BioMath - multidimensional dynamics 1- September 1, 2015

1. **Rabbits and wolves**

   The defense department has developed a strain of weaponized rabbits who fight back against hungry wolves. You've decided to study the dynamics of weaponized rabbit and hungry wolf populations, and the state of New Jersey has kindly granted you an enclosure for your work. Every day, the rabbit population increases at a rate (animals per day) that is a factor of 1.13 times the existing rabbit population, and decreases at rate that is a factor of 1.14 times the existing wolf population (those are some hungry wolves). The wolf population goes up at a rate given by a factor of 1.14 times the existing rabbit population, but goes down by a factor of 0.93 times the wolf population. To make the problem exciting, you introduce an average of 11.7 weaponized rabbits and 3 hungry wolves into the enclosure each day.

   • What is the number of rabbits and wolves at which the population stays fixed?

   • Is that fixed point stable or unstable? Are there any oscillations around it? Are there any other fixed points?

   • Let's call the number of rabbits introduced per day r_{in} and the number of wolves introduced per day w_{in}. Are there values of r_{in} and w_{in} that could change the dynamics from oscillating to non-oscillating or vice-versa? That is, by controlling the constant input rates, can you control whether the dynamics have oscillations?

   • The dynamics change. The rabbits got angrier and the wolves became more timid predators. Now every day the rabbit population is goes up by a rate 0.93 times the existing rabbit population, and decreases by a rate 1.14 times the existing wolf population. Meanwhile, the wolf population goes up by a rate 1.14 times the existing rabbit population, and goes down by a rate 1.13 times the wolf population.

   • If you now introduce 17.7 rabbits/day and 11 wolves/day, what is the new fixed point of the dynamics?

   • Is the new fixed point stable or unstable? If there are oscillations, how does their frequency compare to the frequency in the old dynamics?

2. **Analytical solution to the simple harmonic oscillator**

   We saw in class that the analytical solution to $\dot{x} = \alpha x$ is $x(t) = x(0)e^{\alpha t}$. Often in biological problems we won't find analytical solutions, but it turns out that one problem we saw in class does have an analytical solution.

• Show that $x(t) = a\sin(\sqrt{\alpha}t) + b\cos(\sqrt{\alpha}t)$, where a and b are constants, is a solution to $\ddot{x} = -\alpha x$. The two constants a and b index the family of solutions.

3. **Anglo-French cooperation**

   English people are moving into the French riviera at a rate of 10% per year. In response, 40,000 people flee the riviera each year. 20% of people in the riviera are engaged in a campaign to stop more people moving to the riviera. They reckon their best chance is to deplete the english population, so there is no one left to move in. These 20% of people in the riviera manage to each entice one english person in England per year to move to Spain, where they are never heard from again. In addition, 10% of english people leave England every year to go elsewhere (e.g., Australia). Finally, 100,000 people migrate to england every year from India and the Caribbean.

   • If e represents the number of people in England, and r represents the number of people in the riviera, write a set of differential equations governing the dynamics of the population.

   • Is there a fixed point to these dynamics? Is it stable or unstable? Do the dynamics show oscillations? Answer these questions both by (a) using the Euler rule to simulate the dynamics in Matlab (use 10,000 english people and 40,000 people on the riviera as a starting point); and (b) by finding the eigenvalues and eigenvectors of the dynamics. If the dynamics are oscillatory, how many times round the fixed point do they go every 10 years?

4. **Linear dynamics**

   Say that you have two neurons influencing each other such that

$$\begin{aligned} \dot{x} &= -1.25x + 1.3y \\ \dot{y} &= 1.3x + 0.35y \end{aligned} \tag{1}$$

   • Write the matrix equation for this sustem in the form $\dot{\mathbf{x}} = A\mathbf{x}$

   • Use quiver.m (from the multid.zip package on the wiki) to make a quiver plot for this system.

   • Use add_axes.m from the multid.page package on the wiki to add red lines corresponding to the two eigendirections, and black lines for the cartesion axes.

   • On the same plot, add a line that represents the motion of a trajectory starting at $x = 3, y = -4$. Add another for $x = 3, y = 2$. Use other starting points at will.

• Now use Matlab to compute the eigenvectors and eigenvalues. Use the inverse of V, the matrix of eigenvectors, to transform the original space into one where the horizontal axis is the first eigenvector and the vertical axis is the second eigenvector. In this new eigenvector space, plot the quiver plot, the red lines for the eigendirections, the black lines for the original cartesian axes, and the same trajectories (transformed to the eigenvector space) that you plotted in the original space.

5. **Transforming higher-order linear equations into first-order equations** Change variables (e.g., "$y_1 = x; y_2 = \dot{x}$") so as to rewrite each fo the following 1-dimensional higher order equations into multi-dimensional first-order equations (meaning, we don't want to see any double dots– single dots maximum, and all the single dots on the left of the equals signs!). (Note: each of these three is a separate exercise.)

$$\ddot{x} = -\gamma\dot{x} - \alpha x,$$

$$\dddot{x} = -\dot{x} + x,$$

$$\frac{\mathrm{d}^4 y}{\mathrm{d}t^4} - y = 5.$$

6. **More diagonalization fun**

   • For each of the equations in problem 5, figure out: (a) where is the fixed point? (b) is it stable or unstable? If it is unstable, is it a saddle?

   • For the second equation in problem 5 (namely, $\dddot{x} = -\dot{x} + x$), use Euler numerical integration to make two plots, one for each of $x(t)$ and $\dot{x}(t)$ as functions of time, for time running from $t = 0$ to $t = 100$, using as a starting point $x(0) = 1, \dot{x}(0) = 1, \ddot{x}(0) = 1$.

   Does the ratio of $x(t)$ to $\dot{x}(t)$ approach some asymptotic value? Try using other starting points. Does the asymptotic value of the ratio of $x(t)$ to $\dot{x}(t)$ depend on the starting point?

   Write the equation in form $\dot{\mathbf{x}} = M\mathbf{x}$, where $\mathbf{x}$ is a vector, and compute the eigenvalues and eigenvectors of M. What is the ratio of the first two components of the eigenvector that corresponds to the positive eigenvalue? How does this compare to/explain the asymptotic ratios you found using Euler integration?