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Course: **Mathematics**

# 1 Basic topology

## 1.1 Metric spaces

**Definition 1.1.** A set $X$, whose elements we shall call *points*, is said to be *a metric space* if with any two points $p$ and $q$ of $X$ there is associated a real number $d(p,q)$, called the *distance* from $p$ to $q$, such that

(a) $d(p,q) > 0$ if $p \neq q$ and $d(p,p) = 0$,

(b) $d(p,q) = d(q,p)$,

(c) $d(p,q) \leq d(p,r) + d(r,q)$, for $\forall r \in X$.

Any function with these three properties is called a *distance function*, or *a metric*.

**Example 1.2** (Metric spaces)**.** The following are examples of the metric spaces:

1. the set of real numbers $\mathbb{R}$ with a metric $d(p,q) = |p - q|$,
2. a real plane $\mathbb{R}^2$ with a metric $d(\mathbf{p},\mathbf{q}) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} := \|\mathbf{p} - \mathbf{q}\|$ (Eucledian distance),
3. a real plane $\mathbb{R}^2$ with a metric $d(\mathbf{p},\mathbf{q}) = |p_1 - q_1| + |p_2 - q_2|$ (Manhattan distance),
4. the set of probability distributions defined on the same measurable space with a metric $d(P,Q) = \frac{1}{\sqrt{2}}\left(\int\left(\sqrt{p(x)} - \sqrt{q(x)}\right)^2 dx\right)^{1/2}$ (Hellinger distance).

It is important to observe that every subset $Y$ of a metric space $X$ is a metric space in its own right, with the same distance function. Thus, every subset of a Euclidean space is a metric space.

**Definition 1.3.** By the *segment* $(a,b)$ we mean the set of all real numbers $x$ such that $a < x < b$. By the *interval* $[a,b]$ we mean the set of all real numbers $x$ such that $a \leq x \leq b$.

If $a_i < b_i$ for $i = 1,\ldots,k$, the set of all points $\mathbf{x} = (x_1,\ldots,x_k)$ in $\mathbb{R}^k$ whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ ($1 \leq i \leq k$) is called a *k-cell*. Thus, a 1-cell is an interval, a 2-cell is a rectangle, etc.

If $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, the *open* (or *closed*) *ball* $B$ with center at $\mathbf{x}$ and radius $r$ is defined to be the set of all $\mathbf{y} \in \mathbb{R}^k$ such that $\|\mathbf{y} - \mathbf{x}\| < r$ (or $\|\mathbf{y} - \mathbf{x}\| \leq r$).

We call a set $E \subset \mathbb{R}^k$ *convex* if

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in E$$

whenever $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $0 < \lambda < 1$. For example, balls are convex. It is also easy to see that $k$-cells are convex.

**Definition 1.4.** Let $X$ be a metric space. All points and sets mentioned below are understood to be elements and subsets of $X$.

(a) *A neighborhood* of a point $p$ is a set $N_r(p)$ consisting of all points $q$ such that $d(p,q) < r$. The number $r$ is called the *radius* of $N_r(p)$.

(b) A point $p$ is a *limit point* of the set $E$ if *every* neighborhood of $p$ contains a point $q \neq p$ such that $q \in E$. Example: take a set $A := (0, 1)$. Point 0 is a limit point, because any open interval, say $(-\varepsilon, \varepsilon)$, intersects $A$.

(c) If $p \in E$ and $p$ is not a limit point of $E$, then $p$ is called an *isolated point* of $E$. Example: take a set $A = \{n^{-1} : n \in \mathbb{N}\}$. Each element is an isolated point because you can take a small interval around $n^{-1}$ that avoids the other fractions in the set.

(d) $E$ is *closed* if every limit point of $E$ is a point of $E$. Example: take $A = [0, 1]$. Both 0 and 1 are limit points and both belong to the set $A$. A set $B = (0, 1]$ is not closed because a limit point 0 does not belong to the set.

(e) A point $p$ is an *interior point* of $E$ if there is a neighborhood $N_r(p)$ of $p$ such that $N \subset E$. Example: take a set $A = (0, 1)$. A point 0.5 is an interior point because there is a neighborhood around it, say, $N_{0.1}(0.5)$ that belongs to the set $A$; if $N_{0.1}(0.5) = (0.4, 0.6) := B$, we have $B \subset A$. On the other hand, if $C = [0.5, 1]$, 0.5 is not an interior point of $C$, because there is no neighborhood around it that is a subset of $C$; some points of that neighborhood are outside of $C$.

(f) $E$ is *open* if every point of $E$ is an interior point of $E$.

(g) The *complement* of $E$ (denoted by $E^c$) is the set of all points $p \in X$ such that $p \notin E$.

(h) $E$ is *perfect* if $E$ is closed and if every point of $E$ is a limit point of $E$. Example: take $A = [0, 1]$, which is closed with all points being limit points, so it is perfect. On the other hand, $B = [0, 1] \cup \{3\}$ is not perfect because it contains a point 3, which is not a limit point (it is an isolated point).

(i) $E$ is *bounded* if there is a real number $M$ and a point $q \in X$ such that $d(p, q) < M$ for $\forall p \in E$.

(j) $E$ is *dense in $X$* if every point of $X$ is a limit point of $E$, or a point of $E$ (or both).

Let us note that in $\mathbb{R}^1$ neighborhoods are segments, whereas in $\mathbb{R}^2$ neighborhoods are interiors of circles.

**Theorem 1.5.** *Every neighborhood is an open set.*

*Proof.* Consider neighborhood $E = N_r(p)$, and let $q$ be any point of $E$. Then there is a positive real number $h$ such that

$$d(p, q) = r - h.$$

For all points $s$ such that $d(q, s) < h$, we have then

$$d(p, s) \leq d(p, q) + d(q, s) < r - h + h = r,$$

so that $s \in E$. Thus, $q$ is an interior point of $E$. □

**Theorem 1.6.** *If $p$ is a limit point of a set $E$, then every neighborhood of $p$ contains infinitely many points of $E$.*

*Proof.* Suppose there is a neighborhood $N$ of $p$ which contains only a finite number of points of $E$. Let $q_1, \ldots, q_n$ be those points of $N \cap E$, which are distinct from $p$, and put

$$r = \min_{1 \leq m \leq n} d(p, q_m)$$

The minimum of a finite set of positive numbers is clearly positive, so that $r > 0$.

The neighborhood $N_r(p)$ contains no point $q$ of $E$ such that $q \neq p$, so that $p$ is not a limit point of $E$. This contradiction established the theorem. □

**Corollary 1.7.** *A finite point set has no limit points.*

**Theorem 1.8.** *A set $E$ is open if and only if its complement is closed.*

## 1.2 Compact sets

**Definition 1.9.** By an *open cover* of a set $E$ in a metric space $X$ we mean a collection $\{G_\alpha\}$ of open subsets of $X$ such that $E \subset \cup_\alpha G_\alpha$.

**Definition 1.10.** A subset $K$ of a metric space $X$ is said to be *compact* if every open cover of $K$ contains a *finite subcover*. More explicitly, the requirement is that if $\{G_\alpha\}$ is an open cover of $K$, then there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \ldots \cup G_{\alpha_n}.$$

**Corollary 1.11.** *A set $E$ is compact if it is both closed and bounded.*

## 1.3 Functions

**Definition 1.12.** Consider two sets $A$ and $B$, whose elements may be any objects whatsoever, and suppose that with each element $x$ of $A$ there is associated, in some manner, an element of $B$, which we denote by $f(x)$. Then $f$ is said to be a *function* from $A$ to $B$ (or a *mapping* from $A$ into $B$). The set $A$ is called the *domain* of $f$ (we also say $f$ is defined on $A$), and the elements $f(x)$ are called the *values* of $f$. The set of *all* values of $f$ is called the *range* of $f$.

**Definition 1.13.** If for every $y \in B$ there is at most one $x \in A : f(x) = y$, the function $f$ is said to be a 1-1 (*one-to-one*) mapping of $A$ into $B$. This may also be expressed as follows: $f$ is a 1-1 mapping of $A$ into $B$ provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1 \in A$, $x_2 \in A$.

**Definition 1.14.** Let $A$ and $B$ be two sets and let $f$ be a mapping of $A$ into $B$. If $f(A) = B$, we say that $f$ maps $A$ *onto* $B$. If, additionally, $f$ is 1-1, then $f$ is *one-to-one and onto (bijection)*.

**Definition 1.15.** If there exists a 1-1 mapping of $A$ *onto* $B$, we say that $A$ and $B$ can be put in 1-1 *correspondence*, or that $A$ and $B$ have the same *cardinal number*, or, briefly, that $A$ and $B$ are *equivalent*, and we write $A \sim B$.

**Definition 1.16.** For any positive integer $n$, let $J_n$ be the set whose elements are the integers $1, 2, \ldots, n$; let $J$ be the set consisting of all positive integers. For any set $A$, we say:

(a) $A$ is finite if $A \sim J_n$ for some $n$.

(b) $A$ is *infinite* if $A$ is not finite.

(c) $A$ is *countable* if $A \sim J$.

(d) $A$ is *uncountable* if $A$ is neither finite nor countable.

(e) $A$ is *at most countable* if $A$ is finite or countable.

For two finite sets $A$ and $B$, we evidently have $A \sim B$ if and only if $A$ and $B$ contain the same number of elements (same *cardinality*). For infinite sets, however, the idea of cardinality becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

**Example 1.17.** Let $A$ be the set of all integers. Then $A$ is countable. Consider, the following arrangement of the sets $A$ and $J$:

$$\begin{aligned} A: &\quad 0, 1, -1, 2, -2, \ldots \\ J: &\quad 1, 2, 3, 4, 5, \ldots \end{aligned}$$

We can, in this example, even give an explicit formula for a function $f$ from $J$ to $A$ which sets up a 1-1 correspondence:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

**Remark 1.18.** A finite set cannot be equivalent to one of its proper subsets. That this is, however, possible for infinite sets, is shown by Example 1.17, in which $J$ is a proper subset of $A$.

**Definition 1.19.** In the following, assume that the set $A$ is a subset of $\mathbb{R}$.

(a) If there exists $x \in \mathbb{R}$ such that for every $y \in A$ we have $x \geq y$, then the set $A$ is *bounded from above*.

(b) If there exists $x \in \mathbb{R}$ such that for every $y \in A$ we have $x \leq y$, then the set $A$ is *bounded from below*.

(c) The *supremum* of $A$, denoted as $\sup A$, is the smallest upper bound of the set $A$.

(d) The *infimum* of $A$, denoted as $\inf A$, is the largest lower bound of the set $A$.

We note that the set $A$ is bounded, if it is bounded both from below and from above, which is equivalent to the Definition 1.4(i). If the set $A$ is not bounded from above, then $\sup A = \infty$, and if it is not bounded from below, then $\inf A = -\infty$.