

Exercise session 2

Problem 4

$$y_{it} = \lambda y_{it-1} + \eta_i + v_{it}$$

$$\begin{aligned} y_{it} &= \lambda (\lambda y_{it-2} + \eta_i + v_{it-1}) + \eta_i + v_{it} \\ &= \lambda^2 y_{it-2} + \lambda \eta_i + \lambda v_{it-1} + \eta_i + v_{it} \\ &= \lambda^2 (\lambda y_{it-3} + \eta_i + v_{it-2}) + \lambda \eta_i + \lambda v_{it-1} + \\ &\quad + \eta_i + v_{it} \\ &= \underbrace{\lambda^3 y_{it-3}}_{(3)} + \underbrace{\lambda^2 \eta_i}_{\text{red}} + \underbrace{\lambda^2 v_{it-2}}_{\text{blue}} + \underbrace{\lambda \eta_i}_{\text{red}} + \underbrace{\lambda v_{it-1}}_{\text{blue}} + \\ &\quad + \underbrace{\eta_i}_{\text{red}} + \underbrace{v_{it}}_{\text{blue}} \\ &= \dots \\ &= \lambda^t y_{0i} + \left(\sum_{s=0}^{t-1} \lambda^s \right) \eta_i + \left(\sum_{s=0}^{t-1} \lambda^s v_{it-s} \right) \end{aligned}$$

Bias is driven by $\mathbb{E}[y_{it-1} u_{it}]$, $u_{it} := \eta_i + v_{it}$

$$\hat{\lambda} = \frac{\sum_{i=1}^n y_i^* - \bar{y}_i^*}{\sum_{i=1}^n y_i^* - \bar{y}_i^{**}} = \dots = \frac{-11-}{}$$

$$y_{it-1} = \delta^t y_{i0} + \underbrace{\left(\sum_{s=0}^{t-2} \delta^s \right) \eta_{is}}_{\text{red box}} + \left(\sum_{s=0}^{t-2} \delta^s \nu_{it-s} \right)$$

$$= \delta^t y_{i0} + \left(\frac{1 - \delta^t}{1 - \delta} \right) \eta_{is} + \left(\sum_{s=0}^{t-2} \delta^s \nu_{it-s} \right)$$

$$\sum_{k=0}^{n-1} r^k = \frac{1 - r^n}{1 - r}$$

$$\mathbb{E}[y_{it-1} \nu_{it}] = \underbrace{\mathbb{E}[y_{it-1} \eta_{is}]}_{\text{red}} + \underbrace{\mathbb{E}[y_{it-1} \nu_{it}]}_{=0}$$

$$\Phi = \mathbb{E}\left[\delta^t y_{i0} \eta_{is} + \left(\frac{1 - \delta^t}{1 - \delta} \right) \eta_{is}^2 + \sum_{s=0}^{t-2} \delta^s \eta_{is} \nu_{it-s} \right]$$

$$= \delta^t \mathbb{E}[\eta_{is} y_{i0}] + \left(\frac{1 - \delta^t}{1 - \delta} \right) \sigma_\eta^2 + \mathbb{E}[\eta_{is} \nu_{it-s}]$$

$\mathbb{E}[\eta_{is} \nu_{it-s}] = 0$

$\sigma_\eta^2 \neq 0$

if assume
 $= 0$

$$\mathbb{E}[\eta_{is}^2] = \text{var}[\eta_{is}] = \sigma_\eta^2$$

Problem 5

$$y_{it} = \alpha y_{it-1} + \eta_i + v_{it}, \eta_i \sim \text{iid}(0, \sigma_n^2)$$

$$v_{it} \sim \text{iid}(0, \sigma_v^2)$$

$$y_{i0} = 0 \quad \forall i$$

Derive $\text{var}[y_{it}]$.

$$\begin{aligned} y_{it} &= \alpha y_{i0} + \left(\sum_{s=0}^{t-1} \alpha^s \right) \eta_i + \sum_{s=0}^{t-1} v_{it-s} \alpha^s = \\ &= \left(\sum_{s=0}^{t-1} \alpha^s \right) \eta_i + \sum_{s=0}^{t-1} v_{it-s} \alpha^s. \end{aligned}$$

$$\cdot \text{var}\left[\left(\sum_{s=0}^{t-1} \alpha^s\right)^2 \eta_i\right] = \left(\sum_{s=0}^{t-1} \alpha^s\right)^2 \cdot \sigma_n^2 = \sigma_n^2 \left(\frac{1-\alpha^t}{1-\alpha}\right)^2$$

$$\cdot \text{var}\left[\sum_{s=0}^{t-1} v_{it-s} \alpha^s\right] = \sigma_v^2 \sum_{s=0}^{t-1} \alpha^{2s} = \sigma_v^2 \cdot \left(\frac{1-\alpha^{2t}}{1-\alpha^2}\right).$$

$$\Rightarrow \text{var}[y_{it}] = \sigma_n^2 \left(\frac{1-\alpha^t}{1-\alpha}\right)^2 + \sigma_v^2 \cdot \left(\frac{1-\alpha^{2t}}{1-\alpha^2}\right).$$

Problem 6

$$y_{it} = \phi y_{it-1} + \eta_i + \underline{v_{it}}$$

$$\underline{v_{it}} = w_{it} + b w_{it-1}, w_{it} \sim \text{iid } (0, \sigma_w^2)$$

Show that y_{it-2} is not a valid instrument,
but $y_{it-j}, j \geq 3$ is valid.

Autocovariances $\mathbb{E}[v_{it} v_{it-s}]$:

$$\begin{aligned} \bullet s=0: \mathbb{E}[v_{it} v_{it}] &= \mathbb{E}[v_{it}^2] = \mathbb{E}[(w_{it} + b w_{it-1})^2] \\ &= \mathbb{E}[w_{it}^2 + 2bw_{it}w_{it-1} + b^2 w_{it-1}^2] \\ &= \underbrace{\mathbb{E}[w_{it}^2]}_0 + 2b \underbrace{\mathbb{E}[w_{it}w_{it-1}]}_0 + b^2 \mathbb{E}[w_{it-1}^2] \\ &= \sigma_w^2 + 0 + b^2 \sigma_w^2 = (1+b^2) \sigma_w^2. \end{aligned}$$

$$\begin{aligned} \bullet s=1: \mathbb{E}[v_{it} v_{it-1}] &= \mathbb{E}[(w_{it} + b w_{it-1}) \times \\ &\quad \times (w_{it-1} + b w_{it-2})] = \mathbb{E}[v_{it} v_{it-1}] = \mathbb{E}[v_{it-1} v_{it-2}] \\ &= \mathbb{E}[w_{it}w_{it-1} + w_{it}w_{it-2} \cdot b + b w_{it-1}^2 + b w_{it-1} w_{it-2}] \\ &= \underbrace{\mathbb{E}[w_{it}w_{it-1}]}_0 + b \underbrace{\mathbb{E}[w_{it}w_{it-2}]}_0 \\ &\quad + b \underbrace{\mathbb{E}[w_{it-1}^2]}_0 + b^2 \underbrace{\mathbb{E}[w_{it-1}w_{it-2}]}_0 \\ &= b \sigma_w^2. \end{aligned}$$

$$s=2 : \mathbb{E}[v_{it} v_{it-2}] = \mathbb{E}[(w_{it} + b w_{it-1})(w_{it-2} + b w_{it-2})] = 0$$

$$s=3 = 0$$

$s = \dots$

FD:

$$y_{it} - y_{it-1} = \alpha(y_{it-1} - y_{it-2}) + \underbrace{v_{it} - v_{it-1}}_{:= u_{it}}$$

For y_{it-2} to be valid it should hold that

$$\mathbb{E}[y_{it-2} u_{it}] = 0$$

$$\Leftrightarrow \mathbb{E}[(\dots + v_{it-2})(v_{it} - v_{it-1})] = \dots = \mathbb{E}[v_{it-1} v_{it-2}] = b \sigma_w^2 \neq 0$$

$$y_{it-j}, j \geq 3 : \mathbb{E}[y_{it-3} u_{it}] = 0 ?$$

$$\mathbb{E}[(\dots + v_{it-3})(v_{it} - v_{it-1})]$$

$$= \dots = \mathbb{E}[v_{it-3} v_{it-1}] = \boxed{\mathbb{E}[v_{it-2} v_{it}]} = 0$$

Problem 7

a) W.G:

$$\left(\frac{j_{it}}{k_{it}} - \frac{\bar{j}_c}{\bar{k}_c} \right) = \alpha \left(\frac{j_{it-1}}{k_{it-1}} - \frac{\bar{j}_c}{\bar{k}_c} \right) + \nu_{it} - \bar{\nu}_c$$

$$\frac{\bar{j}_c}{\bar{k}_c} := \frac{1}{T} \sum_{t=1}^T \frac{j_{it}}{k_{it}}$$

F.D.:

$$\left(\frac{j_{it}}{k_{it}} - \frac{j_{it-1}}{k_{it-1}} \right) = \alpha \left(\frac{j_{it-1}}{k_{it-1}} - \frac{j_{it-2}}{k_{it-2}} \right) + \nu_{it} - \bar{\nu}_{it-1}$$

As instrument, use for example $\frac{j_{it-2}}{k_{it-2}}$, 2SLS

$$1) \left(\frac{j_{it-1}}{k_{it-1}} - \frac{j_{it-2}}{k_{it-2}} \right) = \gamma_0 + \gamma_1 \frac{j_{it-2}}{k_{it-2}} + \epsilon_i(t) \sim \text{iid}(0, \sigma_\epsilon^2)$$

run OLS

\Rightarrow

$$2) \left(\frac{j_{it}}{k_{it}} - \frac{j_{it-1}}{k_{it-1}} \right) = \alpha \underbrace{\left(\frac{j_{it-1}}{k_{it-1}} - \frac{j_{it-2}}{k_{it-2}} \right)}_{:= \hat{\gamma}_1 \cdot \left(\frac{j_{it-2}}{k_{it-2}} \right)} + \nu_{it} - \bar{\nu}_{it-1}$$

$$:= \hat{\gamma}_1 \cdot \left(\frac{j_{it-2}}{k_{it-2}} \right)$$

OLS

Exercise session 3

Causal Inference

Problem 1. Case 1

Table 2: Surgery vs therapy

patient	Y_i^1	\tilde{Y}_i^0	δ_i	Y_i	D_i
case 1					
1	4	3	1	4	1
1	6	5	1	6	1
3	1	2	-1	2	0
4	3	6	-3	6	0
case 2					
1	5	3	2	5	1
1	6	5	1	6	1
3	7	6	1	6	0
4	10	8	2	8	0

$$ATE = \frac{1+1-1-3}{4} = -0,5$$

$$\rightarrow SDO = \frac{4+6}{2} - \frac{2+6}{2} = 5-4=1$$

$\Rightarrow SDO \neq ATE \Rightarrow \text{bias}$

$$= 1 = SB + HEB = 0 + 1,5 - 0,5$$

1) Deletion bias:

$$E[\tilde{Y}_i^0 | D_i=1] - E[\tilde{Y}_i^0 | D_i=0] \\ = \frac{3+5}{2} - \frac{2+6}{2} = 0$$

2) Heterogenous effect bias: $(1-\pi)(ATT-ATU) =$

$$ATT = E[\tilde{\delta}_i | D_i=1] = 1 \quad = 0,5(1+2) = 1,5$$

$$ATU = E[\tilde{\delta}_i | D_i=0] = \frac{-1-3}{2} = -2$$

$$\pi = \frac{\#\{D_i=1\}}{N} = \frac{2}{4} = 0,5$$

Case 2

$$ATE = \frac{2+1+1+2}{4} = 1,5$$

$$SDO = \frac{5+6}{2} - \frac{6+8}{2} = -1,5$$

$$ATT = \frac{2+1}{2} = 1,5 \quad ATU = \frac{1+2}{2} = 1,5$$

$$SB: E[\tilde{Y}_i^0 | D_i=1] - E[\tilde{Y}_i^0 | D_i=0] \\ = \frac{3+5}{2} - \frac{6+8}{2} = -3$$

$$HTB: (1-\pi)(ATT-ATU) = \\ = 0,5(1,5-1,5) = 0$$

Problem 2

$y_i = \beta_0 + \beta_1 D_i + \varepsilon_i$. Show that $\hat{\beta}_1 = \frac{1}{N_1} \sum_{i=1}^{N_1} (y_i | D_i=1) - \frac{1}{N_0} \sum_{i=1}^{N_0} (y_i | D_i=0)$

$$\hat{\beta}_1 = \frac{\textcircled{1} \sum_{i=1}^N (D_i - \bar{D})(y_i - \bar{y})}{\textcircled{2} \sum_{i=1}^N (D_i - \bar{D})^2}$$

$$\bar{D} = \frac{1}{N} \sum_{i=1}^N D_i = \frac{N_1}{N}$$

$D_i \in \{0, 1\}$

$$\sum_{i=1}^N D_i = 1+1+0\dots=N_1$$

$$\textcircled{1} \sum_{i=1}^N (D_i - \bar{D})(y_i - \bar{y}) = \sum_{i=1}^N D_i y_i - \bar{D} y_i - \bar{D} y_i + \bar{D} \bar{y}$$

$$\begin{aligned} \bullet \sum_{i=1}^N D_i y_i &= \frac{N_1}{N_1} \underbrace{\sum_{i=1}^N D_i y_i}_{D_i=1} = \frac{N_1}{N_1} \underbrace{\sum_{i=1}^{N_1} y_i}_{D_i=1} = N_1 \cdot \frac{1}{N_1} \underbrace{\sum_{i=1}^{N_1} y_i}_{D_i=1} \\ &= 1 \cdot y_1 + 1 \cdot y_2 + \\ &\quad + 0 \cdot y_3 + \dots = \\ &= y_1 + y_2 + \dots \end{aligned}$$

$$\bullet - \sum_{i=1}^N \bar{D} y_i = - \sum_{i=1}^N \frac{N_1}{N} y_i = - N_1 \cdot \frac{1}{N} \sum_{i=1}^N y_i = - N_1 \cdot \bar{y}$$

$$\bullet - \sum_{i=1}^N D_i \bar{y} = - \underbrace{\sum_{i=1}^N \bar{y}}_{D_i=1} = - N_1 \cdot \bar{y} \Rightarrow \textcircled{1} = N_1 \bar{y}_1 - N_1 \bar{y} - N_1 \bar{y} + N_1 \bar{y}$$

$$\bullet \sum_{i=1}^N \bar{D} \bar{y} = \sum_{i=1}^N \frac{N_1}{N} \bar{y} = N_1 \cdot \bar{y} = N_1 (\bar{y}_1 - \bar{y})$$

HOMEWORK

$$\bar{y} = \frac{1}{N} (N_1 \bar{y}_1 + N_0 \bar{y}_0) \Rightarrow \textcircled{1} = \left(N_1 - \frac{N_1}{N} \right) (\bar{y}_1 - \bar{y}_0)$$

$$\begin{aligned}
 ② &= \sum_{i=1}^n (D_i - \bar{D})^2 = \sum_{i=1}^n D_i^2 - 2\bar{D}_i \bar{D} + \bar{D}^2 = \\
 &= \sum_{i=1}^n D_i^2 - 2 \sum_{i=1}^n D_i \bar{D} + \sum_{i=1}^n \bar{D}^2 \\
 &= N_1 - 2 \sum_{i=1}^{N_1} D_i \frac{N_1}{N} + \sum_{i=1}^{N_1} \frac{N_1^2}{N^2} \\
 &= N_1 - 2 \frac{N_1^2}{N} + N_1 \cdot \frac{N_1^2}{N^2} \\
 &= \boxed{N_1 - \frac{N_1^2}{N}}
 \end{aligned}$$

$$\Rightarrow \hat{p} = \frac{\left(N_1 - \frac{N_1^2}{N}\right)(\bar{y}_1 - \bar{y}_0)}{N_1 - \frac{N_1^2}{N}} = \underbrace{\bar{y}_1 - \bar{y}_0}_{\text{OLS}}, \quad \bar{y}_i = \frac{1}{N_1} \sum_{i=1}^{N_1} y_i, \quad D_i = 1$$

$$\bar{y}_0 = \frac{1}{N_0} \sum_{i=1}^{N_0} y_i, \quad D_i = 0$$

Problem 3

$$y_i = \beta_0 + \beta_1 D_i + \epsilon_i$$

$$\hat{\beta}_1 = \bar{y}_1 - \bar{y}_0 ; \quad \text{var}[\hat{\beta}_1] = \text{var}[\bar{y}_1] + \text{var}[\bar{y}_0] \leftarrow$$

$$= \frac{\hat{\sigma}_1^2}{N_1} + \frac{\hat{\sigma}_0^2}{N_0}, \quad \sigma_k^2, k \in \{0, 1\}$$

$\hat{\sigma}_1^2 \neq \hat{\sigma}_0^2$

Set $x_i := (1 D_i)^T$,

$$\text{var}\left(\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix}\right) = \frac{1}{N} \mathbb{E}[x_i x_i^T] \mathbb{E}[x_i \hat{\epsilon}_i^2] \mathbb{E}[x_i x_i^T]^\top \dots \text{asympt.}$$

$$\Rightarrow \text{var}\left(\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix}\right) = \frac{1}{N} \left(\frac{1}{N} \sum_{i=1}^N x_i x_i^T \right)^{-1} \left(\frac{1}{N} \sum_{i=1}^N x_i \hat{\epsilon}_i^2 \right) \left(\frac{1}{N} \sum_{i=1}^N x_i x_i^T \right)^{-1}$$

variance

$$\hat{\epsilon}_i = y_i - x_i^T \hat{\beta}$$

$$= \frac{1}{N} \left(\frac{1}{N} \sum_{i=1}^N \begin{pmatrix} 1 & D_i \\ D_i & D_i \end{pmatrix} \right)^{-1} \left(\frac{1}{N} \sum_{i=1}^N \begin{pmatrix} 1 & D_i \\ D_i & D_i \end{pmatrix} \hat{\epsilon}_i^2 \right) \times$$

$$\times \left(\frac{1}{N} \sum_{i=1}^N \begin{pmatrix} 1 & D_i \\ D_i & D_i \end{pmatrix} \right)^{-1}$$

$$= \frac{1}{N} \left(\frac{1}{N} \frac{N_1/N}{N/N} \right)^{-1} \left(\frac{\sum_{i=1}^N \hat{\epsilon}_i^2 / N}{\sum_{D_i=1}^N \hat{\epsilon}_i^2 / N} \right) \left(\frac{\sum_{i=1}^N \hat{\epsilon}_i^2 / N}{\sum_{D_i=1}^N \hat{\epsilon}_i^2 / N} \right)$$

$$\times \left(\frac{1}{N} \frac{N_1/N}{N/N} \right)^{-1} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \left(\frac{N}{N_1} \frac{N_1}{N_1} \right)^{-1} \left(\frac{\sum_{i:D_i=1} \hat{\epsilon}_i^2}{\sum_{i:D_i=1} \hat{\epsilon}_i^2} \right) \left(\frac{N}{N_1} \frac{N_1}{N_1} \right)^{-1}$$

$$= \begin{pmatrix} 1/N_0 & -1/N_0 \\ -1/N_0 & -1/N_0 + 1/N_1 \end{pmatrix}^A \times$$

$$\times \begin{pmatrix} \sum_{i=1}^n \hat{\epsilon}_i^2 & \sum_{i:D_i=1} \hat{\epsilon}_i^2 \\ \sum_{i:D_i=1} \hat{\epsilon}_i^2 & \sum_{i:D_i=1} \hat{\epsilon}_i^2 \end{pmatrix}^B \begin{pmatrix} 1/N_0 & -1/N_0 \\ -1/N_0 & 1/(N_0+1/N_1) \end{pmatrix}^C$$

$$\sum_{i=1}^n \hat{\epsilon}_i^2 = \sum_{\substack{i=1 \\ D_i=1}} \hat{\epsilon}_i^2 + \sum_{\substack{i=1 \\ D_i=0}} \hat{\epsilon}_i^2 \Rightarrow \underbrace{AB \times C}_{D} \Rightarrow DC$$

$$= \begin{pmatrix} \sum_{\substack{i=1 \\ D_i=0}} \hat{\epsilon}_i^2 / N_0 \\ \sum_{\substack{i=1 \\ D_i=1}} \hat{\epsilon}_i^2 / N_1 - \sum_{\substack{i=1 \\ D_i=0}} \hat{\epsilon}_i^2 / N_0 \end{pmatrix}^O \times \begin{pmatrix} \sum_{i:D_i=1} \hat{\epsilon}_i^2 / N_1 \\ \sum_{i:D_i=1} \hat{\epsilon}_i^2 / N_1 \end{pmatrix}^P$$

$$\text{var}[\hat{\beta}_0, \hat{\beta}_1]$$

$$\times \begin{pmatrix} 1/N_0 & -1/N_0 \\ -1/N_0 & 1/N_0 + 1/N_1 \end{pmatrix} =$$

$$= \begin{pmatrix} \sum_{i:D_i=0} \hat{\epsilon}_i^2 / N_0^2 \\ -\sum_{i:D_i=0} \hat{\epsilon}_i^2 / N_0^2 \end{pmatrix}$$

$$-\sum_{i:D_i=0} \hat{\epsilon}_i^2 / N_0^2$$

$$\boxed{\frac{\sum_{i:D_i=0} \hat{\epsilon}_i^2}{N_0^2} + \frac{\sum_{i:D_i=1} \hat{\epsilon}_i^2}{N_1^2}}$$

$$\widehat{\text{var}}[\hat{\beta}_0, \hat{\beta}_1] = \frac{\sum_{i:D_i=0} \hat{\epsilon}_i^2}{N_0^2} + \frac{\sum_{i:D_i=1} \hat{\epsilon}_i^2}{N_1^2}$$

$$\hat{\epsilon}_i^2 = y_i - x_i \hat{\beta}_0 \quad |D_i=0 \quad \hat{\sigma}_0^2$$

$$\hat{\epsilon}_i^2 = y_i - x_i \hat{\beta}_1 \quad |D_i=1 \quad \hat{\sigma}_1^2$$

ES4

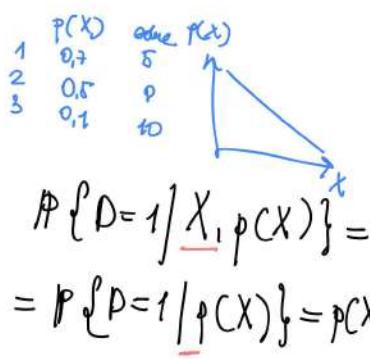
Problem 4

$$p(X_i) := \mathbb{P}\{D_i=1 | X_i\}$$

propensity score

$$= \mathbb{E}[D_i | X_i]$$

i) show that if $p(X)$ is propensity score,

$$X \perp D | p(X).$$


$$\begin{aligned} \mathbb{P}\{D=1 | X, p(X)\} &= \mathbb{E}[D | X, p(X)] \\ &= \mathbb{E}[D | X] = \mathbb{P}\{D=1 | X\} := p(X). \\ \mathbb{P}\{D=1 | p(X)\} &= \mathbb{E}[D | p(X)] = \mathbb{E}[D | p(X), X] \\ \text{ie} \quad &= \mathbb{E}\left[\mathbb{E}[D | p(X), X] | p(X)\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}[D | X]\right] | p(X)\right] \\ &= \mathbb{E}\left[p(X) | p(X)\right] = p(X) \end{aligned}$$

ii) if $\underbrace{y^1, y^0 \perp D/X}_1 \Rightarrow y^1, y^0 \perp D/p(X)$

$$\begin{aligned}
 P\{D=1 | y^1, y^0, p(X)\} &= E[D | y^1, y^0, p(X)] \\
 &\stackrel{\text{LINE}}{=} E_x [E[D | X, y^1, y^0] | y^1, y^0, p(X)] \\
 &\stackrel{1}{=} E_x [E[D | X] | y^1, y^0, p(X)] \\
 &\stackrel{\text{ps def.}}{=} E_x [p(X) | y^1, y^0, p(X)] = p(X) \\
 \Rightarrow P\{D=1 | \underbrace{y^1, y^0, p(X)}_1\} &= P\{D=1 | \underbrace{p(X)}_1\}
 \end{aligned}$$

• homogeneous TE

$$y_i^1 - y_i^0 = \delta_i \equiv \delta \quad \forall i$$

Problem 5.

$$\text{Show that } \delta_{iv} = \frac{\text{cov}[Z_i, Y_v]}{\text{cov}[Z_i, D_v]} \equiv \delta_w = \frac{\mathbb{E}[Y_v | Z_i=1] - \mathbb{E}[Y_v | Z_i=0]}{\mathbb{E}[D_v | Z_i=1] - \mathbb{E}[D_v | Z_i=0]}$$

↑ Wald estimator

Denominator,

Numerator:

$$\begin{aligned}
 \bullet \text{cov}[Z_i, Y_v] &= \mathbb{E}[Z_i Y_v] - \mathbb{E}[Z_i] \mathbb{E}[Y_v] & \text{cov}[Z_i, D_v] &= \\
 &= \mathbb{E}[\mathbb{E}[Z_i Y_v | Z_i=1]] - \mathbb{E}[Z_i] \mathbb{E}[Y_v] &= (\mathbb{E}[D_v | Z_i=1] - \mathbb{E}[D_v | Z_i=0]) \\
 &= \mathbb{E}[P\{Z_i=1\} \mathbb{E}[Y_v | Z_i=1]] - \mathbb{E}[Z_i] \mathbb{E}[Y_v] && \times P\{Z_i=1\} (1 - P\{Z_i=1\}) \\
 &= P\{Z_i=1\} \mathbb{E}[Y_v | Z_i=1] - \mathbb{E}[Z_i] \mathbb{E}[Y_v] \\
 &= P\{Z_i=1\} \mathbb{E}[Y_v | Z_i=1] - P\{Z_i=1\} (\mathbb{E}[Y_v | Z_i=1] \cdot P\{Z_i=1\} + \\
 &\quad \times P\{Z_i=1\} \times (1 - P\{Z_i=1\})) \\
 &= (\mathbb{E}[Y_v | Z_i=1] - \mathbb{E}[Y_v | Z_i=0]) \times P\{Z_i=1\} + \mathbb{E}[Y_v | Z_i=0] \cdot P\{Z_i=0\} \\
 &\quad \times P\{Z_i=1\} \times (1 - P\{Z_i=1\}) \\
 &= 1 - P\{Z_i=1\}
 \end{aligned}$$

$$Y_i = d_0 + d_1 D_i + \varepsilon_i$$

$\in [0, 1]$

$$\Rightarrow \text{cov}[D_i, Y_i] \neq 0$$

$$\text{cov}[Z_i, \varepsilon_i] = 0 \quad \text{and} \quad \text{cov}[Z_i, D_i] \neq 0$$

LATE

	$D_i=1$	$D_i=0$
$Z_i=0$	$D_i=1$: always taken	$D_i=0$: never taken
$D_i=1$	compliance	
$D_i=0$		

$$\begin{aligned} \mathbb{E}[Y_i | Z_i = 0] &= \mathbb{E}[Y_i^0] + \mathbb{E}[(Y_i^1 - Y_i^0) D_i | Z_i = 0] \\ &= \mathbb{E}[Y_i^0] + \mathbb{E}[(Y_i^1 - Y_i^0) | D_i = 1, Z_i = 0] \underbrace{\mathbb{P}\{D_i = 1 | Z_i = 0\}}_{=0} \\ &= \mathbb{E}[Y_i^0]. \end{aligned}$$

Problem 6

Instead of monotonicity, we assume $\mathbb{P}\{D_i = 1 | Z_i = 0\} = 0$. (eligibility rule)

Show that we can identify ATT.

$$\begin{aligned} \mathbb{E}[Y_i | Z_i = 1] &= \mathbb{E}[Y_i^0] + \mathbb{E}[(Y_i^1 - Y_i^0) D_i | Z_i = 1] \\ &= \mathbb{E}[Y_i^0] + \mathbb{E}[\mathbb{E}[Y_i^1 - Y_i^0 | D_i = 1, Z_i = 1]] \\ &= \mathbb{E}[Y_i^0] + \mathbb{E}[(Y_i^1 - Y_i^0) | D_i = 1, Z_i = 1] \underbrace{\mathbb{P}\{D_i = 1 | Z_i = 1\}}_{=1} \\ &= \mathbb{E}[Y_i^0] + \mathbb{E}[(Y_i^1 - Y_i^0) | D_i = 1] \underbrace{\mathbb{P}\{D_i = 1 | Z_i = 1\}}_{\neq 1}. \text{ ATT} = \mathbb{E}[Y_i^1 - Y_i^0 | D_i = 1] \end{aligned}$$

Because $D_i = 1$ is sufficient for $Z_i = 1$