Exercise session 2

Problem 4

$$y_{it} = \alpha \underbrace{y_{it-1}} + \eta_i + v_{it}$$

$$y_{it} = \alpha(\alpha y_{it-2} + \eta_i + v_{it-1}) + \eta_i + v_{it}$$

$$= \alpha^2 \boxed{y_{it-2}} + \alpha\eta_i + \alpha v_{it-1} + \eta_i + v_{it}$$

$$= \alpha^2(\alpha y_{it-3} + \eta_i + v_{it-2}) + \alpha\eta_i + \alpha v_{it-1} + \eta_i + v_{it}$$

$$= \alpha^3 y_{it-3} + \alpha^2\eta_i + \alpha^2 v_{it-2} + \alpha\eta_i + \alpha v_{it-1} + \eta_i + v_{it}$$

$$= \dots$$

$$= \alpha^t y_{0i} + \left(\sum_{s=0}^{t-1} \alpha^s\right)\eta_i + \left(\sum_{s=0}^{t-1} \alpha^s v_{it-s}\right)$$

Bias is driven by $\boxed{\mathbb{E}[\underbrace{y_{it-1}u_{it}}]}$, $u_{it} := \eta_i + v_{it}$

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} y_{i-}^{*} y_{i}^{*}}{\sum_{i=1}^{n} y_{i-}^{*} y_{i-}^{*\prime}} = \dots = \frac{\quad}{-\,\text{"}\,-}$$

$$\boxed{y_{it-1}} = \alpha^{t-1} y_{i0} + \underbrace{\left(\sum_{s=0}^{t-2} \alpha^s\right)}\eta_i + \left(\sum_{s=0}^{t-2} \alpha^s v_{it-s}\right)$$

$$= \alpha^{t-1} y_{i0} + \left(\frac{1-\alpha^{t-1}}{1-\alpha}\right)\eta_i + \left(\sum_{s=0}^{t-2} \alpha^s v_{it-s}\right)$$

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}.$$

$$\mathbb{E}[y_{it-1} u_{it}] = \underbrace{\mathbb{E}[y_{it-1}\eta_i]}_{\circledast} + \underbrace{\mathbb{E}[y_{it-1} v_{it}]}_{=0}$$

$$\circledast = \mathbb{E}\left[\alpha^{t-1} y_{i0}\eta_i + \left(\frac{1-\alpha^{t-1}}{1-\alpha}\right)\eta_i^2 + \underbrace{\sum_{s=0}^{t-2} \alpha^s \eta_i v_{it-s}}_{\mathbb{E}[\eta_i v_{it-s}] = 0}\right]$$

$$= \underbrace{\alpha^{t-1}\mathbb{E}[\eta_i y_{i0}]}_{\text{if assume } = 0} + \boxed{\left(\frac{1-\alpha^{t-1}}{1-\alpha}\right)\sigma_\eta^2}$$

$$\mathbb{E}[\eta_i^2] = \mathrm{var}[\eta_i] = \sigma_\eta^2$$

$$\sigma_\eta^2 \neq 0$$

Made with Goodnotes

Problem 5

$$y_{it} = \alpha y_{it-1} + \eta_i + v_{it}, \quad \eta_i \sim iid(0, \sigma_\eta^2)$$

$$v_{it} \sim iid(0, \sigma_v^2)$$

$$y_{i0} = 0 \ \forall i$$

Derive $var[y_{it}]$.

$$y_{it} = \alpha^t y_{i0} + \left(\sum_{s=0}^{t-1} \alpha^s\right)\eta_i + \sum_{s=0}^{t-1} v_{it-s}\alpha^s \overset{y_{i0}=0}{=}$$

$$= \left(\sum_{s=0}^{t-1} \alpha^s\right)\eta_i + \sum_{s=0}^{t-1} v_{it-s}\alpha^s.$$

- $var\left[\underbrace{\left(\sum_{s=0}^{t-1} \alpha^s\right)^2}\underbrace{\eta_i^2}\right] = \left(\sum_{s=0}^{t-1} \alpha^s\right)^2 \cdot \sigma_\eta^2 = \sigma_\eta^2\left(\frac{1-\alpha^t}{1-\alpha}\right)^2$

- $var\left[\underline{\sum_{s=0}^{t-1} v_{it-s}^2\alpha^{2s}}\right] = \sigma_v^2 \sum_{s=0}^{t-1} \alpha^{2s} = \sigma_v^2 \cdot \left(\frac{1-\alpha^{2t}}{1-\alpha^2}\right).$

$$\Rightarrow var[y_{it}] = \sigma_\eta^2\left(\frac{1-\alpha^t}{1-\alpha}\right)^2 + \sigma_v^2 \cdot \left(\frac{1-\alpha^{2t}}{1-\alpha^2}\right).$$

Problem 6

$$y_{it} = \alpha y_{it-1} + \eta_i + \underline{v_{it}}$$

$$\underline{v_{it} = w_{it} + b\,w_{it-1}}, \quad w_{it} \sim iid(0, \sigma_w^2)$$

Show that y_{it-2} is not a valid instrument, but $y_{it-j},\ j \geq 3$ is valid.

Autocovariances $E[v_{it} v_{it-s}]$:

• $s=0$: $E[v_{it} v_{it}] = E[v_{it}^2] = E[(w_{it} + b w_{it-1})^2]$

$$= E[w_{it}^2 + 2b\,w_{it} w_{it-1} + b^2 w_{it-1}^2]$$

$$= \underbrace{E[w_{it}^2]}_{\sigma} + 2b\,\underbrace{E[w_{it} w_{it-1}]}_{} + b^2 E[w_{it-1}^2]$$

$$= \sigma_w^2 + 0 + b^2 \sigma_w^2 = (1+b^2)\sigma_w^2.$$

• $\boxed{s=1:}$ $E[v_{it} v_{it-1}] = E[(w_{it} + b w_{it-1}) \times$
$\times (w_{it-1} + b w_{it-2})] =$

$$\boxed{E[v_{it} v_{it-1}] = E[v_{it-1} v_{it-2}] = b\sigma_w^2}$$

$$= E[\underset{0}{w_{it} w_{it-1}} + \overset{0}{w_{it} w_{it-2} \cdot b} + b\,w_{it-1}^2 + b^2 w_{it-1} w_{it-2}]$$

$$= \overbrace{E[w_{it} w_{it-1}]}^{0} + b\,\overbrace{E[w_{it} w_{it-2}]}^{0}$$

$$+ b\,\underbrace{E[w_{it-1}^2]}_{} + b^2 \underbrace{E[w_{it-1} w_{it-2}]}_{0}$$

$$= b\sigma_w^2.$$

$\boxed{s=2:}$ $\boxed{\mathbb{E}[v_{it}v_{it-2}] = 0}$ $= \mathbb{E}[(w_{it}+bw_{it-1})(w_{it-2}+bw_{it-3})]$

$s=3$ $= 0$
$s=\dots$

FD:

$$y_{it}-y_{it-1} = \alpha(y_{it-1}-y_{it-2}) + \underbrace{v_{it}-v_{it-1}}_{:=u_{it}}$$

For y_{it-2} to be valid it should hold that

$$\mathbb{E}[y_{it-2}u_{it}] = 0$$

$$\Leftrightarrow \mathbb{E}[(\dots + v_{it-2})(v_{it}-v_{it-1})]$$

$$= \dots = \mathbb{E}[v_{it-1}v_{it-2}] = b\sigma_w^2 \neq 0$$

y_{it-j}, $j \geq 3$: $\mathbb{E}[y_{it-3}u_{it}] = 0$?

$$\mathbb{E}[(\dots + v_{it-3})(v_{it}-v_{it-1})]$$

$$= \dots = \mathbb{E}[v_{it-3}v_{it-1}] = \boxed{\mathbb{E}[v_{it-2}v_{it}] = 0}$$

Problem 7

a) WG:

$$\left(\frac{I_{it}}{K_{it}} - \frac{\bar{I}_i}{\bar{K}_i}\right) = \alpha\left(\frac{I_{it-1}}{K_{it-1}} - \frac{\bar{I}_i}{\bar{K}_i}\right) + v_{it} - \bar{v}_i$$

$$\frac{\bar{I}_i}{\bar{K}_i} := \frac{1}{T}\sum_{t=1}^{T} \frac{I_{it}}{K_{it}}.$$

FD:

$$\left(\frac{I_{it}}{K_{it}} - \frac{I_{it-1}}{K_{it-1}}\right) = \underline{\alpha\left(\frac{I_{it-1}}{K_{it-1}} - \frac{I_{it-2}}{K_{it-2}}\right)} + \boxed{v_{it} - v_{it-1}}$$

As instrument, use for example $\frac{I_{it-2}}{K_{it-2}}$, 2SLS

1) $$\left(\frac{I_{it-1}}{K_{it-1}} - \frac{I_{it-2}}{K_{it-2}}\right) = \gamma_0 + \gamma_1 \frac{I_{it-2}}{K_{it-2}} + \varepsilon_{i(t)} \sim iid(0, \sigma_\varepsilon^2)$$

run OLS

$\Rightarrow$

2) $$\left(\frac{I_{it}}{K_{it}} - \frac{I_{it-1}}{K_{it-1}}\right) = \alpha \underbrace{\left(\frac{I_{it-1}}{K_{it-1}} - \frac{I_{it-2}}{K_{it-2}}\right)}_{:= \underset{\text{OLS}}{\hat{\gamma}_1} \cdot \left(\frac{I_{it-2}}{K_{it-2}}\right)} + v_{it} - v_{it-1}$$

Exercise session 3

Causal inference

Problem 1.

Case 1

Table 2: Surgery vs therapy

patient	Y_i^1	Y_i^0	δ_i	Y_i	D_i
		case 1			
1	4	3	1	4	1
1	6	5	1	6	1
3	1	2	−1	2	0
4	3	6	−3	6	0
		case 2			
1	5	3	2	5	1
1	6	5	1	6	1
3	7	6	1	6	0
4	10	8	2	8	0

$$ATE = \frac{1+1-1-3}{4} = -0{,}5$$

$$\rightarrow SDO = \frac{4+6}{2} - \frac{2+6}{2} = 5-4 = 1$$

$$\Rightarrow \boxed{SDO \neq ATE \Rightarrow \text{bias}}$$

$$= 1 = SB + HEB + ATE = 0 + 1{,}5 - 0{,}5$$

1) Selection bias:

$$E[y_i^0 | D_i = 1] - E[y_i^0 | D_i = 0]$$

$$= \frac{3+5}{2} - \frac{2+6}{2} = 0$$

2) Heterogenous effect bias: $(1-\pi)(ATT - ATU) =$

$$= 0{,}5(1+2) = 1{,}5$$

$$ATT = \mathbb{E}[\delta_i | D_i = 1] = 1$$

$$ATU = \mathbb{E}[\delta_i | D_i = 0] = \frac{-1-3}{2} = -2$$

$$\pi = \frac{\#\{D_i = 1\}}{N} = \frac{2}{4} = 0{,}5$$

Case 2

$$ATE = \frac{2+1+1+2}{4} = 1{,}5$$

$$SDO = \frac{5+6}{2} - \frac{6+8}{2} = -1{,}5$$

$$ATT = \frac{2+1}{2} = 1{,}5 \qquad ATU = \frac{1+2}{2} = 1{,}5$$

SB: $\mathbb{E}[y_i^0 | D_i = 1] - \mathbb{E}[y_i^0 | D_i = 0]$

$$= \frac{3+5}{2} - \frac{6+8}{2} = -3$$

HTB: $(1-\pi)(ATT - ATU) =$

$$= 0{,}5\,(1{,}5 - 1{,}5) = 0$$

Problem 2

$y_i = \beta_0 + \beta_1 D_i + \varepsilon_i$. Show that $\overset{\text{OLS}}{\hat{\beta}_1} = \frac{1}{N_1}\sum_{i=1}^{N_1}(y_i \mid D_i = 1) - \frac{1}{N_0}\sum_{i=1}^{N_0}(y_i \mid D_i = 0)$

$$\hat{\beta}_1 = \frac{① \sum_{i=1}^{N}(D_i - \bar{D})(y_i - \bar{y})}{② \sum_{i=1}^{N}(D_i - \bar{D})^2}, \quad \boxed{\bar{D} = \frac{1}{N}\underbrace{\sum_{i=1}^{N} D_i}_{} = \frac{N_1}{N}}$$

$D_i \in \{0, 1\}$

$$\sum_{i=1}^{N} D_i = 1 + 1 + 0 \ldots = N_1$$

$$① \sum_{i=1}^{N}(D_i - \bar{D})(y_i - \bar{y}) = \sum_{i=1}^{N} D_i y_i - D_i \bar{y} - \bar{D} y_i + \bar{D}\bar{y}$$

- $\sum_{i=1}^{N} D_i y_i = \frac{N_1}{N_1}\underbrace{\sum_{i=1}^{N} D_i y_i}_{} = \frac{N_1}{N_1}\underbrace{\sum_{i=1,\, D_i=1}}_{} y_i = N_1 \cdot \frac{1}{N_1}\underbrace{\sum_{i=1,\, D_i=1}}_{} y_i$

$= 1 \cdot y_1 + 1 \cdot y_2 + 0 \cdot y_3 + \ldots = y_1 + y_2 + \ldots$

$= N_1 \cdot \bar{y}_1$

- $-\sum_{i=1}^{N} \bar{D} y_i = -\sum_{i=1}^{N}\frac{N_1}{N} y_i = -N_1 \cdot \frac{1}{N}\sum_{i=1}^{N} y_i = -N_1 \cdot \bar{y}$

- $-\sum_{i=1}^{N} D_i \bar{y} = -\underbrace{\sum_{i=1,\, D_i=1}}_{} \bar{y} = -N_1 \cdot \bar{y}$

$$\Rightarrow ① = N_1 \bar{y}_1 - N_1 \bar{y} - \cancel{N_1 \bar{y}} + \cancel{N_1 \bar{y}}$$

$$= N_1(\bar{y}_1 - \bar{y})$$

- $\sum_{i=1}^{N} \bar{D}\bar{y} = \sum_{i=1}^{N}\frac{N_1}{N}\bar{y} = N_1 \cdot \bar{y}$

- $N = N_1 + N_0$

HOMEWORK

$$\bar{y} = \frac{1}{N}(N_1 \bar{y}_1 + N_0 \bar{y}_0) \Rightarrow \boxed{① = \left(N_1 - \frac{N_1^2}{N}\right)(\bar{y}_1 - \bar{y}_0)}$$

$$② = \sum_{i=1}^{N}(D_i-\bar{D})^2 = \sum_{i=1}^{N} D_i^2 - 2D_i\bar{D} + \bar{D}^2 =$$

$$= \sum_{i=1}^{N} D_i^2 - 2\sum_{i=1}^{N} D_i\bar{D} + \sum_{i=1}^{N}\bar{D}^2$$

$$= N_1 - 2\underbrace{\sum_{i=1}^{N} D_i}\frac{N_1}{N} + \sum_{i=1}^{N}\frac{N_1^2}{N^2}$$

$$= N_1 - 2\frac{N_1^2}{N} + \cancel{N}\cdot\frac{N_1^2}{N^{\cancel{2}}}$$

$$= \boxed{N_1 - \frac{N_1^2}{N}}$$

$$\Rightarrow \hat{\beta} = \frac{\cancel{\left(N_1 - \frac{N_1^2}{N}\right)}(\bar{y}_1-\bar{y}_0)}{\cancel{N_1 - \frac{N_1^2}{N}}} = \underbrace{\bar{y}_1 - \bar{y}_0}, \quad \bar{y}_1 = \frac{1}{N_1}\sum_{\substack{i=1\\D_i=1}} y_i$$

OLS

$$\bar{y}_0 = \frac{1}{N_0}\sum_{\substack{i=1\\D_i=0}} y_i$$

Problem 3

$$y_i = \beta_0 + \beta_1 D_i + \varepsilon_i$$

$$\hat{\beta}_1 = \bar{y}_1 - \bar{y}_0 \;;\; \boxed{\widehat{var}[\hat{\beta}_1]} = \widehat{var}[\bar{y}_1] + \widehat{var}[\bar{y}_0]$$

$$= \boxed{\frac{\hat{\sigma}_1^2}{N_1} + \frac{\hat{\sigma}_0^2}{N_0}}, \quad \sigma_k^2,\ k \in \{0,1\}$$

$$\sigma_1^2 \neq \sigma_0^2$$

Set $x_i := (1 \; D_i)'$,

$$var\left[\begin{pmatrix}\beta_0 \\ \beta_1\end{pmatrix}\right] = \frac{1}{N}\,\mathbb{E}[x_i x_i']^{-1}\,\mathbb{E}[x_i x_i' \varepsilon_i^2]\,\mathbb{E}[x_i x_i']^{-1} \ldots \text{asympt. variance}$$

$$\Rightarrow \widehat{var}\left[\begin{pmatrix}\hat{\beta}_0 \\ \hat{\beta}_1\end{pmatrix}\right] = \frac{1}{N}\left(\frac{1}{N}\sum_{i=1}^N x_i x_i'\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^N x_i x_i' \hat{\varepsilon}_i^2\right)\left(\frac{1}{N}\sum_{i=1}^N x_i x_i'\right)^{-1} =$$

$$\hat{\varepsilon}_i = y_i - x_i'\hat{\beta}$$

$$= \frac{1}{N}\left(\frac{1}{N}\sum_{i=1}^N \begin{pmatrix}1 & D_i \\ D_i & D_i\end{pmatrix}\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^N \begin{pmatrix}1 & D_i \\ D_i & D_i\end{pmatrix}\hat{\varepsilon}_i^2\right) \times$$

$$\times \left(\frac{1}{N}\sum_{i=1}^N \begin{pmatrix}1 & D_i \\ D_i & D_i\end{pmatrix}\right)^{-1}$$

$$= \frac{1}{N}\begin{pmatrix}1 & N_1/N \\ N_1/N & N_1/N\end{pmatrix}^{-1}\begin{pmatrix}\sum_{i=1}^N \hat{\varepsilon}_i^2/N & \sum_{i=1, D_i=1} \hat{\varepsilon}_i^2/N \\ \sum_{i=1, D_i=1} \hat{\varepsilon}_i^2/N & \sum_{i=1, D_i=1} \hat{\varepsilon}_i^2/N\end{pmatrix} \times$$

$$\times \begin{pmatrix}1 & N_1/N \\ N_1/N & N_1/N\end{pmatrix}^{-1}$$

$$A = \begin{pmatrix}a & b \\ c & d\end{pmatrix} \quad A^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b \\ -c & a\end{pmatrix}$$

$$= \begin{pmatrix}N & N_1 \\ N_1 & N_1\end{pmatrix}^{-1}\begin{pmatrix}\sum_i \hat{\varepsilon}_i^2 & \sum_{i, D_i=1} \hat{\varepsilon}_i^2 \\ \sum_{i, D_i=1} \hat{\varepsilon}_i^2 & \sum_{i, D_i=1} \hat{\varepsilon}_i^2\end{pmatrix}\begin{pmatrix}N & N_1 \\ N_1 & N_1\end{pmatrix}^{-1}$$

$$= \underbrace{\begin{pmatrix} 1/N_0 & -1/N_0 \\ -1/N_0 & -1/N_0 + 1/N_1 \end{pmatrix}}_{A} \times \qquad N = N_1 + N_0$$

$$\times \underbrace{\begin{pmatrix} \sum_i \hat{\varepsilon}_i^2 & \sum_{i,D_i=1} \hat{\varepsilon}_i^2 \\ \sum_{i,D_i=1} \hat{\varepsilon}_i^2 & \sum_{i,D_i=1} \hat{\varepsilon}_i^2 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} 1/N_0 & -1/N_0 \\ -1/N_0 & 1/N_0 + 1/N_1 \end{pmatrix}}_{C}$$

$$\sum_{i=1}^{N} \hat{\varepsilon}_i^2 = \sum_{\substack{i=1 \\ D_i=1}} \hat{\varepsilon}_i^2 + \sum_{\substack{i=1 \\ D_i=0}} \hat{\varepsilon}_i^2$$

$$\Rightarrow \underbrace{AB}_{D} \times C \Rightarrow DC$$

$$= \begin{pmatrix} \sum_{\substack{i=1 \\ D_i=0}} \hat{\varepsilon}_i^2 / N_0 & 0 \\ \sum_{\substack{i=1, \\ D_i=1}} \hat{\varepsilon}_i^2 / N_1 - \sum_{\substack{i=1 \\ D_i=0}} \hat{\varepsilon}_i^2 / N_0 & \sum_{\substack{i=1 \\ D_i=1}} \hat{\varepsilon}_i^2 / N_1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} 1/N_0 & -1/N_0 \\ -1/N_0 & 1/N_0 + 1/N_1 \end{pmatrix} =$$

$$= \begin{pmatrix} \boxed{\sum_{i,D_i=0} \hat{\varepsilon}_i^2 / N_0^2} & -\sum_{i,D_i=0} \hat{\varepsilon}_i^2 / N_0^2 \\ -\sum_{i,D_i=0} \hat{\varepsilon}_i^2 / N_0^2 & \boxed{\dfrac{\sum_{i,D_i=0} \hat{\varepsilon}_i^2}{N_0^2} + \dfrac{\sum_{i,D_i=1} \hat{\varepsilon}_i^2}{N_1^2}} \end{pmatrix}$$

$$\mathrm{var}\left[\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix}\right]$$

$$\widehat{\mathrm{var}}[\hat{\beta}_1] = \frac{\sum_{i,D_i=0} \hat{\varepsilon}_i^2}{N_0^2} + \frac{\sum_{i,D_i=1} \hat{\varepsilon}_i^2}{N_1^2}$$

$$\hat{\varepsilon}_i^2 = y_i - x_i' \hat{\beta} \mid D_i = 0 \quad \hat{\sigma}_0^2$$

$$\hat{\varepsilon}_i^2 = y_i - x_i' \hat{\beta} \mid D_i = 1 \quad \hat{\sigma}_1^2$$