# Post-retirement Financial Planning with Discrete Dynamic Programming: A Practical Approach

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**Abstract**

This paper finds the optimal consumption and asset allocation strategy for an Australian retiree who aims to maintain a level of minimum consumption. I use a discrete dynamic programming algorithm, with historical stock return distribution and a regime switching investment model, while taking into account mortality and realistic age pension means testing. The financial plan produced with this approach is compared to the 4% consumption rule and financial plans produced with alternative assumptions. In numerical examples by looking at the outcome assuming the retiree retired in one of past 86 years and followed these financial plans. Dynamic programming method produces flexible financial plans which are robust to the form of investment return distribution assumed, and provide superior outcome compared to constant consumption and investment rules. Taking into account of the required consumption floor or age pension means testing in dynamic programming process helps to smooth consumption, but provides little improvement in term of average consumption level or financial safety. And the optimal investment strategy is found to depends essentially on current market states only. Base investment strategies on retiree's age, wealth, future market states and transitions provide little improvement in the outcome of the retiree.

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# 1 Summary

This paper analyzes the practical application of dynamic programming method in financial planning for Australian retirees. This is the first literature that applies dynamic programming with predictable investment returns in Australian context, with age pension means testing taken into account. The following features differentiate this paper to previous literatures in this area:

- I discretize wealth as a state variable in the dynamic programming process and obtaine numerical solutions without the need to differentiate the utility function. This provides the flexibility to take many realistic assumptions into account in the dynamic programming process.
- Age pension means testing is taken into calculation in the dynamic programming process.
- Stock returns conform to the actual market history of the ASX All Ordinaries index.
- A regime switching investment model forecasts stock return using cumulative return of the whole trend that goes back to several years.
- The retiree aims to maintain a consumption floor throughout of retirement.
- Various models are evaluated with numerical examples by looking at the outcomes in 86 historical scenarios, by assuming the retiree retired in one of past 86 years.
- Risk aversion is measured by the distribution of ruin age.

Table 1 illustrates the financial advice produced by this dynamic programming method:

| Wealth W | ... | \$182,560 | \$188,928 | \$195,349 | \$201,821 | \$208,344 | \$214,916 | \$221,538 | ... |
|---|---|---|---|---|---|---|---|---|---|
| D(65,W,Good) | ... | \$14,016 | \$14,334 | \$14,334 | \$14,655 | \$14,977 | \$15,301 | \$15,627 | ... |
| D(66,W,Good) | ... | \$14,016 | \$14,334 | \$14,334 | \$14,655 | \$14,977 | \$15,301 | \$15,627 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| v(65,W,Good) | ... | 100% | 95% | 95% | 95% | 100% | 100% | 100% | ... |
| v(66,W,Good) | ... | 95% | 95% | 90% | 95% | 100% | 100% | 100% | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| D(66,W,Bad) | ... | \$14,016 | \$14,334 | \$14,334 | \$14,655 | \$14,977 | \$15,301 | \$15,627 | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| v(66,W,Bad) | ... | 0% | 0% | 0% | 0% | 0% | 0% | 0% | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |

Table 1: Consumption and investment strategy table

The optimal consumption and investment strategy depends on three state variables: age, wealth and market state. Strategy given in Table 1 advise that for example, a retiree currently age 66, should withdraw between $14,334 to $14,655 if her superannuation saving falls between $195,349 to $201,821, and allocates 95% into risky asset if the market state is good and 0% if the market state is bad.[2]

Here we looked at a single female homeowner who just retired at age 65 with a lump sum of $200,000 superannuation savings, and requires at least $24,000 per year in today's dollar to meet minimum consumption requirements. If she follows this financial plan, she would run out of money at age 100 in average, age 97 in the worst case while has an average level of consumption totaling $528,965 throughout of retirement. This is 17% higher than if she had withdrawn a constant 4% of initial wealth and allocated a constant 50% into risky assets. In which case the retiree also run out of money at age 100 in average but has an average level of consumption totaling $450,640, while can run out of money at age 87 in the worst case.

Chart 1 shows the selected wealth paths of these 2 financial plans, outcomes are selected as to represent the worst, the bottom 10%, the average and the top 10% of all 86 historical scenarios.

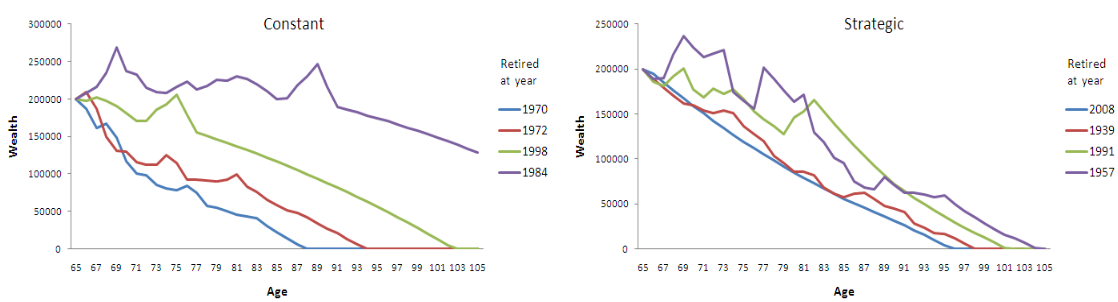


Chart 1: Wealth path of constant drawdown and investment strategy compared to the financial strategy in Table 1 produced with dynamic programming method

Several additional models with various assumptions have been compared, with the aim to find out what are the assumptions that would significantly impact the outcome of retirees in practice. I compare the models based on the outcomes in term of age of ruin and average consumption level in the 86 historical scenarios. The actual outcome is used to define the degree of risk aversion of retirees. Therefore if two different models, by

[2]This is a discrete approximation. When applying the result, linear interpolation is used. At the same time, since the retiree aims to maintain a consumption floor, therefore if the resulted consumption is less than minimum consumption required then I assume the retiree spends the required minimum consumption

adjusting the risk aversion parameters, can produce similar average consumption level and similar distribution of ruin age. Then we conclude that these two models have no much practical difference, although the strategy given by the two models maybe very different.

The main findings are as follows:

- Due to mortality, and the assumed absence of a market for life annuities, optimal amount of consumption is found to be highest at the early stage of retirement then decrease with age. This finding is consistent with the results obtained in Milevsky and Huang (2010).

- The current market state is the most important factor that determines the optimal investment allocation. While age and wealth have little effect on whether an allocation is optimal. This suggests that in practice, investment strategy does not need to be depended on consumption strategy and can be determined without dynamic programming.

- On the other hand, when wealth is high relative to required consumption floor, consumption depends on the outcome of investment strategy, which suggests that a valid investment model is still required in dynamic programming to produce the optimal consumption decision.

- However, when wealth is insufficient relative to required consumption floor. The retiree will just consume a constant amount equal to the required minimum consumption every year. In this case dynamic programming becomes unnecessary as age and wealth affect neither the consumption nor investment decision. This suggests that dynamic programming is most useful as a reserving tool to allocate surplus wealth over retirement hence prevent under or over consumption.

- Financial plans produced with dynamic programming method appears to be robust to the form of investment distributions assumed. It produce similar outcomes whether we assume the investment return to follow the actual historical distribution, normal distribution or simpler. This finding is similar to Brandt (1999). It support previous academic researches which used normal distribution to obtain closed form solutions. At the same time it suggest that dynamic programming method is useful for practical financial planning as we do not need to know the form of future investment return distribution for the financial plans to work, only the conditional mean and variance.

- Optimal investment strategy is found to only critically depend on current market states; market states of the future and transition between states have little effect. So a short term investment allocation strategy can be suitable for a long term investor when investment allocation is reviewed annually. Note however, I have assumed only two assets, which are liquidly traded with insignificant transaction cost, the situation can be very different when taking into account illiquid assets such as properties and life annuities.

- Taking age pension means testing into calculation in the dynamic programming process helps to smooth consumption in retirement, however it does not substantially affect the ruin age or average consumption level.

- I have looked at the case with a utility function that takes into account of the required consumption floor. The result shows that it does not significantly improve the financial safety non the average consumption level. This shows that financial plans produced with dynamic programming method are very flexible, and given the circumstances, the retiree can choose not to follow the recommendations occasionally, and would not significantly affect her financial safety.

This paper is organized as follows: the following section gives an introduction on the topic of post-retirement financial planning. Section 3 set up the problem and assumptions and Section 4 details the solution method using a discrete dynamic programming algorithm, while Section 5 present the investment model used in the process. Section 6 describe the numerical example to be used and models to be compared and Section 7 present the results and discussions.

# 2 Introduction

Post-retirement financial planning is becoming an increasingly important issue, as more and more Australians are retiring with a large lump sum of superannuation savings but without expertise in managing it. However, because of the complexity involved, most financial planners currently relying on "rules of thumb" to manage longevity and investment risks faced by retirees. For example, Otar (2009) suggests annually withdraw a fixed 3% - 4% of initial wealth, when superannuation saving is large or otherwise fully annuitize within the first 2 to 3 years of retirement.

Financial economists oppose these approachs, arguing that decisions made with these "one size fits all" rules are sub-optimal, expose retirees to avoidable risks, and cause retirees to either over or under-spend. (Sharpe, Scott and Watson 2007, Kotlikoff 2008). Alternatively they have addressed the problem using the dynamic programming technique dates back to Samuelson (1969) and Merton (1971). These can optimize consumption and investment every year according to the age and financial position of the retiree.

Some of the recent contributions including Brennan, Schwartz and Lagnado (1997) which recognized the predicability of stock return and incorporated an investment model into the dynamic programming, and forecast stock return with joint stochastic processes of stock return, interest rate, dividend yield and bond yield. Wachter (2002), incorporated consumption into utility function, and a model that predict stock return with joined stochastic processes of stock return and dividend yield, derived solution in closed form. Thorp, Kingston and Bateman (2007) discussed the importance of the objective for the retiree to secure a consumption floor, and applied the dynamic programming model in Australian context. While Milevsky and Huang (2010) take into account of mortality and solved the problem in closed form with deterministic stock return.

Unfortunately the financial planning community has largely ignored these models. One reason is because most of academic literatures present the outcome of their model as formula results and not in practical language. In contrast, many articles in practical financial planning present their model by looking at market histories and outcomes assuming retiree retires in one of the historical years (Bengen 1994, Cooley 1998 and Otar 2009).

Another reason is because most of academic models lack important practical assumptions. In Australia, publicly funded age pension currently provides 80% of retirees income in average, therefore take into account realistic age pension means testing into dynamic programming process is very important for its practical application.

This paper can be an be considered an extension to Brennan, Schwartz and Lagnado (1997) and Thorp, Kingston and Bateman (2007). The aim of this paper is to assess the benefit of using dynamic programming method in practical financial planning for Australian retirees. I solve the dynamic programming problem using a numerical method for an Australian retiree who aims to maintain a level of minimum consumption. With the assumption of historical stock return distribution and a regime switching investment model, while taking into account mortality and realistic age pension means testing.

The financial plan produced with this approach are then compared to the 4% consumption rule and financial plans produced with less realistic assumptions. Through numerical examples by looking at the outcome assuming the retiree retired in one of past 86 years and followed these financial plans. With the aim to find out what are the assumptions that would significantly impact the outcome of retirees in practice. The aim of this paper is not to explain retirees behavior under different models, therefore only the final outcomes are compared. If two models produce similar outcome in term of average level of total consumption and the distribution of ruin age, then they are concluded as have no much practical difference. Although the strategy given by the two models maybe very different due to different assumptions.

## 3 Problem setup and assumptions

Consider financial planning for a retiree who just retired at age $x$, with a lump sum of $W_x$ dollars of superannuation savings, can live at most to age $T$, and has no bequest motive.

The retiree receives no labor income, her consumption $C_t = A_t + D_t$ at any age $t$ consist only of age pension payment $A_t$ and drawdown from her superannuation account $D_t$. Age pension payment $A_t$ is determined by means testing and is therefore a function of $W_t$ and $D_t$, for detail on the calculation of age pension payment see Appendix A.

Her wealth in the superannuation account are invested in a risky asset and a risk free asset. The real return of risk free asset $R$ is assumed to be constant, while the conditional distribution of real return of the risky asset at age $t$, $(\tilde{z}_t|z_x, z_{x+1}...z_{t-1}) = \tilde{z}|M_t$, is conditional on the state of the market $M_t$ at that time, where $M_t$ is predictable from past returns.

The financial planning problem for this retiree is to maximize her utility over retirement, by choosing the amount of superannuation drawdown $D_t$ and proportion of wealth to be allocated to risky asset $v_t$ for every time period, which can be set up as follows:

Find the series $D(t)$ and $v(t)$ to maximize the expected sum of utilities:

$$\max E_x \left[ \sum_{t=x}^{T} tP_x U(C_t, t) | W_x, M_x \right] \tag{1}$$

Subject to the constraints:

1. $W_{t+1} = [W_t - D_t][v_t(\tilde{z}|M_t) + (1 - v_t)R]$
2. $W_t \geq 0$ for all $t$
3. $0 \leq D_t \leq W_t$ for all $t$
4. $0 \leq v_t \leq 1$ for all $t$

Here $tP_x$ denotes the probability a person currently age $x$ will live till at least age $t$. Note that the last constraint implies no borrowing to invest or short selling of risky asset.

I assume the utility function $U(C_t, t)$ of this retiree exhibits constant relative risk aversion (CRRA) and take the following form:

$$U(C_t, t) = v^{t-x} \frac{C_t^{1-\gamma}}{1-\gamma} \tag{2}$$

Where $v$ is the discount rate denoting the time preference of this retiree, while $\gamma$ denote the degree of risk aversion of this retiree, the bigger the $\gamma$, the more risk averse she is.

Note that for simplicity of notations, utility function $U(C_t, t)$ will be denoted as $U[C_t]$ for the rest of this paper.

# 4 Algorithm

The problem can be solved with dynamic programming techniques, as at any time $s$, all past utilities have been realized and the decision to be made by the investor would depend only on his current position $W_s$ and $M_s$, together with the expectation of future, which can be illustrated through the Bellman Equation of Equation (1).

$$\max E_s\left[\sum_{t=x}^{T} tP_s U[C_t]|W_s, M_s\right] = \sum_{t=x}^{s-1} U(C_t) + \max\left(U[C_s] + E_s\left[\sum_{t=s+1}^{T} tP_s U[C_t]|W_s, M_s\right]\right) \quad (3)$$

We can observe that the investor's state is depended on three state variables, age, wealth and market state. Noting that at any age, past decisions made affects only the initial wealth at that age. The problem can be solved as follows:

**Setting A:** The investor's wealth are discretized into one of $N$ groups denoted by $W(n)$ for $n = 1$ to $N$. In this paper I set $N = 151$, $W(n) = 2000 * (n-1)^{1.32}$ so the first wealth group denote wealth less than \$2,000, increase exponentially to the last wealth group which denote wealth greater than \$1,500,000. Note this setting poses the constraint that wealth cannot be negative.

**Setting B:** The distribution of real return of the risky asset $(\tilde{z}|M_t)$ is assumed to be a $L$ point discrete distribution, which can take $L$ total possible values with corresponding probability $P(\tilde{z} = z_l|M_t)$ for $l = 1$ to $L$.

**Setting C:** There are $K$ market states denoted by $M(k)$ for $k = 1$ to $K$. The market states are solely determined by historical realized return of risky assets, as $M_t$ is determined by $z_x$ to $z_{t-1}$, consequently $M_{t+1} = M(M_t, z_t)$ can be determined by $M_t$ and $z_t$.

**Setting D:** From Equation (3), denote:

$$F(s, n, k) = U[C_s] + E_s\left[\sum_{t=s+1}^{T} tP_s U[C_t]|W_s = W(n), M_s = M(k)\right]$$

and $F^*(s, n, k) = \max(F(s, n, k))$, the optimal expected value of the sum of all future utilities at time s, given wealth W(n) and market state M(k) at that time.

**Setting E:** Denote age pension payment as $A(D_t, W(n))$, a function of drawdown and wealth.

At age T, it is optimal to consume all the remaining wealth, therefor for every $W(n)$, $D_T = W(n)$ and

$$F^*(T,n,k) = U[C_T] = U[W(n) + A(W(n), W(n))] \quad \text{for all k} \tag{4}$$

Compute the value of $F^*(T,n,k)$ for every $n$ and $k$ and save as a $N \times K$ table.

Going backwards, from age $T-1$, at any age $t < T$, for every $n$ and $k$, choose the optimal pair of $(D_t|W(n), M(k))$ and $(v_t|W(n), M(k))$ through maximizing:

$$F(t,n,k) = U[D_t + A(D_t, W(n))] + p_t \cdot E_{t,k}\left[F^*(t+1)\right] \tag{5}$$

Where $p_t$ denote the probability a person age $t$ will live for at least one more year, and

$$E_{t,k}\left[F^*(t+1)\right] = \sum_{l=1}^{L} F^*\left(t+1, (W(n)-D_t)(v_t z_l + (1-v_t)R), M(M(k), z_l)\right) \cdot P(\tilde{z} = z_l|M(k)) \tag{6}$$

Note the value of $F^*\left(t+1, (W(n)-D_t)(v_t z_l + (1-v_t)R), M(M(k), z_l)\right)$ can be read from the previous table $F^*(t+1,n,k)$. Where $M(M(k), z_l)$ and $P(\tilde{z} = z_l|M(k))$ are given by an independent investment model. The optimal $D_t$ and $v_t$ that maximizes $F(t,n,k)$ are found using numerical methods, for detail see Appendix B.

For every $n$ and $k$, given the optimal $(D_t|W(n), M(k))$ and $(v_t|W(n), M(k))$, save the value of $F^*(t,n,k)$ as a $N \times K$ table, which is to be used for next step of dynamic programming process.

Follow the same procedure going backward till age $x$ will solve for all optimal decisions for every possible combination of age $t$, wealth $W(n)$ and market state $M(k)$. The optimal decisions are saved in tables $D(t,n,k)$ and $v(t,n,k)$ as Table 1.

# 5 Investment Model

An investment model is required to estimate the current market state $M(k)$, the conditional distribution of stock returns $P(\tilde{z} = z_l|M(k))$ and the transition dynamics between states $M(M(k), z_l)$ in Equation (6).

The basic assumption made here is that the market can be divided into bull trends and bear trends. If we are currently in the beginning of a bull trend then the investment return is expected to be good in the recent future, but if we are currently in the end of a bull trend then the investment return is expected to be bad. Vice versa for the bear trends.

The investment model is estimated with historical yearly real return of ASX All Ordinary Index in the past 87 years. For detail discussion of the data see Appendix C. Chart 2 shows the historical series of ASX All Ordinary Index divided in to bull markets (red) and bear markets (green).

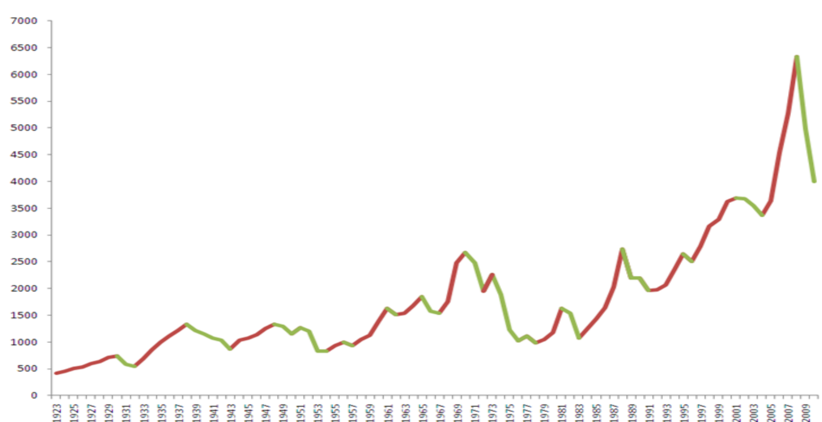


Chart 2: Historical trends of ASX All Ordinary Index

The markets are defined to be a bear market if the real investment return is less than 1% that year, otherwise it is defined to be a bull market. Market states and transitions are determined by past cumulative return since the beginning of current trend, which are summarized as in Table 2. Eight market states are identified in total, for detail on the estimation method see Appendix D.

For example: at the beginning of year 2007, we are in a bull trend which started at year 2004, the cumulative return of the index since 2004 to the end of 2006 is 156.5%, then according to Table 1, current market state is M(7).

| Cumulative return of the index since beginning of current trend | Market States |
|---|---|
| Loss greater than 35% | M(1) |
| Loss between 35% and 25% | M(2) |
| Loss between 25% and 10% | M(3) |
| Loss less than 10% | M(4) |
| Gain less than 20% | M(5) |
| Gain between 20% and 40% | M(6) |
| Gain between 40% and 60% | M(7) |
| Gain greater than 60% | M(8) |

Table 2: Definition of Market states

Transition between market state $M(M_t, z_t)$ can be found as follows, for example: Assume we are currently in state M(7), if current year return is a loss less than 10%, then we will be in state M(4) next year, but if the loss is greater than 10% but less than 25% we will be in state M(3) next year, etc. On the other hand, if current year return is a gain greater than 13.33%, we will be in state M(8) next year, but if we have a gain less than 13.33%, we will stay in M(7).

After defining the market states for the historical data, the conditional distribution of the risky asset return $P(\tilde{z} = z_l | M(k))$ can be easily constructed. For detail of construction of conditional distribution and transition between states see Appendix D.

# 6 Model Evaluation

In this section we consider financial planning for a single female homeowner who just retired at age 65 with a lump sum of \$200,000 superannuation savings, and want her money to last to age 100 in average. I assume she requires at least \$24,000 per year in today's dollar to meet minimum consumption requirements, if a financial plan advice her to spend less than \$24,000, she will spend \$24,000 instead.

This retiree is assumed to be able to live at most to age \$105, and her mortality follow the female mortality rate published in ABS(2009). The real return of risk free asset $R$ is assumed to be 2.07% per annual constant, while the discount rate $v = (1 + R)^{-1}$, which is used in the utility function to denote the time preference of the retiree. For detail discussion of these assumptions see Appendix C

I compare financial plans produced by using different assumptions in the dynamic programming procedure, by analyzing the results by looking at what would have happened if she retired in one of past 86 years and followed this financial plan. All financial plans are compared based on the same average age of running out of money for these 86 scenarios. (which is achieved by adjusting risk aversion parameters). With the aim to identify which assumptions are critical to produce a successful financial plan.

**Plan A: The 4% rule**
Under this financial plan, the retiree withdraws \$8,000 (4% of initial wealth, in real term) every year, while allocates a constant 50% into risky asset.

**Plan B: Complete Dynamic programming model**
This financial plan use consumption and investment strategy derived using the complete procedure described in Section 3 to 5.

**Plan C: Without investment model**
This financial plan is derived by assuming there is only one market state in the dynamic programming process, for detail see Appendix C.

**Plan D: Without age pension**
This financial plan is derived by not taking age pension into calculation in the dynamic programming process.

**Plan E: Considering the minimum consumption requirement**

The CRRA utility used do not consider the need of the retiree to secure a consumption floor, it would be interesting to see if the retiree can be better off if we take this requirement into dynamic programming.

This financial plan is derived by assuming the utility function of this retiree $U(C_t, t)$ take the following form:

$$\begin{cases} v^{t-x} log(C_t), & \text{if } C_t \geq \hat{C} \\ v^{t-x} log(C_t) - \gamma, & \text{if } C_t < \hat{C} \end{cases}$$

This utility function is a modification of the well-know log utility function, except this retiree suffers an utility loss $\gamma$ when the minimum consumption requirement is not met, the bigger the $\gamma$, the more risk averse she is. This utility function reflects the incentive of the retiree to maintain its consumption above the minimum required consumption throughout of retirement, as she suffers negative utility for every year ruined. On the other hand by varying the risk aversion parameter $\gamma$, she can choose a level of risk she's willing to take.

**Plan F: Unrealistic transition and investment distribution**

This financial plan is derived by assuming all market state transit to state M(1) while investment return distribution is set to be a 3-point discrete distribution, for detail see Appendix E.

## 6.1 With $30,000 minimum required consumption

So far we have looked at financial plans assuming the retiree has sufficient superannuation savings to finance the required consumption, however it would be interesting to look at the case when the superannuation saving is considered not enough.

The above 8 plans are reproduced by assuming the retiree requires $30,000 instead of $24,000 for minimum consumption.

In this case not all financial plans have the ability to produce the same average age of ruin. Plan A is now used as a benchmark when comparing models, in which we now assume the retiree to spend a constant $30,000 after age pension, and allocates a constant 30% into risky assets.

# 7 Results and Discussions

Table 3 summaries the outcome of the retiree under each financial plans when the required minimum consumption is \$24,000, assuming she retires in one of past 86 years. In this table the Average total consumptions are calculated as $\frac{\sum_{l=1}^{86}\sum_{t=65}^{105} tP_{65}v^t C_{t,l}^*}{86}$, where $C_{t,l}^*$ represent the realized consumption at time $t$ in scenario $l$.

| Financial Plan | Average age of ruin | Earliest age of ruin | Average Total Consumptions |
|---|---|---|---|
| A | 100 | 87 | \$450,640 |
| B | 100 | 96 | \$528,965 |
| C | 100 | 97 | \$470,392 |
| D | 100 | 96 | \$528,236 |
| E | 100 | 96 | \$528,765 |
| F | 100 | 95 | \$527,583 |

Table 3: Outcomes with \$24,000 minimum consumption

Chart 3 show the selected wealth paths of the financial plans, outcomes are selected as to represent the worst, the bottom 10%, the average and the top 10% of all 86 scenarios.

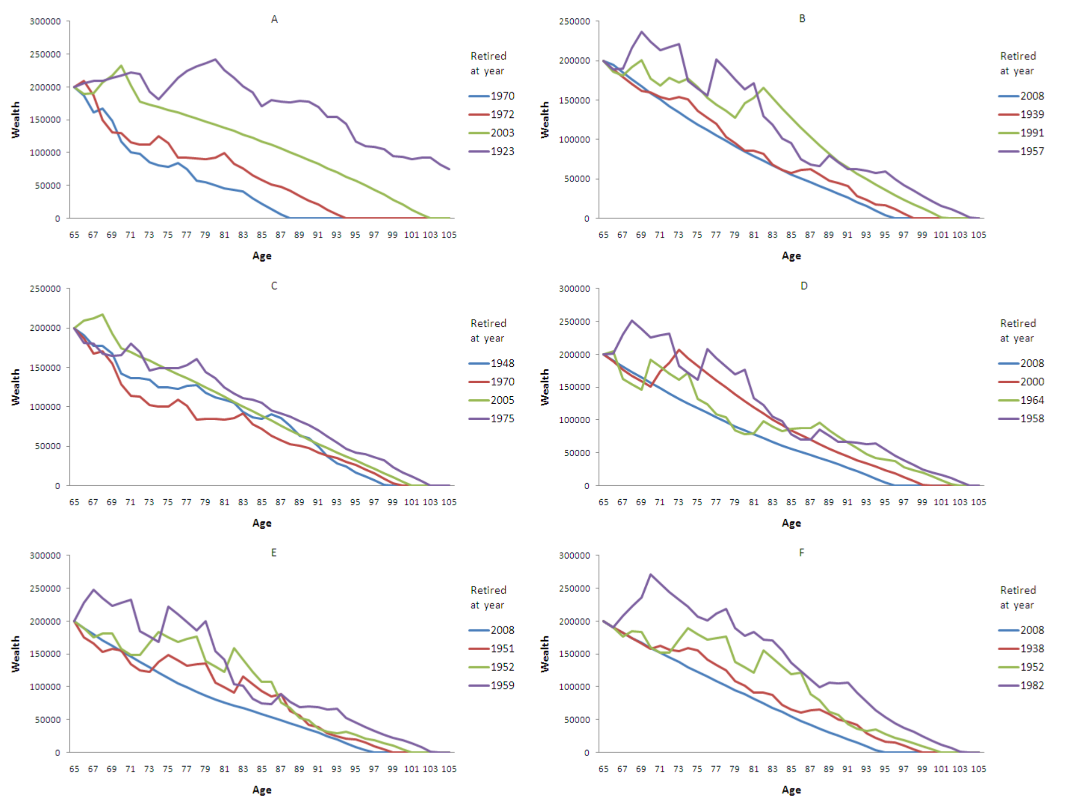


Chart 3: Wealth path with low consumption floor

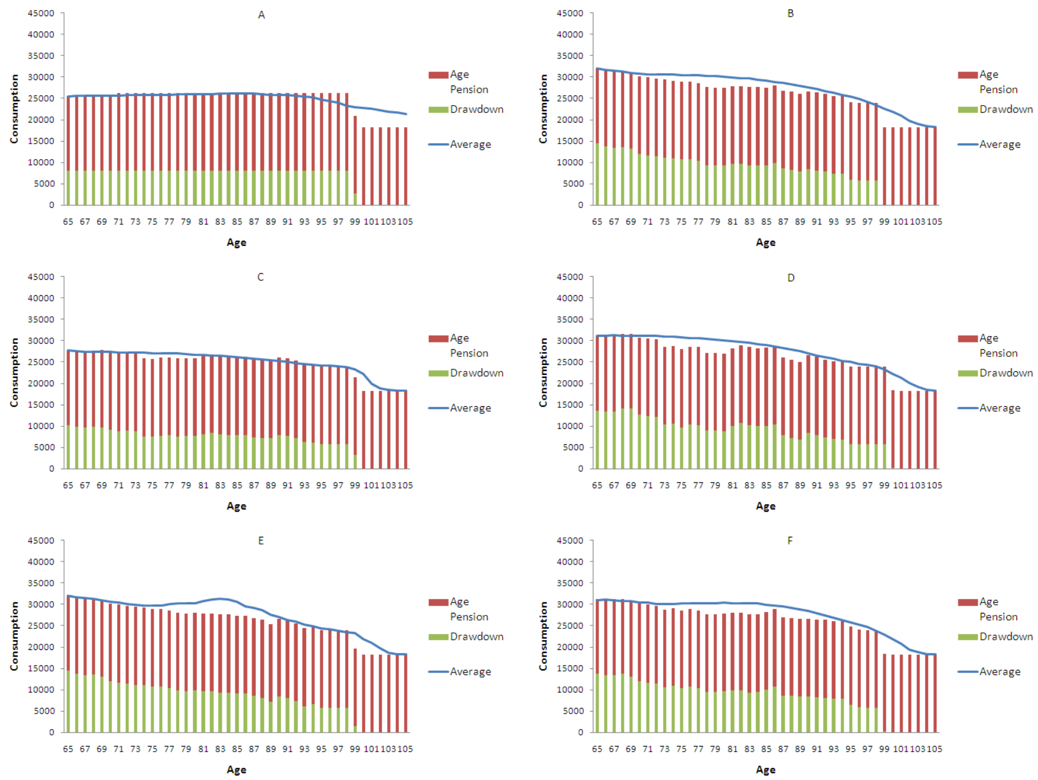


Chart 4: Average and the 1944 consumption path with low consumption floor

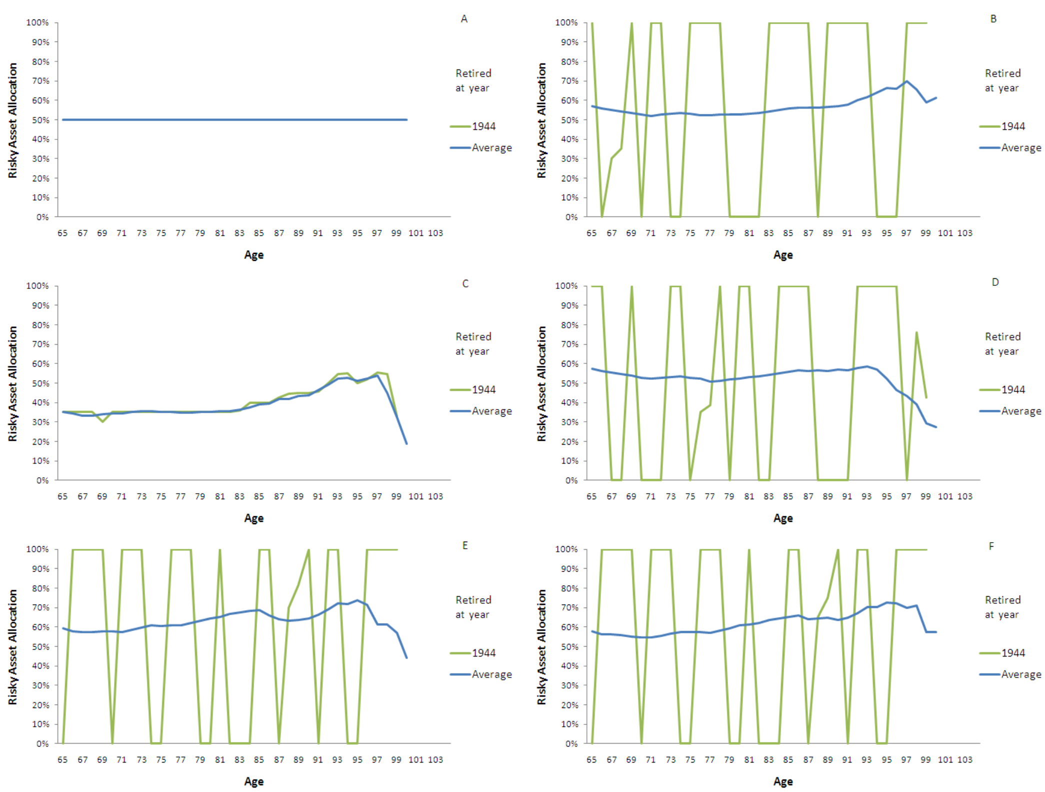


Chart 5: Average and the 1944 investment path with low consumption floor

Chart 4 and 5 shows the average consumption and investment paths, together with the actual paths if the retiree retired at year 1944 (which is chosen to represent an average year).

Comparing model A, B and C, we can see that dynamic programming do have the potential to greatly improve the welfare of the retirees. In model A, when both consumption and investment are held constant through retirement, there are great uncertainties as the retiree can either run out of money by age 87, or leaving a large bequest. In model C, the retiree can choose the optimal consumption depends on age and wealth, to achieve the outcome she desires. In this case she is much safer with the earliest age of running out of money at age 96, and has an average consumption being 4.4% higher compare to model A.

The outcome can be further improved with model B, in which the retiree not only has control over consumption, but also varies investment allocation depends on the market state, which resulted in a further 12.5% improvement in average consumption during retirement.

Age pension payment is not considered in the dynamic programming procedure in model D, the result shows that the retiree is not worse off than Plan B in term of financial safety or average consumption level. However the consumption path is much less smooth, which is expected.

In model E we used a different utility function to take the consumption floor into account in the dynamic programming process. The strategy produced is very different as the aim of retiree has shifted to secure the minimum consumption floor in the future. However it provides not clear benefit in term of outcome, as by varying the risk aversion parameter, similar outcome in term of financial safety and average consumption level can be achieved, as compared to model B.

Model F use unrealistic investment return distributions and transitions in dynamic programming procedure, the outcome however is not much worse off then model B, where the realistic assumptions have been used.

We can soon draw some conclusions from our results, after we look at the following tables and charts. These summarize the results when the retiree requires \$30,000 instead of \$24,000 minimum consumption, in which case her wealth is considered insufficient:

| Financial Plan | Average age of ruin | Earliest age of ruin | Average Total Consumptions |
|---|---|---|---|
| A | 84 | 80 | \$495,354 |
| B | 89.5 | 83 | \$531,388 |
| C | 83 | 76 | \$495,178 |
| D | 89.5 | 83 | \$530,997 |
| E | 89.5 | 83 | \$531,045 |
| F | 89.5 | 82 | \$530,970 |

Table 4: Outcomes with \$30,000 minimum consumption

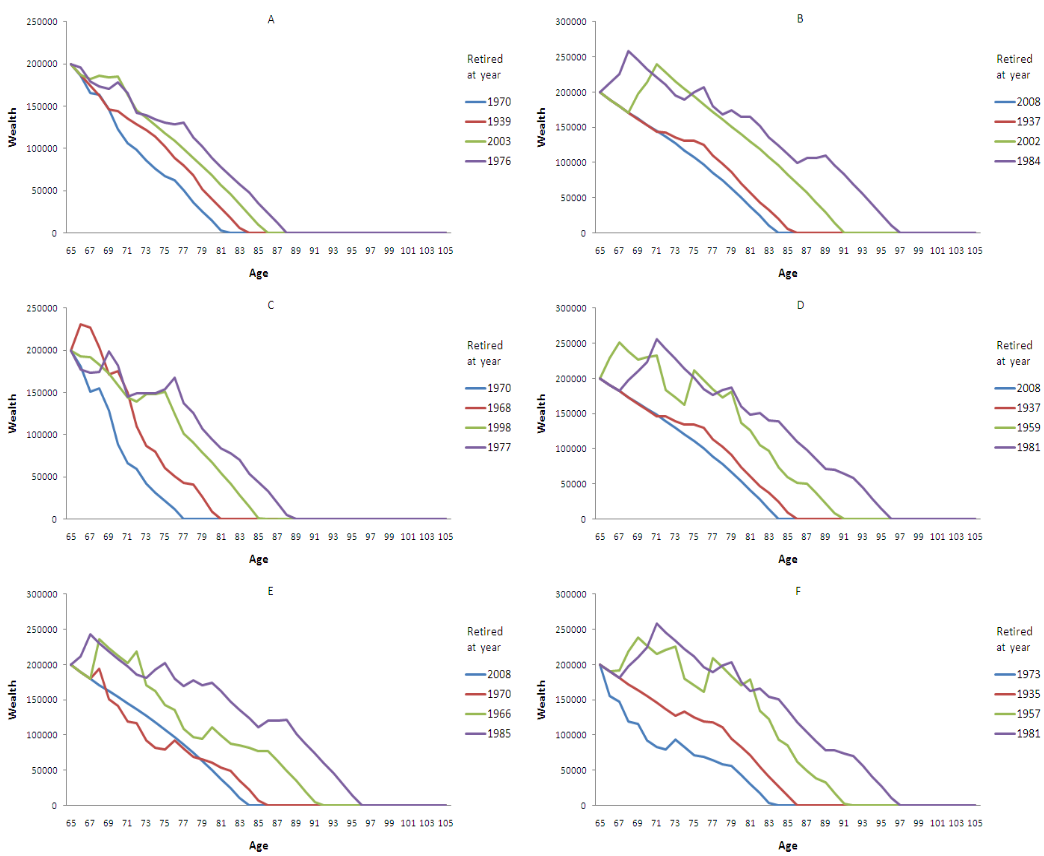


Chart 6: Wealth path with high consumption floor

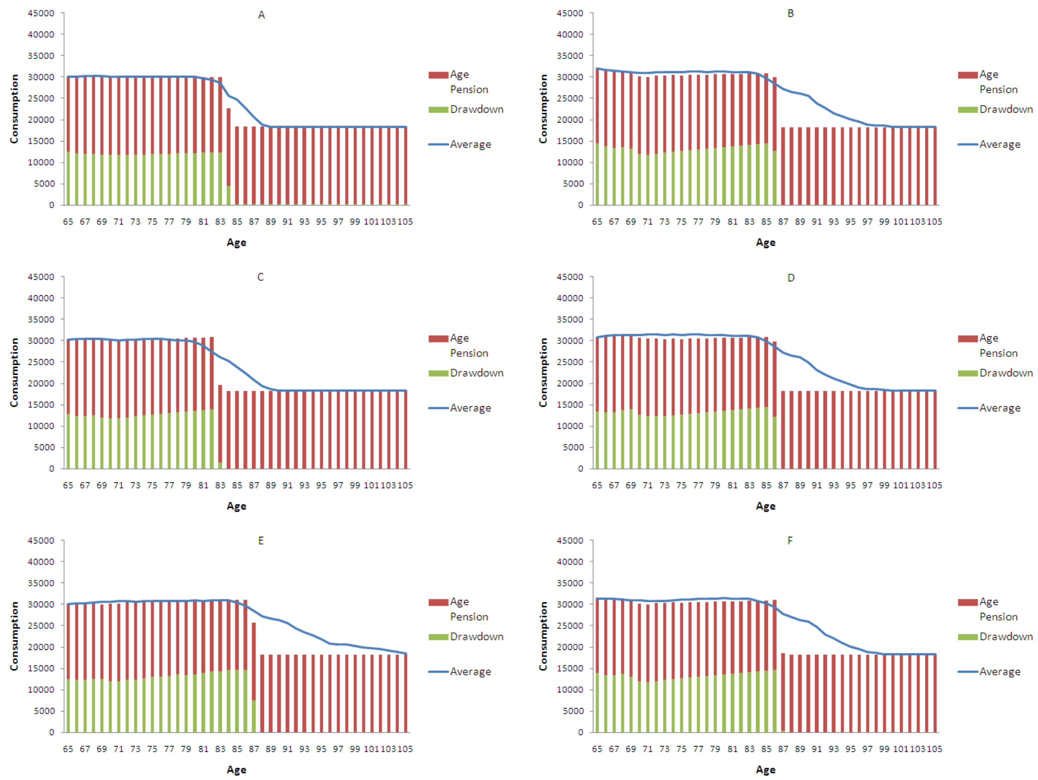

Chart 7: Average and the 1944 consumption path with high consumption floor

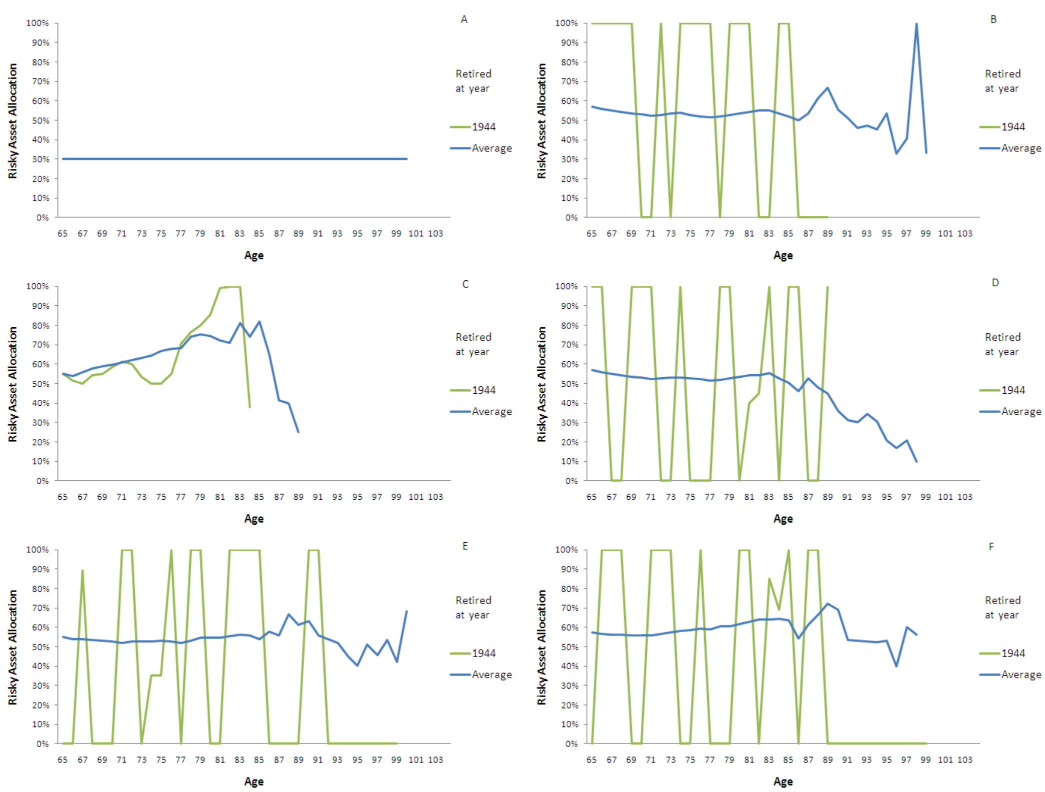

Chart 8: Average and the 1944 investment path with high consumption floor

We can see that in all the models, it is optimal for the retiree consume the constant \$30,000 minimum consumption every year, therefore she has not real control over consumption decision.

It is a surprise to see that model A now outperforms model C, which shows that optimizing investment allocation based on age and wealth in this case actually yield worse outcome than constant investment allocation. Model B still provides superior results, this indicates that optimal investment strategy depends critically on the state of the market, while age and wealth (therefore the consumption decisions) has little effect on whether an investment strategy is optimal.

On the other hand, consumption decisions depends critically on the outcome of investment strategy as we can in from Chart 4, model B implies a higher consumption level than model C, which suggest that a valid investment model is required in dynamic programming process to produce the optimal consumption decision when wealth are sufficient. However we can also conclude that when wealth is insufficient relative to required consumption floor, neither the consumption decision nor investment decision are depended on wealth or age, hence using a dynamic programming approach in this case has little benefit[3]. Therefore dynamic programming method is most useful as a reserving tool to allocate surplus wealth over retirement hence prevent under or over consumption.

Model D provides very similar results to model C in this case, which is also because the retiree have lost control over consumption, which determines the amount of age pension that she receive. This result shows that, taking age pension means testing into calculation in the dynamic programming process helps to smooth consumption in the retirement when there are sufficient wealth, but provide little improvement in average consumption level or financial safety.

Model E again provides very similar results to model C, we can therefore conclude that wether wealth is sufficient relative to the consumption floor or not. There is little practical benefit of including a consumption floor in the utility function. This is because every period, the dynamic programming process consider the optimal decisions for every level of wealth, and if at one period the required minimum consumption is higher

[3] In this case, constant consumption and threshold investment strategy can actually do better, for detail see Appendix D

than the amount of consumption recommended by the financial plan, it only transfer to a lower wealth than expected next period, but at the new period, the financial plan is still optimal given the realized wealth. Note however, we have only looked at a simple case[4] and in real life, what constitute a good outcome is much up to the individual preference of retirees.

Model F also provides similar results to model C regardless wealth is sufficient or not. One implication of this result is that market states of the future and transition between states have little effect on the optimal investment strategy, which suggest short term investment allocation strategy can be suitable for a long term investor when investment allocation is reviewed annually. Note however, I have assumed only two assets, which are liquidly traded with insignificant transaction cost, the situation can be very different when taking into account illiquid assets such as properties and life annuities.

This result also suggest that the form of investment return distribution has little effect on the outcome of financial plan in practice, whether it is assumed to be the actual historical distribution, normal distribution[5] or simpler. This is a surprising result yet seems reasonable here, as the consumption smoothing utility function used results that there are always some wealth reserved for future uncertainties. Therefore the effect of an unexpected investment shock will be spread across all future years and would not make material difference. On the other hand, although the actual investment strategies do depend on the distribution assumptions, it only make some scenarios better off and some worse off, without material difference in average.

This suggest that dynamic programming method is useful for practical financial planning as we do not need to know the form of future investment return distribution for the financial plans to work. However, it is important to note that I have used an investment model which is optimally fitted to historical data, therefore the market states in this paper are predictable. In practice, there is substantial uncertainty about wether an investment model fitted to past data have the ability to forecast the future. We can see in many models assessed the worst outcome turnout to be retiring at year 2008, due to the fact that we used constant return for future investments which are not forecasted with the investment model.

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[4]a more complicated case can be when a retiree requires \$30,000 p.a up to age $t$ and \$24,000 p.a thereafter and wants to solve for the optimal $t$ depends on her age and wealth

[5]see Appendix E for detailed results of normal investment return assumption

# 8 Conclusion

This paper presents a discrete dynamic programming algorithm which has the ability to handle complex real world problems faced by retirees, therefore enables dynamic programming method to be used in practical financial planning. And through numerical examples, I have illustrated the strength and weakness of the dynamic programming methods, and identified the critical and the not so important assumptions.

This paper however have several limitations and can be further improved in the following areas:

- This paper assumed the retiree has choice of two liquid assets, the problem can be very different when illiquid assets are incorporated such as property or life annuities. Extending this assumption is of both academic and practical interest. However, the problem can be very hard to solve in closed form with realistic assumptions, and as discussed in Brennan, Schwartz and Lagnado (1997), we face the curse of dimensionally if we use numerical method, as the holding of each illiquid asset needs to be added as an additional state variable.

- Rice and Higgins (2009) discussed the retirement expenditure patterns, as required expenditure of retirees is found to vary and depends on age and phases of retirement, which should be taken into account when determining financial strategies, we would also expect retirees to have different utility for consumption under different health states, as discussed in Yogo (2009). This may be done with a modification to utility function or an additional state variable denoting health.

- The investment model used in this paper only used historical real stock return to determine the market state, many other financial or economic variables can be used as dividend yield, interest rate, unemployment rate etc. Also the model is optimally fitted to the historical data used therefore the market states in this paper are predictable. To best assess the performance of an investment model, the parameters should be fitted to out of sample data as in Brennan, Schwartz and Lagnado (1997). To do this in this paper, dynamic programming models need to be run separately for each 86 scenarios, which is time consuming. However As we have seen that investment models can be assessed independently to dynamic programming process, it would be interesting to compare various investment models for their ability to forecast the future market, and then choose the best one to be used with dynamic programming method.

I wish this paper to shed some lights in the direction of future researches to continually improve the welfare of retirees.

## Appendix A

Following age pension parameter are accessed at http://www.centrelink.gov.au/ on 30th June 2010:

| Age Pension Assumptions | Singles | |
|---|---|---|
| Full Age Pension Rate | $644.20 | per fortnight |
| **Income Test** | | |
| Threshold | $146.00 | per fortnight |
| Rate of Reduction | $0.50 | per dollar over threshold |
| **Assets Test** | | |
| Threshold: Homeowners | $181,750.00 | |
| Threshold: Non-homeowners | $313,250.00 | |
| Rate of Reduction | $1.50 | per fortnight per $1,000 over threshold |
| Age Pension Age | 65 | |
| Pension Supplement | $56.90 | per fortnight |

Life Expectancy Table

| t | E_t |
|---|---|
| 65 | 21.15 |
| 66 | 20.32 |
| 67 | 19.49 |
| 68 | 18.67 |
| 69 | 17.87 |
| 70 | 17.08 |
| ... | ... |

Consider a single female homeowner who retired at age 65 with a lump sum of $200,000 superannuation savings.

Her entitlement of age pension at age $t$ with wealth $W_t$ at that time, is calculated as follows under the Asset test:

$$\max\left(26 * (644.2 + 56.9) - \max(W_t - 181750, 0) * 1.5 * 26/1000, 0\right)$$

And is calculated as follows under the Income test, given her annual drawdown $D_t$ and life expectancy $E_t$:

$$\max\left(26 * (644.2 + 56.9) - \max(D_t - 200000/E_t/(1 + 3\%)^{t-65} - 26 * 146, 0) * 0.5, 0\right)$$

The actual age pension payment she is entitled is the lower amount, after compared the entitlement under Asset and Income test.

## Appendix B

Here we detail the numerical method used to find the optimal consumption and investment decisions which, maximizes:

Equation (5): $F(t,n,k) = U[D_t + A(D_t, W(n))] + p_t \cdot E_{t,k}\left[F^*(t+1)\right]$

and Equation (6):

$$E_{t,k}\left[F(t+1)\right] = \sum_{l=1}^{L} F^*\left(t+1, (W(n)-D_t)(v_t z_l + (1-v_t)R), M(M(k), z_l)\right) \cdot P(\tilde{z} = z_l | M(k))$$

**Setting** Every period the investor has $c$ choices for drawdown denoted by $D(c)$ and $d$ choices for asset allocation denoted by $v(d)$. In this paper I set $c = 0$ to 500, $D(c) = 24000 * (c/100)^{1.32}$ and $d = 0$ to 20, $v(d) = d/20$ so she can choose to withdraw \$0 to \$200,000 (exponentially increasing) and allocate 0% to 100% (with 5% increment) of remaining wealth into risky asset. Note this setting poses constraints that consumption cannot be greater than wealth and the investor cannot borrow to invest in risky asset.

**Setting** $M(M(k), z_l)$ and $P(\tilde{z} = z_l | M(k))$ are two $87 \times 8$ matrixes estimated with the investment model, for detail see Appendix D.

The codes are programmed in MATLAB. Note that Equation (6) can be maximized first for the optimal investment decision and the result can then be taken into Equation (5) to solve for the optimal drawdown decision. As for every drawdown decision there is an optimal asset allocation decision, therefore asset allocation decisions can be assumed to be made given the drawdown decision.

At age $t$ after after consumption and probability of death, given wealth after consumption $W(n)$ (which replace $W(n) - D_t$ in Equation (6)) and market state $M(k)$, compute the value of Equation (6), for every pair of investment choices $v(d)$ (21 of them) and investment scenarios $z(l)$ (87 of them), save the value as a $87 \times 21$ matrix. Average to a $1 \times 21$ array, and choose the optimal investment allocation $(v_t | W(n), M(k))$ as the one that gives the highest value. This is done for every $W(n)$ and $v(k)$, and a total of $1,208$ $(151 \times 8)$ matrixes are computed for every time period. Save the optimal choices and the optimal value of function in tables $v^*(t,n,k)$ and $F^*(t', n, k)$.

At age $t$ before consumption, given wealth $W(n)$ and market state $M(k)$ compute the value of Equation (5), for every pair of consumption choices $D(c)$ (500 of them), where the value of $E_{t,k}\left[F(t+1)\right]$ can be read from previous table $F^*(t', n, k) = F^*(t', W(n) - D(c), M(k))$. The optimal drawdown $(D_t | W(n), M(k))$ is the one that gives the highest value. Save the optimal choices and the optimal value of function in tables $D^*(t, n, k)$ and $F^*(t, n, k)$. Then follow the same procedure going backward.

Following this method, it takes MATLAB 1 minute to solve the problem, as a total of $84,560$ matrixes or $98,385,560$ functions need to be calculated to produce the complete financial plan as Table 1. An alternative method, would be instead of discretizing consumption and investment choices and compute the function at every point, we can use a non-linear optimizer to find the optimal $v$ and $D$ every time period for every $W(n)$ and $M(k)$. This method have been used by various researchers including Brennan, Schwartz and Lagnado (1997) and Brandt (1999), it is however slower, as it take MATLAB 10 minute to solve the same problem, while providing very similar results.

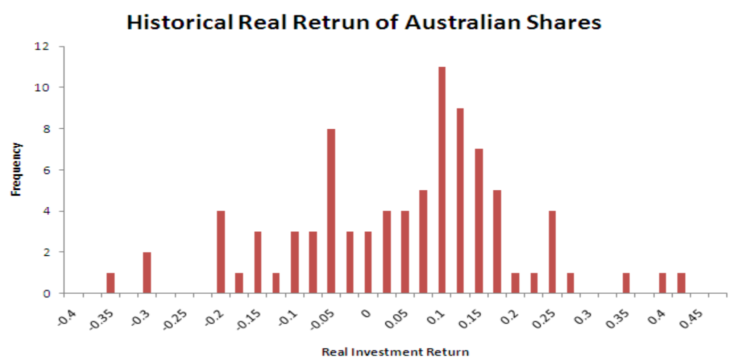
The problem of using the method of discretizing the choices, is that we would face the curse of dimensionality when we want to expand the investment choices to several assets, as the possible combination of investment allocations increase exponentially as the number of asset increases. One solution is to decrease the number of choices for each asset, the question is to what degree can we simplify the choices yet still obtain reasonable results.

**Appendix C**

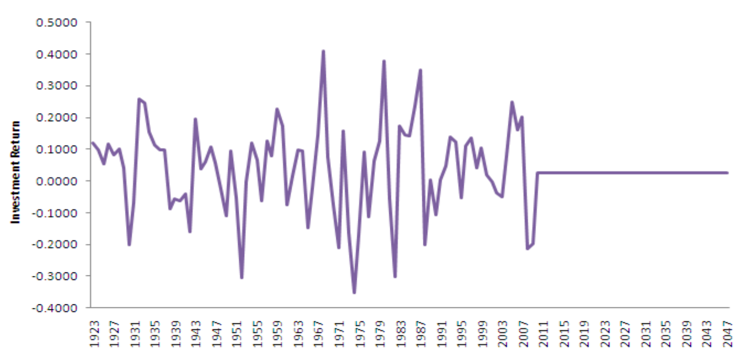
The data used for risky asset return is the ASX All Ordinary Index history financial market data from http://www.wrenresearch.com.au/downloads/, accessed on 20th April 2010. Which is then adjusted by inflation statistics obtained published by RBA (2010B), and adding 2% dividend and subtracting 1% management fee to obtain real investment return.

The real return of risky free asset is assumed to be 2.07% p.a. as the 3 month bank bill rate published by RBA (2010A), adjusted by inflation.

Following chart graphs the historical distribution of investment return of ASX all ordinary index, after adjusted with 2% dividend and 1% management fee. This is the unconditional distribution regarless of market state and is used in the dynamic programming process of Model C.




Following chart graphs the time series of investment returns, after 2009, investment returns are assumed to follow historical average of 2.65% p.a. This series is used for evaluating financial plans when the retiree is assumed to retire in one of historical years.



**Appendix D**

The basic assumption of the investment model is that the market can be divided into bull trends and bear trends. If we are currently in the beginning of a bull trend then the investment return is expected to be good in the recent future, but if we are currently in the end of a bull trend then the investment return is expected to be bad. Vice versa for the bear trends.

The parameters in the investment model are estimated by assuming a threshold strategy as follows:

First define a certain year to be a bear year if the real stock return in that year is less than $a$, otherwise the year is defined to be a bull year. Consecutive bull years constitute a bull trend and consecutive bear years constitute a bear trend, as illustrated in Chart 2.

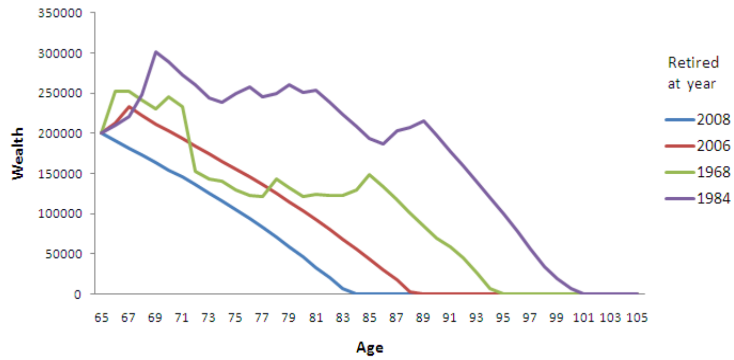
At the beginning of each year, compute the cumulative return since the beginning of latest trend. Define upper threshold $u$ and lower threshold $l$, which constitute a threshold investment strategy, such that we invest $v_{bu}$ into risky asset when the cumulative return is either less than $l$, or greater than 1 but less than $u$, while invest $v_{be}$ into risky asset when the cumulative return is either greater than $u$, or less than 1 but greater than $l$.

We estimate the parameters $a$,$u$,$l$,$v_{bu}$ and $v_{be}$, by considering a financial plan for a single female homeowner who just retired at age 65 with a lump sum of $200,000 superannuation savings. Assume she spend a constant $30,000 p.a. in real term and follows this threshold strategy. The parameters that maximize her average total consumption over retirement in the 86 historical scenarios are are follows:

| | |
|---|---|
| $a$ | 1% |
| $u$ | 160% |
| $l$ | 65% |
| $v_{bu}$ | 85% |
| $v_{be}$ | 0% |

Table 5: Optimal Parameter of Investment Model

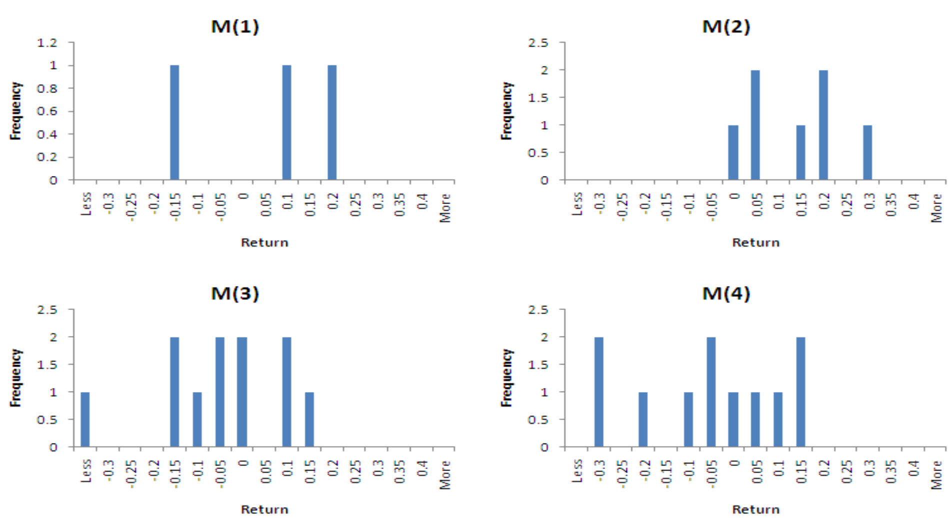
The outcome of the retiree can be illustrated with the following chart:

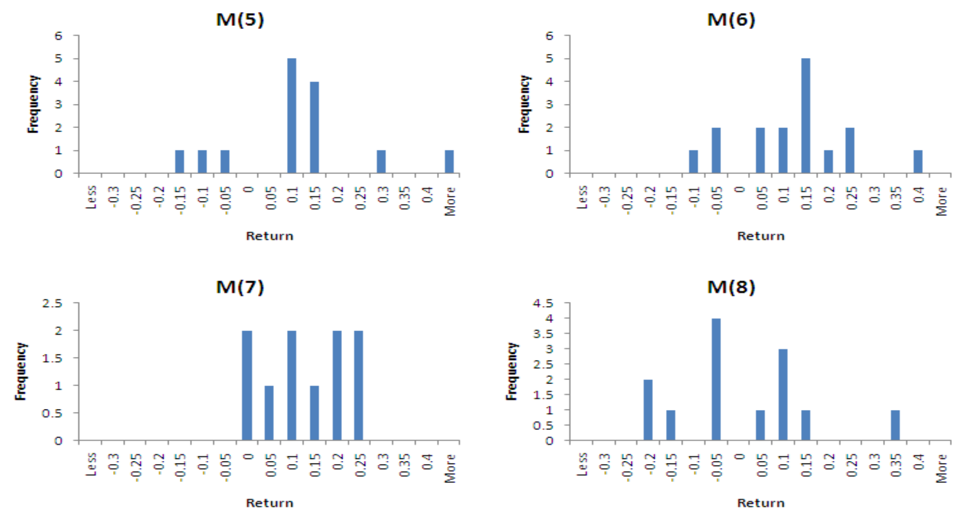



Following this strategy, she would run out of money at age 95 in average, age 82 in the worst case while has an average level of consumption totaling $532,602 throughout of retirement. This is a superior outcome compared to all models discussed in Section 6 and 7. We can therefore conclude that dynamic programming provide little benefit for selecting the optimal investment strategies, in this case with constant consumption.

Market states in Section 5 are chosen, with M(2) present the cumulative return less than the optimal lower threshold of 65%, and M(8) chosen to present the cumulative return greater than the optimal upper threshold of 160%. The other states are needed to model the transition between states and are chosen with judgement.

After the market states are defined we can classify the historical year into corresponding market states and breakdown the unconditional historical distribution shown in Appendix C, into conditional sub distributions as follows:



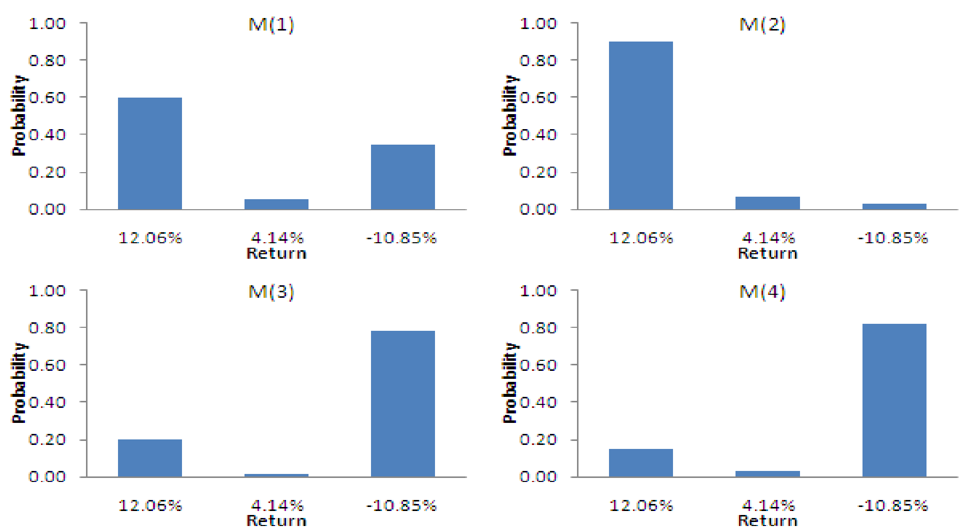


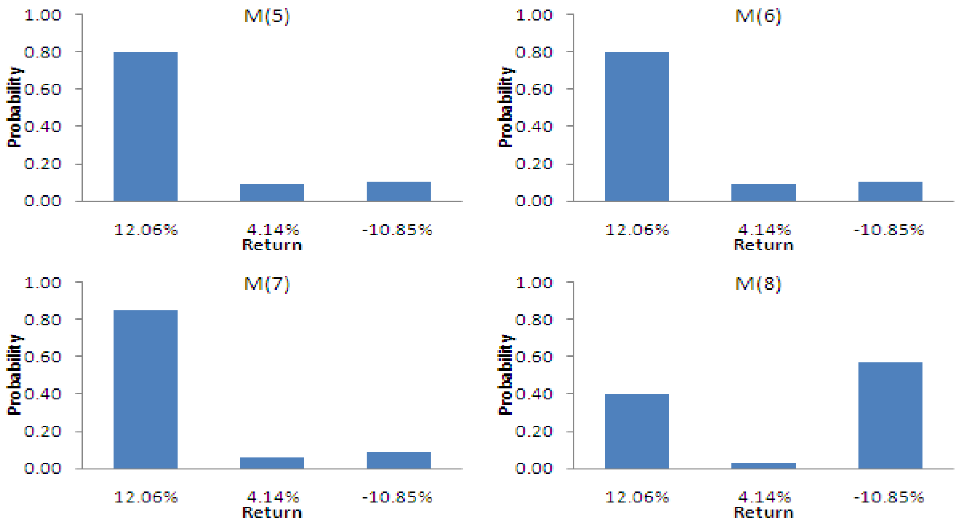

## Appendix E

In this paper I compare dynamic programming model using 4 different investment return assumptions for stock return:

Model B used actual historical distributions as shown in Appendix D.

Model F used simple 3-point discrete distributions with the same mean and variance, but can only take 3 possible values, these distributions are shown as the follows:






I have also tried to replace the historical distributions with normal distributions with the same mean and variance in Model B, the outcomes are however very similar in term of both average total consumption and financial safety, which can be seen with the wealth path:


Outcome with \$24,000 and \$30,000 consumption floor

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