BioMath - Dynamics prerequisites, and 1-D dynamics- August 23, 2013

1. **Imaginary numbers** Let $a = 2+i3$, let $b = 3+i4$, $c = 1-i\frac{3}{2}$, and let $\alpha = \cos 30 + i \sin 30$.

   • Sketch (either in Matlab or by hand) each of these numbers as a point on a plane in which the horizontal axis is the real part of the number and the vertical axis is its imaginary part.

   • Compute the following 6 numbers: $a+b$, ab, the magnitude of a, the magnitude of b, aa^*, and ac. Did the last one have a non-zero imaginary component or not? If not, can you say how this was related to aa^*?

   • Use Matlab to compute $\alpha^0 a$, $\alpha^1 a$, $\alpha^2 a$, $\alpha^3 a$, and $\alpha^4 a$. Using either Matlab or by hand, sketch each of these points on the real-imaginary plane. If in Matlab, "axis equal" might be useful. What do you notice about the effect of multiplying by α?

   • rewrite both a and α in the form $re^{i\theta}$, where r is a real scalar. Now compute $\alpha^0 a$, $\alpha^1 a$, $\alpha^2 a$, $\alpha^3 a$, and $\alpha^4 a$ analytically, i.e., without using Matlab or solving for actual numbers on each step. How does your answer here relate to the answer to the previous bullet point • ?

2. **Taylor expansions, general form.** (This question has a long intro, to serve as notes about Taylor expansions. The actual questions are after the • signs.)

   Any continuous differentiable function can be well approximated by it's *Taylor expansion*, meaning an expansion in powers of x around some point x_0 at which you know the derivatives. Let's go through it.

   Suppose you're thinking of the function $f(x)$. Then at points close to x_0, the first shot at an approximation would be that $f(x)$ is $\approx$ to its value at x_0:

$$\hat{f}(x) \approx f(x_0) \qquad \text{(we'll use } \hat{f} \text{ to mean "approximation of f")}$$

   As you get further away from x_0, that approximation is clearly going to get worse and worse. One thing you could do is fit a straight line that goes through $f(x_0)$, and that is tangent to $f(x)$ at x_0– in other words, a line has the same derivative as $f(x)$ at $x = x_0$. That would provide an approximation that is a little better than plain old $f(x_0)$ for values of x a little bit further away from x_0. If you do that, you're saying that your approximation is:

$$\hat{f}(x) \approx f(x_0) + (x - x_0) \cdot \frac{df}{dx}\Big|_{x_0} \tag{1}$$

   where $\frac{df}{dx}|_{x_0}$ means "some number, i.e., a constant, that is equal to the first derivative of f with respect to x, evaluated at $x = x_0$." The approximation in equation (1) is called a *first-order approximation*.

• Differentiate equation (1) with respect to x, and evaluate it at $x = x_0$ to show that $\hat{f}$ has the same derivative as f at $x = x_0$. Show that $\hat{f}$ also has the same value as f at $x = x_0$.

That got you somewhere, but you could go further. Suppose you wanted to approximate f with a function that, as above, (a) matches the value of f at $x = x_0$; (b) matches the first derivative of f at $x = x_0$; but now, in addition, (c) matches the second derivative of f at $x = x_0$. In other words, we now also want to match the curvature of the function f at the point where $x = x_0$. Then you could construct the following polynomial that has up to quadratic powers of x:

$$\hat{f}(x) \approx f(x_0) + (x - x_0) \cdot \frac{df}{dx}|_{x_0} + \frac{1}{2}(x - x_0)^2 \cdot \frac{d^2 f}{dx^2}|_{x_0} \tag{2}$$

• Show that $\hat{f}$ of equation (2) has the same value as f at $x = x_0$, differentiate it to show that it has the same derivative as f at $x = x_0$, and differentiate it again to show that it has the same second derivative as f at $x = x_0$.

We hope you're beginning to see the pattern here! Let's do one more round, and then we'll go to the general equation. Take the cubic polynomial

$$\hat{f}(x) \approx f(x_0) + (x - x_0) \cdot \frac{df}{dx}|_{x_0} + \frac{1}{2}(x - x_0)^2 \cdot \frac{d^2 f}{dx^2}|_{x_0} + \frac{1}{3!}(x - x_0)^3 \cdot \frac{d^3 f}{dx^3}|_{x_0} \tag{3}$$

• and show that $\hat{f}$ from equation (3) matches the value of f as well as its first, second, and third-order derivatives at $x = x_0$.

Ok, we're ready to go whole hog! Take

$$\hat{f}(x) \approx \sum_{n=0}^{\infty} \frac{1}{n!}(x - x_0)^n \cdot \frac{d^n f}{dx^n}|_{x_0} \tag{4}$$

and show that $\hat{f}$ matches f in value and *every* derivative (i.e., first, second, etc.). Equation (4) is the Taylor expansion of f around $x = x_0$.

3. **Taylor expansions, specific example**

Finally, let's do a specific example in Matlab. Take the function

$$g(x) = \frac{1}{1 + e^{-x^3}} \tag{5}$$

Plot it from $x = -2$ to $x = 2$. Remembering your calculus, you can manually compute the first, second, third, etc., derivatives of this function. But, to keep you focused on Taylor expansions rather than the minutae of keeping track of coefficients as you differentiate, we've done it for you. Download and unzip `gfunc.zip` from the wiki. That contains a file, `gfunc.m`, that computes g and its first through fourth derivatives.

• Write a script that uses `gfunc.m` to plot g and its first, second, third, and fourth-order Taylor approximations around some point x_0. For your first choice of x_0, use $x_0 = -0.35$. Then choose other values, whatever looks interesting to you (for example, $x_0 = +0.5$). Comment on what you observe. Do the approximations get better as you use more terms? Over what range would the approximation be a good one if you used an infinite number of terms? If you used an infinite number of terms, would the choice of x_0 matter?

4. **matrices with a rotational component**

Consider the matrix $M = \begin{pmatrix} -0.5 & 1 \\ -1 & 0 \end{pmatrix}$. Is this a rotation matrix?

• Use Matlab to find the eigenvectors and eigenvalues of M. Is there an imaginary component to them?

• Using diagonalization (i.e., $M = V\Lambda V^{-1}$ where Λ is a diagonal matrix of eigenvalues of M and V is a matrix whose columns are the corresponding eigenvectors), compute the tenth root of M, i.e., $M^{0.1}$.

• Write a program that will plot a red dot for each of a field of initial vectors $\mathbf{x}$, ranging over $x_1 = -5 : 5$ and $y_1 = -5 : 5$ would The next plot should show a blue line for each of these initial vectors, going from $\mathbf{x}$ to $\mathbf{x}' = M^{0.1}\mathbf{x}$, with a red dot at the new position $\mathbf{x}'$. Do you see a rotational component to what this tenth-root of M matrix does to the space?

• Let's make a movie in Matlab. Each frame of the movie will be like the bullet point above, but adding one-tenth of a power of M each time. That is, the very first frame should be the bullet point above, but using M^0; the next frame should use $M^{0.1}$; the next frame $M^{0.2}$; and so on, until we reach M^1 (or, in other words, M itself). After the plots that correspond to each frame, call `pause(0.1)` to pause for 100 msec, so that the movie does not run too fast. This movie will show the effects of building up to the full effect of M– and should help your eye see the rotational component even better!

• Take the point $\mathbf{x} = [1; 0]$. Use Matlab to compute what it is in eigenvector space. (In other words, find α_1 and α_2 such that $\mathbf{x} = \mathbf{v}_1\alpha_1 + \mathbf{v}_2\alpha_2$.) Staying in the eigenvector space, use Matlab to compute $\mathbf{y} = M\mathbf{x}$. Now transform $\mathbf{y}$ back to the original space. Are there any imaginary components to the final answer?