BioMath - Eigenvectors - Exercises - August 26, 2015

This is a very long problem set, ending in some quite advanced problems. Don't worry, you're not really expected to do them all! Do the ones you can.

1. **Eigenvalues and eigenvectors.** For $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

   Remember that an eigenvector for a matrix $\mathbf{A}$ is defined as "any vector such that when it is put through $\mathbf{A}$, it stays in the same direction." In other words, it must satisfy $\mathbf{A}\vec{x} = \lambda\vec{x}$ where λ is a scalar.

   • Show that $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$. (without using Matlab). What is its associated eigenvalue?

   • Find the second eigenvector and eigenvalue (without using Matlab: instead, use the $|M - \lambda I| = 0$ formula to find the eigenvalues, and after finding the second eigenvalue, use the $M\mathbf{v}_i = \lambda_i\mathbf{v}_i$ formula to find the corresponding eigenvector).

   • Use the `[V, D] = eig(M)` function in Matlab to confirm your results.

2. **Eigenvalues and eigenvectors; example of Trace and Determinant.** Without using the Matlab function eig.m, find the eigenvalues and eigenvectors for the following two matrices:

   • $\begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$

   • $\begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$

   For the next one, use the Matlab function called eig.m to obtain the answer.

   • $\begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$.

   • For each of the three matrices, calculate the trace of the matrix (defined as the sum of the diagonal elements) and the determinant of the matrices (for the 2x2 matrices, do it without using Matlab; for the 3x3 matrix, feel free to use Matlab). Compare these to the sum and to the product of the eigenvalues, respectively. What do you see? In problem 7 we'll come back to the general proof and implications of the pattern you noticed.

3. **Matrix powers part 1**

Take the matrix M, and let's say that we use Matlab to find its eigenvectors, packaged as the columns of the matrix V, and its eigenvalues, packaged as the diagonal elements of a diagonal matrix Λ. You'll remember that rewriting M in the eigenvector basis gives us

$$\Lambda = V^{-1} M V \tag{1}$$

Now consider a vector $\mathbf{x}$ that we're going to repeatedly put through the mapping represented by M. That is, we're going to consider

$$\mathbf{z} = M^n \mathbf{x} \tag{2}$$

where n is a large number.

• Rewrite equation (2) in the eigenvector basis (that is, write it in terms of $\mathbf{z}' = V^{-1}\mathbf{z}$ and $\mathbf{x}' = V^{-1}\mathbf{x}$).

• Let $M = \begin{pmatrix} 1.1 & -0.3 \\ 0.3 & 0.35 \end{pmatrix}$. Compute the eigenvalues of M, and use them, together with your answer to the previous bullet point, to figure out whether $\mathbf{z} = M^n\mathbf{x}$ will be small or large as $n \to \infty$. You should find that your answer won't depend on $\mathbf{x}$.

• Repeat the previous bullet point for $M = \begin{pmatrix} 1.3 & -0.2 \\ 0.2 & 0.8 \end{pmatrix}$. The eigenvalues make the answers evident, but without knowing the eigenvalues it is much harder to predict what is going to happen.

• Now consider the general case, where M is an N-dimensional matrix. What happens to the length of $\mathbf{z}$ if all of M's eigenvalues are less than one and $n \to \infty$? What happens to the length of $\mathbf{z}$ if at least one of M's eigenvalues is greater than one and $n \to \infty$?

4. **Matrix powers part 2**

We're going to repeat the previous problem, but using slightly different notation. Nevertheless, the concepts are the same. First, let's remember that the standard rules of matrix multiplication tell us that multiplying a matrix times a vector is distributive. Namely, for any matrix M and vectors $\mathbf{b}$ and $\mathbf{c}$, we have

$$M(\mathbf{b} + \mathbf{c}) = M\mathbf{b} + M\mathbf{c}. \tag{3}$$

The same rules tell us that for any scalar α, we have

$$M(\mathbf{b}\alpha) = \alpha(M\mathbf{b}). \tag{4}$$

Finally, let's remember that the definition of the ith eigenvector of a matrix A is that

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i, \tag{5}$$

in other words, multiplying A by an eigenvector returns the same eigenvector, scaled by its eigenvalue.

Ok, now to the problem: let's say that matrix A has eigenvectors $\{\mathbf{v}_i\}$ and corresponding eigenvalues λ_i.

• Writing a vector $\mathbf{y}$ in terms of the basis set given by the eigenvectors, namely, writing it as

$$\mathbf{y} = \mathbf{v}_1\alpha_1 + \mathbf{v}_2\alpha_2 + \mathbf{v}_3\alpha_3 + ... \tag{6}$$

where the α_i are scalar coefficients, show that the n-th power of A times $\mathbf{y}$ can be calculated as

$$\mathbf{z} = A^n\mathbf{y} = \lambda_1^n\mathbf{v}_1\alpha_1 + \lambda_2^n\mathbf{v}_2\alpha_2 + \lambda_3^n\mathbf{v}_3\alpha_3 + ... \tag{7}$$

You can start by considering $A\mathbf{y}$. Then multiply on the left by A again to consider $A^2\mathbf{y}$. Then again for $A^3\mathbf{y}$. Do you see the pattern, and how it generalizes to $A^n\mathbf{y}$?

5. **Matrix powers part 3**. Let's apply what you've learnt in the last two problems. You can approach this problem using the approach of either problem 3 or problem 4– the two are really the same concept.

   • You are looking at an ongoing dynamic reaction between two molecules, creatin and destroyin. If at time t there are c creatins and d destroyins, then an time $t+1$ there will be $2.4c - 3d$ creatins and $c - 1.1d$ destroyins. Show that as time goes to infinity, the population of creatins and destroyins falls to zero.

   • An enzyme is added to the bath such that the reaction becomes $1.5c - 0.6d$ creatins at time $t+1$ and $0.2c + 0.8d$ destroyins at time $t+1$. Show that as time goes to infinity, the population of creatins and destroyins now grows without bound.

6. **Eigenvector space and regular space – demonstration with Matlab.** Let's continue the study of our creatin/destroyin chemical reaction, where c is the number of creatins, and d the number of destroyins at timestep t. (Note: for the sake of the exercise, we're going to allow negative numbers of proteins.) This time conditions are such that at timestep $t+1$, the number of creatins changes to $1.433c - 0.267d$, and the number of destroyins becomes $0.267c + 0.767d$.

   • Write down the problem in the matrix form $\mathbf{c}(t+1) = M\mathbf{c}(t)$. Then use Matlab to compute the eigenvalues and eigenvectors of M.

• Let's make a "quiver" plot of the dynamics defined by the matrix M– that is, we'll make a field of arrows that shows how points move under $\mathbf{c}(t+1) = M\mathbf{c}(t)$. Let's take each point in a grid defined by all points from c=-300 to x=+300 in steps of 50, and from d=-300 to +300 in steps of 50. For each of these points, find what $c(t+1)$ and $d(t+1)$ would be, and let's call $\delta c = c(t+1) - c(t)$ and let's call $\delta d = d(t+1) - d(t)$. Use the `quiver.m` function to plot an arrow at each of the points, where the base of each arrow is at (c, d), and the arrow points in the direction of $(\delta c, \delta d)$. This field of arrows will be a graphical representation of the dynamics.

• On top of your quiver plot, add a black horizontal line at $d = 0$, and a black vertical line at $c = 0$. These represent your original, or "measurement space" axes. Then, for each of the two eigenvectors, add a line in red that starts at the origin and goes in the direction of the eigenvector. (Feel free to multiply an eigenvector by -1 if that makes for a prettier plot). Make these lines long enough to be visible. These lines represent the eigenvector axes, i.e., the "diagonal space" axes.

• We're now going to consider four starting points for the dynamics: "x", which is 100 creatins and 100 destroyins, "o", which is 200 creatins and 100 destroyins, "d", which is 0 creatins and 200 destroyins, and "m", which is 50 creatins and -150 destroyins. Place each of the symbols for the fours starting points, in red, on this plot. Now, in blue, add symbols at the places where each of the starting points would move to after 1, 2, 3, and 4 timesteps. That is, you should end up with 4 blue "x"s, 4 blue "o"s, etc., etc. Compute these positions directly in the original space, i.e., you do not need to use the eigenvectors and eigenvalues to compute them. What you're looking at are the dynamics for the four starting points.

For the next bullet point, it might help to remember that when diagonalizing with an eigenvector matrix V, we go from physical measurement space to diagonal space using V^{-1}:

$$d = V^{-1}p \tag{8}$$

and we go back from diagonal or eigenvector space to physical measurement space using V:

$$p = Vd. \tag{9}$$

• Make a new figure on which to plot the same data, but now in the eigenvector basis (also known as the "diagonal" space) . That is, on this new plot, the horizontal axis should represent the first eigenvector, and the vertical axis should represent the second eigenvector. Use `quiver.m` to make a quiver plot to show what the dynamics looks like here. On this same plot, add a red line for the horizontal axis and a red line for the vertical axis: these are the "diagonal space" axes. Now add two lines in black that start from the origin. The first one should go in the direction of $V^{-1} * [1;0]'$ and the second should go in the direction of $V^{-1} * [0;1]'$. These lines represent what the old, original, cartesian axes look like in the new eigenvector or "diagonal" space. Now add the blue symbols for what the four starting points move to in the eigenvector space after 1, 2, 3, and 4 timesteps. Compute these eigenvector space positions in two ways: first, by using the diagonal eigenvalue matrix on the starting position in eigenvector space; and second, by taking each of the blue points from the old, original, space, and transforming it into the new space.

• Comment on the relationship between your two plots.

• Without further use of Matlab: where will each of the symbols go to as the number of timesteps goes to infinity?

7. **Trace and Determinant in general** As you will remember, $\det(AB) = \det(BA)$. Let's also remember that if V is a matrix that has all the eigenvectors of M as its columns, and Λ is a diagonal matrix that has all the corresponding eigenvalues of M along the diagonal, then $M = V\Lambda V^{-1}$.

   • What is the determinant of a diagonal matrix (i.e. a matrix with zeros on all the off-diagonals)? Thinking about this geometrically might help – i.e., remember that the determinant is the factor by which a unit volume is changed after going through a matrix, and rmemeber that for a diagonal matrix, what happens to each dimension is independent of the other dimensions. Once you've done that, convince yourself that the determinant of Λ is the product of all the eigenvalues.

   • Show that for any matrix, its determinant is the product of its eigenvalues.

   • Letting Tr stand for trace, i.e., the sum of the diagonal elements, and using the fact that matrix multiplication follows the rule $C = AB \to C_{ij} = \sum_k A_{ik}B_{kj}$, show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

   • Show that for any matrix, its trace is the sum of its eigenvalues.

   • Show, without computing the eigenvalues, that the matrix $M = \begin{pmatrix} 0 & 10 & 23 \\ -4 & 0 & 5 \\ 6 & 2 & 0 \end{pmatrix}$ must have some negative *and* some positive eigenvalues. Assume that all three eigenvalues of M exist.

8. **Powers of matrices** Take the matrix $\begin{pmatrix} 1 & 19 \\ -3 & 2 \end{pmatrix}$. What is its square root? What is its

tenth root? Using the same approach, what is M to the -1 (i.e., $1/M$)? is that the same as the inverse of M? What does this tell you about taking the inverse of a matrix as one of its eigenvalues gets close to 0? You can solve this problem using Matlab, and it becomes particularly interesting when you express M in the basis set of its eigenvectors.

9. **Eigenvectors of symmetric matrices.** Imagine a symmetric matrix $A = A^{\mathrm{T}}$. Show that if two eigenvectors of A have different eigenvalues, then these two eigenvectors will be orthogonal to each other. (Note: "orthogonal" simply means that two vectors are at right angles two each other. "Orthonormal" in addition means that each of the vectors has length 1.)

   Let's say that all of the eigenvalues of A are different. Let's write all the eigenvalues as the elements of a diagonal matrix D, put all the eigenvectors as corresponding columns of a matrix M, and write the diagonalization of A as $A = VDV^{-1}$. Show that V is orthonormal, i.e. $V^{\mathrm{T}}V = I$, where I is the identity matrix.

10. **Eigenvectors of symmetric matrices.** For the matrix

$$A = \begin{pmatrix} 9 & 3 & 6 \\ 3 & 7 & 4 \\ 6 & 4 & 3 \end{pmatrix}$$

   - Use Matlab to calculate the eigenvectors and show that they are orthonormal.