Fourier Analysis

Math Bootcamp 2015

What is frequency space and why do we care?

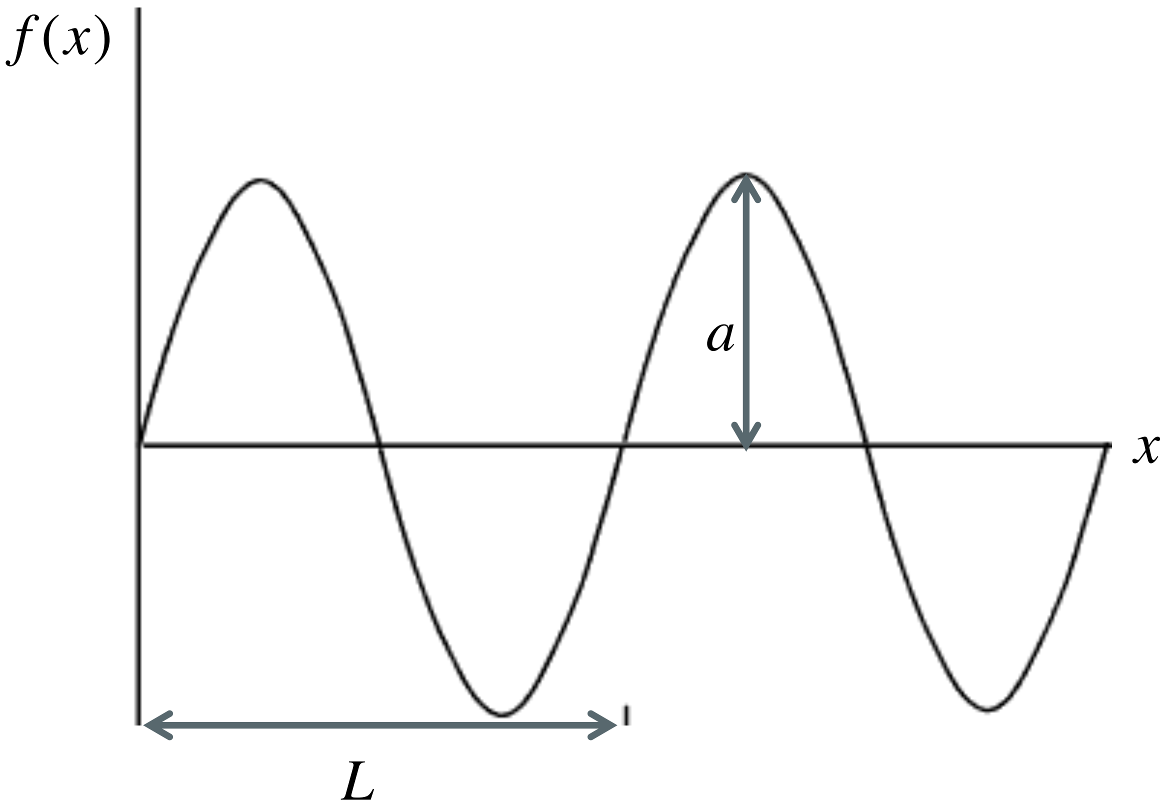
Real Space:
Desk located at
X= 2m, 4m, 6m, …

Frequency Space:
One desk every 2m,
or .5 desks/m

What is frequency space and why do we care?

- Hair cells in the ear respond to different frequencies of sound
- MRI images are made by analyzing the the oscillations of protons in a magnetic field
- Mechanical oscillations depend respond differently to different frequencies

Sin Wave

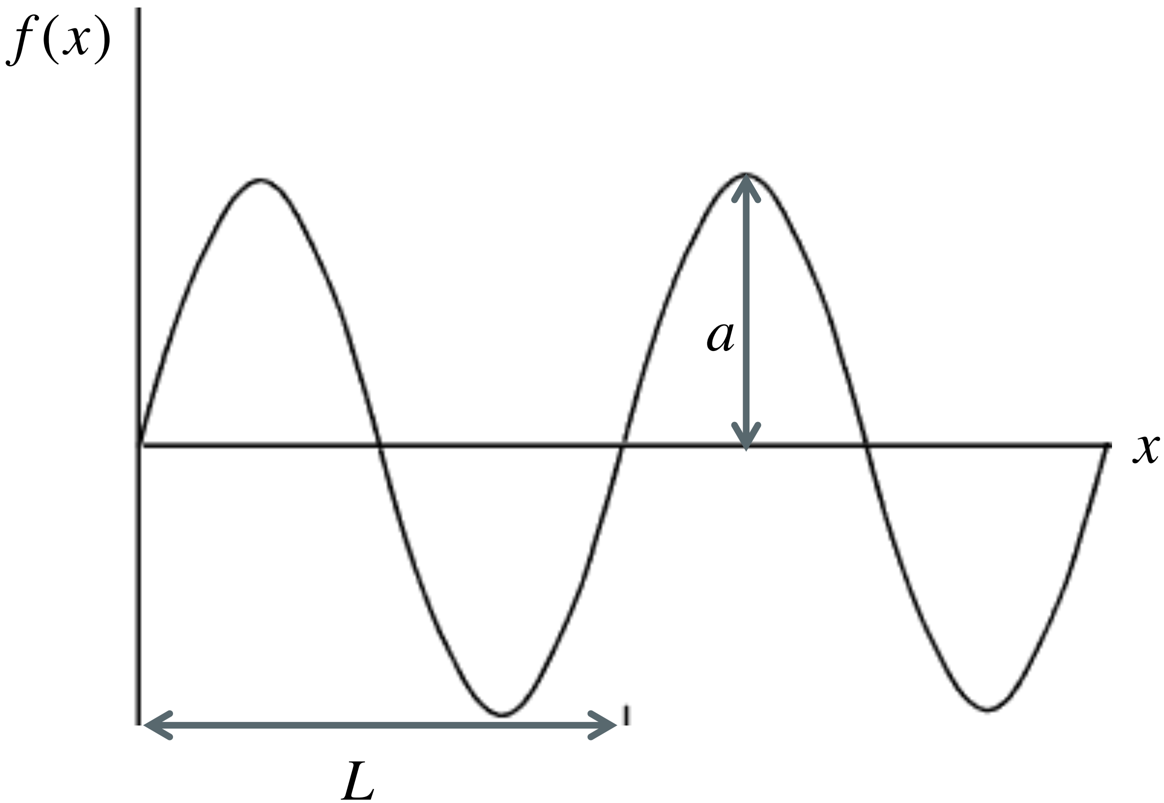


amplitude

$$f(x) = a\sin\left(\frac{2\pi}{L}x\right)$$

period

Sin Wave



amplitude

$$f(x) = a\sin(2\pi\nu x)$$

Frequency

$$\nu = \frac{1}{L}$$

Lets make one in MATLAB

Representing signals as sine waves

We already know how to represent a function as a polynomial:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Can we do this using periodic functions like sine and cosine?

$$f(x) = a_1 \sin\left(\frac{2\pi}{L_1}x\right) + a_2 \sin\left(\frac{2\pi}{L_2}x\right) + a_3 \sin\left(\frac{2\pi}{L_3}x\right) + \cdots$$

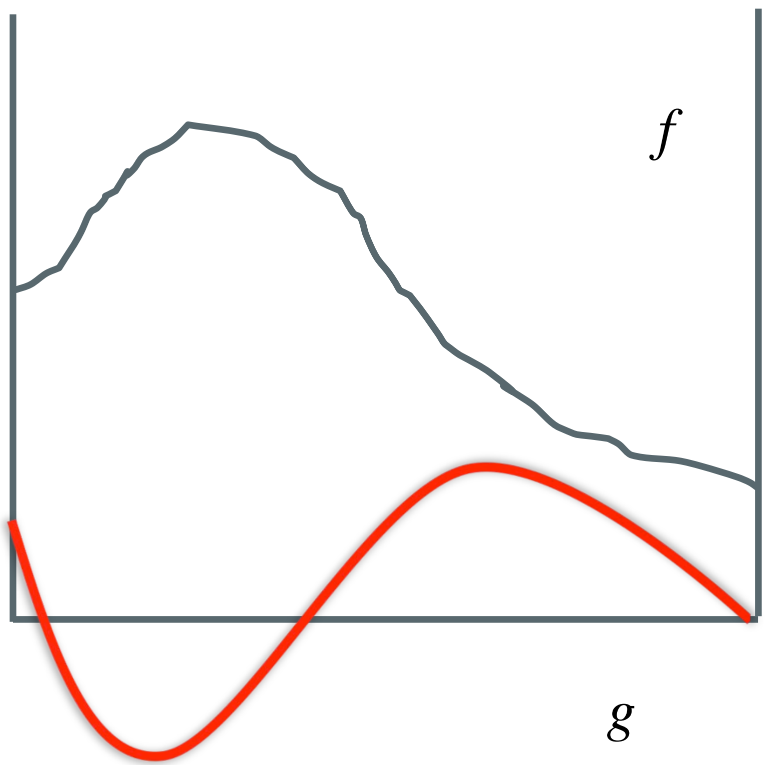
$$+ b_1 \cos\left(\frac{2\pi}{L_1}x\right) + b_2 \cos\left(\frac{2\pi}{L_2}x\right) + b_3 \cos\left(\frac{2\pi}{L_3}x\right) + \cdots$$

Lets play the Wave Game!

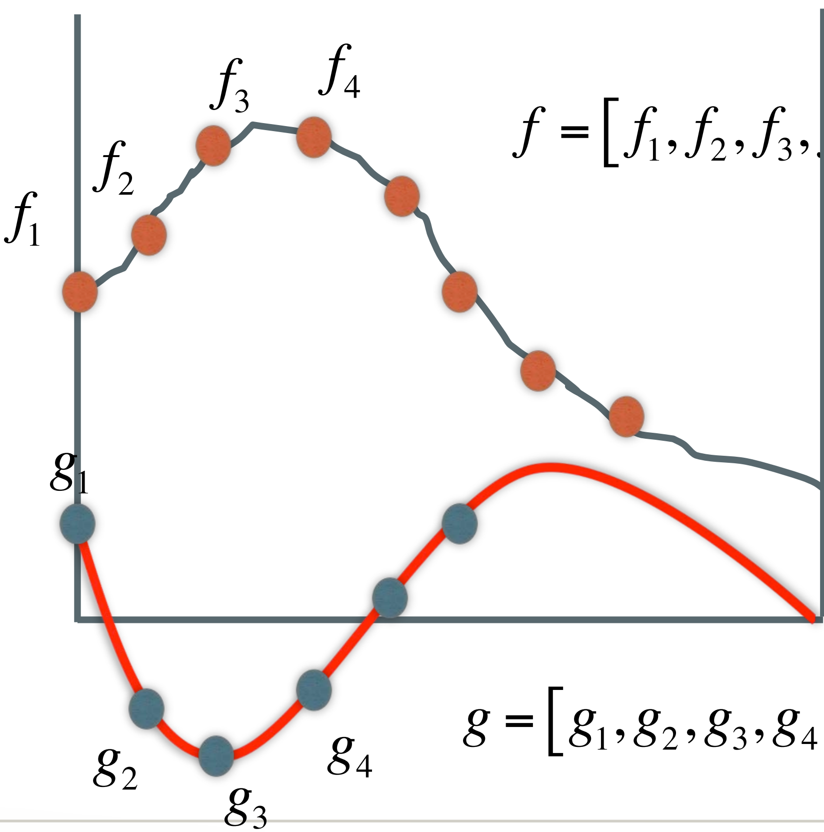
- Check out java program at

https://phet.colorado.edu/en/simulation/fourier

Functions and Vectors



Functions and Vectors

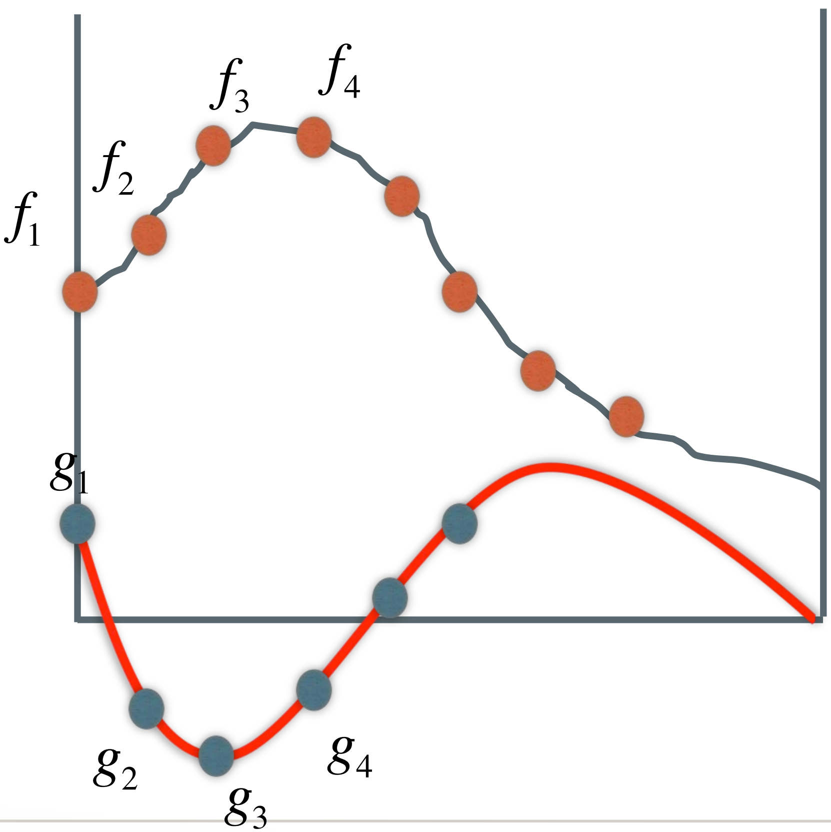


$$f = [f_1, f_2, f_3, f_4, \cdots]$$

$$g = [g_1, g_2, g_3, g_4, \cdots]$$

Functions can be thought of as infinitely long vectors spanning an infinite dimensional space. This allows us to do our linear algebra!

Functions and Vectors



Now we can calculate things like projections

$$f \bullet g = f_1 g_1 + f_2 g_2 + f_3 g_3 + f_4 g_4 + \cdots$$

$$f \bullet g = \int_{-\infty}^{\infty} f \cdot g \, \mathrm{d}x$$

Functions and Vectors

- Functions can be orthogonal too
- Exercise: show sin(x) and cos(x) are orthogonal for x between 0 and 2pi

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

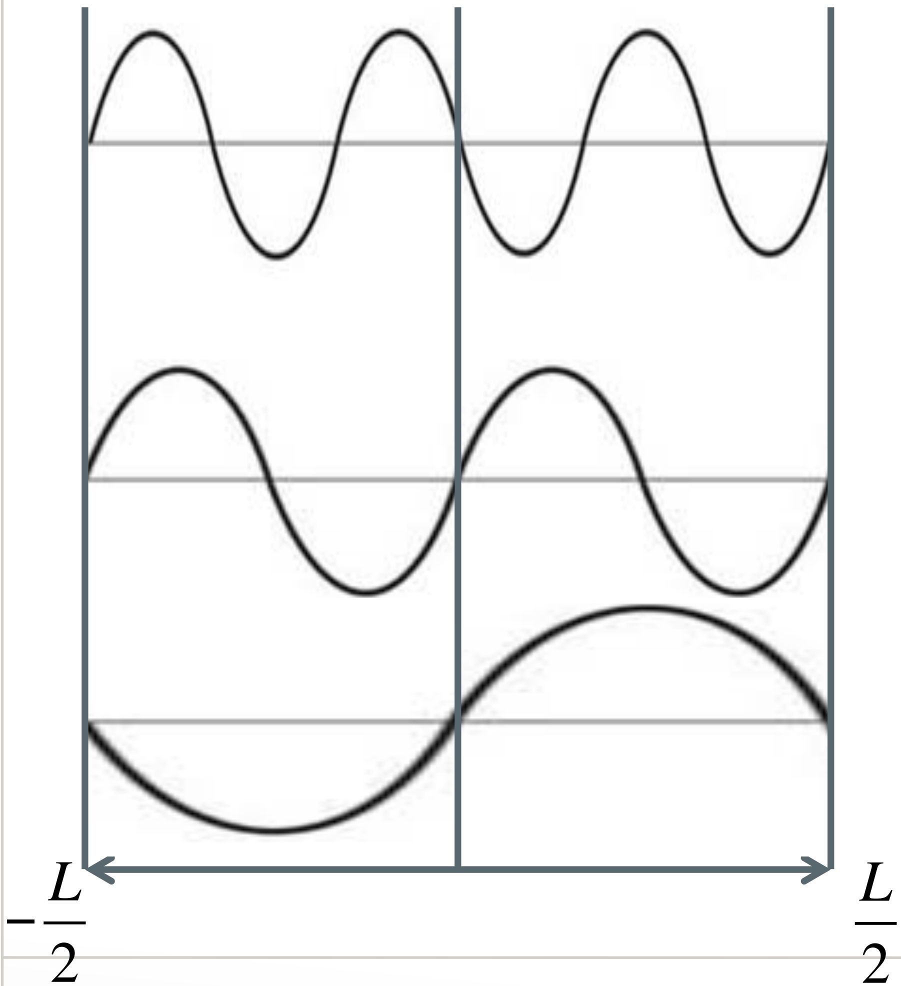
$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

Functions and Vectors

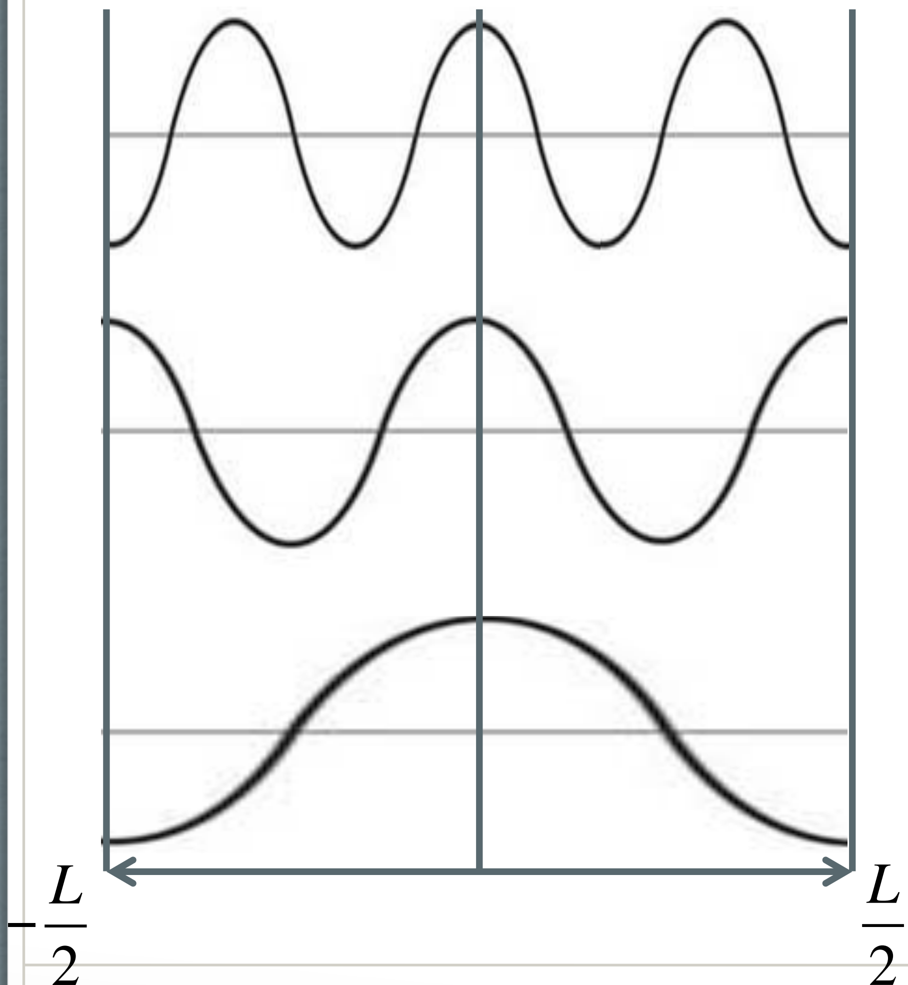
- Functions can be orthogonal too
- Exercise: show sin(x) and cos(x) are orthogonal for x between 0 and 2pi
- We will use sin and cos as a basis represent functions

Sin in a box



Wavelength	$f(x)$
$\frac{L}{3}$	$b\sin\left(2\pi\frac{3}{L}x\right)$
$\frac{L}{2}$	$b\sin\left(2\pi\frac{2}{L}x\right)$
L	$b\sin\left(2\pi\frac{1}{L}x\right)$

Cos in a box



Wavelength	$f(x)$
$\frac{L}{3}$	$a\cos\left(2\pi\frac{3}{L}x\right)$
$\frac{L}{2}$	$a\cos\left(2\pi\frac{2}{L}x\right)$
L	$a\cos\left(2\pi\frac{1}{L}x\right)$

Sin and Cos make a basis

- All of these form an orthogonal basis we can use to describe a function between –L and L
- We can also normalize them so that they are orthonormal

$$a\cos\left(2\pi\frac{1}{L}x\right) \qquad b\sin\left(2\pi\frac{1}{L}x\right)$$

$$a\cos\left(2\pi\frac{2}{L}x\right) \qquad b\sin\left(2\pi\frac{2}{L}x\right)$$

$$a\cos\left(2\pi\frac{3}{L}x\right) \qquad b\sin\left(2\pi\frac{3}{L}x\right)$$

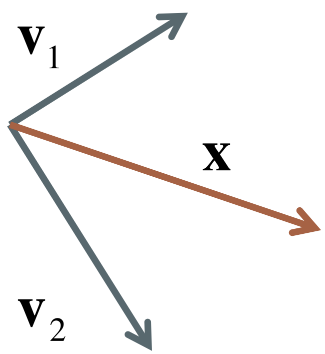
$$\vdots \qquad \vdots$$

$$a\cos\left(2\pi\frac{n}{L}x\right) \quad b\sin\left(2\pi\frac{n}{L}x\right)$$

Projections on to the basis

- Adding up sine and cosine waves, we can reproduce any periodic function, but how much of each component do we need?

- Linear Algebra Flashback:



$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

where

$$c_1 = \mathbf{x} \bullet \mathbf{v}_1$$

$$c_2 = \mathbf{x} \bullet \mathbf{v}_2$$

Projections on to the basis

- Adding up sine and cosine waves, we can reproduce any periodic function, but how much of each component do we need?

$$f = a_0 + a_1 \cos\left(\frac{2\pi}{L}x\right) + a_2 \cos\left(2\frac{2\pi}{L}x\right) + \ldots$$
$$+ b_1 \sin\left(\frac{2\pi}{L}x\right) + b_2 \sin\left(2\frac{2\pi}{L}x\right) + \ldots$$

- What are the values for a and b?

Projections on to the basis

- Adding up sine and cosine waves, we can reproduce any periodic function, but how much of each component do we need?

$$f = a_0 + a_1 \cos\left(\frac{2\pi}{L}x\right) + a_2 \cos\left(\frac{2\pi}{L}2x\right) + \ldots$$
$$+ b_1 \sin\left(\frac{2\pi}{L}x\right) + b_2 \sin\left(\frac{4\pi}{L}2x\right) + \ldots$$

where

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f \cdot \cos\left(\frac{2n\pi}{L}x\right)\mathrm{d}x \qquad b_n = \frac{2}{L}\int_{-L/2}^{L/2} f \cdot \sin\left(\frac{2n\pi}{L}x\right)\mathrm{d}x$$

Example Problem: Make a sawtooth wave in Matlab

Fourier Series to Fourier Transforms

Now lets make everything simpler by making it complex

$$f = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{L}x\right) + b_n \sin\left(\frac{2n\pi}{L}x\right)$$

Recall

$$\exp(ix) = \cos(x) + i\sin(x)$$

Fourier Series to Fourier Transforms

Now lets make everything simpler by making it complex

$$f = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{L} x\right) + b_n \sin\left(\frac{2n\pi}{L} x\right)$$

Complex Form

$$f = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n}{L} x\right) \qquad c_n = \frac{1}{L} \int_{-L/2}^{L/2} f \cdot \exp\left(-\frac{2\pi i n}{L} x\right) \mathrm{d}\,x$$

*Dot products with complex numbers use the complex conjugate

We'll come back to this with MATLAB later…

Fourier Series to Fourier Transforms

Now lets make the box really big

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n}{L} x\right) \qquad c_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) \cdot \exp\left(-\frac{2\pi i n}{L} x\right) \mathrm{d}\, x$$

As L get very large, $\dfrac{n}{L}$ becomes more continuous.

Lets replace it with a continuous variable $\nu = \dfrac{n}{L}$, then $d\nu = \dfrac{1}{L}$

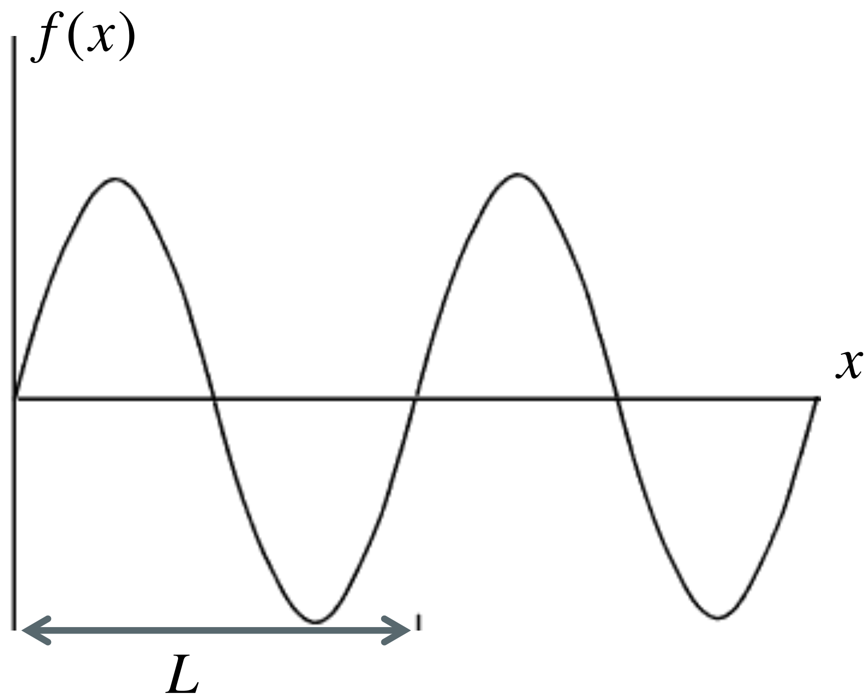
$$f(x) = \sum_{-\infty}^{\infty} c_\nu \exp(i 2\pi \nu x) \qquad c_\nu = d\nu \int_{-\infty}^{\infty} f(x) \cdot \exp(-i 2\pi \nu x) \mathrm{d}\, x$$

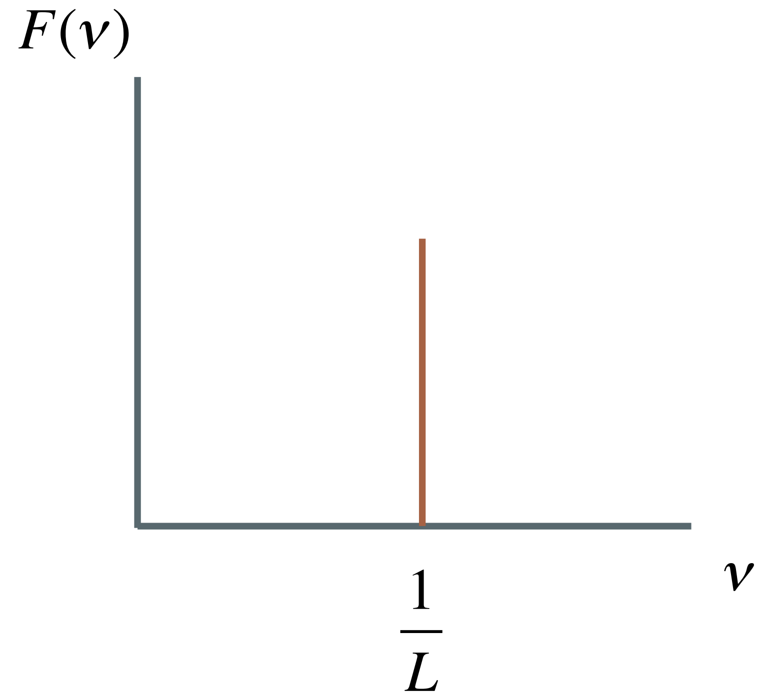
Fourier Series to Fourier Transforms

$$f(x) = \sum_{-\infty}^{\infty} c_\nu \exp\left(i2\pi\nu x\right) \qquad c_\nu = d\nu \int_{-\infty}^{\infty} f(x) \cdot \exp\left(-i2\pi\nu x\right) \mathrm{d}\,x$$

$$f(x) = \int_{-\infty}^{\infty} F(\nu) \cdot e^{i2\pi\nu x} \,\mathrm{d}\,\nu \qquad F(\nu) = \int_{-\infty}^{\infty} f(x) \cdot e^{-i2\pi\nu x} \,\mathrm{d}\,x$$

From space to frequency





$$f(x) = a\sin\left(\frac{2\pi}{L}x\right)$$

$$F(\nu) = \begin{cases} \text{something} & \nu = 1/L \\ 0 & \nu \neq 1/L \end{cases}$$

Fourier Series to Fourier Transforms

- Find the Fourier Transform of cos(x)

$$f(x) = \int_{-\infty}^{\infty} F(\nu) \cdot e^{i2\pi\nu x}\, \mathrm{d}\nu \qquad F(\nu) = \int_{-\infty}^{\infty} f(x) \cdot e^{-i2\pi\nu x}\, \mathrm{d}x$$

Discrete Fourier Transforms in MatLAB

- Fourier Transform of cos(x) in matlab

$$f = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n}{L} x\right) \qquad c_n = \frac{1}{L} \int_{-L/2}^{L/2} f \cdot \exp\left(-\frac{2\pi i n}{L} x\right) \mathrm{d}\, x$$

What does it mean to be complex?

- Example with sin(x) vs sin(x+.1)

Power spectra

- Which frequencies are represented and how much?