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Course: **Mathematics**

1 Basic topology

1.1 Metric spaces

Definition 1.1. A set X, whose elements we shall call *points*, is said to be *a metric space* if with any two points p and q of X there is associated a real number $d(p,q)$, called the *distance* from p to q, such that

(a) $d(p,q) > 0$ if $p \neq q$ and $d(p,p) = 0$,

(b) $d(p,q) = d(q,p)$,

(c) $d(p,q) \leq d(p,r) + d(r,q)$, for $\forall r \in X$.

Any function with these three properties is called a *distance function*, or *a metric*.

Example 1.2 (Metric spaces)**.** The following are examples of the metric spaces:

1. the set of real numbers $\mathbb{R}$ with a metric $d(p,q) = |p-q|$,
2. a real plane $\mathbb{R}^2$ with a metric $d(\mathbf{p},\mathbf{q}) = \sqrt{(p_1-q_1)^2 + (p_2-q_2)^2} := \|\mathbf{p}-\mathbf{q}\|$ (Eucledian distance),
3. a real plane $\mathbb{R}^2$ with a metric $d(\mathbf{p},\mathbf{q}) = |p_1-q_1| + |p_2-q_2|$ (Manhattan distance),
4. the set of probability distributions defined on the same measurable space with a metric $d(P,Q) = \frac{1}{\sqrt{2}}\left(\int \left(\sqrt{p(x)} - \sqrt{q(x)}\right)^2 dx\right)^{1/2}$ (Hellinger distance).

It is important to observe that every subset Y of a metric space X is a metric space in its own right, with the same distance function. Thus, every subset of a Euclidean space is a metric space.

Definition 1.3. By the *segment* (a,b) we mean the set of all real numbers x such that $a < x < b$. By the *interval* $[a,b]$ we mean the set of all real numbers x such that $a \leq x \leq b$.

If $a_i < b_i$ for $i = 1,\ldots,k$, the set of all points $\mathbf{x} = (x_1,\ldots,x_k)$ in $\mathbb{R}^k$ whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ $(1 \leq i \leq k)$ is called a *k-cell*. Thus, a 1-cell is an interval, a 2-cell is a rectangle, etc.

If $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, the *open* (or *closed*) *ball* B with center at $\mathbf{x}$ and radius r is defined to be the set of all $\mathbf{y} \in \mathbb{R}^k$ such that $\|\mathbf{y}-\mathbf{x}\| < r$ (or $\|\mathbf{y}-\mathbf{x}\| \leq r$).

We call a set $E \subset \mathbb{R}^k$ *convex* if

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in E$$

whenever $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $0 < \lambda < 1$. For example, balls are convex. It is also easy to see that k-cells are convex.

Definition 1.4. Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(a) *A neighborhood* of a point p is a set $N_r(p)$ consisting of all points q such that $d(p,q) < r$. The number r is called the *radius* of $N_r(p)$.

(b) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$. Example: take a set $A := (0, 1)$. Point 0 is a limit point, because any open interval, say $(-\varepsilon, \varepsilon)$, intersects A.

(c) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E. Example: take a set $A = \{n^{-1} : n \in \mathbb{N}\}$. Each element is an isolated point because you can take a small interval around n^{-1} that avoids the other fractions in the set.

(d) E is *closed* if every limit point of E is a point of E. Example: take $A = [0, 1]$. Both 0 and 1 are limit points and both belong to the set A. A set $B = (0, 1]$ is not closed because a limit point 0 does not belong to the set.

(e) A point p is an *interior point* of E if there is a neighborhood $N_r(p)$ of p such that $N \subset E$. Example: take a set $A = (0, 1)$. A point 0.5 is an interior point because there is a neighborhood around it, say, $N_{0.1}(0.5)$ that belongs to the set A; if $N_{0.1}(0.5) = (0.4, 0.6) := B$, we have $B \subset A$. On the other hand, if $C = [0.5, 1]$, 0.5 is not an interior point of C, because there is no neighborhood around it that is a subset of C; some points of that neighborhood are outside of C.

(f) E is *open* if every point of E is an interior point of E.

(g) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.

(h) E is *perfect* if E is closed and if every point of E is a limit point of E. Example: take $A = [0, 1]$, which is closed with all points being limit points, so it is perfect. On the other hand, $B = [0, 1] \cup \{3\}$ is not perfect because it contains a point 3, which is not a limit point (it is an isolated point).

(i) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for $\forall p \in E$.

(j) E is *dense in X* if every point of X is a limit point of E, or a point of E (or both).

Let us note that in $\mathbb{R}^1$ neighborhoods are segments, whereas in $\mathbb{R}^2$ neighborhoods are interiors of circles.

Theorem 1.5. *Every neighborhood is an open set.*

Proof. Consider neighborhood $E = N_r(p)$, and let q be any point of E. Then there is a positive real number h such that

$$d(p, q) = r - h.$$

For all points s such that $d(q, s) < h$, we have then

$$d(p, s) \leq d(p, q) + d(q, s) < r - h + h = r,$$

so that $s \in E$. Thus, q is an interior point of E. □

Theorem 1.6. *If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.*

Proof. Suppose there is a neighborhood N of p which contains only a finite number of points of E. Let $q_1, \ldots, q_n$ be those points of $N \cap E$, which are distinct from p, and put

$$r = \min_{1 \leq m \leq n} d(p, q_m)$$

The minimum of a finite set of positive numbers is clearly positive, so that $r > 0$.

The neighborhood $N_r(p)$ contains no point q of E such that $q \neq p$, so that p is not a limit point of E. This contradiction established the theorem. □

Corollary 1.7. *A finite point set has no limit points.*

Theorem 1.8. *A set E is open if and only if its complement is closed.*

1.2 Compact sets

Definition 1.9. By an *open cover* of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$.

Definition 1.10. A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite subcover*. More explicitly, the requirement is that if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \ldots \cup G_{\alpha_n}.$$

Corollary 1.11. *A set E is compact if it is both closed and bounded.*

1.3 Functions

Definition 1.12. Consider two sets A and B, whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B, which we denote by $f(x)$. Then f is said to be a *function* from A to B (or a *mapping* from A into B). The set A is called the *domain* of f (we also say f is defined on A), and the elements $f(x)$ are called the *values* of f. The set of *all* values of f is called the *range* of f.

Definition 1.13. If for every $y \in B$ there is at most one $x \in A : f(x) = y$, the function f is said to be a 1-1 (*one-to-one*) mapping of A into B. This may also be expressed as follows: f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1 \in A$, $x_2 \in A$.

Definition 1.14. Let A and B be two sets and let f be a mapping of A into B. If $f(A) = B$, we say that f maps A *onto* B. If, additionally, f is 1-1, then f is *one-to-one and onto (bijection)*.

Definition 1.15. If there exists a 1-1 mapping of A *onto* B, we say that A and B can be put in 1-1 *correspondence*, or that A and B have the same *cardinal number*, or, briefly, that A and B are *equivalent*, and we write $A \sim B$.

Definition 1.16. For any positive integer n, let J_n be the set whose elements are the integers $1, 2, \dots, n$; let J be the set consisting of all positive integers. For any set A, we say:

(a) A is finite if $A \sim J_n$ for some n.

(b) A is *infinite* if A is not finite.

(c) A is *countable* if $A \sim J$.

(d) A is *uncountable* if A is neither finite nor countable.

(e) A is *at most countable* if A is finite or countable.

For two finite sets A and B, we evidently have $A \sim B$ if and only if A and B contain the same number of elements (same *cardinality*). For infinite sets, however, the idea of cardinality becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

Example 1.17. Let A be the set of all integers. Then A is countable. Consider, the following arrangement of the sets A and J:

$$\begin{aligned} A &: \quad 0, 1, -1, 2, -2, \dots \\ J &: \quad 1, 2, 3, 4, 5, \dots \end{aligned}$$

We can, in this example, even give an explicit formula for a function f from J to A which sets up a 1-1 correspondence:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Remark 1.18. A finite set cannot be equivalent to one of its proper subsets. That this is, however, possible for infinite sets, is shown by Example 1.17, in which J is a proper subset of A.

Definition 1.19. In the following, assume that the set A is a subset of $\mathbb{R}$.

(a) If there exists $x \in \mathbb{R}$ such that for every $y \in A$ we have $x \geq y$, then the set A is *bounded from above*.

(b) If there exists $x \in \mathbb{R}$ such that for every $y \in A$ we have $x \leq y$, then the set A is *bounded from below*.

(c) The *supremum* of A, denoted as $\sup A$, is the smallest upper bound of the set A.

(d) The *infimum* of A, denoted as $\inf A$, is the largest lower bound of the set A.

We note that the set A is bounded, if it is bounded both from below and from above, which is equivalent to the Definition 1.4(i). If the set A is not bounded from above, then $\sup A = \infty$, and if it is not bounded from below, then $\inf A = -\infty$.

2 Sequences and limits

Definition 2.1. By a *sequence*, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$ for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, x_3, \ldots$. The values of f, that is, the elements x_n, are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a *sequence in A*, or a *sequence of elements of A*.

Note that the terms $x_1, x_2, x_3, \ldots$ of a sequence need not be distinct.

Since every countable set is the range of a 1-1 function defined on J, we may regard every countable set as the range of a sequence of distinct terms. Speaking more loosely, we may say that the elements of any countable set can be "arranged in a sequence".

Definition 2.2. For a given sequence $\{x_n\}$, if $x_{n+1} > x_n$ for $\forall n \in J$, then the sequence is *increasing*. If $x_{n+1} < x_n$ for $\forall n \in J$, then the sequence is *decreasing*. If $x_{n+1} \geq x_n$ for $\forall n \in J$, then the sequence is *non-decreasing*. If $x_{n+1} \leq x_n$ for $\forall n \in J$, then the sequence is *non-increasing*.

If at least one of these four conditions is satisfied, the sequence is called *monotonic*.

Example 2.3. We give examples of different sequences below.

(a) A sequence that is defined via a formula for the nth term: $x_n = \left(\dfrac{2}{3}\right)^n$.

(b) A sequence that is defined recursively (Fibonacci sequence): $x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$, and $x_1 = x_2 = 1$.

(c) A sequence $x_n = (-1)^n$.

(d) A sequence $x_n = 2^n$.

Note that the sequence (a) is decreasing with n, while the sequence (b) is non-decreasing with n. The sequence (c) is non-monotonic.

Definition 2.4. A sequence $\{x_n\}$ in a metric space X is said to *converge* if there is a point $x \in X$ with the following property: for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(x_n, x) < \varepsilon$.

In this case, we also say that $\{x_n\}$ converges to x, or that x is the limit of $\{x_n\}$, and we write $x_n \to x$, or

$$\lim_{n\to\infty} x_n = x.$$

If $\{x_n\}$ does not converge, it is said to *diverge*.

We recall that the set of all points x_n $(n = 1, 2, 3, \ldots)$ is the *range* of $\{x_n\}$. The range of a sequence may be a finite set, or it may be infinite. The sequence $\{x_n\}$ is said to be *bounded* if its range is bounded. In the Example 2.3, (a) and (c) are bounded sequences, while (b) and (d) are not.

Example 2.5. Show that $\lim_{n\to\infty}(\frac{2}{3})^n = 0$.

We need to show that for a given $\varepsilon > 0$, after some $n \in J$, the distance between the elements of the sequence and the limit 0 is smaller than ε. In other words, that there exists some N such that for all n larger than N we have $d(x_n, 0) < \varepsilon$. Taking the absolute value, we have $|(\frac{2}{3})^n| < \varepsilon$ for $\forall n \geq N$, and rewriting

$$\begin{aligned} \left(\frac{2}{3}\right)^n &< \varepsilon, \\ \log\left(\frac{2}{3}\right)^n &< \log\varepsilon, \\ n\log\left(\frac{2}{3}\right) &< \log\varepsilon, \\ n &> \frac{\log\varepsilon}{\log 2/3}. \end{aligned}$$

Denote the smallest integer larger than a as $\lceil a \rceil$. Then, one can take $N = \lceil n \rceil$, and for all $n \geq N$, the inequality $n > \frac{\log\varepsilon}{\log 2/3}$ is satisfied. Then, 0 is a limit of $\left(\frac{2}{3}\right)^n$.

Theorem 2.6. *Every bounded, monotonic sequence converges.*

Example 2.7. Show that the sequence

$$x_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} = \sum_{k=1}^{n} \frac{1}{k!}$$

converges.

To show that the sequence converges, we use the Theorem 2.6, hence, it is sufficient to show that the sequence is monotonic and bounded. To show monotonicity, note that

$$x_{n+1} = x_n + \frac{1}{(n+1)!} > x_n,$$

so $\{x_n\}$ is increasing and hence monotonic. To show that it is bounded, note that

$$\frac{1}{n!} = \frac{1}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} = \frac{1}{2 \cdot 3 \cdot \ldots \cdot n} \leq \frac{1}{2 \cdot 2 \cdot \ldots \cdot 2} = \frac{1}{2^{n-1}},$$

with strict inequality for $n > 1$. $x_1 = 1$ is finite, hence does not contradict boundedness. For $n > 1$, we have

$$x_n < 1 + \frac{1}{2^1} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} = \frac{1 - (1/2)^n}{1 - 1/2} = 2 - \left(\frac{1}{2}\right)^{n-1} < 2.$$

Because each element of the sequence x_n for $\forall n > 1$ is bounded by 2, the sequence is bounded.

2.1 Limit laws (i)

Corollary 2.8. *Let $\{x_n\}$ and $\{y_n\}$ are convergent sequences, and let c be a constant. Then,*

(a) $\lim\limits_{n\to\infty} (x_n + y_n) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$.

(b) $\lim\limits_{n\to\infty} (x_n - y_n) = \lim\limits_{n\to\infty} x_n - \lim\limits_{n\to\infty} y_n$.

(c) $\lim\limits_{n\to\infty} cx_n = c \lim\limits_{n\to\infty} x_n$.

(d) $\lim\limits_{n\to\infty} c = c$.

(e) $\lim\limits_{n\to\infty} (x_n y_n) = \lim\limits_{n\to\infty} x_n \lim\limits_{n\to\infty} y_n$.

(f) $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = \dfrac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$ *if* $\lim\limits_{n\to\infty} y_n \neq 0$.

(g) $\lim\limits_{n\to\infty} x_n^p = \left(\lim\limits_{n\to\infty} x_n\right)^p$ *if* $p > 0$ *and* $x_n > 0$.

Example 2.9. Find the limit of $\{x_n\}$, where

$$x_n = \frac{2n^3 + n^2 - 7n}{n^3 + 2n + 2}.$$

Rewrite the nth term of the sequence as

$$\frac{2 + n^{-1} - 7n^{-2}}{1 + 2n^{-2} + 2n^{-3}}.$$

The limit of the numerator and the denominator respectively is

$$\lim_{n\to\infty} \left(2 + \frac{1}{n} - \frac{7}{n^2}\right) = 2, \quad \lim_{n\to\infty} \left(1 + \frac{2}{n^2} + \frac{2}{n^3}\right) = 1,$$

so that $\lim\limits_{n\to\infty} x_n = 2$.

Definition 2.10. Given a sequence $\{x_n\}$, consider a sequence $\{n_k\}$ of natural numbers, such that $n_1 < n_2 < n_3 < \ldots$. Then the sequence $\{x_{n_i}\}$ is called a *subsequence* of $\{x_n\}$. If $\{x_{n_i}\}$ converges, its limit is called a *subsequential limit* of $\{x_n\}$.

The sequence $\{x_n\}$ converges to x if and only if every subsequence of $\{x_n\}$ converges to x.

Example 2.11. Consider a sequence $x_n = (-1)^n$ that we know to be divergent. Now, consider two sequences of natural numbers, $\{n_k\} = \{1, 3, 5, \ldots\}$ and $\{m_k\} = \{2, 4, 6, \ldots\}$. The subsequence corresponding to $\{n_k\}$ is $\{-1, -1, -1, \ldots\}$ with the limit -1, and the subsequence corresponding to $\{m_k\}$ is $\{1, 1, 1, \ldots\}$ with the limit 1. Hence, it is possible for subsequences to converge even though the whole sequence does not.

2.2 Upper and lower limits

Definition 2.12. Let $\{x_n\}$ be a sequence of real numbers with the following property: for every real M there is an integer N such that $n \geq N$ implies $x_n \geq M$. We then write

$$x_n \to +\infty.$$

Similarly, if for every real M there is an integer N such that $n \geq N$ implies $x_n \leq M$, we write

$$x_n \to -\infty.$$

Definition 2.13. Let $\{x_n\}$ be a sequence or real numbers. Let E be the set of numbers x such that $x_{n_k} \to x$ for some subsequence $\{x_{n_k}\}$. This set E contains all subsequential limits as defined in the Definition 2.10, plus possibly the numbers $+\infty$, $-\infty$.

Put

$$x^\star = \sup E, \quad x_\star = \inf E.$$

The numbers $x^\star$ and $x_\star$ are called the *upper* and *lower limits* of $\{x_n\}$. We use the notation

$$\limsup_{n\to\infty} x_n = x^\star, \quad \liminf_{n\to\infty} x_n = x_\star.$$

Theorem 2.14. *If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then*

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n,$$
$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n.$$

3 Continuity

3.1 Limits of functions

Definition 3.1. Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or

$$\lim_{x\to p} f(x) = q$$

if there is a point $q \in Y$ with the following property: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta.$$

The symbols d_X and d_Y refer to the distances in X and Y, respectively.

If X and/or Y are replaced by the real line, the complex plane, or by some Euclidean space $\mathbb{R}^k$, the distances d_X, d_Y are of course replaced by absolute values, or by appropriate norms.

Corollary 3.2. *If f has a limit at p, this limit is unique.*

Definition 3.3. One can also define *one-sided (left-sided and right-sided limits)* by manipulating the definition such that it considers not all x in the δ-neighborhood of p but those x that are smaller (or larger) than p:

$$\lim_{x\to p^-} f(x) = q,$$
$$\lim_{x\to p^+} f(x) = q.$$

Theorem 3.4. *It holds that* $\lim_{x\to p} f(x) = q$ *if and only if* $\lim_{x\to p^-} f(x) = \lim_{x\to p^+} f(x) = q$.

3.2 Limit laws (ii)

Corollary 3.5. *If $\lim_{x\to p} f(x)$ and $\lim_{x\to p} g(x)$ exist and c is a constant, then*

(a) $\lim\limits_{x\to p}(f(x)+g(x)) = \lim\limits_{x\to p} f(x) + \lim\limits_{x\to p} g(x)$.

(b) $\lim\limits_{x\to p}(f(x)-g(x)) = \lim\limits_{x\to p} f(x) - \lim\limits_{x\to p} g(x)$.

(c) $\lim\limits_{x\to p}(cf(x)) = c\lim\limits_{x\to p} f(x)$.

(d) $\lim\limits_{x\to p} c = c$.

(e) $\lim\limits_{x\to p} x = p$.

(f) $\lim\limits_{x\to p}(f(x)g(x)) = \lim\limits_{x\to p} f(x)\lim\limits_{x\to p} g(x)$.

(g) $\lim\limits_{x\to p}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to p} f(x)}{\lim\limits_{x\to p} g(x)}$ *if* $\lim\limits_{x\to p} g(x) \neq 0$.

(h) $\lim\limits_{x\to p}(f(x))^n = \left(\lim\limits_{x\to p} f(x)\right)^n$, $n \in \mathbb{N}$.

Definition 3.6. We write $f(x) \to +\infty$ as $x \to p$, or

$$\lim_{x\to p} f(x) = +\infty,$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(x) > \varepsilon$ for every x for which $0 < |x-p| < \delta$. An example of such a function is $f(x) = x^{-1}$ with a limit $\lim_{x\to 0} f(x)$.

3.3 Continuous functions

Definition 3.7. Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is said to be *continuous at p* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for all points $x \in E$ for which $d_X(x,p) < \delta$.

If f is continuous at every point of E, then f is said to be *continuous on E*. It should be noted that f has to be defined at the point p in order to be continuous at p.

We now turn to compositions of functions. A brief statement of the following theorem is that a continuous function of a continuous function is continuous.

Theorem 3.8. *Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps the range of f, $f(E)$, into Z, and h is the mapping of E into Z defined by*

$$h(x) = g(f(x)) \quad (x \in E).$$

If f is continuous at point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at p.

This function h is called the *composition* or the *composite* of f and g. The notation

$$h = g \circ f$$

is frequently used in this context.

Example 3.9. Consider two functions $f(x) = \frac{x}{2}$ and $g(x) = x^2$. We have

(a) $f \circ g = f(g(x)) = \dfrac{g(x)}{2} = \dfrac{x^2}{2}$.

(b) $g \circ f = g(f(x)) = \left(\dfrac{x}{2}\right)^2 = \dfrac{x^2}{4}$.

(c) $g \circ g = g(g(x)) = (x^2)^2 = x^4$.

Theorem 3.10. *Let f and g be functions defined on the same interval. If $f(x)$ and $g(x)$ are continuous at p, so are $f(x)+g(x)$ and $f(x)g(x)$. If $g(p) \neq 0$, $f(x)/g(x)$ is also continuous at p.*

4 Differentiation

In this section we shall confine our attention to *real* functions defined on intervals or segments.

Definition 4.1. Let f be defined (and real-valued) on $[a,b]$. For any $x \in [a,b]$ form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, t \neq x),$$

and define

$$f'(x) = \lim_{t \to x} \phi(t), \tag{1}$$

provided that this limit exists.

We thus associate with the function f a function f' whose domain is the set of points x at which the limit (1) exists; f' is called the *derivative of* f.

If f' is defined at a point x, we say that f is *differentiable* at x. If f' is defined at every point of a set $E \subset [a,b]$, we say that f is differentiable on E.

It is possible to consider right-hand and left-hand limits in (1); this leads to the definition of right-hand and left-hand derivatives. In particular, at the endpoints a and b, the derivative, if it exists, is a right-hand or left-hand derivative respectively.

If f is defined on a segment (a,b) and if $a < x < b$, then $f'(x)$ is defined by (4.1) and (1), as above. But $f'(a)$ and $f'(b)$ are not defined in this case.

Theorem 4.2. *Let f be defined on $[a,b]$. If if is differentiable at a point $x \in [a,b]$, then f is continuous at x.*

Proof. As $t \to x$, we have

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \to f'(x) \cdot 0 = 0.$$

□

The converse of this theorem is not true.

Example 4.3. Consider two functions,

$$f(x) = \begin{cases} x, & x < 0, \\ x^2, & x \geq 0, \end{cases} \quad g(x) = \begin{cases} 0, & x \leq 0, \\ 1 & x > 0. \end{cases}$$

The function $g(x)$ is discontinuous at 0, hence it is not differentiable. The function $f(x)$ is continuous at 0, but not differentiable. To show this, note

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^-} \frac{x - 0}{x} = 1 \neq \lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{x^2 - 0}{x} = 0.$$

Because one-sided derivatives are not equal, the derivative at 0, $f'(0)$, does not exist.

Theorem 4.4. *Suppose f and g are defined on $[a,b]$ and are differentiable at a point $x \in [a,b]$. Then $f + g$, $f \cdot g$, and f/g are differentiable at x, and*

(a) $(f+g)'(x) = f'(x) + g'(x)$.

(b) $(f \cdot g)(x) = f'(x)g(x) + f(x)g'(x)$.

(c) $(f/g)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$, $g(x) \neq 0$.

Example 4.5. The derivative of any constant is clearly zero. If f is defined by $f(x) = x$, then $f'(x) = 1$. Repeated application of *(b)* and *(c)* then shows that $f(x) = x^n$ is differentiable, and that its derivative is $f'(x) = nx^{n-1}$, for any integer n. Thus, every polynomial is differentiable and so is every rational function, except at the points where the denominator is zero.

Example 4.6. Consider $f(x) = x^2, g(x) = 1 + x$. Then we have

$$\begin{aligned} f'(x) &= 2x, \\ g'(x) &= 1, \\ (f+g)'(x) &= (x^2 + 1 + x)' = 2x + 1, \\ (f \cdot g)'(x) &= (x^2 \cdot (1+x))' = 2x \cdot (1+x) + x^2 = 2x + 3x^2, \\ \left(\frac{f(x)}{g(x)}\right)' &= \frac{2x(1+x) - x^2}{(1+x)^2} = \frac{2x + x^2}{(1+x)^2}. \end{aligned}$$

The following theorem is known as the "chain rule" for differentiation. It deals with differentiation of composite functions and is probably the most important theorem about derivatives.

Theorem 4.7. *Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If*

$$h(t) = g(f(t)) \quad (a \leq t \leq b),$$

then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x).$$

Example 4.8. Consider two functions, $f(x) = \frac{x}{2}$ and $g(x) = x^2$, and their composite function $h(x) = \left(\frac{x}{2}\right)^2$. Then,

$$\begin{aligned} f'(x) &= \frac{1}{2}, \\ g'(x) &= 2x, \\ h'(x) &= g'(f(x))f'(x) = \frac{x}{2}. \end{aligned}$$

4.1 Mean value theorems

Definition 4.9. Let f be a real function defined on a metric space X. We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \varepsilon$.

Local minima are defined likewise. Our next theorem is the basis of many applications of differentiation.

Theorem 4.10. *Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$. The analogous statement for local minima is also true.*

Proof. Choose δ in accordance with Definition 4.9, so that

$$a < x - \delta < x < x + \delta < b.$$

If $x - \delta < t < x$, then

$$\frac{f(t) - f(x)}{t - x} \geq 0.$$

Letting $t \to x$, we see that $f'(x) \geq 0$.

If $x < t < x + \delta$, then

$$\frac{f(t) - f(x)}{t - x} \leq 0,$$

which shows that $f'(x) \leq 0$. Hence, $f'(x) = 0$. □

The following result is usually referred to as the mean value theorem:

Theorem 4.11. *If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 4.12. *Suppose f is differentiable in (a, b).*

(a) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.

(b) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

(c) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.

4.2 o and $\mathcal{O}$ notation

Suppose we have a function $f(x)$ with $f(a) = 0$ and we want to consider how quickly the function goes to zero around a. Then ideally, we would want to find a simple function g (for example, $g(x) = (x-a)^n$) which also vanishes at a such that g and f are almost equal around a. The "small-o" and "big-o" notation expresses this notion, but only states that f goes to zero faster than g.

Definition 4.13. We say
$$f(x) = \mathcal{O}(g(x))$$
as $x \to a$ if there exists a constant M such that $|f(x)| \leq M\,|g(x)|$ in some punctured neighborhood of a, that is for $x \in (a-\delta, a+\delta)\backslash\{a\}$ for some value of δ.

We say
$$f(x) = o(g(x))$$
as $x \to a$ if $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$. This implies that there exists a punctured neighborhood of a on which g does not vanish.

Example 4.14. The first two examples are derived from Taylor polynomials, the rest can be checked directly:

a) $e^x = 1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \mathcal{O}(x^4)$ as $x \to 0$,

b) $\dfrac{1}{1-x} = 1 + x + x^2 + \mathcal{O}(x^3) = 1 + x + x^2 + o(x^2)$ as $x \to 0$,

c) $|x^3| = \mathcal{O}(x^3) = o(x^2)$ as $x \to 0$,

d) $\cosh(x) = \mathcal{O}\left(e^x\right) = o\left(e^{\frac{5}{4}x}\right)$ as $x \to 0$,

e) $\dfrac{1}{\sin(x)} = \mathcal{O}\left(\dfrac{1}{x}\right) = o\left(\dfrac{1}{x^{\frac{3}{2}}}\right)$ as $x \to 0$.

Theorem 4.15. *The following holds:*

(a) $f(x) = \mathcal{O}\left(f(x)\right)$.

(b) If $f(x) = o\left(g(x)\right)$ *then* $f(x) = \mathcal{O}\left(g(x)\right)$.

(c) If $f(x) = \mathcal{O}\left(g(x)\right)$ *then* $\mathcal{O}\left(f(x) + g(x)\right) = \mathcal{O}\left(g(x)\right)$.

(d) If $f(x) = \mathcal{O}\left(g(x)\right)$ *then* $o\left(f(x) + g(x)\right) = o\left(g(x)\right)$.

(e) Let $c \neq 0$, *then* $c \cdot \mathcal{O}\left(g(x)\right) = \mathcal{O}\left(g(x)\right)$ *and* $c \cdot o\left(g(x)\right) = o\left(g(x)\right)$.

(f) $\mathcal{O}\left(f(x)\right)\mathcal{O}\left(g(x)\right) = \mathcal{O}\left(f(x)g(x)\right)$.

(g) $o\left(f(x)\right)\mathcal{O}\left(g(x)\right) = o\left(f(x)g(x)\right)$.

(h) If $g(x) = o(1)$ *then* $\frac{1}{1+o(g(x))} = 1 + o\left(g(x)\right)$, *and* $\frac{1}{1+\mathcal{O}(g(x))} = 1 + \mathcal{O}\left(g(x)\right)$.

In the case when functions $f(\cdot)$ and $g(\cdot)$ are polynomials these rules simplify to the following.

Corollary 4.16. *Around 0 we have*

a) $x^a = \mathcal{O}(x^b)$ *for all* $b \leq a$, *and* $x^a = o(x^b)$ *for all* $b < a$.

b) $\mathcal{O}(x^a) + \mathcal{O}(x^b) = \mathcal{O}\left(x^{\min(a,b)}\right)$, $o(x^a) + o(x^b) = o\left(x^{\min(a,b)}\right)$, *and*
$$\mathcal{O}(x^a) + o(x^b) = \begin{cases} o(x^b), & b < a, \\ \mathcal{O}(x^a), & b \geq a. \end{cases}$$

c) For $c \neq 0$, $c \cdot \mathcal{O}(x^a) = \mathcal{O}(x^a)$, *and* $c \cdot o(x^a) = o(x^a)$.

d) $x^b\mathcal{O}(x^a) = \mathcal{O}(x^{a+b})$, *and* $x^b o(x^a) = o(x^{a+b})$.

e) $\mathcal{O}(x^a)\mathcal{O}(x^b) = \mathcal{O}(x^{a+b})$, $\mathcal{O}(x^a)o(x^b) = o(x^{a+b})$, *and* $o(x^a)o(x^b) = o(x^{a+b})$.

4.3 Differentiation of functions of several variables

So far, we have focused on functions of one variable; a straightforward extension of the differentiation ideas to functions of several variables involves *partial derivatives*.

Definition 4.17. Let $f : \mathbb{R}^n \to \mathbb{R}$. Then for each x_i at each point $x = (x_1, \ldots, x_n)$ in the domain of f, the *partial derivative* of f at x is

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

provided that this limit exists.

5 Integration

Definition 5.1. Let f be a function defined on $[a, b]$. Divide the interval $[a, b]$ into n subintervals of equal width, $\Delta x = (b - a)/n$. Let $x_0, x_1, \ldots, x_n$ be the endpoints of these subintervals, and let $x_1^\star, \ldots, x_n^\star$ be any points in these subintervals, so that $x_i^\star$ lies in the ith subinterval $[x_{i-1}, x_i]$. The *definite integral* of f from a to b is

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^\star)\Delta x$$

provided that the limit exists. If it does, we say that f is *integrable* on $[a, b]$.

Sometimes instead of definite integrals, we work with indefinite integrals.

Definition 5.2. *Indefinite integral* (or *antiderivative*) of the function f is defined as

$$\int f(x)dx = F(x),$$

such that $F'(x) = f(x)$.

Note that if $F(x)$ is the antiderivative of $f(x)$, then $F(x) + C$ is also the antiderivative of $f(x)$ for any constant C. Thus, an indefinite integral represents the whole family of functions.

Theorem 5.3. *If f is continuous on $[a, b]$, or if f has only a finite number of jumpdiscontinuities, then f is integrable on $[a, b]$.*

Corollary 5.4. *Let f and g be integrable on $[a, b]$, and k be a constant. Then we have*

(a) $\displaystyle\int_a^b k dx = k(b - a).$

(b) $\displaystyle\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$

(c) $\displaystyle\int_a^b kf(x)dx = k\int_a^b f(x)dx.$

(d) $\displaystyle\int_a^b (f(x) - g(x))\, dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$

(e) $\displaystyle\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$ *for some* $c \in [a, b]$.

(f) $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx.$

(g) $\displaystyle\int_a^b f(x)dx \geq \int_a^b g(x)dx$ *if* $f(x) \geq g(x)$ *for all* $x \in [a, b]$.

Lemma 5.5 (Integration by parts). *Let f and g be integrable, and assume $f'(x)$ and $g'(x)$ exist for all x. Then,*

$$\int f(x)g'(x)dx + \int g(x)f'(x)dx = f(x)g(x).$$

For definite integrals defined on $[a,b]$, it holds that

$$\int_a^b f(x)'g(x)dx + \int_a^b f(x)g(x)'dx = (f(x)g(x))\Big|_a^b.$$

Example 5.6. Consider $\int x \sin x dx$. Pick $f(x) = x$, $g(x) = -\cos x$. Then

$$\int x\sin x dx = -x\cos x - \int -\cos x dx = -x\cos x + \sin x + C.$$

Example 5.7. Consider $\int_0^\pi e^x \sin x dx$. First, pick $f(x) = e^x$ and $g(x) = -\cos x$. Then

$$\int_0^\pi e^x \sin x dx = (e^x(-\cos x))\Big|_0^\pi + \int_0^\pi e^x \cos x dx.$$

Let us integrate by parts again. Now pick $f(x) = e^x$ and $g(x) = \sin x$. Then

$$\int_0^\pi e^x \sin x dx = (e^x(-\cos x))\Big|_0^\pi + (e^x \sin x)\Big|_0^\pi - \int_0^\pi e^x \sin x dx.$$

Regrouping, we have

$$\begin{aligned}\int_0^\pi e^x \sin x dx &= \frac{1}{2}\left((e^x(-\cos x))\Big|_0^\pi + (e^x \sin x)\Big|_0^\pi\right)\\ &= \frac{1}{2}\left[e^\pi(-\cos\pi) - e^0(-\cos 0)\right] + \frac{1}{2}\left[e^\pi \sin\pi - e^0 \sin 0\right]\\ &= \frac{e^\pi + 1}{2}.\end{aligned}$$

Lemma 5.8 (Integration by substitution). *If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then*

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

If g' is continuous on $[a,b]$, and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Example 5.9. Consider $\int x^3\cos(x^4+2)dx$. Let $u = x^4 + 2$, then $du = 4x^3 dx$ and $dx = du/4x^3$. So we have

$$\int x^3\cos(x^2+4)dx = \frac{1}{4}\int \cos u du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^2+4) + C.$$

Example 5.10. Consider $\int_1^2 \frac{1}{(3-5x)^2}dx$. Let $u = g(x) = 3 - 5x$, then $du = -5dx$, and $dx = -du/5$. The lower bound is $x = 1$, hence, $u = g(1) = -2$, and the upper bound is $x = 2$, hence, $u = g(2) = -7$. We have

$$\int_1^2 \frac{1}{(3-5x)^2}dx = -\frac{1}{5}\int_{-2}^{-7}\frac{1}{u^2}du = -\frac{1}{5}\left(-\frac{1}{u}\right)\Big|_{-2}^{-7} = \frac{1}{14}.$$

Definition 5.11. If $\int_a^t f(x)dx$ exists for every $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

provided this limit exists.

If $\int_t^b f(x)dx$ exists for every $t \le b$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided this limit exists. Improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called *convergent* if the corresponding limit exists, and *divergent* if the limit does not exist.

Definition 5.12. If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

provided this limit exists.

If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

provided this limit exists.

The improper integral $\int_a^b f(x)dx$ is called *convergent* if the corresponding limit exists, and *divergent* if the limit does not exist.

If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example 5.13. Consider $\displaystyle\int_0^3 \frac{1}{x-1}dx$. First, note that $\dfrac{1}{x-1}$ is not defined at $x = 1$, so it is discontinuous at $x = 1$. Then

$$\int_0^3 \frac{1}{x-1}dx = \int_0^1 \frac{1}{x-1}dx + \int_1^3 \frac{1}{x-1}dx = \lim_{t\to1^-} \int_0^t \frac{1}{x-1}dx + \lim_{t\to1^+} \int_t^3 \frac{1}{x-1}dx.$$

Consider the first term:

$$\lim_{t\to1^-} \int_0^t \frac{1}{x-1}dx = \lim_{t\to1^-} \log|x-1|\Big|_0^t = \lim_{t\to1^-} (\log|t-1| - \log|-1|) = \infty.$$

The first term diverges, hence, the whole integral diverges. Note that if we fail to take discontinuity at $x = 1$ into account, we get

$$\int_0^3 \frac{1}{x-1}dx = \log|x-1|\Big|_0^3 = \log 2,$$

which is incorrect.

Theorem 5.14. *Suppose f is continuous on $[-a,a]$.*

(a) If f is even, $f(-x) = f(x)$, then $\displaystyle\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$.

(b) If f is odd, $f(-x) = -f(x)$, then $\displaystyle\int_{-a}^a f(x)dx = 0$.

Theorem 5.15. *Suppose f is continuous on $[a,b]$.*

(a) If $g(x) = \displaystyle\int_a^x f(t)dt$, then $g'(x) = f(x)$.

(b) $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

5.1 Differential equations and their systems

Consider an equation that contains y', for example, $y' = x^3$. It is called a *differential equation*, and you might think of y as of some function of x: $y = f(x)$. So the objective is to find y such that its derivative is x^3. It is straightforward that $y = \frac{1}{4}x^4 + C$ (where C is some constant) solves the differential equation $y' = x^3$. In fact, $\frac{1}{4}x^4 + C$ is the *general solution* (for any C it solves $y = x^3$, thus, it represents a family of solutions).

Now consider that, in addition to the equation $y' = x^3$, we are given some *initial condition*, for example, $y(0) = 0.5$. Then we need only solutions that satisfy this initial condition: $y = \frac{1}{4}x^4 + C$ such that $y(0) = 0.5$. Note that $y(0) = C$, then the *particular solution* to this equation that satisfies the given initial condition is $y = \frac{1}{4}x^4 + 0.5$. Here, we focus on two types of the differential equations.

Definition 5.16. The differential equation $y' = f(x, y)$ is *separable* if we can factorize $f(x, y) = g_1(x)g_2(y)$.

In this case, it holds that

$$\frac{dy}{dx} = g_1(x)g_2(y), \tag{2}$$

$$\frac{dy}{g_2(y)} = g_1(x)dx, \tag{3}$$

$$\int \frac{dy}{g_2(y)} = \int g_1(x)dx. \tag{4}$$

Example 5.17. Consider $y' = \dfrac{x^2}{y^2}$. It is separable, with $g_1(x) = x^2$ and $g_2(y) = y^{-2}$. Now, using (4) we have

$$\begin{aligned}
\int y^2 dy &= \int x^2 dx,\\
\frac{1}{3}y^3 + C_1 &= \frac{1}{3}x^3 + C_2,\\
y^3 &= x^3 + C,\\
y &= \sqrt[3]{x^3 + C},
\end{aligned}$$

which is the general solution. Now, suppose there is an initial condition $y(0) = 2$. Then, $\sqrt[3]{0 + C} = 2 \Rightarrow C = 8$. Hence, the corresponding particular solution is $y = \sqrt[3]{x^3 + 8}$.

Definition 5.18. A differential equation $y' = f(x, y)$ is *first-order linear* if we can represent it by $y' + P(x)y = Q(x)$, where $P(x)$ and $Q(x)$ are some continuous functions.

After, it is useful to multiply both sides by $\exp\left(\int P(x)dx\right)$.

Example 5.19. Consider $y' + 3x^2y = 6x^2$. It is not separable but it is first-order linear. Multiply both sides by e^{x^2} (note that $\exp\left(\int P(x)dx\right) = e^{x^3}e^C$ but $e^C > 0$ is a non-zero constant, so we can omit it):

$$\begin{aligned}
\frac{dy}{dx}e^{x^3} + y3x^2e^{x^3} &= 6x^2e^{x^3},\\
\frac{d(e^{x^3} \cdot y)}{dx} &= 6x^2e^{x^3},\\
\int \frac{d(e^{x^3} \cdot y)}{dx}dx &= \int 6x^2e^{x^3},\\
e^{x^3}y &= 2e^{x^3} + C,\\
y &= 2 + \frac{C}{e^{x^3}}.
\end{aligned}$$

6 Topics in optimization

6.1 Unconstrained optimization

Definition 6.1. Let $F : U \to \mathbb{R}$ be a real-valued function of n variables, whose domain U is a subset of $\mathbb{R}^n$.

(a) A point $\mathbf{x}^\star \in U$ is a *maximum* of F on U if $F(\mathbf{x}^\star) \geq F(\mathbf{x})$ for all $\mathbf{x} \in U$.

(b) $\mathbf{x}^\star \in U$ is a *strict maximum* if $\mathbf{x}^\star$ is a maximum and $F(\mathbf{x}^\star) > F(\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{x}^\star$ in U.

(c) $\mathbf{x}^\star \in U$ is a *local maximum* of F if there is a ball $B_r(\mathbf{x}^\star)$ such that $F(\mathbf{x}^\star) \geq F(\mathbf{x})$ for all $\mathbf{x} \in B_r(\mathbf{x}^\star) \cup U$.

In other words, a point $\mathbf{x}^\star$ is a local maximum if there are no nearby points at which F takes on a larger values. Of course, a maximum is always a local maximum. If we want to emphasize that a point $\mathbf{x}^\star$ is a maximum of F on the whole domain U, not just a local maximum, we call $\mathbf{x}^\star$ a global maximum of F on U.

Reversing the inequalities in the above definitions leads to the definitions of a global minimum, a strict global minimum, and a local minimum, respectively.

Theorem 6.2. *Let $F : U \to \mathbb{R}$ be a C^1 function defined on a subset U of $\mathbb{R}^n$. If $\mathbf{x}^\star$ is a local maximum or minimum of F in U and if $\mathbf{x}^\star$ is an interior point of U, then*

$$\frac{\partial F}{\partial x_i}(\mathbf{x}^\star) = 0, \quad i = 1, \ldots, n.$$

The first-order condition for a point $x^\star$ to be a maximum or minimum of a function f of one variable is that $f'(x^\star) = 0$, that is, that $x^\star$ be a *critical point* of f. This condition requires that $x^\star$ not be an endpoint of the interval under consideration, in other words, that $x^\star$ lie in the interior of the domain of f. The same first order condition works for a function F of n variables. However, a function of n variables has n first derivatives: the partials $\partial F/\partial x_i = 0$ at $\mathbf{x}^\star$. The n-dimensional analogue of $f'(x^\star) = 0$ is that each $\partial F/\partial x_i = 0$ at $\mathbf{x}^\star$. In this case, $\mathbf{x}^\star$ is an *interior point* of the domain of F if there is a whole ball $B_r(\mathbf{x}^\star)$ about $\mathbf{x}^\star$ in the domain of F.

Example 6.3. To find the local maxima and minima of $F(x, y) = x^3 - y^3 + 9xy$, one computes the first order partial derivatives and sets them equal to zero:

$$\frac{\partial F}{\partial x} = 3x^2 + 9y = 0, \quad \frac{\partial F}{\partial y} = -3y^2 + 9x = 0. \tag{5}$$

The first equation yields $y = -\frac{1}{3}x^2$. Substitute into the second equation to get

$$0 = -3y^2 + 9x = -\frac{1}{3}x^4 + 9x.$$

This equation is equivalent to $27x - x^4 = x(27 - x^3) = 0$ whose solutions are $x = 0$ and $x = 3$. Substituting these solutions into $y = -\frac{1}{3}x^2$ implies that the solutions to (5) are the two points $(0, 0)$ and $(3, -3)$. At this stage, we can conclude that the only candidates for a local maximum or minimum of F are these two points. We are unable to say whether either of these two is a maximum or a minimum.

Definition 6.4. We say that the n-vector $\mathbf{x}^\star$ is a *critical point* of a function $F(x_1, \ldots, x_n)$ if $\mathbf{x}^\star$ satisfies

$$\frac{\partial F}{\partial x_i}(\mathbf{x}^\star) = 0, \quad i = 1, \ldots, n.$$

The critical points of $F(x, y) = x^3 - y^3 + 9xy$ in Example 6.3 are $(0, 0)$ and $(3, -3)$. To determine whether either of these of these critical points is a maximum or a minimum, we need to use a condition on the second derivatives of F.

Definition 6.5. A C^2 function of n variables has n^2 second order partial derivatives at each point in its domain and it is natural to combine then into an $n \times n$ matrix, called *the Hessian* of F:

$$D^2F(\mathbf{x}^\star) = \begin{pmatrix} \frac{\partial^2 F}{\partial x_1^2}(\mathbf{x}^\star) & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_1}(\mathbf{x}^\star) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_1 \partial x_n}(\mathbf{x}^\star) & \cdots & \frac{\partial^2 F}{\partial x_n^2}(\mathbf{x}^\star) \end{pmatrix}.$$

Since cross-partial derivatives are equal for a C^2 function, $D^2F(\mathbf{x}^\star)$ is a symmetric matrix.

Theorem 6.6. *Let $F : U \to \mathbb{R}$ be a C^2 function whose domain is an open set U in $\mathbb{R}^n$. Suppose that $\mathbf{x}^\star$ is a critical point of F in that it satisfies Definition 6.4.*

(a) If the Hessian $D^2F(\mathbf{x}^\star)$ is a negative definite symmetric matrix, then $\mathbf{x}^\star$ is a strict local maximum of F.

(b) If the Hessian $D^2F(\mathbf{x}^\star)$ is a positive definite symmetric matrix, then $\mathbf{x}^\star$ is a strict local minimum of F.

(c) If $D^2F(\mathbf{x}^\star)$ is indefinite, then $\mathbf{x}^\star$ is neither a local maximum nor a local minimum of F.

The second order condition for a critical point $x^\star$ of a function f on $\mathbb{R}$ to be a maximum is that the second derivative $f''(x^\star)$ be negative. The corresponding condition for a function F of n variables is that the second derivative $D^2F(\mathbf{x}^\star)$ be *negative definite* as a symmetric matrix at the critical point $\mathbf{x}^\star$. Similarly, the second order sufficient condition for a critical point of a function f of one variable to be a local minimum is that $f''(x^\star)$ be positive; the analogous second order condition fo an n-dimensional critical point $\mathbf{x}^\star$ to be a local minimum is that the Hessian of F at $\mathbf{x}^\star$, $D^2F(\mathbf{x}^\star)$, be *positive definite*.

Definition 6.7. A critical point $\mathbf{x}^\star$ of F for which the Hessian $D^2F(\mathbf{x}^\star)$ is indefinite is called a *saddle point* of F.

A saddle point $\mathbf{x}^\star$ is a minimum of F in some directions and a maximum of F in other directions.

6.2 Constrained optimization

Theorem 6.8. *Let f and h be C^1 functions of two variables. Suppose that $\mathbf{x}^\star = (x_1^\star, x_2^\star)$ is a solution of the problem*

$$\max f(x_1, x_2) \quad \textit{subject to} \quad h(x_1, x_2) = c.$$

Suppose further that $(x_1^\star, x_2^\star)$ is not a critical point of h. Then, there is a real number $\mu^\star$ such that $(x_1^\star, x_2^\star, \mu^\star)$ is a critical point of the Lagrangian function

$$\mathcal{L}(x_1, x_2, \mu) := f(x_1, x_2) - \mu\,(h(x_1, x_2) - c)\,.$$

In other words, at $(x_1^\star, x_2^\star, \mu^\star)$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0, \quad \frac{\partial \mathcal{L}}{\partial x_2} = 0, \quad \frac{\partial \mathcal{L}}{\partial \mu} = 0.$$

Remark 6.9. If we were minimizing f instead of maximizing f on the constrained set C_h, we would have used the same arguments that we used in the proof of Theorem 6.8. In other words, the conclusion of Theorem 6.8 holds whether we are maximizing f or minimizing f on C_h.

Example 6.10. To solve the maximization problem

$$\max f(x_1, x_2) = x_1 x_2, \quad \text{subject to} \quad h(x_1, x_2) := x_1 + 4x_2 = 16,$$

we use Theorem 6.8. Since the gradient $\nabla h(x_1, x_2) = \begin{pmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, h has no critical points and the constraint qualification is satisfied. Form the Lagrangian

$$\mathcal{L}(x_1, x_2, \mu) = x_1 x_2 - \mu(x_1 + 4x_2 - 16),$$

and set its partial derivatives equal to zero:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= x_2 - \mu = 0, \\ \frac{\partial \mathcal{L}}{\partial x_2} &= x_1 - 4\mu = 0, \\ \frac{\partial \mathcal{L}}{\partial \mu} &= -(x_1 + 4x_2 - 16) = 0. \end{aligned}$$

From the first two equations,

$$\mu = x_2 = \frac{1}{4} x_1,$$

and therefore, $x_1 = 4x_2$. Substituting into the third equation,

$$4x_2 + 4x_2 = 16, \quad \text{or} \quad x_2 = 2.$$

We conclude that the solution is

$$x_1 = 8, \quad x_2 = 2, \quad \mu = 2.$$

Theorem 6.8 states that the only candidate for a solution is $x_1 = 8$, $x_2 = 2$.

The statement of the general theorem for maximizing a function of n variables constrained by m equality constraints is a straightforward generalization of Theorem 6.8.

Theorem 6.11. *Let $f, h_1, \ldots, h_m$ be C^1 functions of n variables. Consider the problem of maximizing (or minimizing) $f(\mathbf{x})$ on the constrained set*

$$C_{\mathbf{h}} := \{\mathbf{x} = (x_1, \ldots, x_n) : h_1(\mathbf{x}) = a_1, \ldots, h_m(\mathbf{x}) = a_m\}.$$

Suppose that $x^\star \in C_{\mathbf{h}}$ and that $\mathbf{x}^\star$ is a local maximum or minimum of f on $C_{\mathbf{h}}$. Suppose further that $\mathbf{x}^\star$ satisfies the nondegenerate constraint qualification at $\mathbf{x}^\star$. Then, there exist $\mu_1^\star, \ldots, \mu_m^\star$ such that $(x_1^\star, \ldots, x_n^\star, \mu_1^\star, \ldots, \mu_m^\star) := (\mathbf{x}^\star, \mu^\star)$ is a critical point of the Lagrangian

$$\mathcal{L}(\mathbf{x}, \mu) := f(\mathbf{x}) - \mu_1(h_1(\mathbf{x}) - a_1) - \ldots - \mu_m(h_m(\mathbf{x}) - a_m).$$

In other words, at $(\mathbf{x}^\star, \mu^\star)$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= 0, \quad \ldots, \quad \frac{\partial \mathcal{L}}{\partial x_n} = 0, \\ \frac{\partial \mathcal{L}}{\partial \mu_1} &= 0, \quad \ldots, \quad \frac{\partial \mathcal{L}}{\partial \mu_m} = 0. \end{aligned}$$

Theorem 6.12. *Suppose that f and g are C^1 functions on $\mathbb{R}^2$ and that $(x^\star, y^\star)$ maximizes f on the constraint set $g(x,y) \leq b$. If $g(x^\star, y^\star) = b$, suppose that at $(x^\star, y^\star)$*

$$\frac{\partial g}{\partial x} \neq 0 \quad \textit{or} \quad \frac{\partial g}{\partial y} \neq 0.$$

In any case, form the Lagrandian function

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda \cdot (g(x, y) - b).$$

Then, there is a multiplier $\lambda^\star$ such that

(a) $\dfrac{\partial \mathcal{L}}{\partial x}(x^\star, y^\star, \lambda^\star) = 0,$

(b) $\dfrac{\partial \mathcal{L}}{\partial y}(x^\star, y^\star, \lambda^\star) = 0,$

(c) $\lambda^\star \cdot (g(x^\star, y^\star) - b) = 0,$

(d) $\lambda^\star \geq 0,$

(e) $g(x^\star, y^\star) \leq b.$