# BioMath - Vectors and Matrices - Exercises 1 - August 24, 2015

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*Definition:* Rule for multiplying two matrices.

If $C = AB$ then $C_{ij}$, the element in the $i^{\mathrm{th}}$ row, $j^{\mathrm{th}}$ column of $C$, is equal to the dot product of the $i^{\mathrm{th}}$ row of $A$ and the $j^{\mathrm{th}}$ column of $B$.

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1. **Matrix multiplication.** • Multiplying it out by hand, find $C = AB$ where

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$$
$$B = \begin{pmatrix} 6 & 5 \\ 3 & 2 \end{pmatrix}$$

   • Again by hand, find $D = BA$. Are $C$ and $D$ the same or different?

2. **Matrix multiplication.** • Multiplying it out by hand, find $C = AB$ where

$$A = \begin{pmatrix} 2 & 2 \\ 4 & 3 \\ 3 & 3 \end{pmatrix}$$
$$B = \begin{pmatrix} 6 & 5 \\ 3 & 2 \end{pmatrix}$$

   • Does $BA$ exist?

3. **Matrix multiplication.** It has been proposed that integration over time, in the sense of calculus, is an important operation carried out in many systems of the brain (e.g., Major and Tank, 2004). In one well-known example, it is thought that decisions are formed after integrating evidence over time (Gold and Shadlen, 2007). Let the vector $\mathbf{v}$ represent a time series: $v_i$ represents the input to a neural system at the $i^{th}$ time step.

• show that the integral of $\mathbf{v}$ over time, namely $s_i = \Sigma_{j \leq i} v_j$, can be computed by $\mathbf{s} = L\mathbf{v}$ where $L$ is a matrix that has ones on the diagonal and in every element to the lower left of the diagonal, but zeroes everywhere else.

4. **Orthonormal basis sets.** The vector $\mathbf{x}$, defined in standard cartesian coordinates, is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

Find its new coordinates in the basis set given by

$$\hat{\mathbf{b}}_1 = \left[\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right]^T \quad \hat{\mathbf{b}}_2 = \left[0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right]^T \quad \hat{\mathbf{b}}_3 = \left[\frac{5}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right]^T \tag{1}$$

Start by first showing that the basis set is orthonormal (all vectors are length one, and they're all orthogonal to each other). Then find the projections of $\mathbf{x}$ onto each of the basis vectors.

5. **Orthonormal basis sets and matrix multiplication .** Take an orthonormal basis set, $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3, ...,$ and write it as a matrix $B$ where the $i^{\text{th}}$ row of $B$ is the $i^{\text{th}}$ basis vector. Using the rules of matrix multiplication and what you know about dot products, show that $BB^T = I$ where $I$ is a square matrix with ones along the diagonal and zeros everywhere else. (Take a moment to convince yourself that for any vector $\mathbf{x}$, it is true that $I\mathbf{x} = \mathbf{x}$. The matrix $I$ is the matrix generalization of the scalar number 1: multiplying by it leaves you right where you started.)

As you'll see in the next problem below, $BB^T = I$ means that for the *particular* case of a matrix with orthonormal rows, its inverse is simply its transpose. However, the inverse being the transpose does not hold for matrices in general, it only holds for matrices whose rows are orthonormal.[1]

6. **Inverses.** *Defintion:* Thinking of a matrix $A$ as a mapping from a field of points into a new set of points, let's consequently think of an inverse of a matrix as the reverse mapping, written as $A^{-1}$. This means that if we go through mapping $A$, and then through its inverse $A^{-1}$, we should get back to where we started. In other words, for any vector $\mathbf{x}$,

$$A^{-1}A\mathbf{x} = \mathbf{x} \tag{2}$$

[1]It turns out that if a matrices rows are orthonormal to each other, then its columns are also orthonormal to each other. That is, $BB^T = I \quad \Rightarrow \quad B^TB = I$. Can you prove that?

Because this is true for any vector $\mathbf{x}$, it can be shown that

$$A^{-1}A = I \tag{3}$$

• *Now to the exercise:* Remembering also the order of multiplication in linear algebra– or, in other words, remembering that that $AB\mathbf{x}$ means "put $\mathbf{x}$ through the $B$ mapping, then through the $A$ mapping",– show that the inverse of a product of matrices is the reverse product of the inverses. That is, show that $(AB)^{-1} = B^{-1}A^{-1}$.

7. **Orthonormal basis sets and inverses.** Remembering the definition if sine and cosine, in regular two-dimensional cartesian coordinates, what are the coordinates of a vector that is unit length and is $\theta$ degrees from the horizontal? Let's call this $\hat{\mathbf{b}}_1$. And how about another unit length vector, $\hat{\mathbf{b}}_2$ that is at $\theta + 90$ degrees? Since these two vectors are both unit length and are at 90 degrees to each other, they are an orthonormal basis set. Arrange the two vectors into a matrix $R$ that takes an original point $\mathbf{x}^T = [x_1 \;\; x_2]$ and maps it onto coordinates using the basis set $\{\hat{\mathbf{b}}_1, \; \hat{\mathbf{b}}_2\}$.

   • Remembering that $R$ represents a mapping onto an orthonormal basis set, is there a quick way to find it's inverse? (Hint: problem 5 will help you.) What is that inverse? Check your answer by multiplying $R$ times $R^{-1}$. Does your proposed inverse match what you expect from replacing $\theta$ with $-\theta$ in the definition of $R$?

8. **Inverses and matrix multiplication.** Take the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   • show that

$$B = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{4}$$

   has the properties:

$$AB = I \quad \text{and} \quad BA = I$$

   or in other words, $B = A^{-1}$.

   • What happens when $ad = bc$?

   • Consider the case $d = 1$. When $ad = bc$, what do the two separate points $[1 \quad -c]$ and $[2 \quad -2c]$ get mapped to? What does this tell you about being able to compute the inverse of $A$?

• Show that when $ad = bc$, the two vectors formed by the rows of $A$ are parallel to each other; show also that the two vectors formed by the columns of $A$ are parallel to each other. This means that neither the rows nor the columns can form the basis to describe a full 2-d space.

9. **Inverse of a rotation matrix**. Let's take our 2D rotation matrix again

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Use the formula in equation (4) from problem 8 to compute the inverse of $R$. What is the value of $ad - bc$ in this case? Does your answer here match what you found in problem 7?

10. **Dot Product.** Let $\mathbf{x}^T = (1, 3)$ and $\mathbf{y}^T = (0, 2)$. • Assuming these vectors are described in an orthonormal basis, what is their dot product $\mathbf{x}^T \cdot \mathbf{y}$? Sketch the two vectors on cartesian axes. If it helps you, use Matlab to do the plotting, but you can do it by hand if you wish.

We're going to illustrate the invariance of dot products under rotation. Now consider the rotation matrix

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

As we've discussed, this takes points and maps them onto a new, orthonormal, basis set, with basis vectors described by the rows of $R$.

• For $\theta = 60°$, find $\mathbf{x}' = R\mathbf{x}$ and $\mathbf{y}' = R\mathbf{y}$. Sketch $\mathbf{x}'$ and $\mathbf{y}'$ in cartesian axes. What does the new sketch look like? What do you expect the dot product to be, given the definition $\mathbf{a}^T \cdot \mathbf{c} = |a||c|\cos\theta$?

• Using the fact thet they are described in an orthonormal basis, compute the dot product of $\mathbf{x}'$ and $\mathbf{y}'$, and compare it to the original before rotation.

• Now compute the dot product of $\mathbf{x}'$ and $\mathbf{y}'$ for an arbitrary angle $\theta$. Does it change with $\theta$?

11. **Using matrix inverses to solve a set of linear equations.** You want to assay the expression level of three proteins, matlabin, algebrain, and vectorin. One of your colleagues has made a mouse that has fluorescent tags on the three proteins. However, for reasons of their own, on the matlabin they put three red fluorescent molecules, two green ones, and four blue ones. On the algebrain they put two red ones and two blue ones. And on the vectorin they put one red one, three green ones, and three blue ones. You take a slice of the tissue you're interested

in, and measure the total fluorescence on each of the red, green, and blue channels. Let's call this measurement the vector $\mathbf{x} = [r, g, b]^T$. Let's assume that the fluorescence luminosity per molecule is the same for the RFP, GFP, and BFP, and equals 1. (That is, if all you had was a matlabin with red fluorescence, and nothing else, and if $r = 1$, then you would infer that there is one matlabin molecule in your sample.) Let's call a vector representing the number of matlabin, algebrain, and vectorin molecules $\mathbf{y} = [m, a, v]^T$. Write down the matrix equation relating $\mathbf{x}$ and $\mathbf{y}$. Given $\mathbf{x}$, how do we find the number of matlabin, algebrain, and vectorin molecules?