Coupled hopf oscillator compiled

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1 Phase encoding in coupled hopf oscillators

Consider two hopf oscillators $z_1 and z_2$ coupled as shown in eqns (21) and (22). We are exploring under stable condition what would be the phase difference of the two oscillators.

$$\dot{z_1} = (\mu - |z_1|^2)z_1 + i\omega z_1 + Ae^{i\phi}z_2 \tag{1}$$

$$\dot{z_2} = (\mu - |z_2|^2)z_2 + i\omega z_2 + Ae^{-i\phi}z_1 \tag{2}$$

Let $z_1 = r_1 e^{i\theta_1}$ and $z2 = r_2 e^{i\theta_2}$

$$\dot{z_1} = \dot{r_1}e^{i\theta_1} + r_1 e^{i\theta_1} i\dot{\theta_1} = e^{i\theta_1}(\dot{r_1} + ir_1\dot{\theta_1}) \tag{3}$$

$$\dot{z_2} = \dot{r_2}e^{i\theta_2} + r_2 e^{i\theta_2} i\dot{\theta_2} = e^{i\theta_2}(\dot{r_1} + ir_1\dot{\theta_2}) \tag{4}$$

substituting (21) and (22) in (23) and (24) respectively,

$$e^{i\theta_1}(\dot{r1} + ir_1\dot{\theta_1}) = (\mu - r_1^2)r_1 e^{i\theta_1} + i\omega_1 r_1 e^{i\theta_1} + Ae^{i\phi}r_2 e^{i\theta_2} \tag{5}$$

$$(\dot{r1} + ir_1\dot{\theta_1}) = (\mu - r_1^2)r_1 + i\omega_1 r_1 + Ar_2 e^{i\theta_2 - \theta_1 + \phi} \tag{6}$$

similarly,

$$(\dot{r2} + ir_2\dot{\theta_2}) = (\mu - r_2^2)r_2 + i\omega_2 r_2 + Ar_1 e^{i\theta_1 - \theta_2 - \phi} \tag{7}$$

Equating real terms and imaginary terms independently

$$\dot{r1} = (\mu - r_1^2)r_1 + Ar_2 cos(\theta_2 - \theta_1 + \phi) \tag{8}$$

$$\dot{\theta_1} = \omega_1 + A\frac{r_2}{r_1}sin(\theta_2 - \theta_1 + \phi) \tag{9}$$

$$\dot{r2} = (\mu - r_2^2)r_2 + Ar_1 cos(\theta_1 - \theta_2 - \phi) \tag{10}$$

$$\dot{\theta_2} = \omega_2 + A\frac{r_1}{r_2}sin(\theta_1 - \theta_2 - \phi) \tag{11}$$

Let ψ be defined as $\psi = \theta_1 - \theta_2$

$$\begin{aligned}
\dot{\psi} &= \dot{\theta_1} - \dot{\theta_2} && (12)\\
&= (\omega_1 - \omega_2) + A\frac{r2}{r1}sin(-(\psi - \phi)) - A\frac{r1}{r2}sin(\psi - \phi) && (13)\\
&= (\omega_1 - \omega_2) - \frac{Ar_2^2 sin(\psi - \phi) - Ar_1^2 sin(\psi - \phi)}{r_1 r_2} && (14)\\
&= (\omega_1 - \omega_2) - \frac{Asin(\psi - \phi)}{r_1 r_2}(r_1^2 + r_2^2) && (15)
\end{aligned}$$

Assuming that $\omega_1 = \omega_2$ and $r_1 = r_2$, equation (34) becomes

$$\dot{\psi} = 2Asin(\phi - \psi) \tag{16}$$

When stable, $\dot{\psi} = 0$, this implies that

$$\phi - \psi = 0 \tag{17}$$

$$\psi = \phi \tag{18}$$

$$\theta_1 - \theta_2 = \phi \tag{19}$$

In case of training a coupled hopf oscillator, initialize the training with weak lateral coupling and strong input so that the two oscillators driven out of phase by an amount equal to the phase difference of applied inputs.
Let the coupling of two oscillators defined in equations (21) and (22) be defined by a weight W such that $W = Ae^{i\phi}$. The update of this weight is done as follows

$$\Delta W = -W + z_1 z_2^* \tag{20}$$

since initially the input drives them out of phase, $z_1 z_2^*$ will tune the weight W into $Ae^{i\phi}$, where ϕ is the phase difference between the two inputs given to the two oscillators. Now when the inputs and removed and these weights are retained, the phase difference $\psi = \theta_1 - \theta_2$ between the two oscillators would stay at ϕ as shown in eqn (39).

Complex Hopf with power

Consider two hopf oscillators $z_1$andz_2 coupled as shown in eqns (21) and (22). We are exploring under stable condition what would be the phase relationship of the two oscillators.

$$\dot{z_1} = (\mu - |z_1|^2)z_1 + i\omega_1 z_1 + Ae^{i\frac{\phi}{\omega_2}}(z_2)^{\frac{\omega_1}{\omega_2}} \tag{21}$$

$$\dot{z_2} = (\mu - |z_2|^2)z_2 + i\omega z_2 + Ae^{i\frac{-\phi}{\omega_1}}(z_1)^{\frac{\omega_2}{\omega_1}} \tag{22}$$

Let $z_1 = r_1 e^{i\theta_1}$ and $z2 = r_2 e^{i\theta_2}$

$$\dot{z_1} = \dot{r_1}e^{i\theta_1} + r_1 e^{i\theta_1} i\dot{\theta_1} = e^{i\theta_1}(\dot{r_1} + ir_1\dot{\theta_1}) \tag{23}$$

$$\dot{z_2} = \dot{r_2}e^{i\theta_2} + r_2e^{i\theta_2}i\dot{\theta_2} = e^{i\theta_2}(\dot{r_1} + ir_1\dot{\theta_2}) \tag{24}$$

substituting (21) and (22) in (23) and (24) respectively,

$$e^{i\theta_1}(\dot{r1} + ir_1\dot{\theta_1}) = (\mu - r_1^2)r_1e^{i\theta_1} + i\omega_1r_1e^{i\theta_1} + Ae^{i\frac{\phi}{\omega_2}}r_2^{\frac{\omega_1}{\omega_2}}e^{i\theta_2\frac{\omega_1}{\omega_2}} \tag{25}$$

$$(\dot{r1} + ir_1\dot{\theta_1}) = (\mu - r_1^2)r_1 + i\omega_1r_1 + Ar_2^{\frac{\omega_1}{\omega_2}}e^{i(\frac{\phi}{\omega_2}+\theta_2\frac{\omega_1}{\omega_2}-\theta_1)} \tag{26}$$

similarly,

$$(\dot{r2} + ir_2\dot{\theta_2}) = (\mu - r_2^2)r_2 + i\omega_2r_2 + Ar_1^{\frac{\omega_2}{\omega_1}}e^{i(\frac{-\phi}{\omega_1}+\theta_1\frac{\omega_2}{\omega_1}-\theta_2)} \tag{27}$$

Equating imaginary terms

$$\dot{\theta_1} = \omega_1 + A(\frac{r_2}{r_1})^{\frac{\omega_1}{\omega_2}}sin(\frac{\phi}{\omega_2} + \theta_2\frac{\omega_1}{\omega_2} - \theta_1)) \tag{28}$$

$$\dot{\theta_2} = \omega_2 + A(\frac{r_2}{r_1})^{\frac{\omega_2}{\omega_1}}sin(\frac{-\phi}{\omega_1} + \theta_1\frac{\omega_2}{\omega_1} - \theta_2)) \tag{29}$$

The relationship between two oscillators of intrinsic frequency ω_1 and ω_2 and phase θ_1 and θ_2 respectively holds the following relation.

$$\frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2} = constant \tag{30}$$

$$\theta_1\omega_2 - \theta_2\omega_1 = constant \tag{31}$$

This constant can be any real number. But what if there is a coupling between the two oscillators such that the weight connection between the two oscillators are defined by

$$W_{12} = Ae^{\frac{i\phi}{\omega_2}} \tag{32}$$

This is the kind of coupling that we observe in equation (21).
For understanding this, let ψ be defined as $\psi = \omega_2\theta_1 - \omega_1\theta_2$
We know that ψ should be a constant and hence $\dot{\psi} = 0$

$$\begin{aligned} \dot{\psi} &= \omega_2\dot{\theta_1} - \omega_1\dot{\theta_2} & (33)\\ &= A(\frac{r2}{r1})^{\frac{\omega_1}{\omega_2}}sin(\frac{\phi}{\omega_2} + \theta_2\frac{\omega_1}{\omega_2} - \theta_1)) - A(\frac{r_2}{r_1})^{\frac{\omega_2}{\omega_1}}sin(\frac{-\phi}{\omega_1} + \theta_1\frac{\omega_2}{\omega_1} - \theta_2)) & (34) \end{aligned}$$

Assuming that $r_1 = r_2 = 1$, equation (34) becomes

$$\dot{\psi} = A(sin(\frac{\phi + \theta_2\omega_1 - \theta_1\omega_2}{\omega_2}) - sin(\frac{-\phi + \theta_1\omega_2 - \theta_2\omega_1}{\omega_1})) \tag{35}$$

$$\dot{\psi} = A(sin(\frac{\phi - \psi}{\omega_2}) - sin(\frac{-\phi + \psi}{\omega_1})) \tag{36}$$

$\dot{\psi} = 0$, implies that

$$\phi - \psi = 0 \tag{37}$$

$$\psi = \phi \tag{38}$$

$$\omega_2\theta_1 - \omega_1\theta_2 = \phi \tag{39}$$

Thus when the oscillators are coupled using the weight connection as in equation (32) to get a coupled equation as in equation (21), we always get a phase relationship which is related to the phase of the weight connection.
i.e, when ($W_{12} = Ae^{\frac{i\phi}{\omega_2}}$),
$\omega_2\theta_1 - \omega_1\theta_2 = \phi$