Gudonov Method, we get the results in Table 3.2.

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
0.1	0.99761	0.99108	0.97299	0.93298	0.86039	0.74555	0.04565	0.00409	0.00172	0.00060	0.00017	$4.1e-5$	$8.1e-6$	$1.3e-6$	$1.8e-7$
0.2	0.96306	0.93239	0.8878	0.8286	0.75462	0.66357	0.08365	0.01438	0.00918	0.00535	0.00282	0.00133512	0.000564	0.00021	$7.2e-5$
0.5	0.79668	0.7591	0.71921	0.6768	0.63174	0.58208	0.16711	0.06023	0.05123	0.04278	0.03495	0.02783	0.02151	0.01606	0.01153

Table 3.2: Data obtained using Gudonov method when the speed breaker is placed after a signal.

Comparing the two tables, we can clearly see the effect of the speed breaker on the road which is slowing the cars down.

3.1.1 Visualization

Images for $t = 0$ and $t = 0.490$ for both normal road and road with a speed breaker are as follows (See Figures 3.2, 3.3 and 3.4):

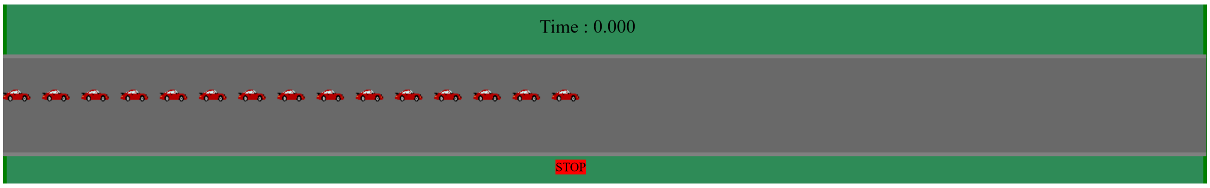

Figure 3.2: Initial traffic density at $t = 0$.

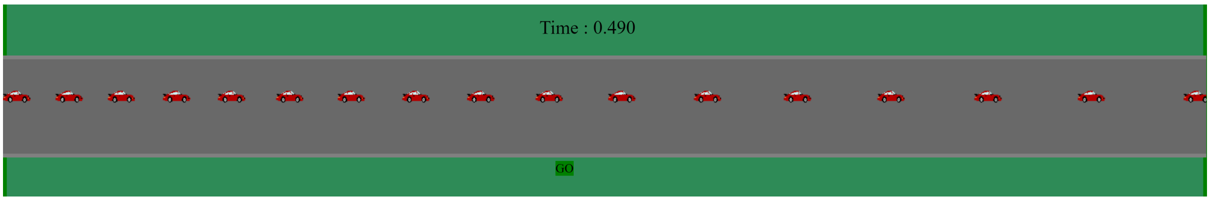

Figure 3.3: Traffic density pattern at $t = 0.490$ when there is no speed breaker on the road.

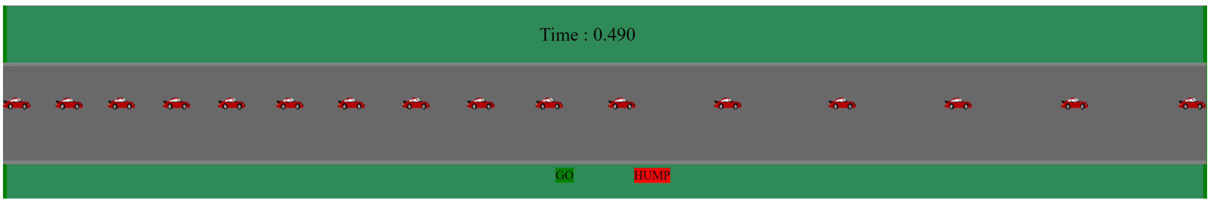

Figure 3.4: Traffic density pattern at $t = 0.490$ when there is a speed breaker on the road.

3.2 Before Traffic Signal

When the speed breaker is placed before the signal and the signal is red, we would expect the cars to start piling up near the signal at a slow pace.
See the characteristic curves for this problem in Figure 3.5.

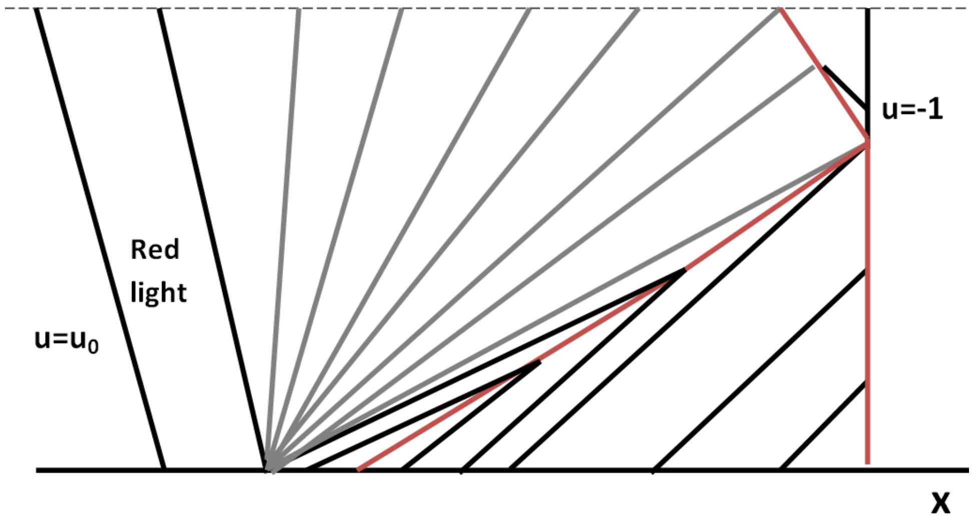

Figure 3.5: Characteristic curves for the speed breaker problem when it is positioned before a traffic signal

In this problem let the signal to be at $x = \bar{x}$ between x_8 and x_9. Since the signal is red we take the boundary condition to be $\rho(\bar{x},t) = 1$. Using Gudonov method gave us the result in Table 3.3.

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
0.1	0.997614	0.991088	0.972994	0.932982	0.860396	0.745552	0.254448	0.245552	0	0	0	0	0	0	0
0.2	0.963067	0.932396	0.887866	0.828682	0.754628	0.663573	0.336427	0.663573	0	0	0	0	0	0	0
0.5	0.797068	0.760879	0.727441	0.716018	0.77849	0.905034	0.981652	0.999048	0	0	0	0	0	0	0

Table 3.3: Data obtained using Gudonov method when there is no speed breaker on the road

Now suppose a speed breaker is added on the road at x_7. Modeling this situation with the same initial and boundary conditions as above, we get the results in Table 3.4

t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
0.1	0.997614	0.991088	0.972994	0.932982	0.860396	0.745552	0.0456528	0.0434724	0	0	0	0	0	0	0
0.2	0.963067	0.932396	0.887866	0.828682	0.754628	0.663573	0.0836517	0.163483	0	0	0	0	0	0	0
0.5	0.796685	0.759141	0.719211	0.676881	0.631743	0.582083	0.167119	0.82881	0	0	0	0	0	0	0

Table 3.4: Data obtained using Gudonov method when the speed breaker is placed before a signal

In this case again, we can see the effect of the slow down section on the road which is slowing the tailback being formed.

3.2.1 Visualization

Images for $t = 0$ and $t = 0.5$ for both normal road and road with a speed breaker are as follows (See Figures 3.6, 3.7 and 3.8):

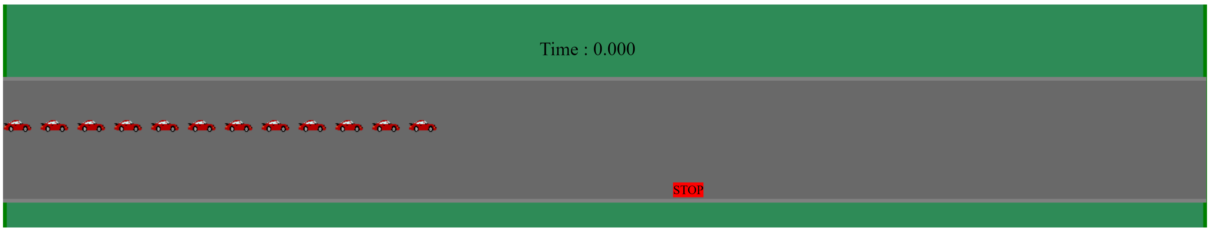

Figure 3.6: Initial traffic density at $t = 0$.

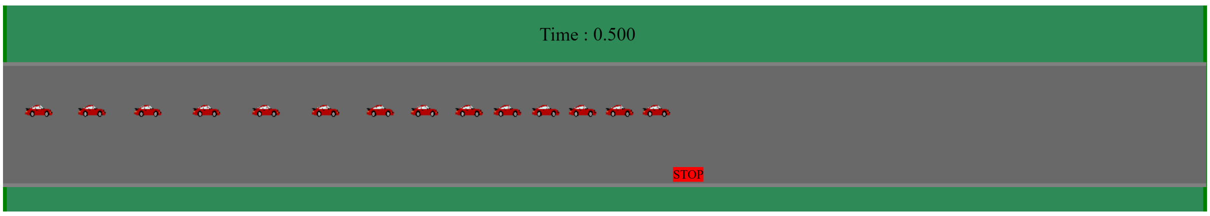

Figure 3.7: Traffic density pattern at $t = 0.5$ when there is no speed breaker on the road.

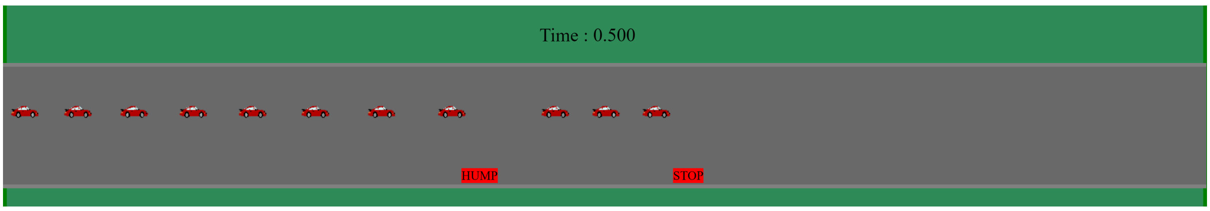

Figure 3.8: Traffic density pattern at $t = 0.5$ when there is a speed breaker before the signal.

Chapter 4

Summary

Recently, traffic problems have attracted considerable attention. In this project, we used the L.W.R traffic flow model to study traffic properties. The first chapter introduced the model. Following this, we studied the method of characteristics which can be applied to analytically solve the model. We also discussed various numerical schemes that can be applied. In the second chapter, we examined in detail the behaviour of traffic near a signal. We analyzed the how the traffic density changes when the signal changes colour. Finally, in the last chapter, we examined the traffic pattern when there is a speed breaker on the road.

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