BioMath - Vectors and Matrices - Exercises part 2 - August 19, 2014

1. **Solving a set of linear equations.** (*This problem is from yesterday– do it if you didn't get a chance to do it yesterday.*). You want to assay the expression level of three proteins, matlabin, algebrain, and vectorin. One of your colleagues has made a mouse that has fluorescent tags on the three proteins. However, for reasons of their own, on the matlabin they put three red fluorescent molecules, two green ones, and four blue ones. On the algebrain they put two red ones and two blue ones. And on the vectorin they put one red one, three green ones, and three blue ones. You take a slice of the tissue you're interested in, and measure the total fluorescence on each of the red, green, and blue channels. Let's call this measurement the vector $\mathbf{x} = [r, g, b]$. Let's assume that the fluorescence luminosity per molecule is the same for the RFP, GFP, and BFP, and equals 1. (That is, if all you had was a matlabin with red fluorescence, and nothing else, and if $r = 1$, then you would infer that there is one matlabin molecule in your sample.) Given $\mathbf{x}$, how do we find the number of matlabin, algebrain, and vectorin molecules?

2. **Matrix times vector** .

   Let's say that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

   • Then show that $A\mathbf{x}$ can be expressed using the columns of A as:

$$A\mathbf{x} = \begin{pmatrix} a \\ c \end{pmatrix} x_1 + \begin{pmatrix} b \\ d \end{pmatrix} x_2 \tag{1}$$

   Let's do the same again, but more generally, for the N-dimensional case. Remember that $\mathbf{y} = M\mathbf{x}$ can be written as $y_i = \Sigma_j M_{ij} x_j$ (here $\mathbf{x}$ and $\mathbf{y}$ are column vectors with elements x_i and y_i, respectively, and M_{ij} is the element in the i^{th} row and j_{th} column of M). Ok, let's say that the j^{th} column of a matrix M is called $\mathbf{M}_j$.

   • Show that

$$M\mathbf{x} \;=\; \mathbf{M}_1 x_1 + \mathbf{M}_2 x_2 + ... \;=\; \sum_i \mathbf{M}_i x_i \tag{2}$$

3. **Changing bases, general case.** We talked a lot about changing bases when the new basis is orthonormal. Let's look now at the more general case. We'll stay in a relatively low number

of dimensions, but our new basis vectors will no longer be length 1 or orthogonal to each other. And hopefully you'll see how the result here applies in N dimensions.

Let's say you have a 3-dimensional vector $\mathbf{x}$, represented in Cartesian basis coordinates as

$$\mathbf{x} = \begin{pmatrix} x_1^{old} \\ x_2^{old} \\ x_3^{old} \end{pmatrix}$$

Now suppose that you want to change the basis in which this vector is represented to the three axes defined by vectors $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$. That is, we want to express $\mathbf{x}$ as

$$\mathbf{x} = \mathbf{v_1} x_1^{new} + \mathbf{v_2} x_2^{new} + \mathbf{v_3} x_3^{new} \tag{3}$$

where x_1^{new}, x_2^{new}, and x_3^{new} are the scalars that tell us the coordinates of $\mathbf{x}$ in the new basis.

• Arrange x_1^{new}, x_2^{new} and x_3^{new} into a column vector $\mathbf{x}^{new}$. Then figure out how to package the basis vectors into a matrix V so that you can rewrite equation (3) as the matrix equation $\mathbf{x}^{old} = V\mathbf{x}^{new}$ (the answer to problem 2 might help you here). Now solve for $\mathbf{x}^{new}$– your solution, in other words, tells you how to find the new coordinates starting from the old coordinates.

• You'll remember that in the case that the basis vectors $\{\mathbf{v}_i\}$ were orthonormal, we could package them as rows of a matrix B, and that in that case $\mathbf{x}^{new} = B\mathbf{x}^{old}$. You'll also remember that in that case, B^{-1} is B^T. How does that relate to your solution in the previous bullet point?

4. **Expressing a matrix in a new basis.** Consider the equation

$$\mathbf{y} = M\mathbf{x}$$

As we saw in lecture, we can think of this equation as a *mapping* of all the points $\mathbf{x}$ into a corresponding set of points $\mathbf{y}$. This mapping is a spatial relationship. How would we describe this spatial relationship if we change the basis in which $\mathbf{x}$ and $\mathbf{y}$ are described? That is, just like a vector is a geometric entity that is mapped onto a new position by a matrix, a matrix itself is a geometric entity– it is a spatial mapping. And it, too, gets remapped by a matrix. Here we'll explore what happens to the geometric entity M under a mapping induced by another matrix B. Here we go.

• Take the matrix

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}$$

Now, MATLAB EXERCISE: Write a program that will plot a red dot for each of a field of initial vectors $\mathbf{x}$. The next plot should show a blue line for each of these initial vectors, going from $\mathbf{x}$ to $\mathbf{x}' = M\mathbf{x}$, with a red dot at the new position $\mathbf{x}'$.

Now consider two vectors $\mathbf{x}$ and $\mathbf{y}$ after being mapped into a new position by the matrix B:

$$\mathbf{x}' = B\mathbf{x} \quad \mathbf{y}' = B\mathbf{y}$$

• Is there a matrix M' such that $\mathbf{y}' = M'\mathbf{x}'$ for any $\mathbf{x}$ and $\mathbf{y}$? Find an expression for M'.

• Go back to the matrix

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}$$

What is the corresponding matrix M' if you rotate space 30° anticlockwise? It might help you to use the formula for the 2-by-2 rotation matrix

$$B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

• MATLAB EXERCISE, PART 2. Now run your program for showing the mapping that a matrix achieves again, but this time don't use M, instead use M'. Does the mapping correspond to what you expected?

• For extra fun, rewrite your code so it can run the original mapping, and then the new mapping, for any rotation angle θ. Have fun playing with θ!

5. **Matrix multiplication.** Take the matrices

$$M = \begin{pmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

• Compute $M\Lambda$. How would you describe each column?

Now suppose that you had N equations of the form $M\mathbf{v}_i = \lambda_i \mathbf{v}_i$, where i runs from 1 to N, and N is the number of rows and columns of M. Write the set of vectors $\mathbf{v}_i$ into a matrix

$$V = \begin{pmatrix} | & | & | \\ \mathbf{v_1} & \ldots & \mathbf{v_N} \\ | & | & | \end{pmatrix}$$

and then • show that the set of [N equations of the form $M\mathbf{v}_i = \lambda_i \mathbf{v}_i$] can all be written together in a single matrix equation

$$MV = V\Lambda$$

where Λ is a matrix with $\lambda_1, \lambda_2, ...\lambda_N$ as the diagonal entries and zeros everywhere else.

The result of this exercise will be useful for the lecture tomorrow.

6. **Calculating a determinant.** without using Matlab, find the determinants of the following two matrices:

   • A = $\begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$

   • B = $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

7. **Determinants of products of matrices**

   • Using the matrices from problem 6, compute $M_1 = AB$ and $M_2 = BA$. What is the determinant of M_1? What is the determinant of M_2? How do these relate to the values of the determinant of A and the determinant of B?

8. **Determinants and inverses**

   You are looking at tissue that has two fluorescent proteins in it. Protein A is emitting 1 blue photon/sec, and α red photons per second, where $\alpha \leq 2$. Protein B is emitting 1 blue photon/sec and 2 red photons/sec. You observe b blue photons/sec and r red photons/sec. How many proteins A and how many proteins B are there? If $\alpha = 2$, can you obtain a unique answer?

   • Compute the determinant of $M = \begin{pmatrix} 1 & 1 \\ \alpha & 2 \end{pmatrix}$. What happens to this determinant as α approaches 2? When $\alpha = 2$, is M invertible?

9. **Determinants after spatial transforms**

   From problem 4, take the matrix

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}$$

and its version M' after rotating space $30°$ anticlockwise.

• What is the determinant of M? What is the determinant of M'?

• Remembering that $|AB| = |BA|$ show that, in general, if a space is transformed by a matrix T, the determinant of a matrix M in the original space is equal to the determinant of the transformed matrix M'. What does this mean in geometric terms?