ASTE 546
HW 12

Part 1: Prove that the two Gauss' Laws in Maxwell's equations are superfluous since they can be recovered from Faraday's and Ampere's laws.

Gauss' Laws:

1. $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ 2. $\nabla \cdot \vec{B} = 0$

Farraday's Law: $\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$

Ampere's Law: $\nabla \times \vec{B} = \mu_0 \left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$

where μ_0 - vacuum permeability

$\vec{j}$ - charge density $\vec{j} = \sum q_s n_s \vec{v}_s$

Starting with amperés law, taking the divergence of both sides:

$$\nabla \cdot (\nabla \times \vec{B}) = \nabla \cdot \left(\mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$0 = \underbrace{\mu_0 (\nabla \cdot \vec{j})} + \mu_0 \varepsilon_0 \left(\nabla \cdot \frac{\partial \vec{E}}{\partial t} \right)$$

we know from the continuity equation:

$$Q = \int_V \rho \, dV$$

that $\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$

$$0 = -\mu_0 \frac{\partial \rho}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \underbrace{\left(\nabla \cdot \vec{E} \right)}$$

$\longrightarrow$ commutation of partial derivatives

$$0 = \frac{\mu_0}{\varepsilon_0}\left(\frac{\partial}{\partial t}(\nabla \cdot \vec{E}) - \frac{\partial \rho}{\varepsilon_0 \partial t}\right)$$

$$0 = \frac{\mu}{\varepsilon_0} \frac{\partial}{\partial t}\left(\nabla \cdot E - \frac{\rho}{\varepsilon_0}\right)$$

if $\frac{\mu_0}{\varepsilon_0} \neq 0$ then to hold:

$$\frac{\partial}{\partial t}\left(\nabla \cdot E - \frac{\rho}{\varepsilon_0}\right) = 0$$

$$\int \frac{\partial}{\partial t}\left(\nabla \cdot E - \frac{\rho}{\varepsilon_0}\right) dt = \int 0 \, dt$$

$$\nabla \cdot E - \frac{\rho}{\varepsilon_0} = C$$

$$\boxed{\nabla \cdot E = \frac{\rho}{\varepsilon_0} + C}$$

Farraday's Law:

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

take div of both sides: $\nabla \cdot (\nabla \times E) = \nabla \cdot \left(-\frac{\partial B}{\partial t}\right)$

$$0 = -\frac{\partial}{\partial t}(\nabla \cdot B)$$

$$\int 0 \, dt = -\int \frac{\partial}{\partial t}(\nabla \cdot B) \, dt$$

$$C = -(\nabla \cdot B)$$

$$\boxed{\nabla \cdot B = -C}$$

Part 2:

current density $\vec{j} = \frac{1}{\eta}\left[-\nabla\phi + \frac{K}{e}\nabla T_e + \frac{KT_e}{en}\nabla n\right]$

a) show you can derive this equation by starting w/ electron momentum equation.

$$m_e n_e\left(\frac{\partial \vec{V}_e}{\partial t} + \vec{V}_e \cdot \nabla \vec{V}_e\right) = -e n_e \vec{E} - \nabla p_e + \vec{R}$$

where $\vec{R} = \frac{(e n_e \vec{J})}{\sigma}$ and $p_e = n_e K T_e$

$$m_e n_e\left(\frac{\partial \vec{V}_e}{\partial t} + \vec{V}_e \cdot \nabla \vec{V}_e\right) = -e n_e \vec{E} - \nabla(n_e K T_e) + \left(\frac{e n_e \vec{J}}{\sigma}\right)$$

$$'' \quad = -e n_e \vec{E} - (K T_e \nabla n_e + K n_e \nabla T_e) + \frac{e n_e \vec{J}}{\sigma}$$

$$m_e n_e\left(\frac{\partial \vec{V}_e}{\partial t} + \vec{V}_e \cdot \nabla \vec{V}_e\right) = n_e\left(-e\vec{E} - \frac{K T_e \nabla n_e}{n_e} - K\nabla T_e + \frac{e\vec{j}}{\sigma}\right)$$

$$m_e\left(\frac{\partial \vec{V}_e}{\partial t} + \vec{V}_e \cdot \nabla \vec{V}_e\right) + e\vec{E} + \frac{K T_e \nabla n_e}{n_e} + K\nabla T_e = \frac{e\vec{j}}{\sigma}$$

$$\frac{m_e \sigma}{e}\underbrace{\left(\frac{\partial \vec{V}_e}{\partial t} + \vec{V}_e \cdot \nabla \vec{V}_e\right)} + \sigma\vec{E} + \frac{\sigma K T_e \nabla n_e}{e\, n_e} + \frac{\sigma K \nabla T_e}{e} = \vec{J}$$

to further simplify we must assume steady state

$$\frac{\partial \vec{V}_e}{\partial t} + \vec{V}_e \cdot \nabla \vec{V}_e = 0$$

which gives $\sigma\left(\vec{E} + \frac{K T_e \nabla n_e}{e\, n_e} + \frac{K\nabla T_e}{e}\right) = \vec{j}$

where $\vec{E} = -\nabla\phi$ and $\sigma = \frac{1}{\eta}$

(electrostatic)

we get: $$\boxed{\vec{j} = \frac{1}{\eta}\left[-\nabla\phi + \frac{K\nabla T_e}{e} + \frac{KT_e\nabla n_e}{e n_e}\right]}$$

where quasineutrality gives $n_e = n_i = n$

b) Derive a Poisson-like equation for plasma potential

$$\nabla^2\phi = f(\phi, n_e, T_e, \sigma)$$

by utilizing: $\nabla\cdot\vec{j} = 0 \quad \nabla\sigma \neq 0$

given the above equation for $\vec{j}$ we have:

$$\nabla\cdot j = \nabla\cdot\left[\sigma\left(-\nabla\phi + \frac{K\nabla T_e}{e} + \frac{KT_e\nabla n_e}{e n_e}\right)\right] = 0$$

$$0 = \left.\begin{array}{l} \nabla\cdot(-\sigma\nabla\phi) \\ + \nabla\cdot\left(\frac{\sigma K\nabla T_e}{e}\right) \\ + \nabla\cdot\left(\frac{\sigma KT_e\nabla n_e}{n_e}\right) \end{array}\right\} \begin{array}{l} -(\nabla\sigma\nabla\phi + \sigma\nabla^2\phi) \\ + \left(\frac{\nabla\sigma K\nabla T_e}{e} + \frac{\sigma K\nabla^2 T_e}{e}\right) \\ + \left(\frac{\nabla\sigma KT_e\nabla n_e}{n_e} + \frac{\sigma KT_e\nabla^2 n_e}{n_e}\right) \end{array}$$

$$\nabla^2\phi = \frac{1}{\sigma}\left[\frac{\nabla\sigma\nabla\phi}{\sigma} - \frac{\nabla\sigma K\nabla T_e}{e} - \frac{\sigma K\nabla^2 T_e}{e} - \frac{\nabla\sigma KT_e\nabla n_e}{n} - \frac{\sigma KT_e\nabla^2 n_e}{n_e}\right]$$

if we assume constant temperature T_e = constant we can simplify further to get

$$\nabla^2\phi = \frac{1}{\sigma}\left[\nabla\sigma\nabla\phi - \frac{\nabla\sigma KT_e\nabla n_e}{n_e} - \frac{\sigma KT_e\nabla^2 n_e}{n_e}\right]$$

c) To discretize we start with (in 1D) $\frac{d^2\phi}{dx^2} \approx \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(\Delta x)^2}$

similarly: $\frac{d\phi}{dx} \approx \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}$

$$\frac{d\sigma}{dx} \approx \frac{\sigma_{i+1} - \sigma_{i-1}}{2\Delta x}$$

$$\frac{dn_e}{dx} \approx \frac{n_{e\,i+1} - n_{e\,i-1}}{2\Delta x}$$

$$\frac{dT_e}{dx} \approx \frac{T_{e\,i+1} - T_{e\,i-1}}{2\Delta x}$$

$$\frac{d^2 n_e}{dx^2} \approx \frac{n_{e,i+1} - 2n_{e,i} + n_{e\,i-1}}{(\Delta x)^2}$$

$$\frac{d^2 T_e}{dx^2} \approx \frac{T_{e\,i+1} - 2T_{e,i} + T_{e\,i-1}}{(\Delta x)^2}$$

plug in to above equation...

d) given $j = \sigma\left[-\nabla\phi + \frac{k}{e}\nabla T_e + \frac{kT_e}{en}\nabla n\right]$

$\frac{d\phi}{dx} = 0$ would not be enough to ensure no current at the boundary.

$\frac{dn_e}{dx}$ and $\frac{dT_e}{dx}$ would also have to $= 0$