Summer 2020

# Post-Modern Portfolio Theory

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MPT

Modified MPT

PMPT

# *Post-Modern Portfolio Theory*

Cassy DeBacco

July 27, 2020

# How Should I Invest My Money?

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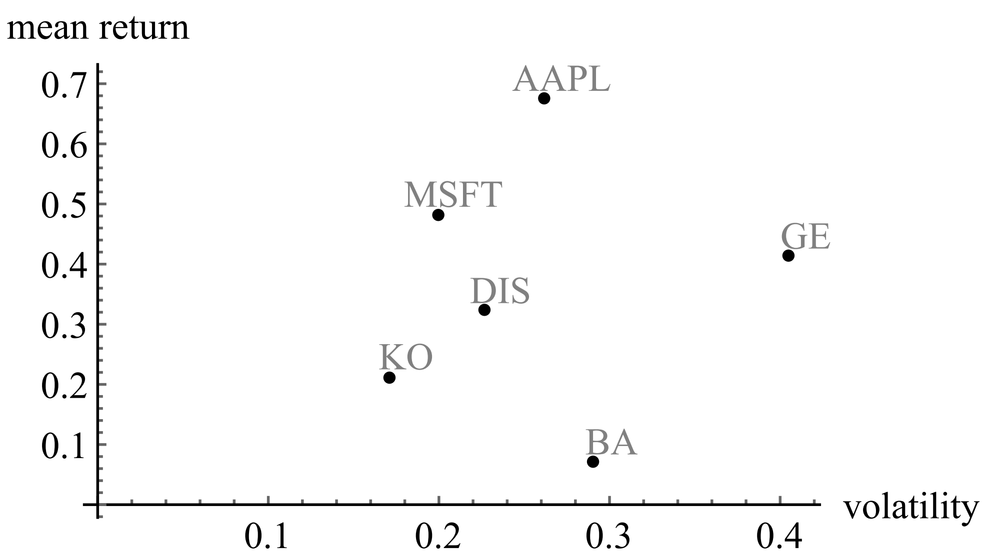

# Modern Portfolio Theory

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The mean and variance for portfolio

$$\mathcal{P} = w_1 X_1 + \ldots + w_n X_n$$

is $\mu_{\mathcal{P}} = \boldsymbol{\mu}^T \boldsymbol{w}$ and $\sigma_{\mathcal{P}}^2 = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$.

Modern Portfolio Theory (MPT) solves the quadratic program

$$\begin{aligned} &\text{Minimize } \sigma_{\mathcal{P}}^2 \\ &\text{such that } \mu_{\mathcal{P}} \geq \alpha \\ &\text{and } \mathbf{1}^T \boldsymbol{w} = 1. \end{aligned}$$

# Solution with Shorting Allowed



If short sales are allowed, then the optimal weights for a minimum variance portfolio are

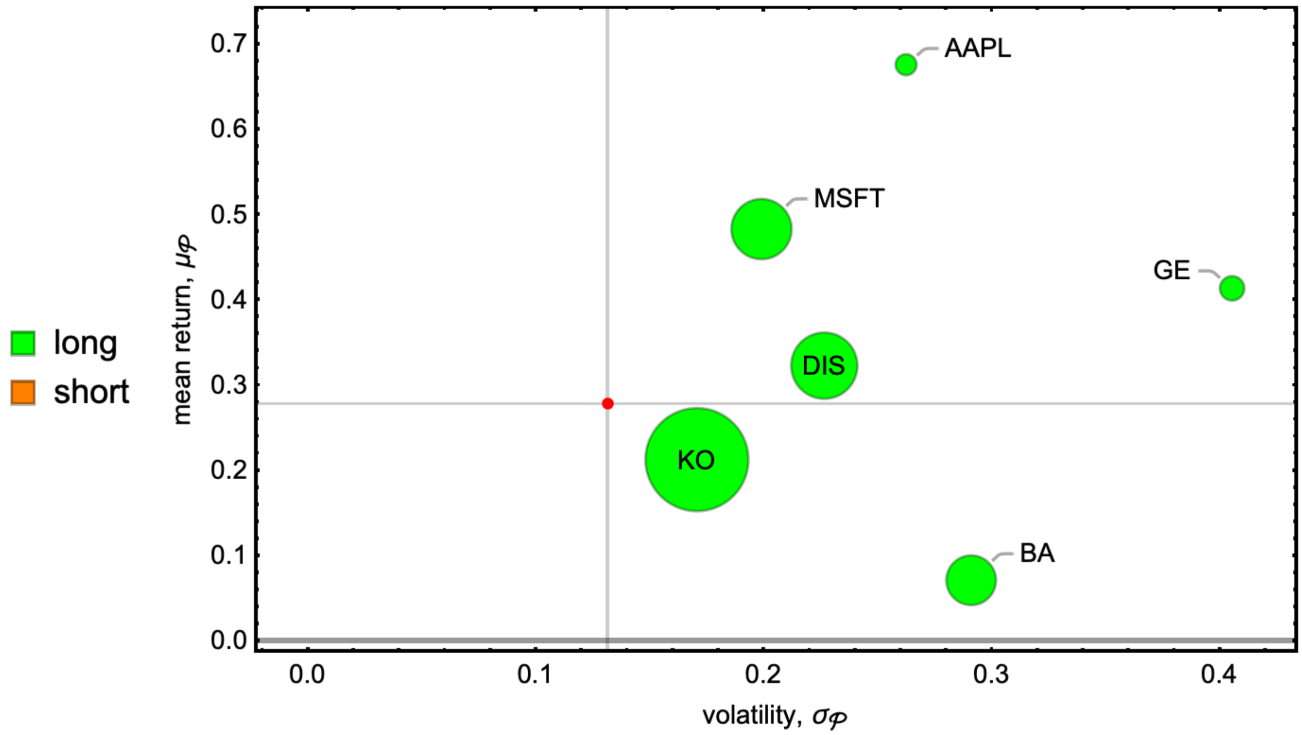
$$\boldsymbol{w} = \frac{1}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}} \boldsymbol{\Sigma}^{-1} \mathbf{1}.$$

The corresponding mean and variance are

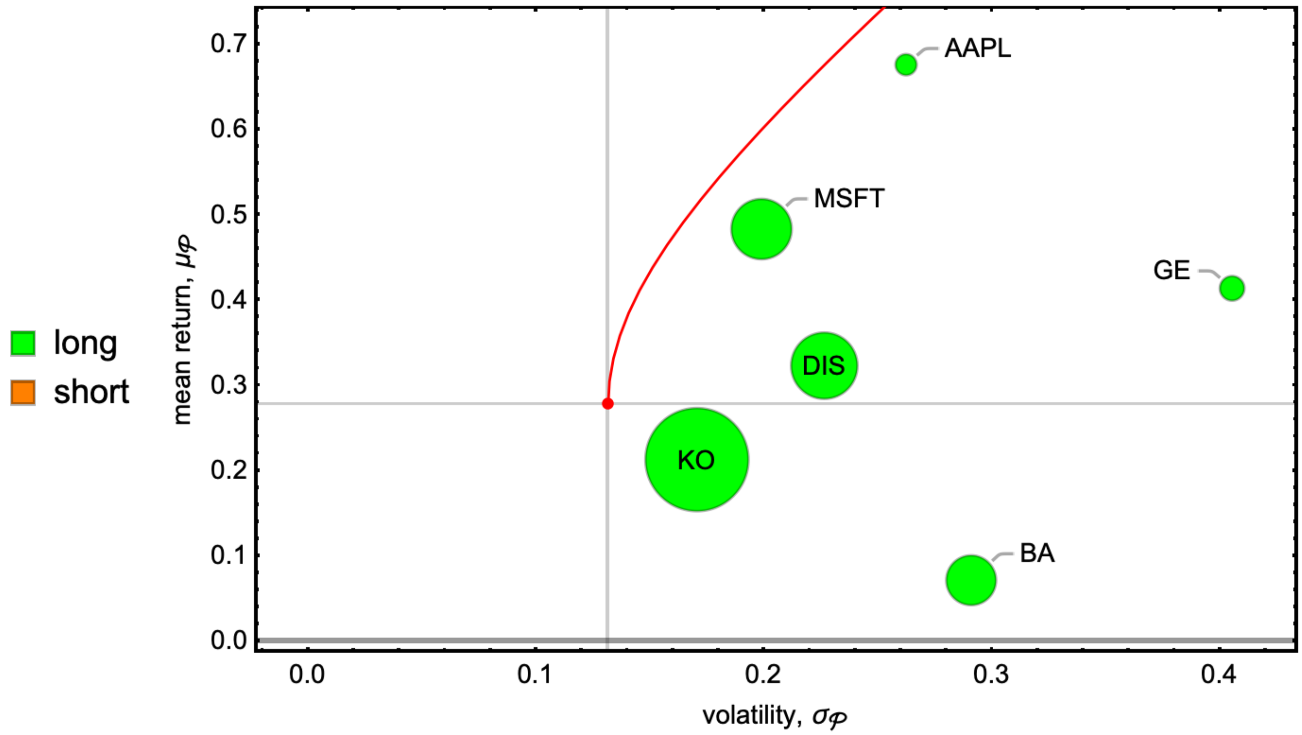
$$\mu_{\mathcal{P}} = \frac{\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}} \quad \text{and} \quad \sigma^2_{\mathcal{P}} = \frac{1}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}}.$$

# Minimum Variance Portfolio





# Efficient Frontier



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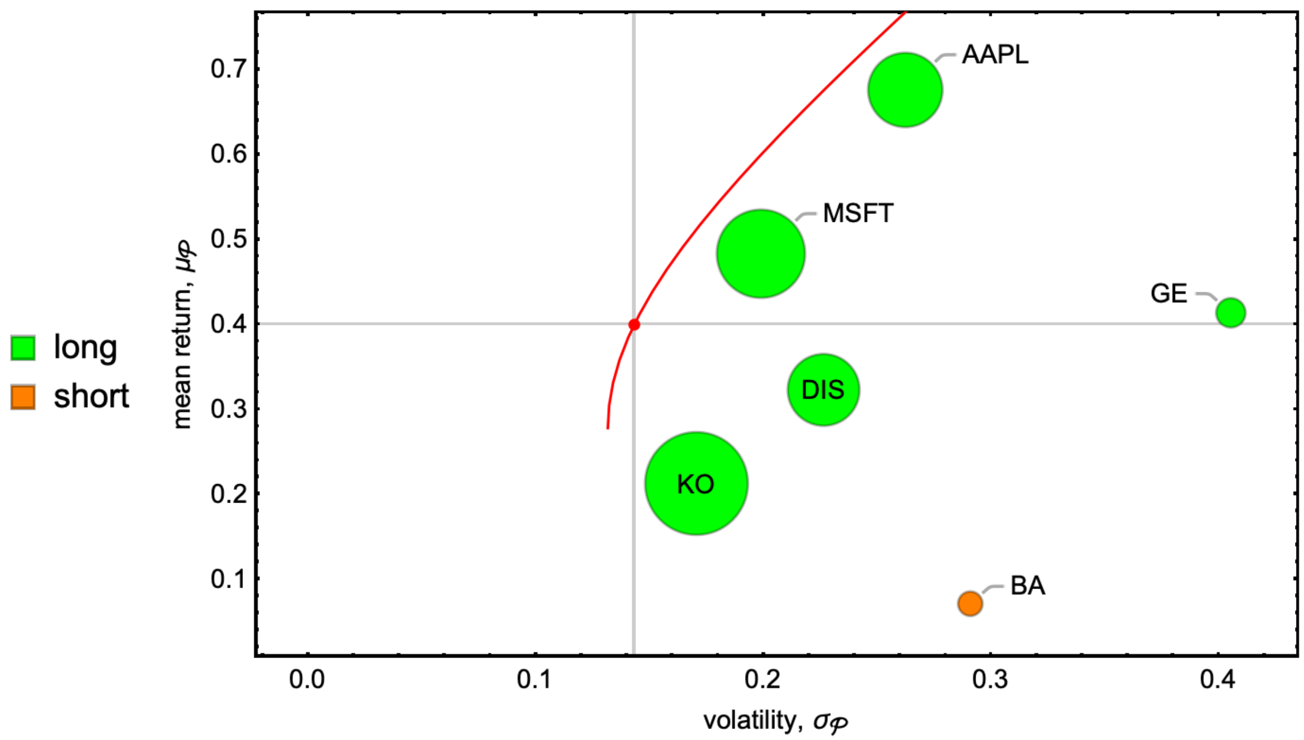

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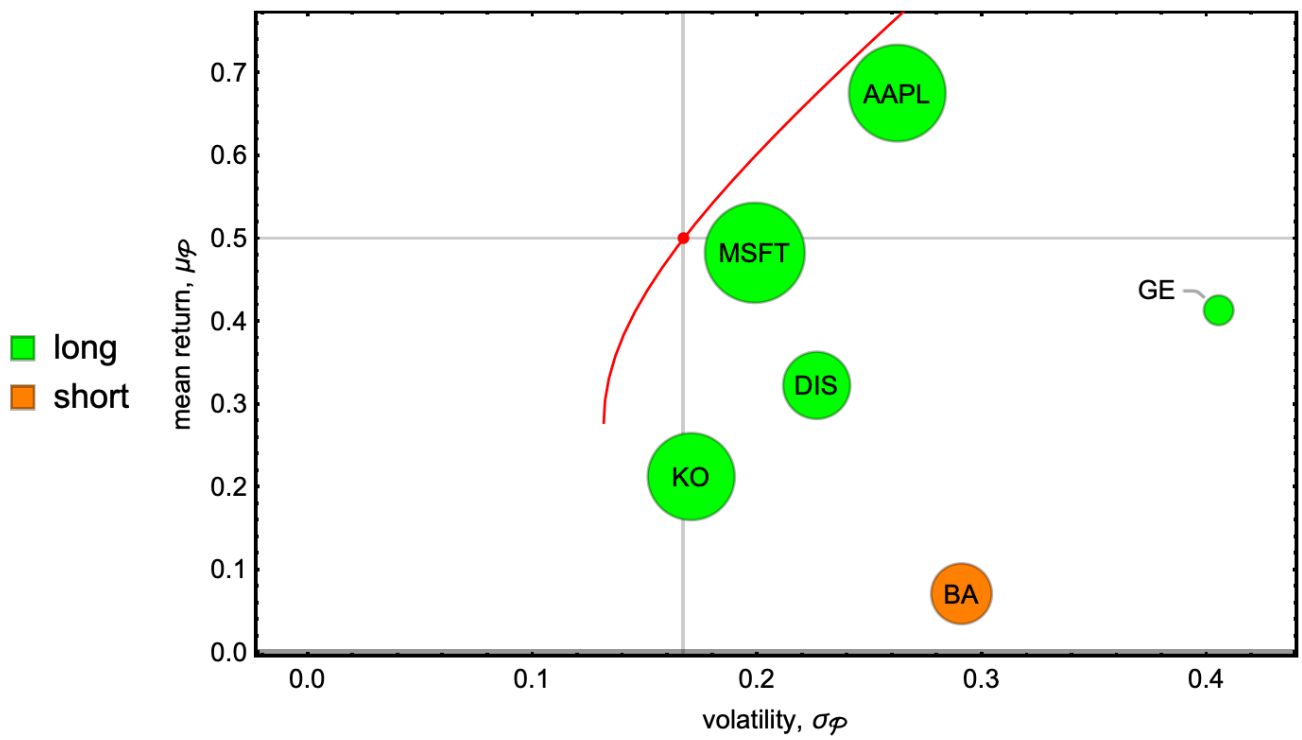

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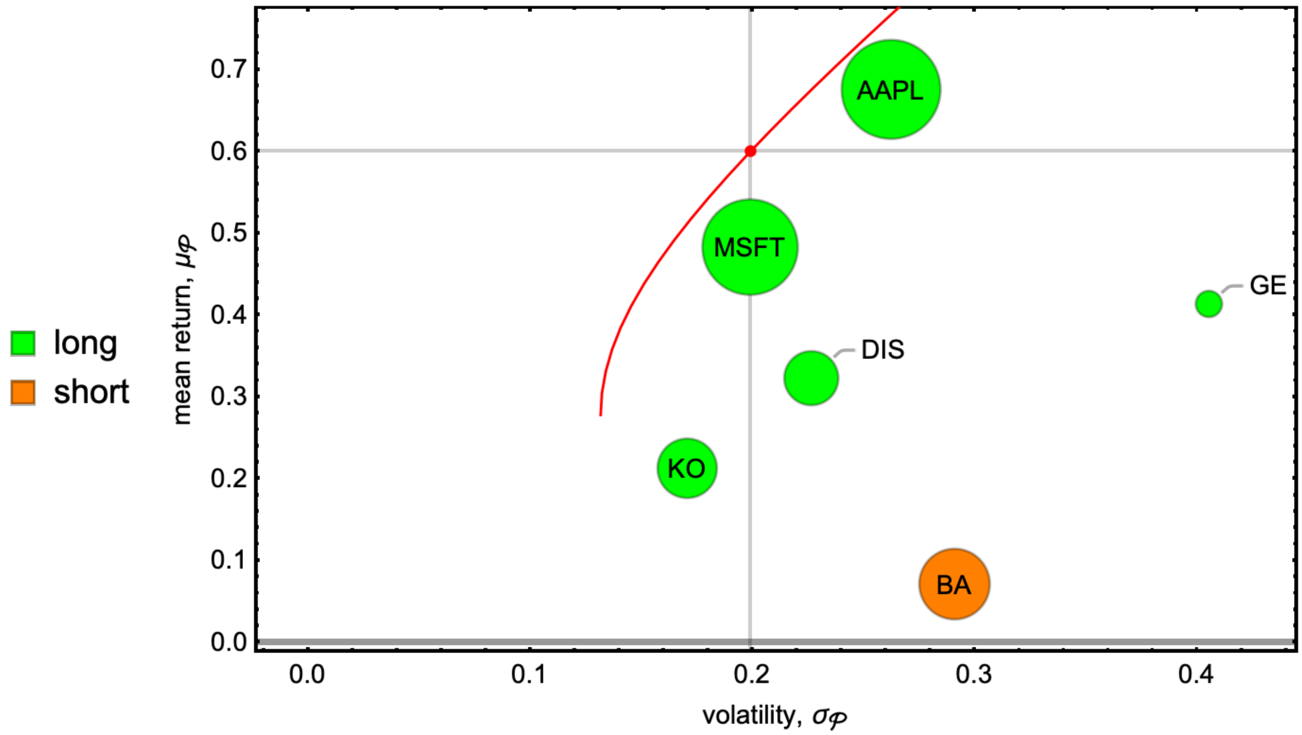

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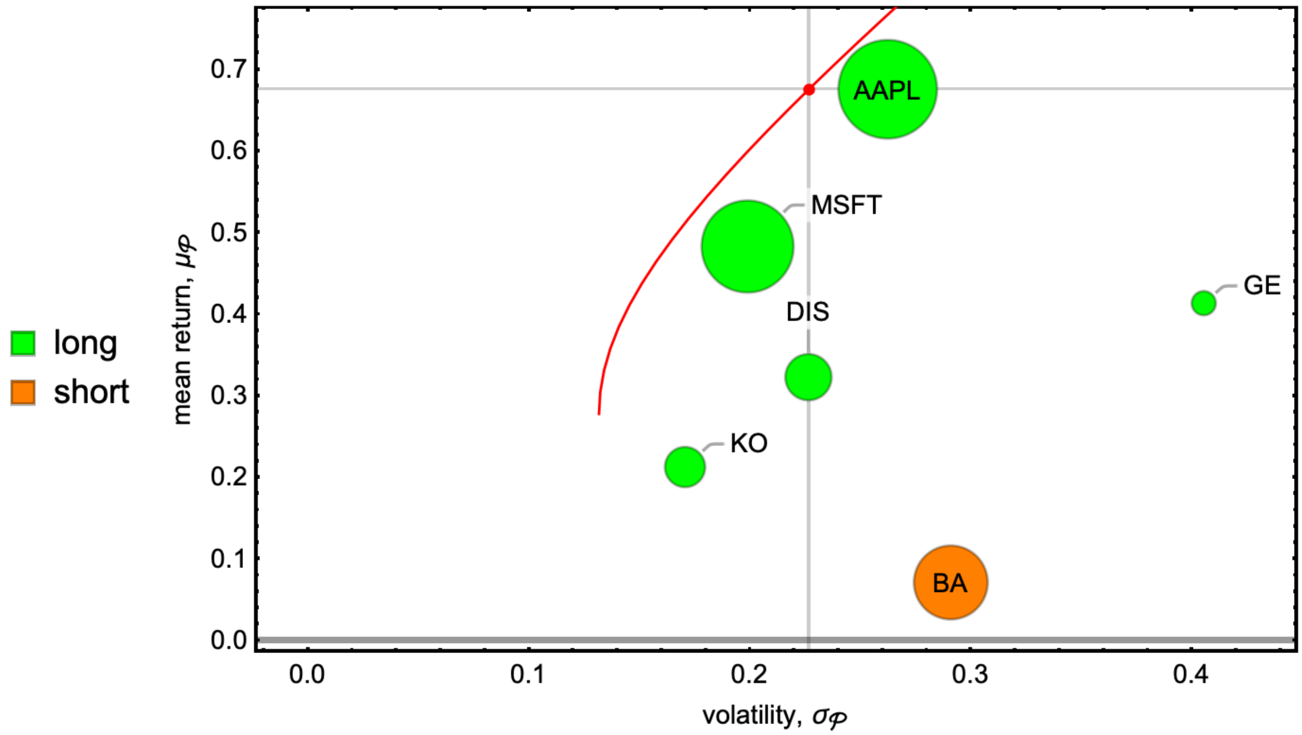
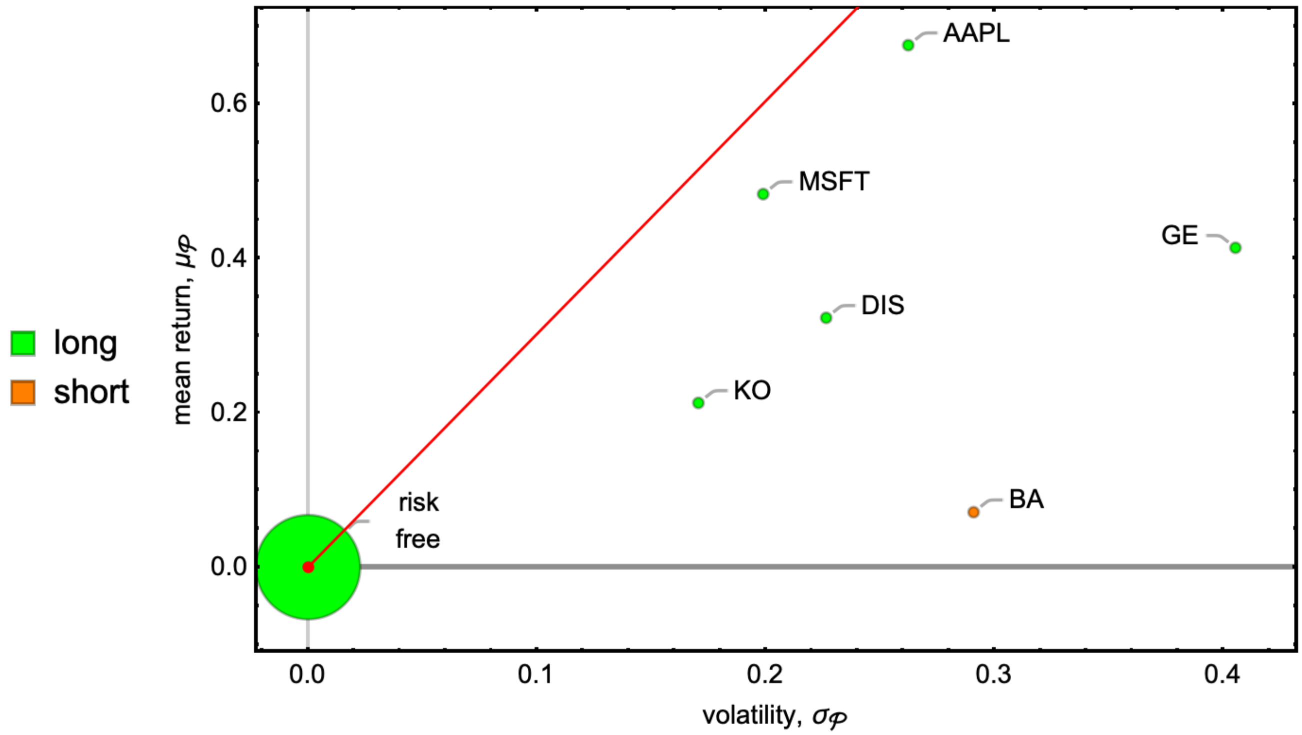


# Efficient Frontier with a Risk-Free Asset





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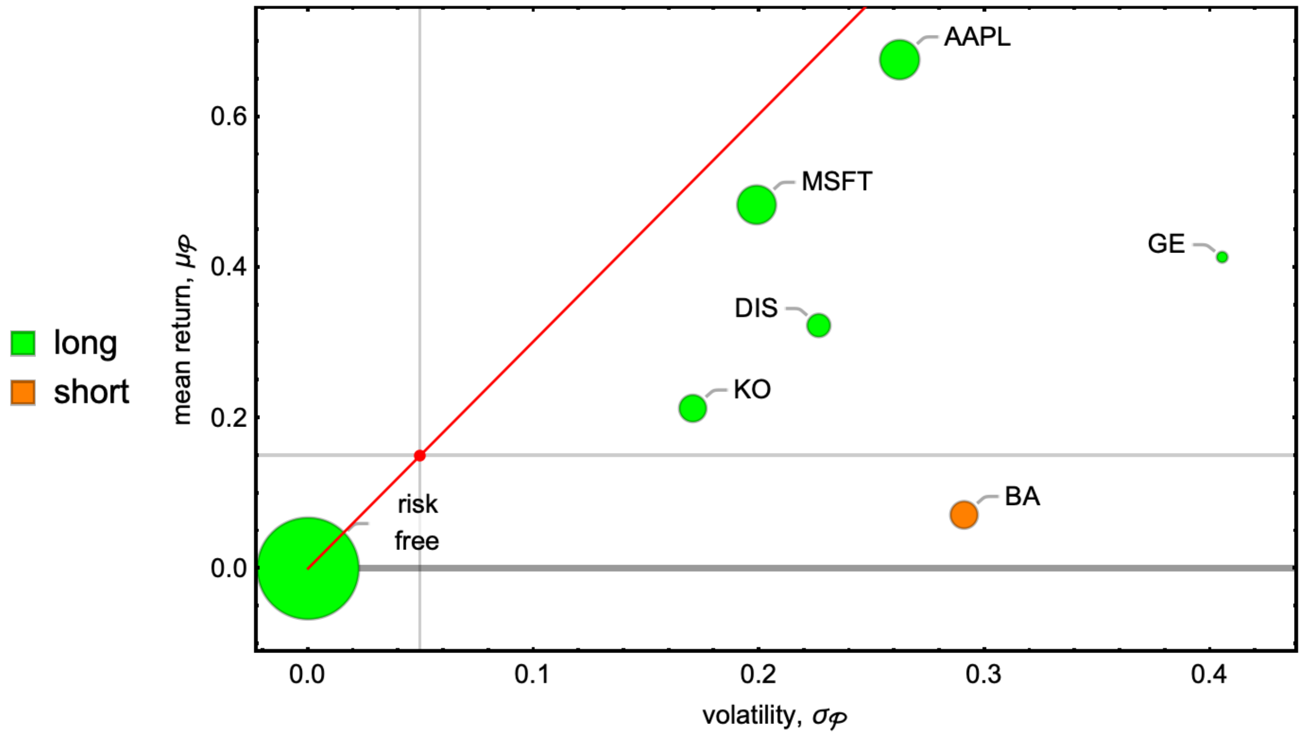

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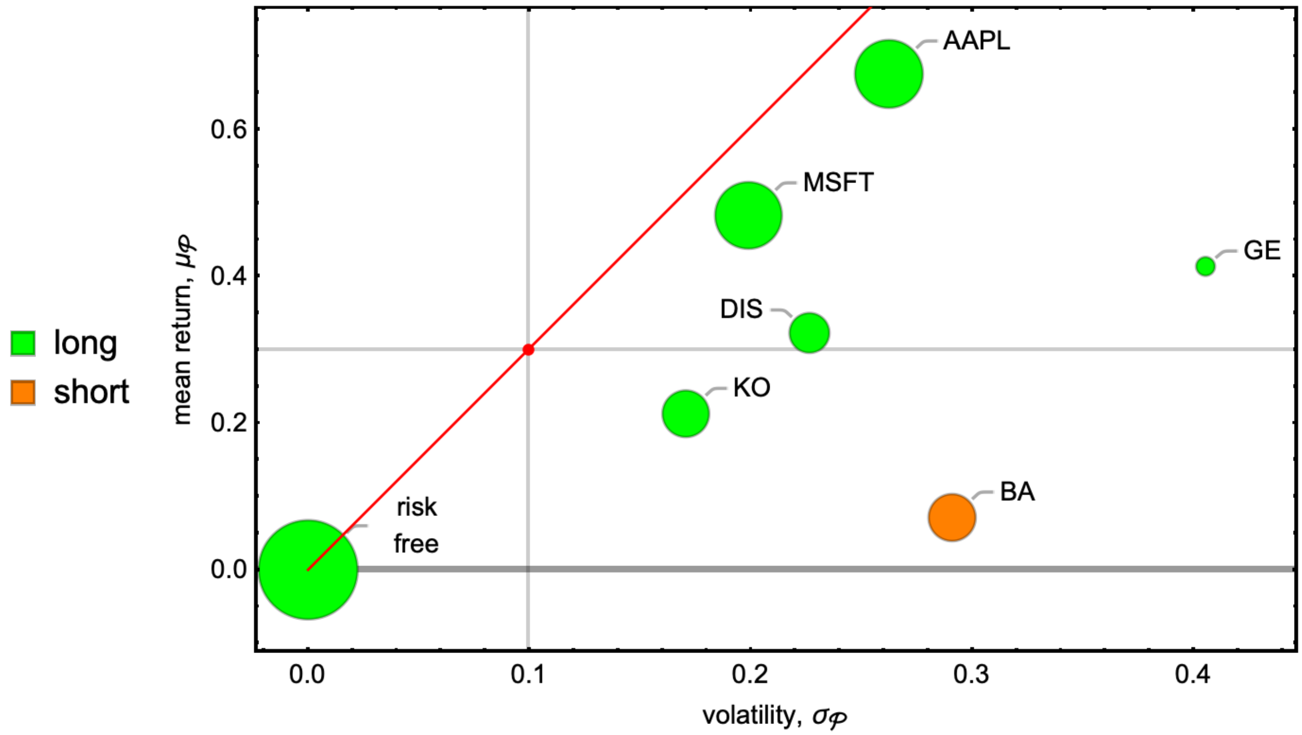

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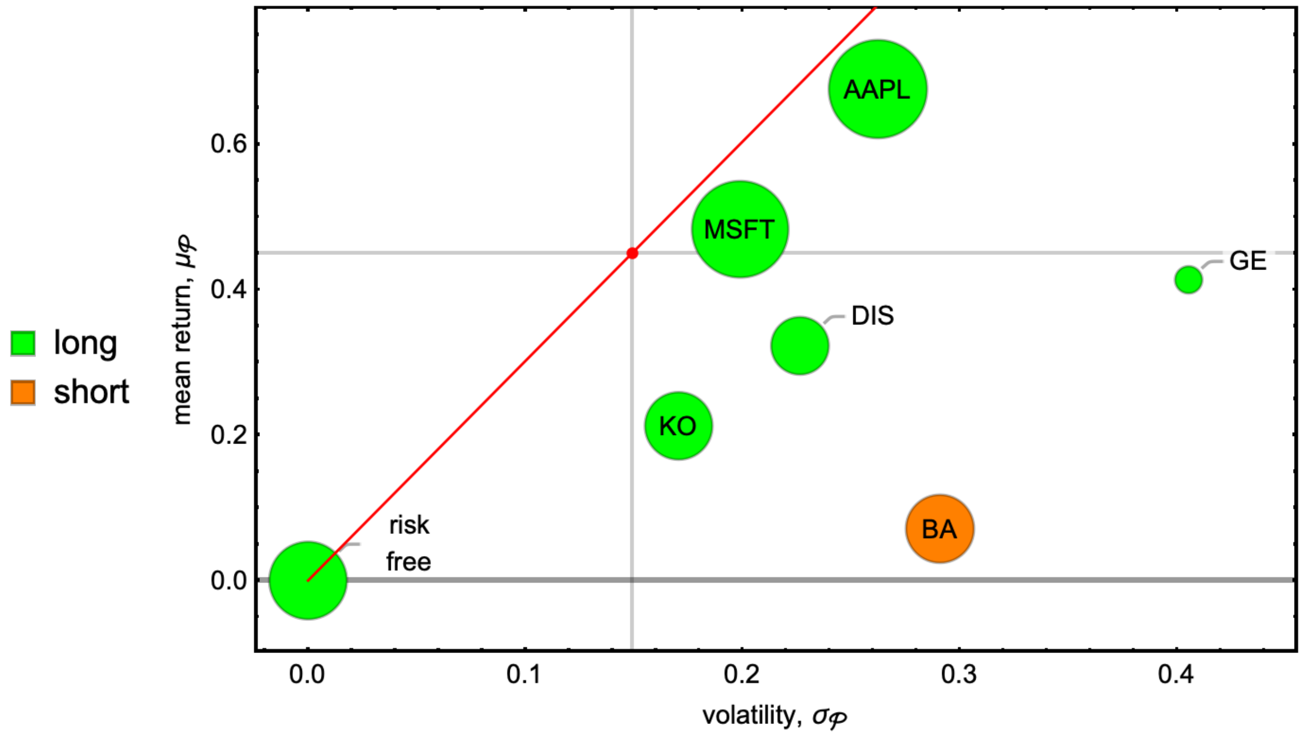

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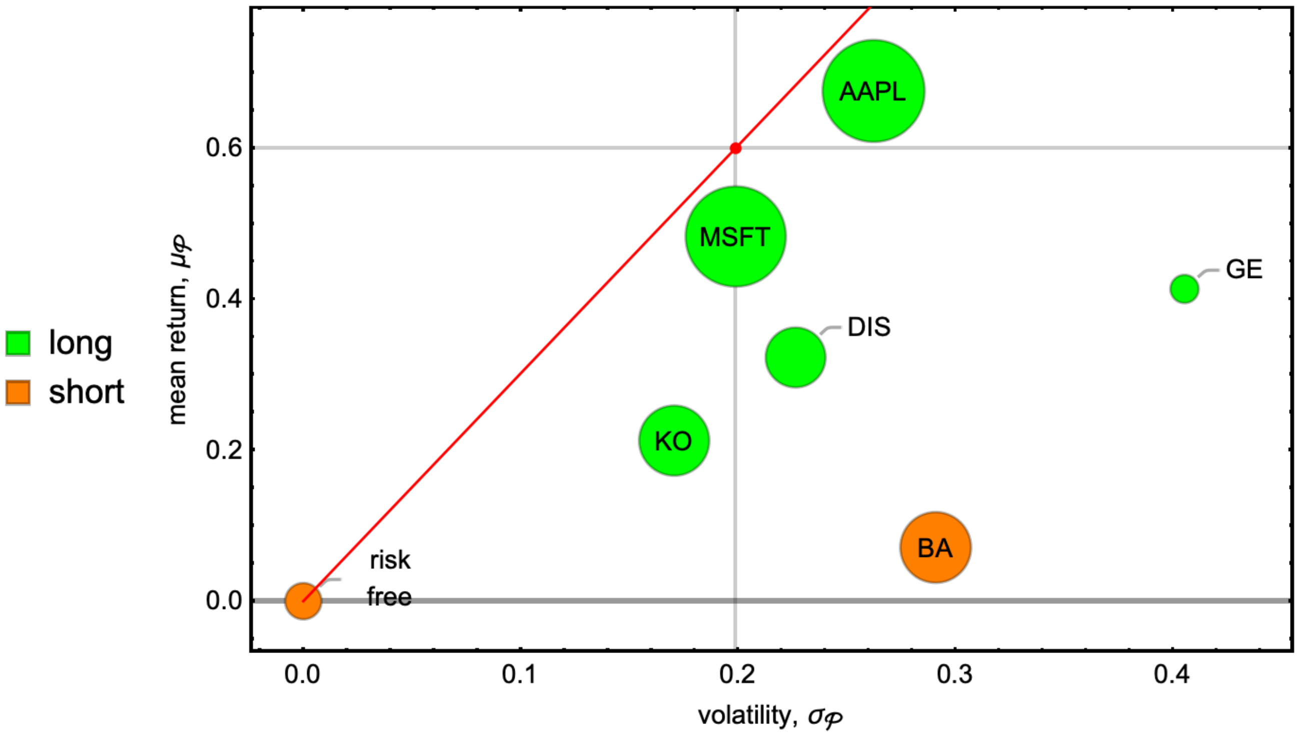

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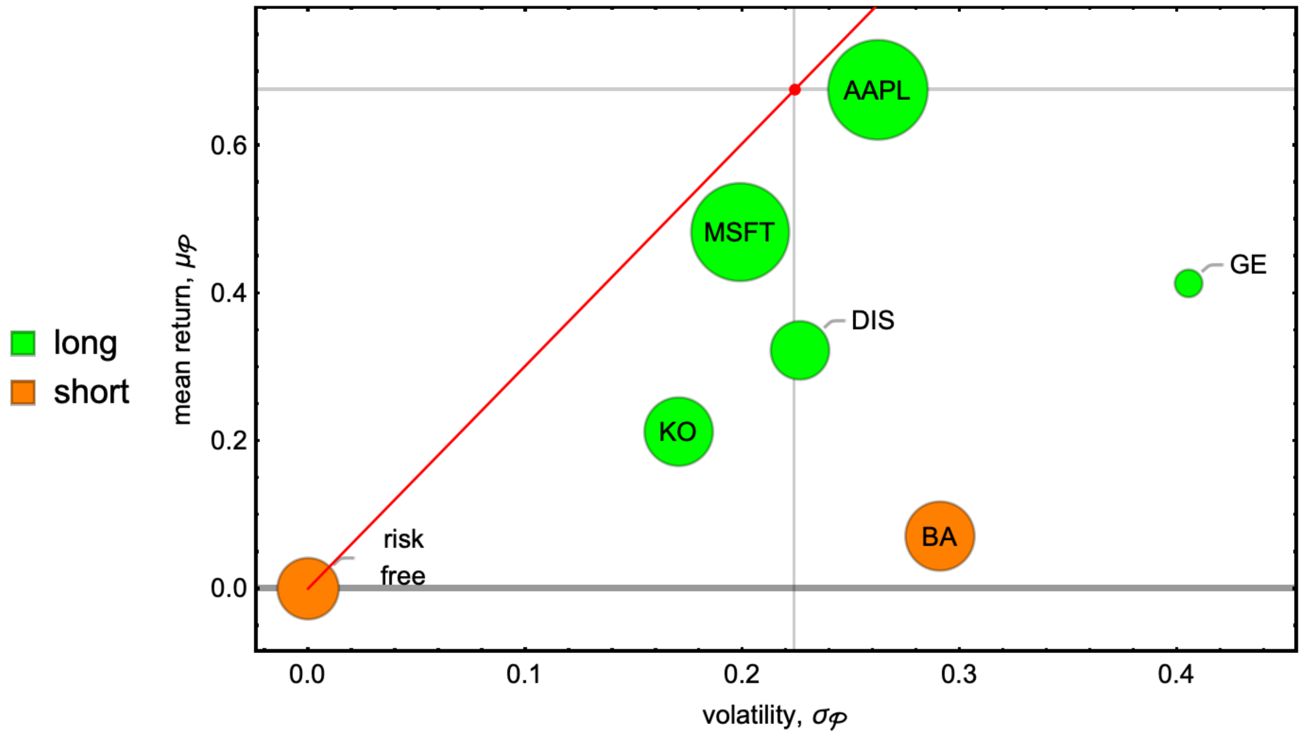


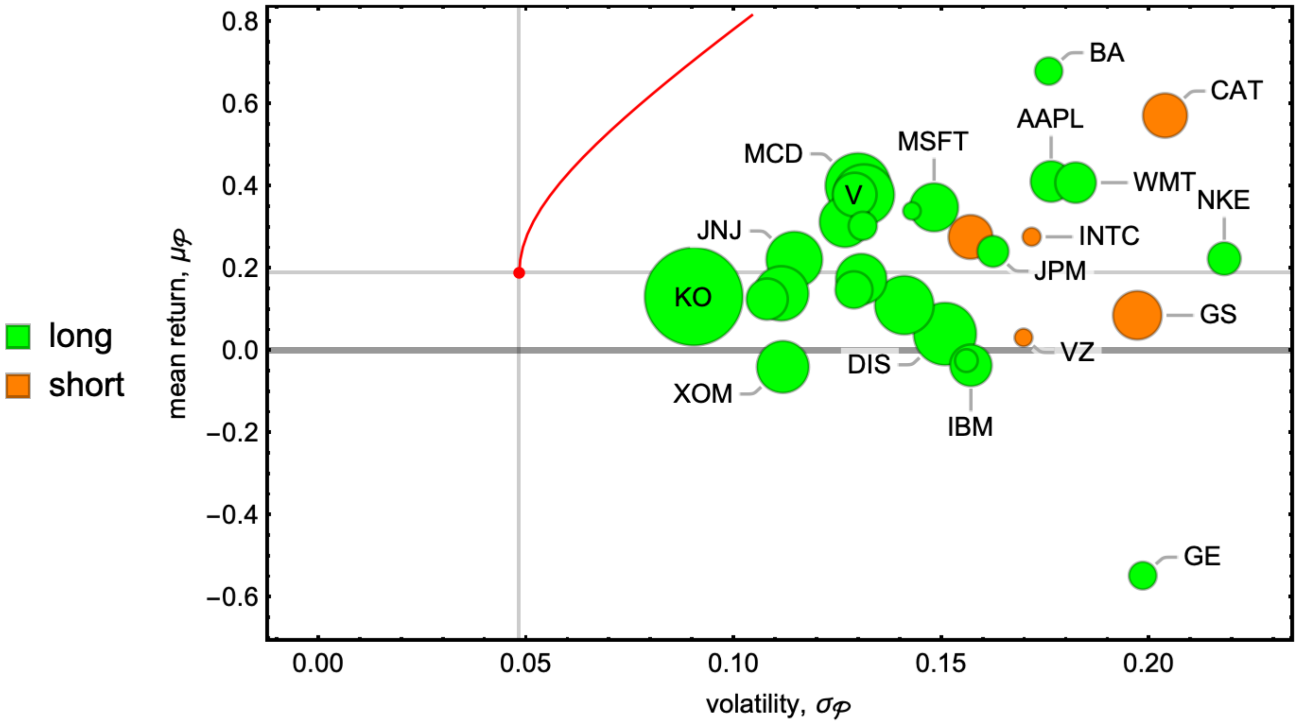
# Some Shortcomings of MPT



- MPT can produce crazy portfolios
- Variance as a measure of risk is debatable
- Proceeds from short sales cannot be reinvested in practice

# MPT Can Produce Crazy Portfolios

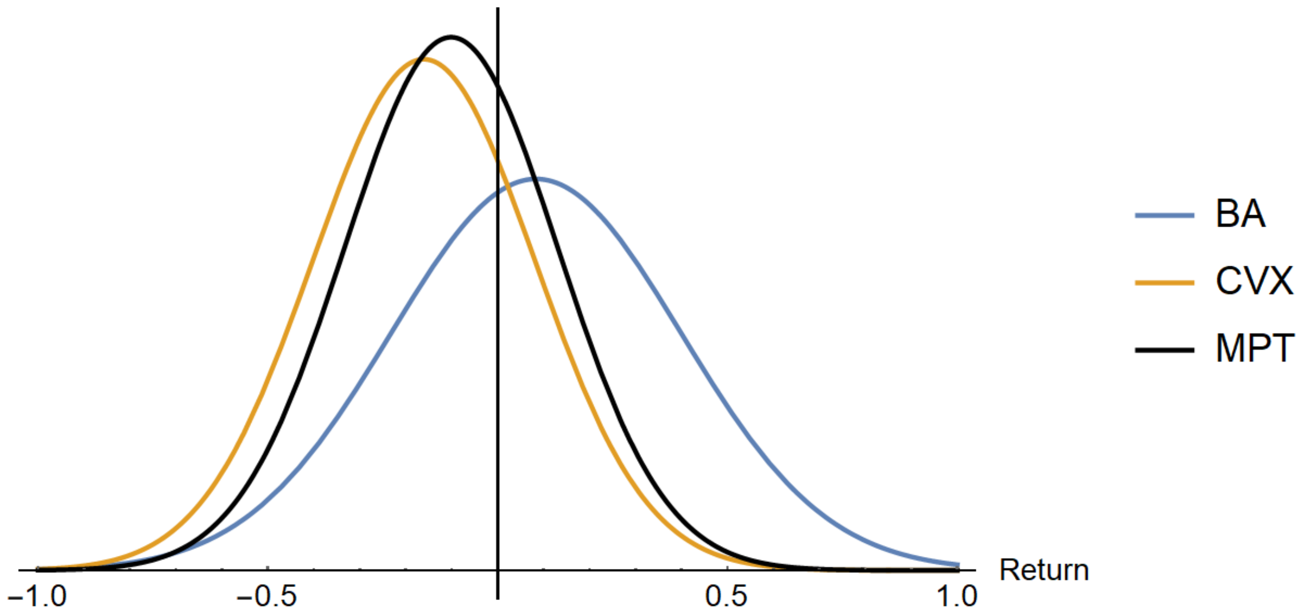





Dow 30 returns for 2017

# Is Smaller Variance Really What We Want?

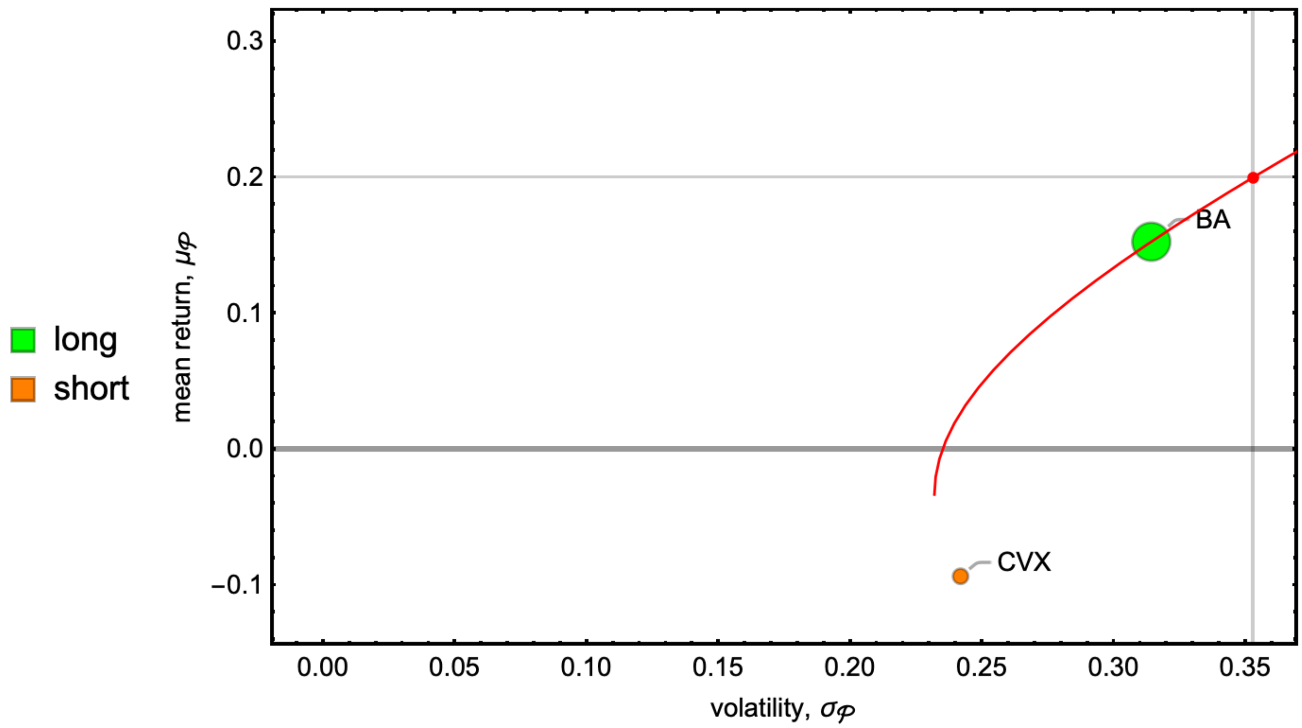





BA, CVX, and Portfolio Normal Distribution

# Proceeds from the Short Sale are Reinvested






Based on Data for BA and CVX in 2018

119% BA and -19% CVX

# Modified MPT



For $n$ assets, modify MPT to minimize $\sigma_{\mathcal{P}}^2$ such that

$$\underbrace{\boldsymbol{\mu}^T \boldsymbol{w} + 1.5r \sum_{i=1}^{n} R(-w_i)}_{\mu_{\mathcal{P}}} \geq \alpha$$

$$\sum_{i=1}^{n} [w_i] = 1,$$

where

$$[x] = \begin{cases} x & x \geq 0 \\ -0.5x & x < 0 \end{cases} \quad \text{and} \quad R(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

MPT

Modified MPT

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# Example Illustrating Constraints

Principal: $\$1000$

Spot Prices: Asset 1: $\$100$ Asset 2: $\$100$

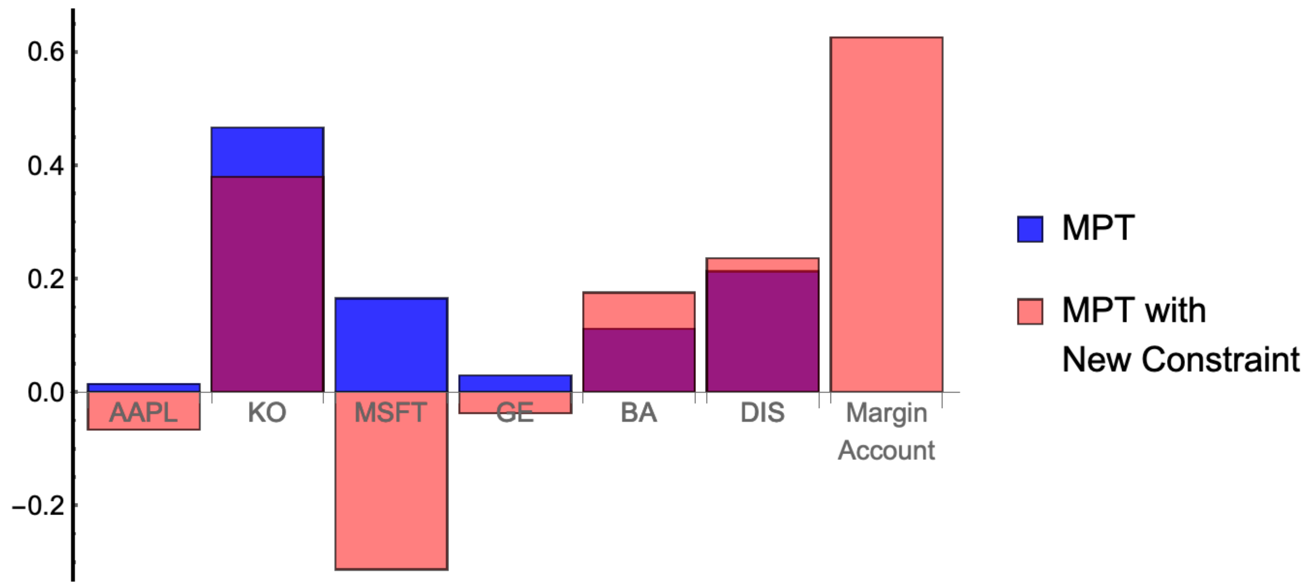
Long $\$500$ in asset 1 ($w_1 = 0.5$) and short $\$1000$ of asset 2 ($w_2 = -1$).

$$w_1 + 0.5(-w_2) = 0.5 + 0.5(+1) = 1.$$

$$\text{Margin:} \quad \underbrace{\$0}_{\substack{\text{deposit}\\ \text{for asset 1}}} + \underbrace{\$500}_{\substack{\text{deposit}\\ \text{for asset 2}}} + \underbrace{\$1000}_{\substack{\text{proceeds from}\\ \text{short sale}}} = \$1500$$

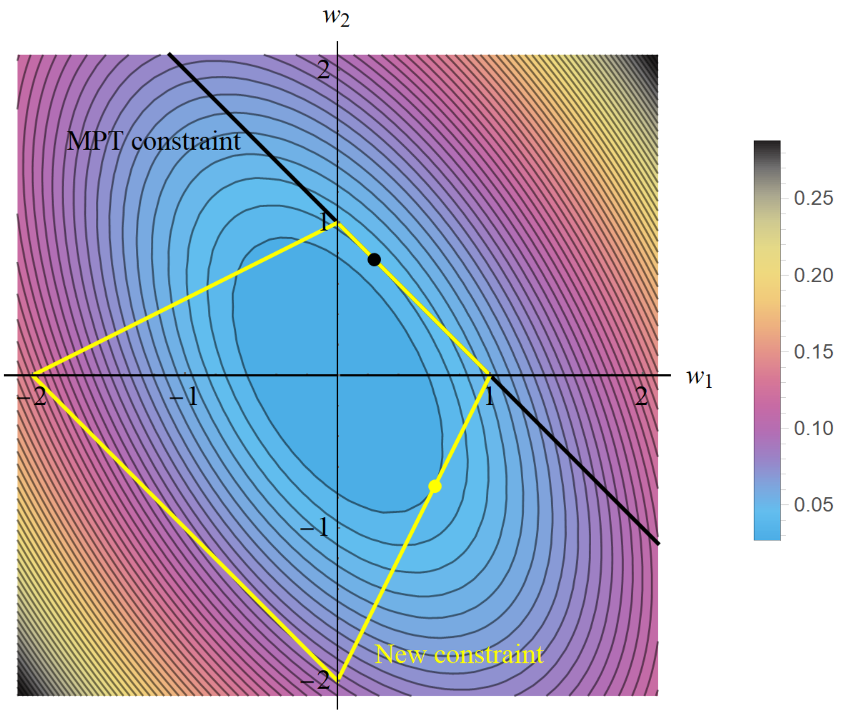
$$1.5\left[R(-w_1) + R(-w_2)\right] = 1.5\left[R(-0.5) + R(1)\right] = 1.5$$

# Example




Weights for 2019 Portfolio with and without the new constraint

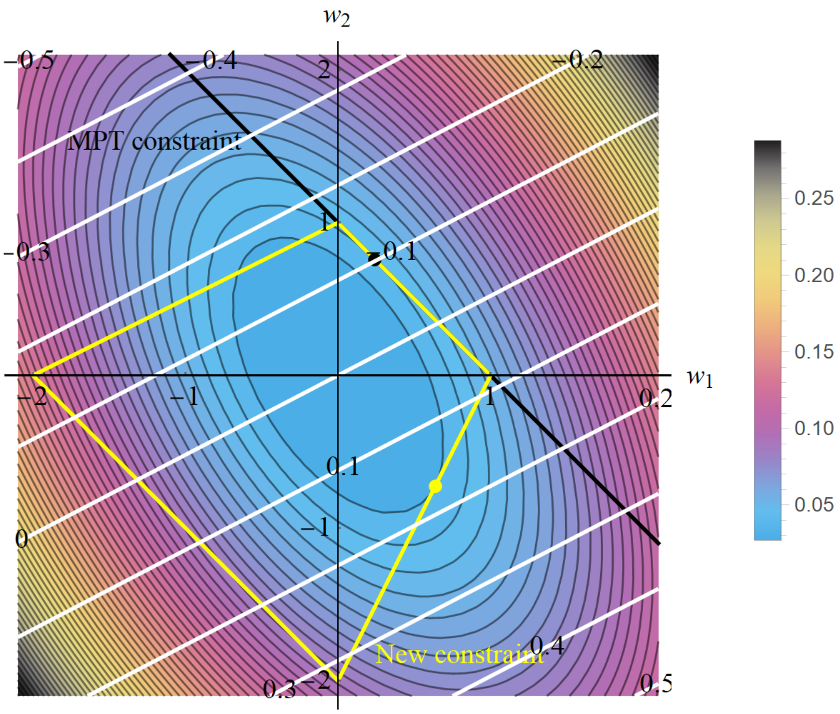
# Compare with MPT (2 Assets)




Based on Data for BA and CVX in 2018.

# Compare with MPT (2 Assets)

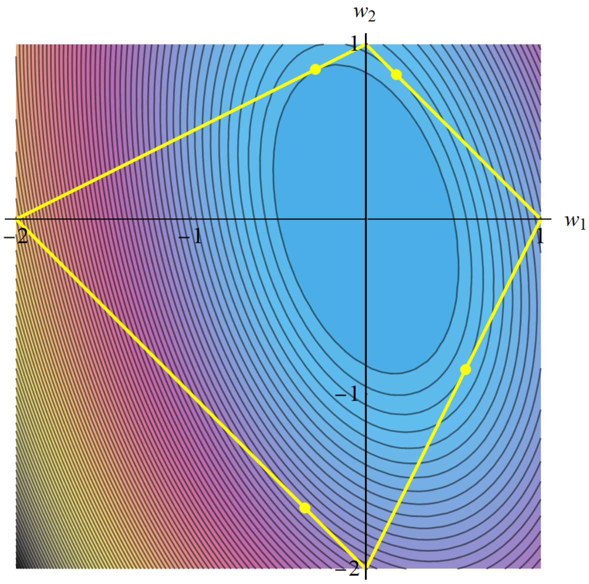


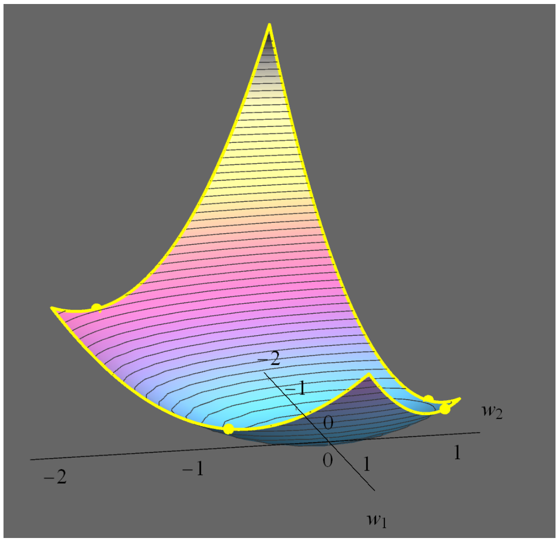



Based on Data for BA and CVX in 2018.

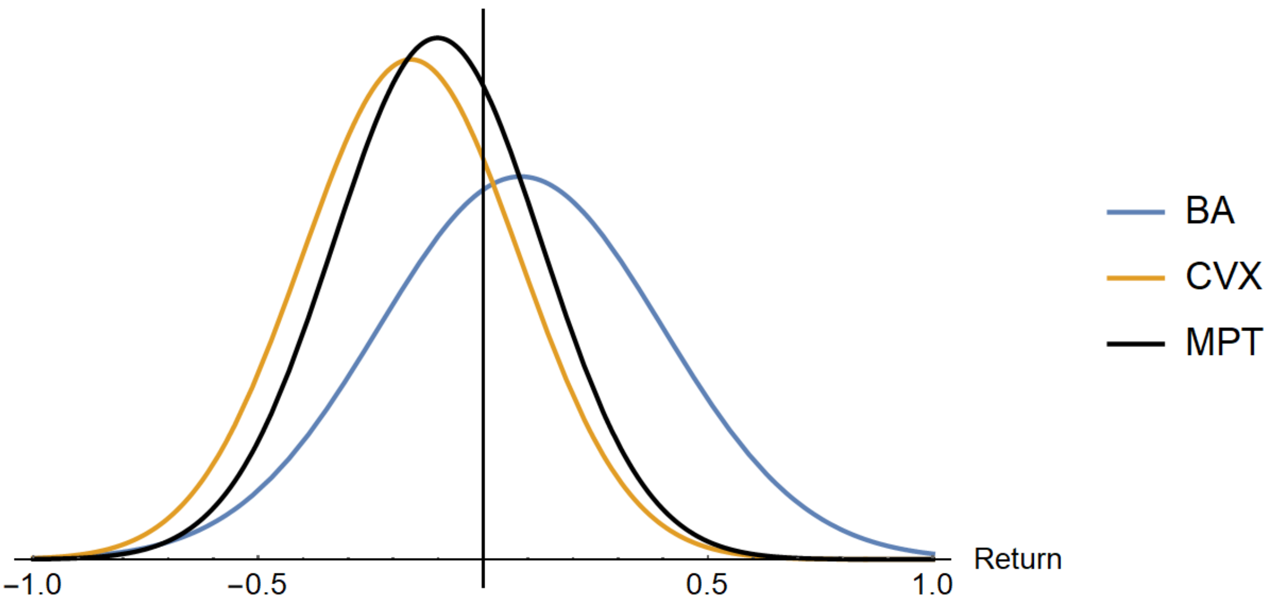
# Solution

Solutions must be found numerically, but the existence of many similar local minima can make the global minimum difficult to find.



# Probability Density Function for Returns




Based on Data for BA and CVX in 2018.

# Probability Density Function for Returns

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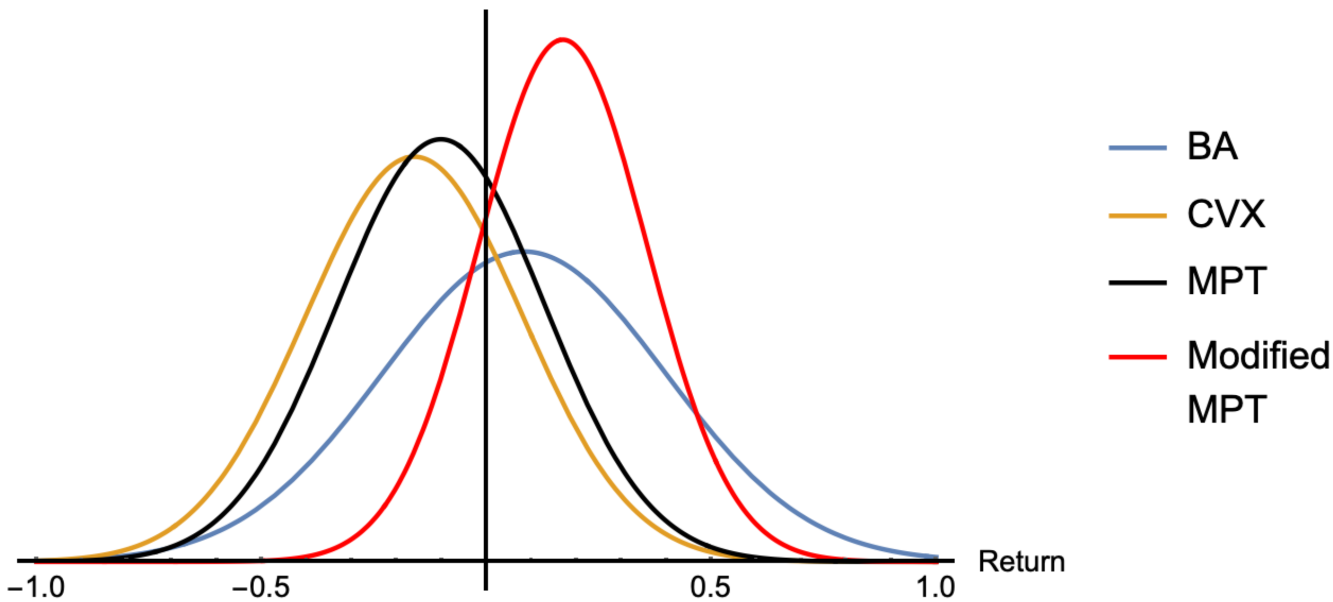
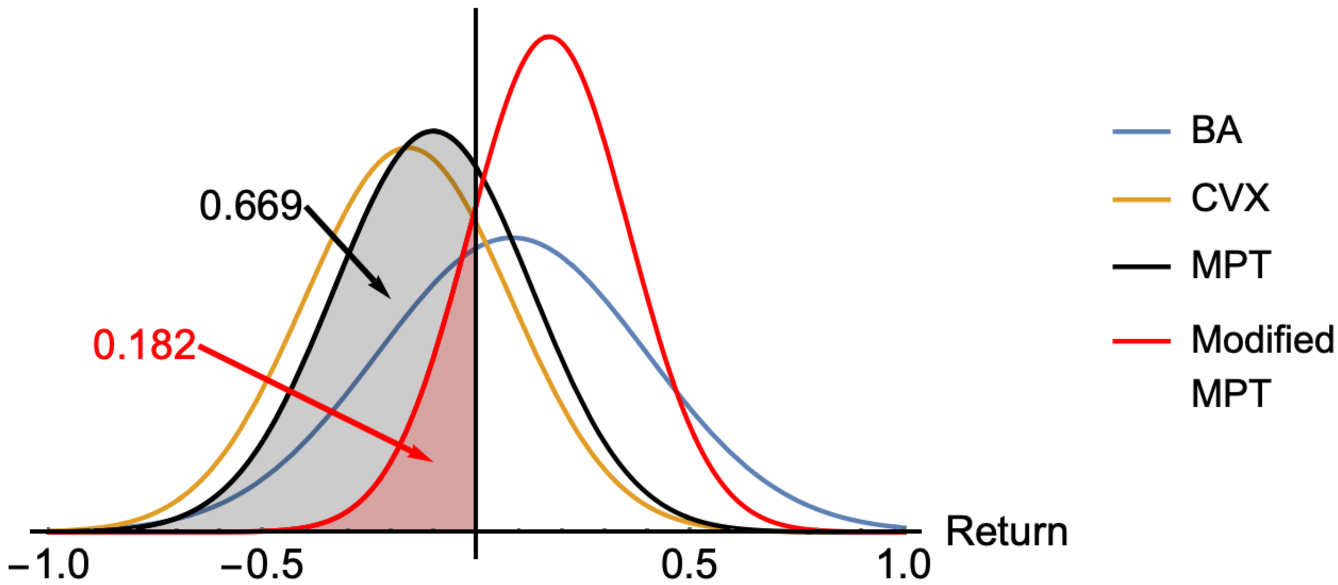



Based on Data for BA and CVX in 2018.

# Value at Risk






Based on Data for BA and CVX in 2018.

# Value at Risk

If the asset returns are normally distributed, then

$$\begin{aligned}
\text{Value at Risk} &= P(\text{portfolio return} < 0) \\
&= P\left(\frac{\text{return} - \mu_p}{\sigma_p} < \frac{-\mu_p}{\sigma_p}\right) \\
&= P\left(Z < -\frac{\mu_{\mathcal{P}}}{\sigma_{\mathcal{P}}}\right) \\
&= \Phi\left(-\frac{\mu_{\mathcal{P}}}{\sigma_{\mathcal{P}}}\right),
\end{aligned}$$

where $\Phi$ is the CDF for the standard normal distribution.

Note that minimizing the value at risk is equivalent to maximizing $\frac{\mu_{\mathcal{P}}}{\sigma_{\mathcal{P}}}$ or minimizing $\frac{\sigma_{\mathcal{P}}}{\mu_{\mathcal{P}}}$, the coefficient of variation for the portfolio.

# Post-Modern Portfolio Theory (PMPT)

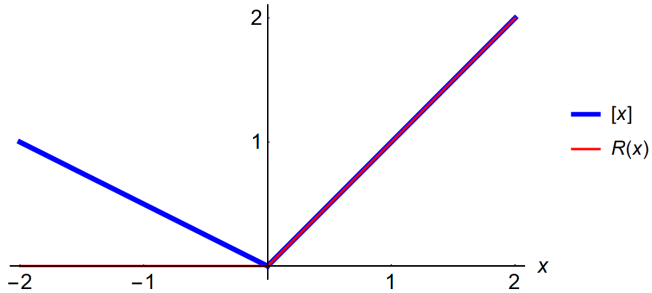
MPT

Modified MPT

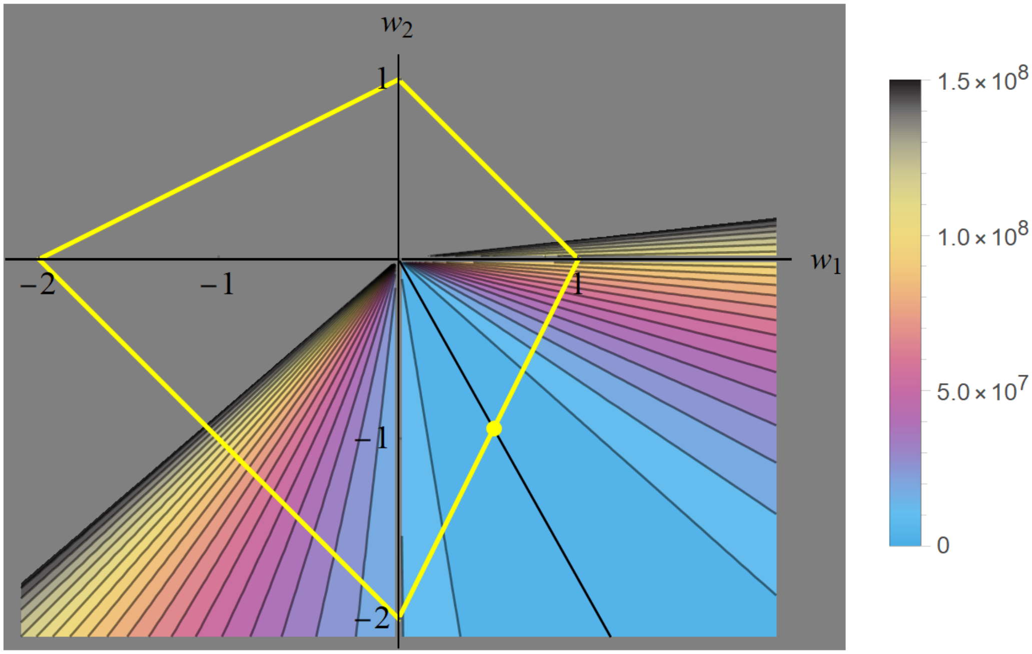
PMPT

Minimize $\dfrac{\sigma_{\mathcal{P}}}{\mu_{\mathcal{P}}} = \dfrac{\sqrt{\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}}}{\boldsymbol{\mu}^T \boldsymbol{w} + 1.5r \sum_{i=1}^{n} R(-w_i)}$ such that $\mu_{\mathcal{P}} \geq \alpha$ and $\displaystyle\sum_{i=1}^{n} [w_i] = 1$, where

$$[x] = \begin{cases} x & x \geq 0 \\ -0.5x & x < 0 \end{cases} \quad \text{and} \quad R(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}.$$



# Optimization Problem for PMPT




Based on Data for BA and CVX in 2018.

# Almost Closed-Form Solution

$$\boldsymbol{w} = \boldsymbol{\Sigma}^{-1}\left[\boldsymbol{\mu} - 1.5rH(-\boldsymbol{w})\right]t, \quad t > 0,$$

where

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

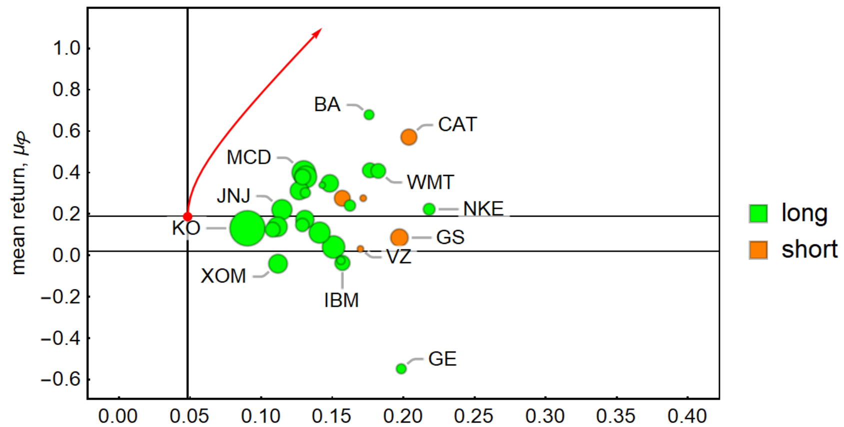
is the Heaviside step function and $t$ is chosen so that $\sum_{i=1}^{n}[w_i] = 1.$

If $r > 0$, then we can use fixed point iteration to quickly find the solution for any $r$ using

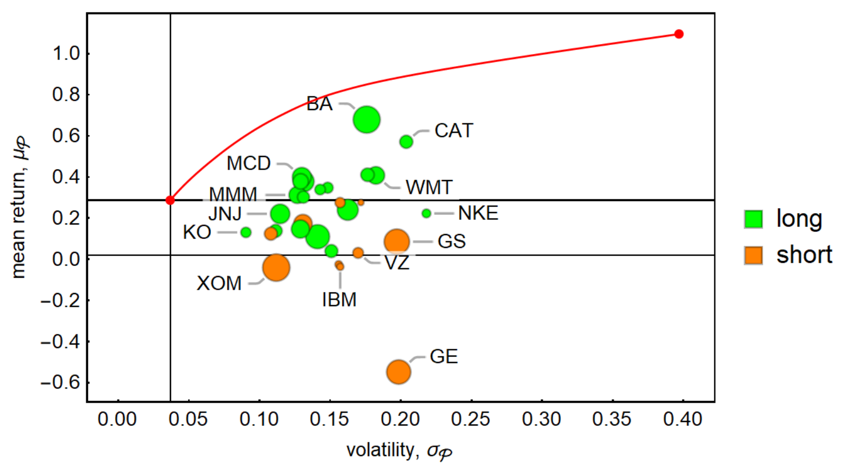
$$\boldsymbol{w}_k = \boldsymbol{\Sigma}^{-1}\left[\boldsymbol{\mu} - 1.5rH(-\boldsymbol{w}_{k-1})\right]t, \quad \boldsymbol{w}_0 = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}t_0.$$

# MPT vs PMPT

MPT



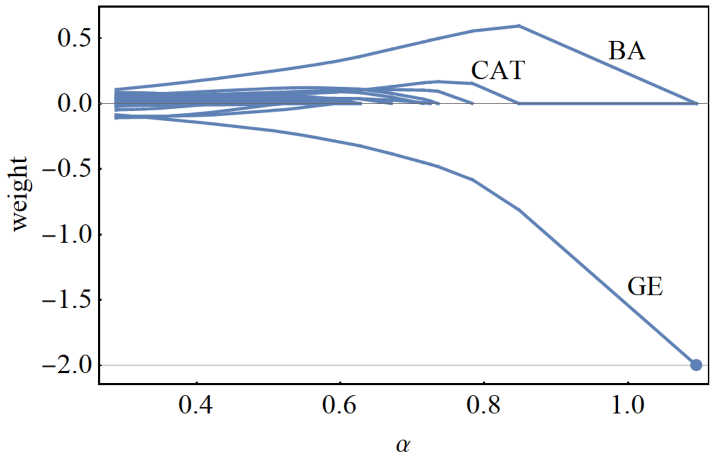

PMPT



# Questions?

# PMPT Weights

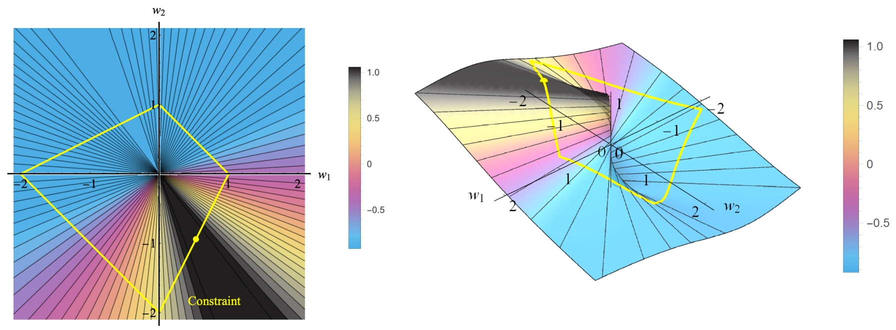
- MPT
- Modified MPT
- PMPT



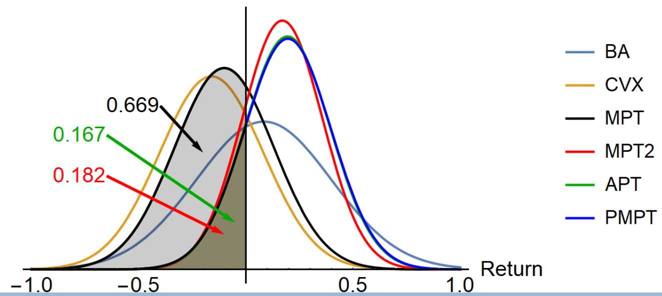

Note that the weights are piecewise linear in $\alpha$ and, except for GE, they achieve exact values of zero for larger values of $\alpha$.

# $r = 0.02$

It looks like the contours are still lines even with $r > 0$. Perhaps we can get a closed form solution here too?




Setting $r = 0.02$ doesn't seem to do much to the portfolio in this case.



# Back Testing



There may not be room for this in your presentation, but it's the kind of thing that a person from the audience may ask, so it might be good to prepare a slide on it.

Pictures comparing the results for all three models would be good to have.