with

$$a_k = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{k\pi}{L}x\right)\,dx.$$

As the initial datum is symmetric with respect to $x = L/2$, and using the elementary identity

$$\sin(k\pi - \alpha) = (-1)^{k+1}\sin\alpha,$$

we find

$$a_k = \begin{cases} \dfrac{4}{L}\displaystyle\int_0^{L/2} \frac{2a}{L}\,x\sin\left(\frac{k\pi}{L}x\right)\,dx & k \text{ odd} \\ 0 & k \text{ even}\,, \end{cases}$$

thus, with $k = 2h+1$, we have

$$\begin{aligned} a_{2h+1} &= \\ &= \frac{8a}{L^2}\left[\frac{-Lx}{(2h+1)\pi}\cos\left(\frac{(2h+1)\pi}{L}x\right)\Big|_0^{L/2} + \frac{L}{(2h+1)\pi}\int_0^{L/2}\cos\left(\frac{(2h+1)\pi}{L}x\right)dx\right] \\ &= \frac{8a}{L^2}\left[-\frac{L^2}{2(2h+1)\pi}\cos\left((2h+1)\frac{\pi}{2}\right) + \frac{L^2}{(2h+1)^2\pi^2}\sin\left((2h+1)\frac{\pi}{2}\right)\right] \\ &= \frac{8a}{(2h+1)^2\pi^2}(-1)^h. \end{aligned}$$

Altogether

$$u(x,t) = \frac{8a}{\pi^2}\sum_{h=0}^{+\infty}\frac{(-1)^h}{(2h+1)^2}\cos\left(\frac{c\pi(2h+1)}{L}t\right)\sin\left(\frac{(2h+1)\pi}{L}x\right).$$

This expression says that only modes that are symmetric with respect to $x = L/2$ can be activated by the initial profile. Note how the series converges uniformly in $[0, L]$ (Weierstrass criterion), but the second derivatives in x and t cannot be computed by swapping derivation and series, because g is not regular, having a non-smooth point at $x = L/2$. The solution thus found is classical only formally; the correct way to interpret it is in a proper (distributional) weak sense (see Chap. 6).

Problem 4.2.2 (Reflection of waves)**.** *Consider the problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L,\ t > 0 \\ u(x,0) = g(x), \quad u_t(x,0) = 0 & 0 \le x \le L \\ u(0,t) = u(L,t) = 0 & t \ge 0. \end{cases}$$

a) *Define suitably the datum g outside the interval $[0, L]$, and use d'Alembert's formula to represent the solution as superposition of traveling waves.*

b) *Examine the physical meaning of the result and the relationship with the method of separation of variables.*

Solution. a) The idea is to extend the Cauchy data to the whole $\mathbb{R}$, so that the corresponding global Cauchy problem has a solution vanishing on the lines $x = 0$, $x = L$. If we restrict this solution to $[0, L] \times \{t > 0\}$ we will find what we want. We indicate by $\tilde{g}$ and $\tilde{h}$ the extended data and set out to solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in \mathbb{R},\ t > 0 \\ u(x, 0) = \tilde{g}(x) & x \in \mathbb{R} \\ u_t(x, 0) = \tilde{h}(x) & x \in \mathbb{R} \end{cases}$$

via d'Alembert's formula

$$u(x,t) = \frac{1}{2}\,[\tilde{g}(x - ct) + \tilde{g}(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(s)\,ds.$$

As u automatically satisfies the vibrating string equation (at least formally), we should choose $\tilde{g}$ and $\tilde{h}$ so to satisfy the initial/boundary conditions. In our case the simplest extension of h is $\tilde{h} = 0$. As for g, we must have

$$\begin{cases} u(x, 0) = \tilde{g}(x) = g(x) & 0 \le x \le L \\ u(0, t) = \dfrac{1}{2}\,[\tilde{g}(-ct) + \tilde{g}(ct)] = 0 & t > 0 \\ u(L, t) = \dfrac{1}{2}\,[\tilde{g}(L - ct) + \tilde{g}(L + ct)] = 0 & t > 0. \end{cases}$$

Therefore, for any s,

$$\tilde{g}(s) = -\tilde{g}(-s), \qquad \tilde{g}(L + s) = -\tilde{g}(L - s). \tag{4.3}$$

The first condition implies that $\tilde{g}$ *must* be an *odd* function. Moreover

$$\tilde{g}(s + 2L) = \tilde{g}(L + (L + s)) = -\tilde{g}(L - (L + s)) = -\tilde{g}(-s) = \tilde{g}(s),$$

so $\tilde{g}$ is $2L$-*periodic*. Then we may define $\tilde{g}$ to be the $2L$-periodic function whose restriction to $[-L, L]$ is given by

$$\tilde{g}(s) = \begin{cases} g(s) & 0 < s < L \\ -g(-s) & -L < s < 0. \end{cases}$$

The solution of the initial problem is then

$$u(x,t) = \frac{1}{2}\,[\tilde{g}(x - ct) + \tilde{g}(x + ct)] \qquad \text{for } 0 \le x \le L,\ t \ge 0. \tag{4.4}$$

• *Physical meaning of* (4.4). Let us divide the strip $[0, L] \times (0, +\infty)$ in regions separated by characteristic segments as in Fig. 4.1.

We analyse (4.4) starting from points $P = (x_0, t_0)$ in region 1. The *direct* and *inverse* characteristic emanating from P meet the x-axis at $a = x_0 - ct_0$ and $b = x_0 + ct_0$

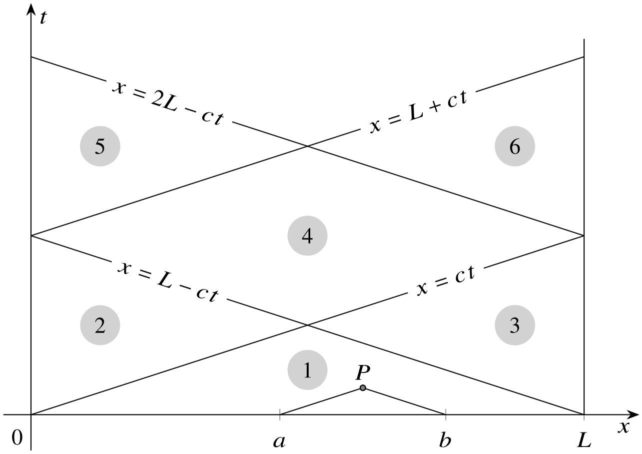

Fig. 4.1 Regions in the xt-plane for Problem 4.2.2

respectively. Since a, b lie in $[0, L]$,

$$u(x_0, t_0) = \frac{1}{2}\left[\tilde{g}(x_0 + ct_0) + \tilde{g}(x_0 - ct_0)\right] = \frac{1}{2}\left[g(x_0 + ct_0) + g(x_0 - ct_0)\right]$$

and the perturbation at P is the average of a *direct* and an *inverse* wave determined by the datum g (at a and b).

Take now $P = (x_0, t_0)$ in region 2 (Fig. 4.2). The point $b = x_0 + ct_0$, foot of the *inverse* characteristic emanating from P, belongs to $[0, L]$, so $\tilde{g}(x_0 + ct_0) = g(x_0 + ct_0)$. The *direct* characteristic through P meets the x-axis at $a = x_0 - ct_0 < 0$. But $\widetilde{g}$ is odd, so $\tilde{g}(a) = -\widetilde{g}(-a) = -g(-a)$ i.e.

$$\tilde{g}(x_0 - ct_0) = -g(-x_0 + ct_0).$$

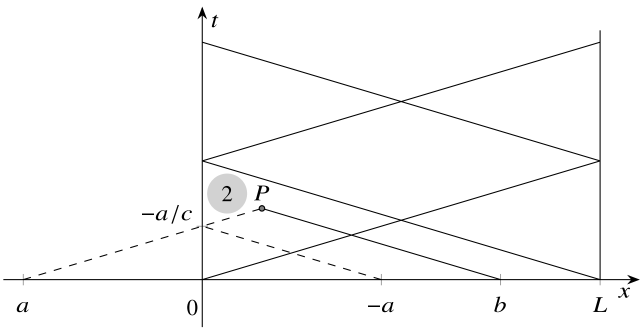

Fig. 4.2 Reflection of an inverse wave

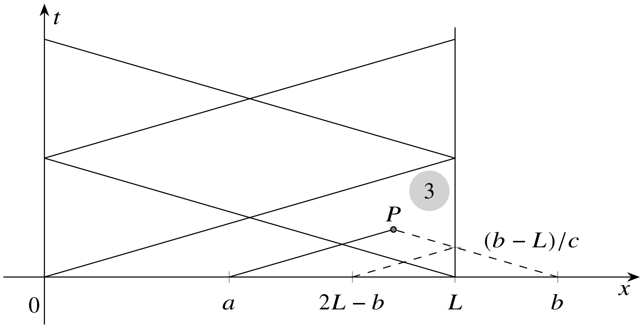

Fig. 4.3 Reflection of a direct wave

This means that the value $\tilde{g}(x_0 - ct_0)$ comes from an *inverse* wave determined by the datum at $-a \in [0, L]$ that reaches the left end at $t = -a/c$, is reflected, changes sign, and then maintains its value until t_0. Thus $u(x_0, t_0)$ is *the superposition of an inverse wave from b and an inverse wave from −a, reflected at* $(0, -a/c)$ *and with opposite sign.*

The argument is similar for any $P = (x_0, t_0)$ in region 3. The point $a = x_0 - ct_0$, foot of the *direct* characteristic through P, belongs to $[0, L]$, so $\tilde{g}(x_0 - ct_0) = g(x_0 - ct_0)$. The *inverse* characteristic from P intersects the x-axis at $b = x_0 + ct_0 > L$. The second relation in (4.3) implies

$$\tilde{g}(x_0 + ct_0) = -\widetilde{g}(2L - x_0 - ct_0) = -g(2L - x_0 - ct_0).$$

This time $\tilde{g}(x_0 - ct_0)$ arises from a *direct* wave determined by the datum at $2L - b$, that reaches the right endpoint at time $t = (b - L)/c$, gets reflected, changes sign and stays constant until t_0. Hence $u(x_0, t_0)$ is *the superposition of a direct wave from a and a direct wave from* $2L - b$*, reflected at* $(0, (b - L)/c)$ *and with opposite sign* (Fig. 4.3).

Figure 4.4 should clarify the meaning of u in other regions. For instance, at a point in region 5 the wave is the superposition of a direct wave that is reflected and changes sign at the right end, and a direct wave that is reflected twice, changing sign first at the right end then at the left end.

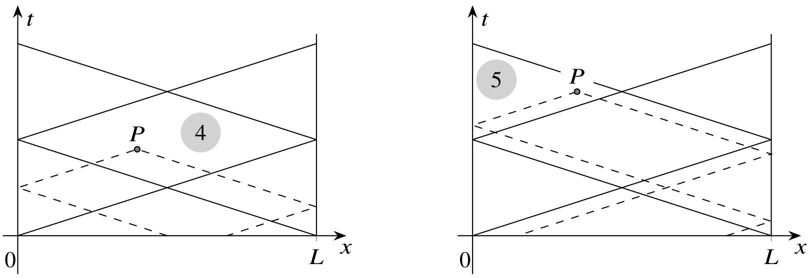

Fig. 4.4 More reflections

• *Relation with the method of separation of variables*. Formula (4.4) can be obtained also by separating variables. In fact, if we proceed as in Problem 4.2.1, and assuming that g is expanded in series of sines on $[0, L]$, the solution is

$$u(x,t) = \sum_{k=1}^{+\infty} a_k \cos\left(\frac{ck\pi}{L}t\right) \sin\left(\frac{k\pi}{L}x\right), \quad \text{with } a_k = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{k\pi}{L}x\right)\,dx. \tag{4.5}$$

The a_k are the Fourier coefficients of the $2L$-periodic odd function $\tilde{g}$, which coincides with g on $[0, L]$. With the notation introduced, this means

$$\sum_{k=1}^{+\infty} a_k \sin\left(\frac{k\pi}{L}s\right) = \tilde{g}(s), \qquad \text{for any } s \in \mathbb{R}. \tag{4.6}$$

On the other hand, known trigonometric addition formulas transform (4.5) into

$$u(x,t) = \sum_{k=1}^{+\infty} a_k \frac{1}{2}\left[\sin\left(\frac{k\pi}{L}x + \frac{ck\pi}{L}t\right) + \sin\left(\frac{k\pi}{L}x - \frac{ck\pi}{L}t\right)\right] =$$

$$= \frac{1}{2}\left[\sum_{k=1}^{+\infty} a_k \sin\left(\frac{k\pi}{L}(x+ct)\right) + \sum_{k=1}^{+\infty} a_k \sin\left(\frac{k\pi}{L}(x-ct)\right)\right],$$

and from (4.6) we deduce (4.4).

Problem 4.2.3 (Equipartition of energy)**.** *Let u denote the solution to the following global Cauchy problem for the vibrating string:*

$$\begin{cases} \rho u_{tt} - \tau u_{xx} = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R} \\ u_t(x,0) = h(x) & x \in \mathbb{R}. \end{cases}$$

Assume g and h are regular functions that vanish outside a compact interval $[a, b]$. Prove that after a sufficiently long time T

$$E_{cin}(t) = E_{pot}(t) \qquad \textit{for any } t \geq T.$$

Solution. Let us recall the expressions for the kinetic and potential energy for small transverse vibrations of an infinite elastic string:

$$E_{cin} = \frac{1}{2}\int_{\mathbb{R}} \rho u_t^2\,dx, \qquad E_{pot} = \frac{1}{2}\int_{\mathbb{R}} \tau u_x^2\,dx,$$

where ρ is the linear density of mass, τ the tension (constant along the string), and

$c = \sqrt{\tau/\rho}$ is the wave speed along the string. D'Alembert's formula reads

$$u(x,t) = F(x+ct) + G(x-ct),$$

where

$$F(s) = \frac{1}{2}\left[g(s) + \frac{1}{c}\int_0^s h(v)\,dv\right], \qquad G(s) = \frac{1}{2}\left[g(s) - \frac{1}{c}\int_0^s h(v)\,dv\right].$$

So we have

$$u_x(x,t) = F'(x+ct) + G'(x-ct), \qquad u_t(x,t) = c\left[F'(x+ct) - G'(x-ct)\right],$$

and then

$$\begin{aligned} E_{pot} &= \frac{1}{2}\int_{\mathbb{R}} \tau\left[F'(x+ct) + G'(x-ct)\right]^2 dx = \\ &= \frac{1}{2}\int_{\mathbb{R}} \tau\left[\left(F'(x+ct)\right)^2 + \left(G'(x-ct)\right)^2 + 2F'(x+ct)G'(x-ct)\right] dx \\ E_{cin} &= \frac{1}{2}\int_{\mathbb{R}} c^2\rho_0\left[F'(x+ct) - G'(x-ct)\right]^2 dx = \\ &= \frac{1}{2}\int_{\mathbb{R}} \tau\left[\left(F'(x+ct)\right)^2 + \left(G'(x-ct)\right)^2 - 2F'(x+ct)G'(x-ct)\right] dx. \end{aligned}$$

To prove the claim it suffices, for t large enough, that the product

$$F'(x+ct)G'(x-ct) = \frac{1}{4}\left[g'(x+ct) + \frac{1}{c}h(x+ct)\right]\cdot\left[g'(x-ct) - \frac{1}{c}h(x-ct)\right]$$

vanishes identically. We exploit the fact that the data are zero outside $[a,b]$: if $F'(x+ct) \neq 0$ then

$$a < x + ct < b; \tag{4.7}$$

in the same way $G'(x-ct) \neq 0$ forces

$$a < x - ct < b. \tag{4.8}$$

Therefore $F'(x+ct)G'(x-ct) \neq 0$ implies, by subtracting (4.7) and (4.8), $a - b < 2ct < b - a$. Consequently, if

$$t > T = \frac{b-a}{2c},$$

the product is zero and the kinetic energy equals the potential energy.

Problem 4.2.4 (Global Cauchy problem – impulses). *Find the formal solution to the problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R} \\ u_t(x,0) = h(x) & x \in \mathbb{R} \end{cases}$$

with the following initial data:

a) $g(x) = 1$ *if* $|x| < a$, $g(x) = 0$ *if* $|x| > a$; $h(x) = 0$.

b) $g(x) = 0$; $h(x) = 1$ *if* $|x| < a$, $h(x) = 0$ *if* $|x| > a$.

Solution. a) As h is identically zero, d'Alembert's formula reads

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right].$$

We then need to distinguish the regions in the plane where $|x \pm ct| \gtrless a$. The possible cases are described below (see the corresponding regions in Fig. 4.5 starting from the right):

- $x > a + ct$. *A fortiori*, then, $x > a - ct$, and $u(x,t) = 0$.
- $\max\{a - ct, -a + ct\} < x < a + ct$. Here $g(x - ct) = 1$ and $g(x + ct) = 0$. Therefore $u(x,t) = 1/2$.
- $\min\{a - ct, -a + ct\} < x < \max\{a - ct, -a + ct\}$. Both contributions are positive and $u(x,t) = 1$.
- $-a + ct < x < a - ct$ (so $t < a/c$). Both contributions are positive and $u(x,t) = 1$.
- $a - ct < x < -a + ct$ (so $t > a/c$). Both contributions vanish and $u(x,t) = 0$.
- $-a - ct < x < \min\{a - ct, -a + ct\}$. Now $g(x - ct) = 0$ and $g(x + ct) = 1$, so $u(x,t) = 1/2$.
- $x < -a - ct$. This implies $x < -a + ct$ and $u(x,t) = 0$.

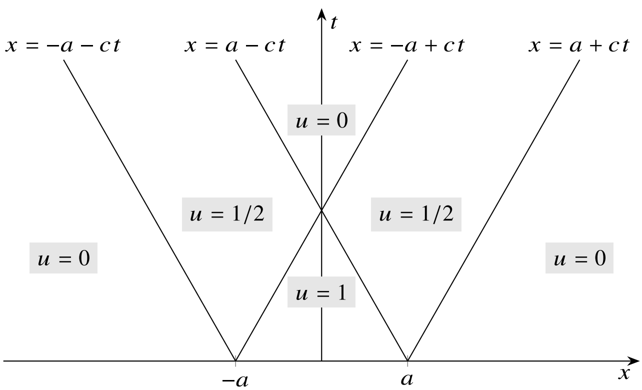

Fig. 4.5 Solution of Problem 4.2.4 a)

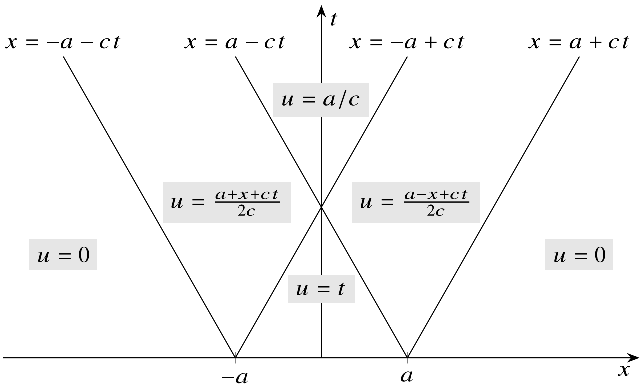

Fig. 4.6 Solution of Problem 4.2.4 b)

b) This time

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds,$$

and arguing case by case as before, we obtain (Fig. 4.6):

- $x > a + ct$. Then $u(x,t) = 0$.
- $\max\{a - ct, -a + ct\} < x < a + ct$. We have

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{a} ds = \frac{a - x + ct}{2c}.$$

- $-a + ct < x < a - ct$ (so $t < a/c$). We have

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} ds = t.$$

- $a - ct < x < -a + ct$ (hence $t > a/c$). Here

$$u(x,t) = \frac{1}{2c}\int_{-a}^{a} ds = \frac{a}{c}.$$

- $-a - ct < x < \min\{a - ct, -a + ct\}$. Then

$$u(x,t) = \frac{1}{2c}\int_{-a}^{x+ct} ds = \frac{a + x + ct}{2c}.$$

- $x < -a - ct$. It follows $u(x,t) = 0$.

Therefore $w_k(t) = 0$, $k \geq 2$, while

$$w_1(t) = \frac{L}{\pi} \int_0^t \sin(l\tau)\, g(t-\tau)\, d\tau.$$

From (4.10) we find

$$u(x,t) = \frac{L}{\pi} \sin\left(\frac{\pi}{L}x\right) \left(\int_0^t \sin\left(\frac{\pi}{L}\tau\right) g(t-\tau)\, d\tau \right).$$

The string reacts to the forcing term by vibrating with the first fundamental mode, whose amplitude depends upon the convolution integral.

Problem 4.2.6 (Semi-infinite string with fixed end). *Consider the problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x > 0,\, t > 0 \\ u(x,0) = g(x),\ u_t(x,0) = h(x) & x \geq 0 \\ u(0,t) = 0 & t \geq 0, \end{cases}$$

with g, h *regular,* $g(0) = 0$.

a) *Extend suitably the initial data to* $\mathbb{R}$ *and use d'Alembert's formula to write a representation formula for the solution.*

b) *Interpret the solution in the case* $h(x) = 0$ *and*

$$g(x) = \begin{cases} \cos(x-4) & |x-4| \leq \dfrac{\pi}{2} \\ 0 & \textit{otherwise.} \end{cases}$$

Solution. a) Let us look for $\tilde{g}$ and $\tilde{h}$, defined on $\mathbb{R}$ and extending g and h to $x < 0$. The d'Alembert solution

$$u(x,t) = \frac{1}{2}\left[\tilde{g}(x+ct) + \tilde{g}(x-ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \tilde{h}(s)\, ds \tag{4.15}$$

must satisfy

$$u(0,t) = 0$$

for any $t > 0$, and therefore we have the necessary condition

$$\frac{1}{2}\left[\tilde{g}(ct) + \tilde{g}(-ct)\right] + \frac{1}{2c} \int_{-ct}^{ct} \tilde{h}(s)\, ds = 0.$$

The easiest way to satisfy this condition is to require the $\tilde{g}$- and $\tilde{h}$-summands to vanish separately, which happens if we extend g and h in an odd way.

Therefore the solution is given by (4.15) with

$$\tilde{g}(s) = \begin{cases} g(s) & s \geq 0 \\ -g(-s) & s < 0 \end{cases}$$

and

$$\tilde{h}(s) = \begin{cases} h(s) & s \geq 0 \\ -h(-s) & s < 0. \end{cases}$$

b) As $h = 0$, the solution reduces to

$$u(x,t) = \frac{1}{2}\left[\tilde{g}(x+ct) + \tilde{g}(x-ct)\right].$$

The initial datum can be understood as the superposition of two sinusoidal waves (with compact support and amplitude $1/2$) that at $t = 0$ start to travel in opposite directions with speed c. Since $x + ct$ is always positive, we have

$$\tilde{g}(x+ct) = g(x+ct)$$

for any (x,t), while

$$x - ct \geq 0 \text{ for any } x \in \left[4 - \frac{\pi}{2}, 4 + \frac{\pi}{2}\right] \text{ if } t \leq \frac{8-\pi}{2c},$$

$$x - ct \leq 0 \text{ for any } x \in \left[4 - \frac{\pi}{2}, 4 + \frac{\pi}{2}\right] \text{ if } t \geq \frac{8+\pi}{2c}.$$

Therefore we distinguish several intervals of time:

- $0 < t < \frac{\pi}{2c}$. The impulses start as opposite, but continue to interact in a neighbourhood of the point $x = 4$.
- $\frac{\pi}{2c} < t < \frac{8-\pi}{2c}$. Same as above, but the impulses do not interfere with each other.
- $\frac{8-\pi}{2c} < t < \frac{8+\pi}{2c}$. The impulse heading left reaches the fixed end and is reflected, turning upside down[1] (and interfering with itself).
- $t > \frac{8+\pi}{2c}$. The impulse moving leftwards has turned completely upside down. The string profile is given by two impulses of same shape, one positive and one negative, at a distance of 8, travelling towards the right at speed c.

These phases are shown in Fig. 4.7.

Problem 4.2.7 (Forced vibrations of a semi-infinite string)**.** *A semi-infinite string is initially at rest along the axis $x \geq 0$, and fixed at $x = 0$. An external force $f = f(t)$ sets it in motion.*

a) *Write the mathematical model governing the vibrations.*

b) *Solve the problem using the Laplace transform in t, assuming that the transform of u is bounded as s tends to $+\infty$.*

[1] Special case of Problem 4.2.2 (page 219).

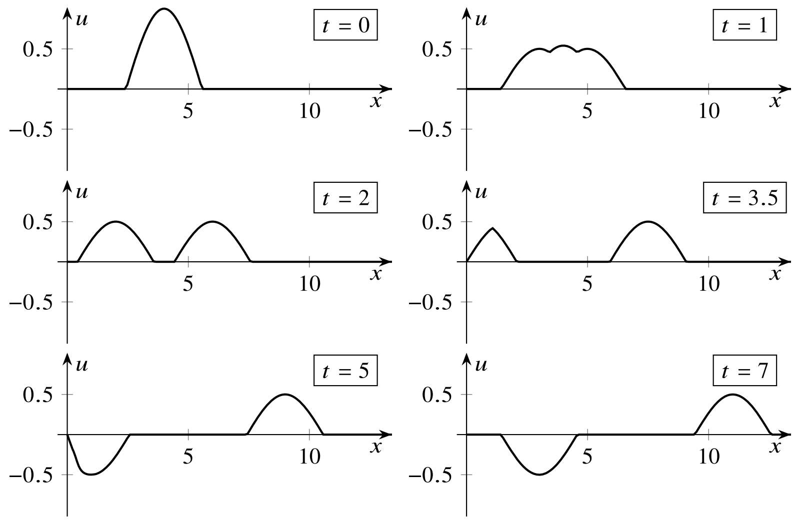

Fig. 4.7 The solution to Problem 4.2.6 ($c = 1$) at different values of t

Solution. a) Indicating with $u(x,t)$ the string profile at time t, the model reads

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(t) & x > 0,\, t > 0 \\ u(x,0) = u_t(x,0) = 0 & x \geq 0 \\ u(0,t) = 0 & t \geq 0. \end{cases}$$

b) Set

$$U(x,s) = \mathcal{L}(u(x,\cdot))(s) = \int_0^{+\infty} u(x,t)e^{-st}\,dt, \qquad s \geq 0,$$

the t-Laplace transform of u. Transforming the equation and using the initial conditions we find[2]

$$-c^2 U_{xx}(x,s) + s^2 U(x,s) = F(s), \tag{4.16}$$

(where $F = \mathcal{L}(f)$), a second order ODE for the function $x \mapsto U(x,s)$, with constant coefficients. The general integral of the homogeneous equation is

$$A(s)e^{-sx/c} + B(s)e^{sx/c} \qquad (s \geq 0)\,.$$

As $F(s)$ doe not depend on x it is easy to check that

$$F(s)/s^2$$

[2] Recall that $\mathcal{L}(u_t) = sU(x,s) - u(x,0)$, $\mathcal{L}(u_{tt}) = s^2U(x,s) - su(x,0) - u_t(x,0)$.

with fundamental frequency

$$g_0 = \frac{c}{2L}.$$

In conclusion, the *open* pipe *produces a sound of double frequency*, essentially because the closed end allows for twice as many wavelengths inside the pipe, and therefore halves the frequency.

4.2.2 Canonical forms. Cauchy and Goursat problems

Problem 4.2.11 (Characteristics and general solution)**.** *Determine the type of the following linear equation of order two (in two variables)*

$$2u_{xx} + 6u_{xy} + 4u_{yy} + u_x + u_y = 0$$

and compute its characteristics. Reduce it to canonical form and find the general solution.

Solution. As

$$3^2 - 2 \cdot 4 = 1 > 0$$

the equation is hyperbolic. The principal part factorises as

$$2u_{xx} + 6u_{xy} + 4u_{yy} = 2(\partial_x - 2\partial_y)(\partial_x - \partial_y)u.$$

Using the techniques of Chap. 3 we solve

$$\phi_x - 2\phi_y = 0$$

to get the family of (real) characteristics $\phi(x, y) = 2x - y =$ constant. Moreover,

$$\psi_x - \psi_y = 0$$

gives the characteristic family $\psi(x, y) = x - y =$ constant.

Another way to proceed would be to look for $y = y(x)$ and to solve the characteristics ODE

$$2\left(\frac{dy}{dx}\right)^2 - 6\frac{dy}{dx} + 4 = 0,$$

giving

$$\frac{dy}{dx} = 1 \quad \text{or} \quad \frac{dy}{dx} = 2$$

so $y - x = c_1$ or $y - 2x = c_2$.

To write the equation in normal form we change coordinates by setting

$$\begin{cases} \xi = 2x - y \\ \eta = x - y \end{cases} \qquad \text{i.e.} \qquad \begin{cases} x = \xi - \eta \\ y = \xi - 2\eta. \end{cases}$$

Set now $U(\xi, \eta) = u(\xi - \eta, \xi - 2\eta)$, so that $u(x, y) = U(2x - y, x - y)$, and then

$$u_x = 2U_\xi + U_\eta, \quad u_y = -U_\xi - U_\eta,$$

$$u_{xx} = 4U_{\xi\xi} + 4U_{\xi\eta} + U_{\eta\eta}, \quad u_{xy} = -2U_{\xi\xi} - 2U_{\xi\eta} - U_{\eta\eta}, \quad u_{yy} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}.$$

The equation for U is $4U_{\xi\eta} + U_\xi = 0$, or equivalently

$$\left(U_\xi\right)_\eta = \frac{1}{4}U_\xi.$$

Integrating in η first, we obtain $U_\xi(\xi, \eta) = e^{\eta/4} f(\xi)$ with f arbitrary (and regular), and then, integrating in ξ, we find

$$U(\xi, \eta) = e^{\eta/4} F(\xi) + G(\eta),$$

for some regular functions F, G. Returning to the original variables, the general solution reads

$$u(x, y) = e^{(x-y)/4} F(2x - y) + G(x - y).$$

Problem 4.2.12 (Euler-Tricomi equation)**.** *Determine the characteristics of the Tricomi equation*

$$u_{tt} - t u_{xx} = 0.$$

Solution. First of all we note that the equation is hyperbolic, so the characteristics are real, only if $t > 0$ (it is parabolic for $t = 0$, a set with empty interior and not characteristic at any point). In the hyperbolic situation, the Tricomi operator factorises as

$$u_{tt} - t u_{xx} = (\partial_t - \sqrt{t}\partial_x)(\partial_t + \sqrt{t}\partial_x)u,$$

and the characteristics are $\phi(x, t) =$ constant, $\psi(x, t) =$ constant, with

$$\phi_t - \sqrt{t}\phi_x = 0 \qquad \psi_t + \sqrt{t}\psi_x = 0.$$

The methods of the previous chapter lead to the general solution of these first-order equations:

$$\phi(x, t) = F\left(3x + 2t^{3/2}\right) \qquad \psi(x, t) = G\left(3x - 2t^{3/2}\right)$$

with F, G arbitrary. Therefore the characteristic curves have equation

$$3x \pm 2t^{3/2} = \text{constant} \qquad \text{for } t \geq 0.$$

Problem 4.2.13 (Cauchy problem)**.** *Solve, if possibile, the Cauchy problem*

$$\begin{cases} u_{yy} - 2u_{xy} + 4e^x = 0 & (x, y) \in \mathbb{R}^2 \\ u(x, 0) = \varphi(x) & x \in \mathbb{R} \\ u_y(x, 0) = \psi(x) & x \in \mathbb{R}. \end{cases}$$

Solution. First let us try to put the equation in normal form in order to write the general integral, and then we shall discuss the initial condition. It is easy to see that the equation is hyperbolic and its principal part decomposes

$$u_{yy} - 2u_{xy} = \partial_y(\partial_y - 2\partial_x)u.$$

Thus we can compute immediately the two characteristic families

$$x = \text{constant} \qquad \text{and} \qquad x + 2y = \text{constant}.$$

To reduce to normal form we change the coordinates by putting

$$\begin{cases} \xi = x \\ \eta = x + 2y \end{cases} \qquad \text{i.e.} \qquad \begin{cases} x = \xi \\ y = \dfrac{-\xi + \eta}{2}. \end{cases}$$

Set $u(x, y) = U(x, x + 2y)$, so

$$u_y = 2U_\eta, \quad u_{xy} = 2U_{\xi\eta} + 2U_{\eta\eta}, \quad u_{yy} = 4U_{\eta\eta}.$$

The equation for U is

$$U_{\xi\eta} = e^\xi,$$

hence

$$U_\eta = e^\xi + f(\eta) \quad \text{and} \quad U(\xi, \eta) = \eta e^\xi + F(\eta) + G(\xi)$$

with F and G arbitrary. Back in the original variables, the general integral is

$$u(x, y) = 2ye^x + F(x + 2y) + G(x)$$

where the summand xe^x has been absorbed into the function G.

Now we come to the Cauchy conditions. To begin with, these are given on the straight line $y = 0$, which is not characteristic at any of its points, so the problem is well posed (at least locally). Substituting, we have

$$\begin{cases} \varphi(x) = u(x, 0) = F(x) + G(x) \\ \psi(x) = u_y(x, 0) = 2e^x + 2F'(x). \end{cases}$$

The second relation gives

$$F(x) = -e^x + \frac{1}{2}\int_c^x \psi(s)\, ds \qquad (c \text{ arbitrary}),$$

whence

$$G(x) = \varphi(x) + e^x - \frac{1}{2}\int_c^x \psi(s)\, ds$$

and finally

$$u(x,y) = 2ye^x - e^{x+2y} + \frac{1}{2}\int_c^{x+2y} \psi(s)\,ds + \varphi(x) + e^x - \frac{1}{2}\int_c^x \psi(s)\,ds =$$
$$= \left(1 + 2y - e^{2y}\right)e^x + \varphi(x) + \frac{1}{2}\int_x^{x+2y} \psi(s)\,ds.$$

Problem 4.2.14 (Characteristics and general integral)**.** *Determine the characteristics of*

$$t^2 u_{tt} + 2tu_{xt} + u_{xx} - u_x = 0.$$

Reduce the equation to canonical form and find the general solution.

Solution. The equation is parabolic, since the principal part decomposes as

$$t^2 u_{tt} + 2tu_{xt} + u_{xx} = (t\partial_t + \partial_x)^2 u.$$

Thus the unique characteristic family is $\phi(x,t) = \text{constant}$, where ϕ solves the first-order equation

$$t\phi_t + \phi_x = 0. \tag{4.21}$$

Using the methods of the previous chapter, we compute

$$\phi(x,t) = g\left(te^{-x}\right),$$

g arbitrary, so $\phi(x,t) = \text{constant}$ reads

$$te^{-x} = \text{constant}.$$

Let $\psi = \psi(x)$ be a smooth function, to be chosen later on, such that $\psi' > 0$. Thus, its inverse ψ^{-1} is well defined, and we can set

$$\begin{cases} \xi = te^{-x} \\ \eta = \psi(x) \end{cases} \qquad \text{i.e.} \qquad \begin{cases} x = \psi^{-1}(\xi) \\ t = \xi \exp[\psi^{-1}(\xi)]. \end{cases}$$

Define $U(\xi,\eta) = u(\psi^{-1}(\xi), \xi \exp[\psi^{-1}(\xi)])$, or in other words

$$u(x,y) = U(te^{-x}, \psi(x)).$$

Then

$$u_x = -te^{-x}U_\xi + \psi' U_\eta, \qquad u_t = e^{-x}U_\xi,$$
$$u_{xx} = te^{-x}U_\xi + t^2e^{-2x}U_{\xi\xi} - 2\psi' te^{-x}U_{\xi\eta} + (\psi')^2 U_{\eta\eta} + \psi'' U_\eta,$$
$$u_{xt} = -e^{-x}U_\xi - te^{-2x}U_{\xi\xi} + \psi' e^{-x}U_{\xi\eta}, \qquad u_{tt} = e^{-2x}U_{\xi\xi}.$$

Substituting into the original equation, we get

$$(\psi')^2 U_{\eta\eta} + (\psi'' - \psi')U_\eta = 0.$$

Now, if we pick $\eta = \psi(x) = e^x$, the second summand vanishes, and $\psi' > 0$. Therefore U solves $U_{\eta\eta} = 0$, and then

$$U(\xi, \eta) = F(\xi) + G(\xi)\eta,$$

with F and G arbitrary. Returning to the original variables, we finally find

$$u(x,t) = F(te^{-x}) + G(te^{-x})e^x.$$

Problem 4.2.15 (A maximum principle)**.** *Let $u = u(x,t)$ be a function such that*

$$Lu = u_{tt} - u_{xx} \le 0 \tag{4.22}$$

on the characteristic (triangular) domain

$$T = \{(x,t) : x > t, x + t < 1, t > 0\}.$$

Assume that $u \in C^2\left(\overline{T}\right)$ and prove that:

***a)** If $u_t(x,0) \le 0$, for $0 \le x \le 1$, then*

$$\max_{\overline{T}} u = \max_{0 \le x \le 1} u(x,0).$$

***b)** If $u_t(x,0) < 0$, for $0 \le x \le 1$, or $Lu < 0$ on T, then*

$$u(x,t) < \max_{0 \le x \le 1} u(x,0), \quad \text{for any } (x,t) \in T.$$

Solution. a) Fix a point C in $\overline{T}$ and consider the characteristic triangle T_C of vertices A, B, C, as in Fig. 4.9. We integrate on T_C the inequality $Lu \le 0$:

$$\int_{T_C} (u_{tt} - u_{xx})\, dxdt \le 0.$$

From Green's formula

$$\int_{T_C} (u_{tt} - u_{xx})\, dxdt = \int_{\partial^+ T_C} (-u_t\, dx - u_x\, dt)$$