Introduction: Multiple objective optimization

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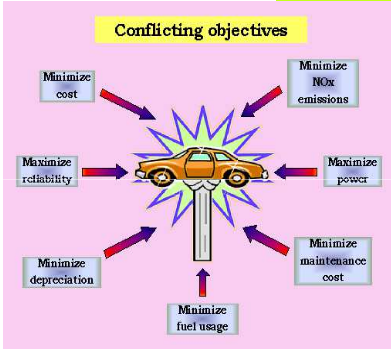
- Solution of problems with **two or more objectives** to be satisfied simultaneously

$$\text{Minimize/Maximize } f_i(\mathbf{x}) \quad \text{for } i = 1, 2, \ldots, n$$

s.t:

$$g_j(\mathbf{x}) \le 0 \qquad j = 1, 2, \ldots, J$$
$$h_k(\mathbf{x}) = 0 \qquad k = 1, 2, \ldots, K$$



- **Objectives are in conflict** with each other and are expressed in different units

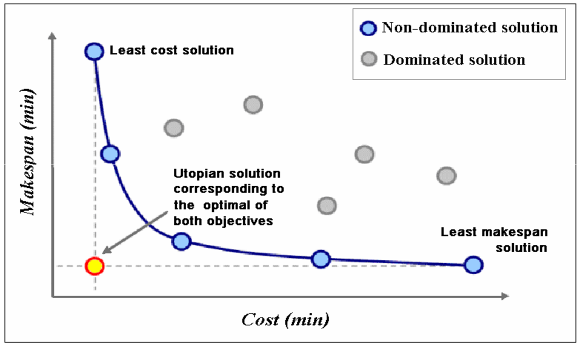
Complexity of Solving Multiple Objective Problems (MOPs)

i) Multiple, conflicting objectives

ii) Highly complex search space

f_1 : makespan $\quad f_2$:cost

Production planning example:



$$\text{Min } \{ f_1, f_2 \}$$

s.t.

$$g_j(\mathbf{x}) \le 0 \qquad j = 1, 2, \ldots, J$$
$$h_k(\mathbf{x}) = 0 \qquad k = 1, 2, \ldots, K$$

Methods Used to Solve Multiple Objective Problems

Methods used to formulate multiple objective problems

- Utility theory
- Weighted sum method
- Value function methods
- Goal programming
- ε-Constraint method
- Math programming
  - Lexicographic method
- Pareto optimality

Combine all objectives into a single objective Solve for a single solution

Sequential objectives

Set of non-dominated solutions

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Metaheuristics used to solve Multi-objective problems

- Simulated Annealing
  - Multi Objective Simulated Annealing (MOSA)
- Tabu Search method
  - Multiobjective Tabu Search (MOTS)
  - Multinomial Tabu Search (MTS)
- Genetic Algorithms
  - Vector Evaluated Genetic Algorithm (VEGA)
  - Multi Objective Genetic Algorithm (MOGA)
  - Nondominated Sorting Genetic Algorithm (NSGA)
  - Strength Pareto Evolutionary Algorithm (SPEA)

Single Solution Methods vs. Pareto Optimal Solution Methods

Original Problem

$$\max\{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_s(\mathbf{x})\}$$
$$s.t. \quad g_i(\mathbf{x}) \le b_i \quad i = 1, 2, \ldots, n$$

Weighted sum method

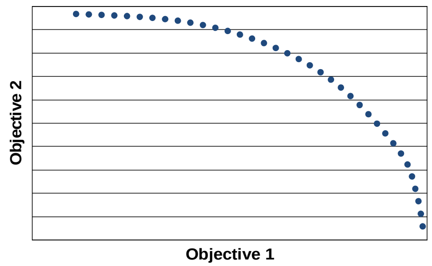
$$\max w_1 \overline{f_1}(\mathbf{x}) + w_2 \overline{f_2}(\mathbf{x}) + \ldots + w_s \overline{f_s}(\mathbf{x})$$
$$s.t. \quad g_i(\mathbf{x}) \le b_i \quad i = 1, 2, \ldots, n$$

Single solution

Easy to implement
Value of weights dictates the solution obtained

Pareto optimality

Straight-forward with two objectives
No best solution: Which solution to select when dealing with >2 objectives?



Literature review: General approaches to solve Multi-objective optimization problems

1) Combine the objective functions into an overall aggregated objective function

→ ***One*** single solution

2) Obtain a set of non-dominated Pareto-optimal

A ***large***

Goal programming:

$$\min \sum_{i=1}^{k} w_i \, |f_i(\mathbf{x}) - T_i|$$

where:

$\mathbf{x} \in X$

Weighted sum method:

$$\min \sum_{i=1}^{n} w_i f_i(\mathbf{x})$$

where:

$$0 \le w_i \le 1$$
$$\sum_{i=1}^{n} w_i = 1 \qquad i \in \{0,1,\ldots,n\}$$

Utility theory:

$$U(x_1,\ldots,x_n) = \lambda_1 U_1(x_1) + \ldots + \lambda_n U_n(x_n)$$

Goal programming:

Charnes, A. and Cooper , W. W. (1961). Management models and Industrial Applications of Linear Programming, vol.1. John Wiley, New York

Ijiri, Y. (1965). Management Goals and Accounting for Control. North Holland, Amsterdam.

Weighted sum method:

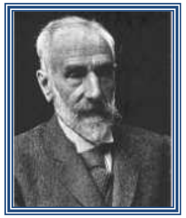
Zadeh (1963). Optimality and Nonscalar-Valued Performance Criteria. IEEE Transactions on Automatic Control, AC-8(1):59-60

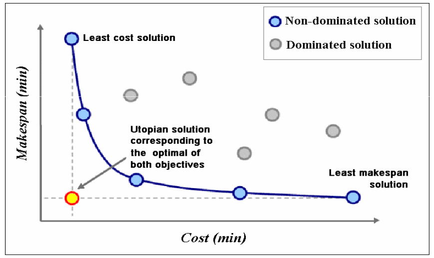
Utility theory:

Keeney, R.L. and Raiffa H. (1976). Decisions with Multiple Objectives: Preferences and Tradeoffs. John Wiley & Sons.

Steuer, R.E. (1989). Multiple Criteria Optimization: Theory, Computation, and Application. Reprint Edition, Krieger Publishing Company, Malabar, Florida.

Notion of Optimality in MOPs





- Having several objective functions, the notion of "optimum" changes, because in MOPs, we are really trying to find good compromises (or "trade-offs") rather than a single solution as in global optimization. The notion of "optimum" that is most commonly adopted is that originally proposed by Francis Y. Edgeworth in 1881.

Notion of Optimality in MOPs (cont'd…)

- This notion was later generalized by Vilfredo Pareto (in 1896).
- Although some authors call *Edgeworth-Pareto optimum* to this notion, the most commonly accepted term: *Pareto optimum*.

Pareto Optimality and Pareto Dominance

A Pareto-optimal solution is a solution that is not dominated by any other solution

Dominance criterion

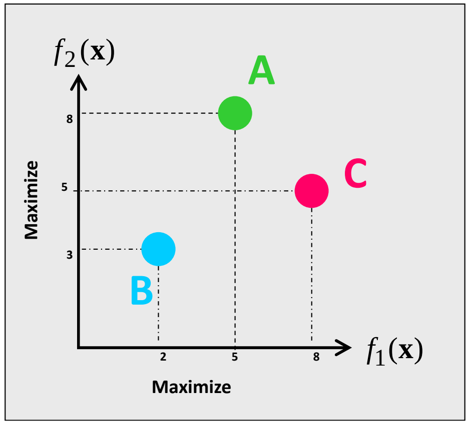
In a minimization problem:

A solution $\mathbf{x}_1$ dominates a solution $\mathbf{x}_2$, if and only if the two following conditions are true:

- $\mathbf{x}_1$ is no worse than $\mathbf{x}_2$ in all objectives,
  i.e $f_i(\mathbf{x}_1) \le f_i(\mathbf{x}_2)$ $\forall$ *i*,
- $\mathbf{x}_1$ is strictly better than $\mathbf{x}_2$ in at least one objective,
  i.e $f_i(\mathbf{x}_1) < f_i(\mathbf{x}_2)$ for at least one *i*.

Pareto Dominance (cont'd…)

$$\text{Maximize} \quad \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}),\ f_2(\mathbf{x}))$$



A dominates B

C dominates B

A and C are non-dominated with each other.

Pareto dominance criterion

Example : Optimization problem with 3 objectives

1. **Maximize Reliability**
2. **Minimize Cost**
3. **Minimize Weight**

Each alternative is evaluated according the objective functions:

Pareto Dominance criterion for a minimization problem:

A solution $\mathbf{x}_1$ dominates a solution $\mathbf{x}_2$, if and only if the two following conditions are true:

- $\mathbf{x}_1$ is no worse than $\mathbf{x}_2$ in all objectives, i.e $f_i(\mathbf{x}_1) \le f_i(\mathbf{x}_2)$ $\forall$ *i*,
- $\mathbf{x}_1$ is strictly better than $\mathbf{x}_2$ in at least one objective, i.e $f_i(\mathbf{x}_1) < f_i(\mathbf{x}_2)$ for at least one *i*

Choices	Rel	Cost	Weight
A.	- 0.9	60	30
B.	- 0.8	30	20
C.	- 0.7	40	20
D.	- 0.6	50	30

How many of these four solution vectors are nondominated solutions ?

Pareto dominance & Dominance count

	Rel	Cost	Weight
A.	- 0.9	60	30
B.	- 0.8	30	20
C.	- 0.7	40	20
D.	- 0.6	50	30

Comparing A against the rest of the solutions

A-B	√	×	×
A-C	√	×	×
A-D	√	×	√

A does not dominate any of the solutions

Comparing B against the rest of the solutions

B-A	×	√	√
B-C	√	√	√
B-D	√	√	√

B dominates C

B dominates D

Comparing C against the rest of the solutions

C-A	×	√	√
C-B	×	×	√
C-D	√	√	√

C dominates D

Comparing D against the rest of the solutions

D does not dominate any of the solutions

Nondominated solutions

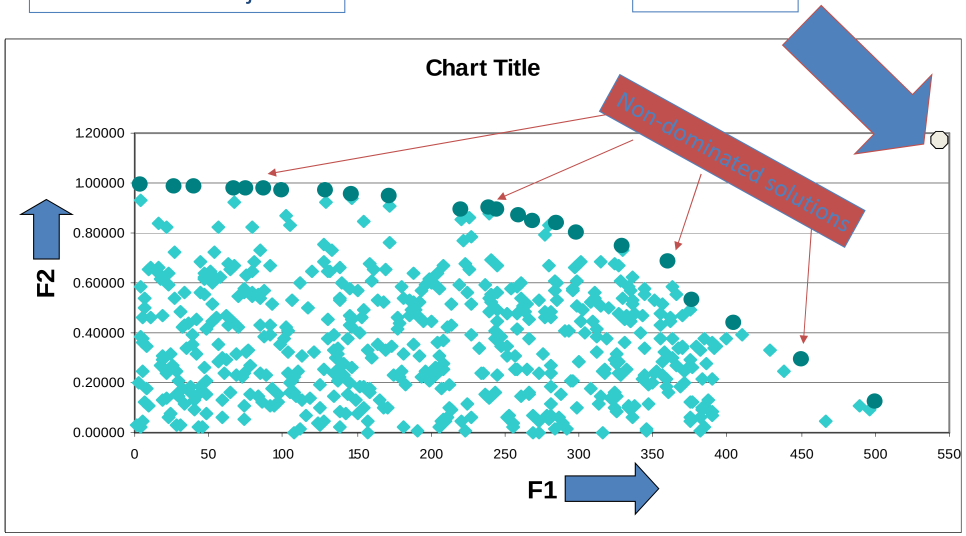
	Rel	Cost	Weight
A.	- 0.9	60	30
B.	- 0.8	30	20

Sol.	Dominance count
A.	0
B.	2
~~C.~~	~~1~~
~~D.~~	~~0~~

What is Pareto Optimality?

Maximize both objectives

Ideal Solution



Obtain a set of non-dominated Pareto-optimal solutions (MOEAs)

Algorithm	Paper reference	Description
VEGA (vector evaluated genetic algorithm)	Schaffer, J. D. (1985). Multiple Objective Optimization with Vector Evaluated Genetic Algorithms. In Genetic Algorithms and their Applications: Proceedings of the First International Conference on Genetic Algorithms, 93-100. Hillsdale, New Jersey.	Fractions of succeeding populations are selected based on separate objective performance
MOGA (multi-objective genetic algorithm)	Fonseca, C. M. and Fleming, P.J. (1993). Genetic Algorithms for Multiobjective Optimization: Formulation, Discussion and Generalization. *Proceedings of the Fifth International Conference on Genetic Algorithms*, 416-423. San Mateo California.	Incorporates niching and mating restrictions
NPGA (niched-Pareto genetic algorithm)	Horn, J., Nafpliotis, N. and Goldberg, D.E. (1994). A Niched Pareto Genetic Algorithm for Multiobjective Optimization. *In Proceedings of the First IEEE Conference on Evolutionary Computation*, IEEE World Congress on Computational Intelligence, 1, 82-87, Piscataway, New Jersey. IEEE Service Center.	Specialized Pareto domination tournaments
NSGA (nondominated sorting genetic algorithm)	Srinivas, N. and Deb, K. (1995). Multiobjective optimization using nondominated sorting in genetic algorithms. *Evolutionary Computation*, 2(3):221–248.	Assigns and shares dummy fitness in each front
SPEA (strength Pareto evolutionary algorithm)	Zitzler, E. and Thiele, L. (1999). Multiobjective Evolutionary Algorithms: A Comparative Case Study and the Strength Pareto Approach. *IEEE Transactions on Evolutionary Computation*, **3**(4), 257-271.	Actively uses secondary population in fitness assignment and selection
NSGA-II (fast elitist NSGA)	Deb, K., Agrawal, S., Pratap, A. and Meyarivan, T. (2000). A fast and elitist multi-objective genetic algorithm: NSGA-II. *IEEE Transactions on Evolutionary Computation*, 6(2):182-197.	Uses elitism and a crowded comparison operator that keeps diversity without specifying any additional parameters
PAES (Pareto-Archived Evolutionary Strategy)	Knowles, J. D. and Corne, D. W. (2000). Approximating the Nondominated Front Using the Pareto Archived Evolution Strategy. *Evolutionary Computation*, 8(2): 149-172.	It uses (1+1) evolution strategy (i.e., a single parent that generates a single offspring)

Many successful applications of MOEAs

Over the last decade, the applications of multiple objective optimization have grown steadily in many areas:

- **Design of groundwater remediation systems.**
- **Shape optimization.**
- **Fault-tolerant systems design.**
- **Computational fluid dynamics.**
- **Supersonic jet design.**
- **Design of control systems.**
- **Power systems**
- **Financial applications (e.g., optimal selection of investment portfolios).**
- **etc.**