is defined implicitly by

$$F(\varphi(x,y,z),\psi(x,y,z))=0,$$

where $F=F(h,k)$ is an arbitrary C^1 function such that $F_h\varphi_z+F_k\psi_z\neq 0$.

3.2 Solved Problems

- **3.2.1 – 3.2.11** : Conservation laws and applications.
- **3.2.12 – 3.2.21** : Characteristics for linear and quasilinear equations.

3.2.1 Conservation laws and applications

Problem 3.2.1 (Burgers equation, shock waves)**.** *Study the global Cauchy problem for Burgers equation*

$$\begin{cases} u_t+uu_x=0 & x\in\mathbb{R},\ t>0 \\ u(x,0)=g(x) & x\in\mathbb{R} \end{cases}$$

where

$$\textit{a)}\quad g(x)=\begin{cases} 1 & x<-1 \\ -1/2 & -1<x<1 \\ -1 & x>1, \end{cases} \qquad \textit{b)}\quad g(x)=\begin{cases} 0 & x\le 0,\ x>1 \\ 2x & 0\le x<1. \end{cases}$$

Solution. a) The Burgers equation is a conservation law of the type

$$u_t+q(u)_x=0$$

with $q(u)=u^2/2$ and $q'(u)=u$.

The characteristic emanating from the point $(\xi,0)$ on the xt-plane, along which the solution is constant and equals $g(\xi)$, has equation

$$x=q'(g(\xi))t+\xi=g(\xi)t+\xi=\begin{cases} t+\xi & \xi<-1 \\ -\frac{1}{2}t+\xi & -1<\xi<1 \\ -t+\xi & \xi>1. \end{cases}$$

As q' is increasing (q is convex) and g has decreasing discontinuities, the characteristic slopes decrease when crossing the datum discontinuities. Then the characteristics then intersect, for small times, near $x=-1$ and also $x=1$ (Fig. 3.1).

Therefore from both points we have shock waves $x=s(t)$, which can be determined using the Rankine-Hugoniot condition

$$s'(t)=\frac{q(u^+(s(t),t))-q(u^-(s(t),t))}{u^+(s(t),t)-u^-(s(t),t)}=\frac{1}{2}[u^+(s(t),t)+u^-(s(t),t)].$$

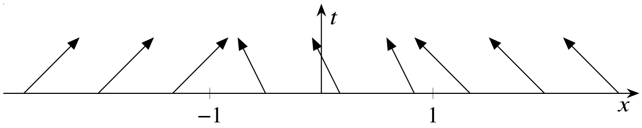

Fig. 3.1 Characteristics for Problem 3.2.1 a) (small times)

Near $(x,t) = (-1,0)$ we have $u^- \equiv 1$, $u^+ \equiv -1/2$, so

$$\begin{cases} s_1'(t) = \frac{1}{4} \\ s_1(0) = -1 \end{cases} \qquad \text{whence } x = s_1(t) = \frac{1}{4}t - 1.$$

Similarly, near $(x,t) = (1,0)$, $u^- \equiv -1/2$, $u^+ \equiv -1$ and

$$\begin{cases} s_2'(t) = -\frac{3}{4} \\ s_2(0) = 1 \end{cases} \qquad \text{whence } x = s_2(t) = -\frac{3}{4}t + 1.$$

Consequently, for small times, the solution $u(x,t)$ equals $-1/2$ for

$$\frac{1}{4}t - 1 < x < -\frac{3}{4}t + 1.$$

As t increases, this interval gets smaller, until it disappears for $t = 2$ (and $x = -1/2$). At this point the two shock waves collide, and the surviving characteristics carry the datum $u^- \equiv 1$ (left) and $u^+ \equiv -1$ (right); this generates a third shock curve $x = s_3(t)$, where

$$\begin{cases} s_3'(t) = 0 \\ s_3(2) = -\frac{1}{2} \end{cases} \qquad \text{thus} \quad x = s_3(t) = -\frac{1}{2}.$$

Overall, the only entropic solution is (Fig. 3.2)

$$u(x,t) = \begin{cases} 1 & x < \min\left(\frac{1}{4}t - 1, -\frac{1}{2}\right) \\ -\frac{1}{2} & \frac{1}{4}t - 1 < x < -\frac{3}{4}t + 1 \\ -1 & x < \max\left(-\frac{3}{4}t + 1, -\frac{1}{2}\right). \end{cases}$$

b) In this case the characteristics are

$$x = \begin{cases} \xi & \xi \le 0,\ \xi > 1 \\ 2\xi t + \xi & 0 \le \xi < 1. \end{cases}$$

In particular, $u(x,t) \equiv 0$ as $x \le 0$, $t \ge 0$. When $0 < \xi < 1$, if t is small, the implicit solution is given in implicit form by

$$u = g\left(x - q'(u)t\right) = 2(x - ut),$$

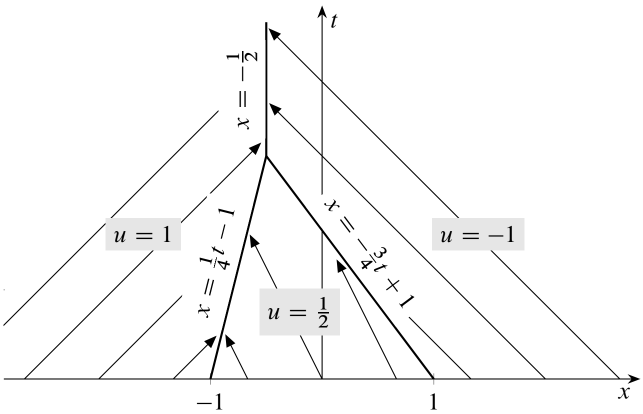

Fig. 3.2 Characteristic lines for Problem 3.2.1 a)

whence

$$u(x,t) = \frac{2x}{2t+1}.$$

Alternatively, from the characteristic $x = 2\xi t + \xi$ we find

$$\xi = \frac{x}{2t+1}, \qquad \text{and hence } u(x,t) = g(\xi) = \frac{2x}{2t+1}.$$

As before, the decreasing discontinuity of g at $x = 1$, plus the convexity of q, cause the formation of a shock wave $x = s(t)$ satisfying the Rankine-Hugoniot condition. Since here $u^-(x,t) = 2x/(2t+1)$, $u^+ \equiv 0$, we have

$$\begin{cases} s_1'(t) = \dfrac{s(t)}{2t+1} \\ s_1(0) = 1. \end{cases}$$

The (ordinary) equation is linear and with separate variables. Integrating and imposing the initial condition gives $s(t) = \sqrt{2t+1}$. The required solution is thus (Fig. 3.3)

$$u(x,t) = \begin{cases} 0 & x \le 0,\ x > \sqrt{2t+1} \\ \dfrac{2x}{2t+1} & 0 \le x < \sqrt{2t+1}. \end{cases}$$

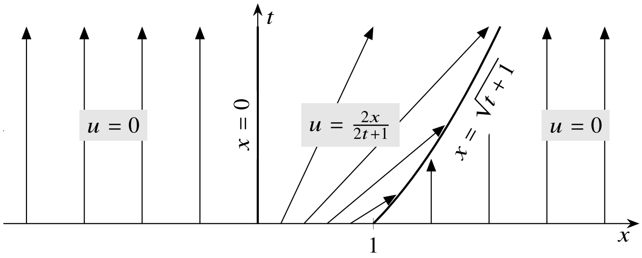

Fig. 3.3 Characteristic lines for Problem 3.2.1 b)

Problem 3.2.2 (Burgers equation, rarefaction vs. shock)**.** *Solve the problem*

$$\begin{cases} u_t + uu_x = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

where

$$\textbf{\textit{a)}} \quad g(x) = \begin{cases} 1 & x \leq -1 \\ -x & -1 \leq x < 0 \\ 1 & x > 1, \end{cases} \qquad \textbf{\textit{b)}} \quad g(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 1 \\ 0 & x > 1. \end{cases}$$

Solution. a) As in the previous problem the characteristics are

$$x = q'(g(\xi))t + \xi = g(\xi)t + \xi = \begin{cases} t + \xi & \xi \leq -1 \text{ or } \xi > 0 \\ -\xi t + \xi & -1 \leq \xi < 0. \end{cases}$$

This time, though, g has an increasing discontinuity at $x = 0$; since $q(u) = u^2/2$ is convex (and hence q' is increasing), the slope of the characteristic has an increasing jump when ξ crosses 0 from left to right. Hence we expect that a region of the xt-plane will not be met by any characteristic. In this case the only entropic solution in this region is a rarefaction wave. On the other hand the characteristics corresponding to $-1 \leq \xi < 0$ form a family of straight lines through the point $(x,t) = (0,1)$. Consequently, for $t < 1$, the solution is constructed by taking

$$\xi = \frac{x}{1-t} \qquad \text{and consequently } u(x,t) = g(\xi) = -\frac{x}{1-t}.$$

So for $t < 1$, no other characteristic enters the sector between the characteristics $x = 0$ and $x = t$, and the solution is given by a rarefaction wave. In general, a rarefaction wave centred at (x_0, t_0) has equation

$$u\,(x,t) = R\left(\frac{x - x_0}{t - t_0}\right) \qquad \text{where } R\,(y) = \left(q'\right)^{-1}(y)\,.$$

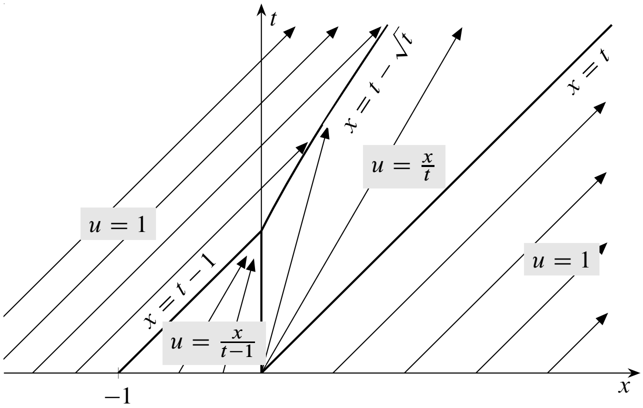

Fig. 3.4 Characteristic lines for Problem 3.2.2 a)

Since here $R(y) = y$, we find

$$u(x,t) = \frac{x}{t}, \qquad 0 \le x \le t,\ t < 1.$$

Alternatively we may put $\xi = 0$ and $g(\xi) = u(x,t)$ in the characteristics equation to get

$$x = u(x,t)t$$

and hence $u = x/t$. Note that a rarefaction wave is constant along the straight lines through the origin, also called characteristics.

When $t > 1$ the characteristics carrying $u^- \equiv 1$ hit the rarefaction characteristics, along which $u^+(x,t) = x/t$, and generate a shock curve Γ satisfying

$$\begin{cases} s_1'(t) = \dfrac{s(t)}{2t} \\ s_1(1) = 0. \end{cases}$$

This gives $s(t) = t - \sqrt{t}$. Note that Γ does not meet the characteristic $x = t$. Finally, we have (Figs. 3.4 and 3.5)

$$u(x,t) = \begin{cases} 1 & x \le t - 1 \text{ for } t < 1 \\ 1 & x < t - \sqrt{t} \text{ for } t \ge 1 \\ x/(t-1) & t - 1 \le x \le 0 \text{ for } t < 1 \\ x/t & \max(0, t - \sqrt{t}) < x \le t \\ 1 & x \ge t. \end{cases}$$

b) The function g has an increasing jump at $x = 0$ and a decreasing one at $x = 1$. Since q is convex, we expect a rarefaction wave at $(0,0)$; after some time, the latter inter-

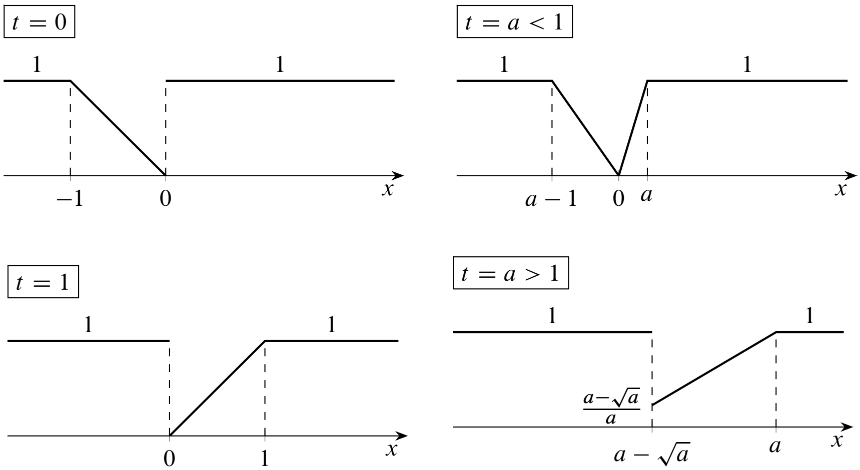

Fig. 3.5 Solution to Problem 3.2.2 a), at various times

acts with a shock wave emanating from $(1, 0)$. The characteristic line from the point $(\xi, 0)$ is

$$x = \xi + q'(g(\xi))t = \xi + g(\xi)t = \begin{cases} \xi & \xi < 0 \text{ or } \xi > 1 \\ t + \xi & 0 < \xi < 1. \end{cases}$$

By varying ξ we deduce immediately the following properties for the solution (Fig. 3.6):

- $u(x, t)$ equals 0 when $x < 0$ (vertical characteristics).
- The characteristics $x = 0$ and $x = t$ bound the region occupied by a rarefaction wave centred at the origin, at least until some instant time t_0 to be determined.
- From $(1, 0)$ starts a shock wave; on the right $u(x, t)$ is 0, while on the left, at least until t_0, $u(x, t)$ equals 1.
- For times larger than t_0 the shock interacts on the left with the rarefaction wave.

If we argue as in the previous situation, the rarefaction wave is

$$u(x, t) = \frac{x}{t}, \qquad 0 \le x \le t.$$

Concerning the shock wave, for small t we have $u^+ = 0$ and $u^- = 1$, so

$$\begin{cases} s'(t) = \frac{1}{2} \\ s(0) = 1, \end{cases} \qquad \text{and thus } x = s(t) = \frac{1}{2}t + 1.$$

What we have said holds until the characteristic $x = t$ intersects the shock curve, that is up to $t_0 = 2$. For later times we still have a shock wave with $u^+ = 0$, but now

$$u^-(s,t) = \frac{s}{t},$$

corresponding to the value of u carried by the rarefaction wave. Therefore

$$\begin{cases} s'(t) = \dfrac{s(t)}{2t} \\ s(2) = 2. \end{cases}$$

The ODE is linear, and with separated variable, and has one solution

$$s(t) = \sqrt{2t}.$$

Summarising,

$$u(x,t) = \begin{cases} 0 & x \le 0 \\ \frac{x}{t} & 0 \le x < \min\left(t, \sqrt{2t}\right) \\ 1 & t \le x < \frac{1}{2}t + 1, \text{ with } t < 2 \\ 0 & x > \max\left(\frac{1}{2}t + 1, \sqrt{2t}\right). \end{cases}$$

The shock speed is $1/2$ until $t = 2$ and then becomes negative, $-1/2t^{3/2}$. The strength equals the jump value of u across the shock, i.e. 1 until $t = 2$, and then fades to zero as $t \to \infty$ (Fig. 3.6).

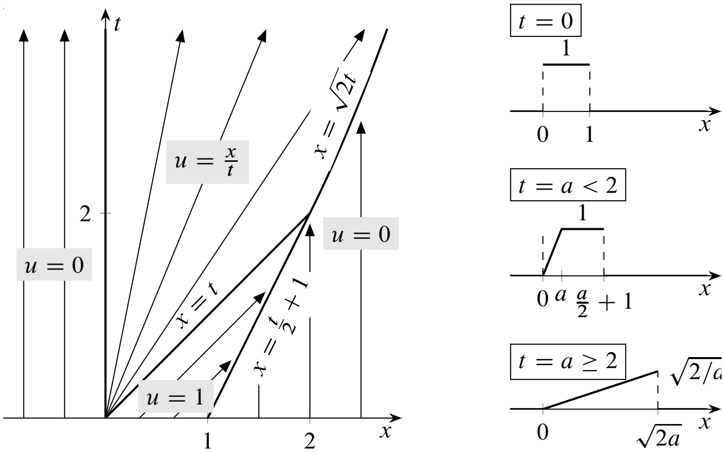

Fig. 3.6 Problem 3.2.2 b): characteristics and shock wave (*left*); solution at various times (*right*)

Problem 3.2.3 (Non-extendability). *Consider the Cauchy problem:*

$$\begin{cases} u_t + u^2 u_x = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = x & x \in \mathbb{R}. \end{cases}$$

a) *Check whether the family of characteristics admits an envelope.*

b) *Find an explicit formula for the solution and discuss whether it may be extended to the whole half plane $\{t > 0\}$.*

Solution. a) The PDE is written as conservation law with $q(u) = u^3/3$, $q'(u) = u^2$. Note how the initial datum $g(x) = x$ is *unbounded* when $x \to \pm\infty$. The characteristic from $(\xi, 0)$ has equation:

$$x = \xi + q'(g(\xi))t = \xi + \xi^2 t.$$

To establish whether this family, depending on ξ, admits an envelope, we must solve for x and t the system

$$\begin{cases} x = \xi + \xi^2 t \\ 0 = 1 + 2\xi t. \end{cases}$$

The second equation is just the first one differentiated with respect to ξ. The parameter ξ can be eliminated and we find that the envelope lies in the quadrant $x < 0$, $t > 0$ and coincides with the hyperbola $4xt = -1$ (Fig. 3.7).

b) The solution $u = u\,(x,t)$ is defined implicitly by

$$u = g\left(x - q'\,(u)\,t\right)$$

at least for small times. In our case, since $g\,(x) = x$, we find

$$u = x - u^2 t.$$

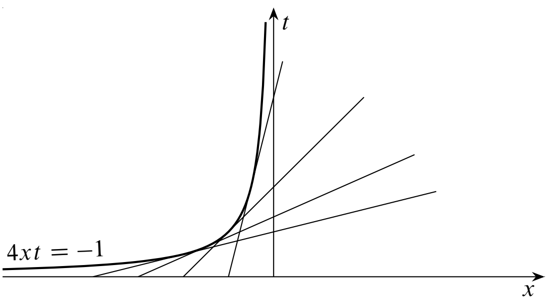

Fig. 3.7 Envelope of characteristics for Problem 3.2.3

Solving for u, we get

$$u^{\pm}(x,t) = \frac{-1 \pm \sqrt{1+4xt}}{2t}, \ \ x \geq -\frac{1}{4t}.$$

Let us determine $\lim_{t\to 0^+} u^{\pm}(x,t)$. For given x:

$$\lim_{t\to 0^+} u^{-}(x,t) = \lim_{t\to 0^+} \frac{-1 - \sqrt{1+4xt}}{2t} = -\infty,$$

while

$$\lim_{t\to 0^+} u^{+}(x,t) = \lim_{t\to 0^+} \frac{-1 + \sqrt{1+4xt}}{2t} = \lim_{t\to 0^+} \frac{4xt}{2t\left(1+\sqrt{1+4xt}\right)} = x.$$

Only u^+ satisfies the initial condition, and is therefore the unique solution, defined in the region $\{x \geq -1/4t\}$ and regular inside. This region is bounded above by the envelope of the characteristics, which becomes a barrier beyond which the characteristics do not carry initial data. Moreover, since the initial datum tends to $-\infty$ as $\xi \to -\infty$, and the characteristics tend to flatten horizontally, there is no coherent way to extend the definition of u beyond the envelope, in the quadrant $x < 0$, $t > 0$.

On the contrary, the formula

$$u(x,t) = \frac{-1 + \sqrt{1+4xt}}{2t}$$

defines the solution on $x \geq 0$, $t \geq 0$ as well.

Problem 3.2.4 (A traffic model, vehicle path)**.** *The following problem models what happens at a traffic light:*

$$\begin{cases} \rho_t + v_m\left(1 - \frac{2\rho}{\rho_m}\right)\rho_x = 0 & x \in \mathbb{R},\ t > 0 \\ \rho(x,0) = \begin{cases} \rho_m & x < 0 \\ 0 & x > 0, \end{cases} \end{cases}$$

where ρ is the density of cars, ρ_m the maximum density, v_m the maximum speed allowed. Determine the solution and calculate:

a) *The density of cars at the light for any $t > 0$.*

b) *The time taken by a car placed at $x_0 < 0$ at time $t = 0$ to get past the light.*

Solution. a) The equation is written as conservation law with

$$q(\rho) = \rho v(\rho) = v_m \rho \left(1 - \frac{\rho}{\rho_m}\right)$$

where $v(\rho)$ is the speed when the cars are in an area of density ρ. The characteristic through $(\xi, 0)$ is

$$x = v_m \left(1 - \frac{2\rho(\xi, 0)}{\rho_m}\right) t + \xi.$$

When $\xi < 0$ we find

$$x = -v_m t + \xi.$$

Thus in the region $x < -v_m t$ we have $\rho(x, t) = \rho_m$. When $\xi > 0$

$$x = v_m t + \xi$$

and if $x > v_m t$ we have $\rho(x, t) = 0$. In the sector $-v_m t \leq x \leq v_m t$ we can join the values ρ_m and 0 with a rarefaction wave centred at the origin. Setting

$$q'(\rho) = v_m \left(1 - \frac{2\rho}{\rho_m}\right) = y$$

we can find the inverse function

$$R(y) = (q')^{-1}(y) = \frac{\rho_m}{2}\left(1 - \frac{y}{v_m}\right),$$

and the rarefaction wave is

$$\rho(x, t) = R\left(\frac{x}{t}\right) = \frac{\rho_m}{2}\left(1 - \frac{x}{v_m t}\right).$$

To sum up, the solution is

$$\rho(x, t) = \begin{cases} \rho_m & x < -v_m t \\ \frac{\rho_m}{2}\left(1 - \frac{x}{v_m t}\right) & -v_m t \leq x \leq v_m t \\ 0 & x > v_m t. \end{cases}$$

Therefore the vehicle density at the traffic light is

$$\rho(0, t) = \frac{\rho_m}{2},$$

constant in time.

b) In the present model the speed of a vehicle at x at time t depends only on the density:

$$v(\rho) = v_m \left(1 - \frac{\rho}{\rho_m}\right).$$

Denote by $x = x(t)$ the law of motion of the car, with $x(0) = x_0 < 0$. Initially the car does not move, until time t_0, with $x_0 = -v_m t_0$; after t_0 the car moves within the region of the rarefaction wave as long as $x(t) < v_m t$, in particular before it reaches the traffic light; after that it moves with constant speed v_m. Therefore, after t_0 and before reaching

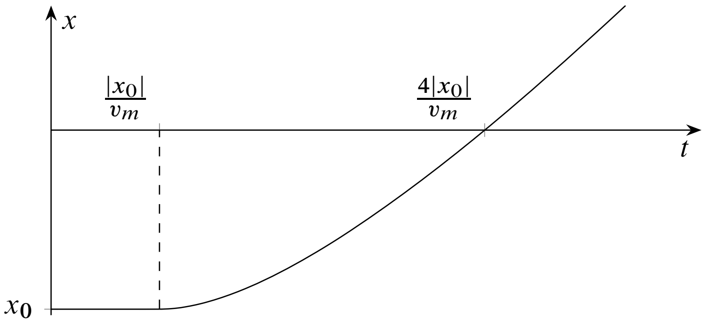

Fig. 3.8 Problem 3.2.4: path of the car starting from $x = x_0 < 0$ at time $t = 0$. The traffic light is reached at time $t = 4|x_0|/v_m$

the light, x solves the Cauchy problem

$$\begin{cases} x'(t) = v(\rho(x(t), t)) = \frac{v_m}{2}\left(1 + \frac{x(t)}{v_m t}\right) \\ x(t_0) = -v_m t_0. \end{cases}$$

Integrating the (linear) equation gives

$$x(t) = v_m(t - 2\sqrt{t_0 t}),$$

and hence $x(t) = 0$ for $t = 4t_0 = 4|x_0|/v_m$ (Fig. 3.8).

Problem 3.2.5 (Traffic model; normalised density)**.** *Let ρ be the vehicle density in the model of Problem 3.2.4. Normalise the density by setting $u(x,t) = \rho(x,t)/\rho_m$, so that $0 \le u \le 1$. Check that u solves*

$$u_t + v_m(1 - 2u)u_x = 0, \qquad x \in \mathbb{R},\ t > 0. \tag{3.8}$$

Determine the solution to (3.8) *with initial condition*

$$u(x,0) = g(x) = \begin{cases} 1/3 & x \le 0 \\ 1/3 + 5x/12 & 0 \le x \le 1 \\ 3/4 & x \ge 1. \end{cases}$$

Solution. The initial datum $g = g(x)$ is shown in Fig. 3.9, left.

Elementary computations show that u satisfies eq. (3.8). We have $q'(u) = v_m(1-2u)$ and hence $q(u) = v_m(u - u^2)$. The characteristic issued from $(\xi, 0)$ is

$$x = \xi + v_m(1 - 2g(\xi))t, \tag{3.9}$$

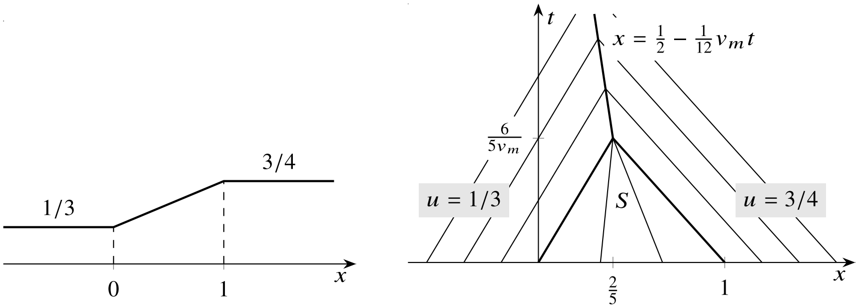

Fig. 3.9 Initial datum and characteristics for Problem 3.2.5

i.e.

$$x = \xi + \frac{1}{3} v_m t \quad \text{for} \quad \xi \le 0$$
$$x = \xi + \left(\frac{1}{3} - \frac{5}{6} \xi \right) v_m t \quad \text{for} \quad 0 \le \xi \le 1$$
$$x = \xi - \frac{1}{2} v_m t \quad \text{for} \quad \xi \ge 1.$$

We can see that the characteristics meet, creating a shock wave. The starting point of it is the point with smallest time coordinate, at which the characteristics intersect for $0 \le \xi \le 1$. In this case the characteristics form a pencil depending on the parameter ξ, and the pencil base point, where all characteristics with $0 \le \xi \le 1$ meet, is

$$(x_0, t_0) = \left(\frac{2}{5}, \frac{6}{5v_m} \right).$$

This is shown in Fig. 3.9, right.

The shock curve Γ, of equation, say, $x = s(t)$, is thus emanating from $(2/5, 6/(5v_m))$. On the right of Γ $u^+ = 3/4$, while on the left $u^- = 1/3$. The Rankine-Hugoniot condition gives

$$s'(t) = \frac{q(u^+) - q(u^-)}{u^+ - u^-} = -\frac{1}{12} v_m.$$

Since $s\left(6/(5v_m)\right) = 2/5$ we get the straight line

$$s(t) = \frac{1}{2} - \frac{1}{12} v_m t.$$

Thus we have found the (entropic) solution, for $t > t_0 = 6/(5v_m)$.

Suppose now $t < t_0$. To compute the solution in the region

$$S = \left\{ (x, t) : 0 \le t < \frac{6}{5v_m}, \ \frac{1}{3} v_m t \le x \le 1 - \frac{1}{2} v_m t \right\},$$

bounded by the characteristics from $\xi = 0$ and $\xi = 1$, we solve for ξ the characteristics equation. We get

$$\xi = \frac{6x - 2v_m t}{6 - 5v_m t}, \qquad 0 \le \xi \le 1,$$

from which, u being constant along characteristics,

$$u(x,t) = g(\xi) = \frac{1}{3} + \frac{5}{12}\frac{6x - 2v_m t}{6 - 5v_m t} = \frac{4 + 5x - 5v_m t}{2(6 - 5v_m t)} \qquad \text{in } S.$$

Another way to proceed would be to use the formula

$$u = g\left(x - v_m(1 - 2u)t\right)$$

which gives u in implicit form. Substituting the expression of g in the internal $0 < x < 1$ we find

$$u = \frac{1}{3} + \frac{5}{12}\left(x - (1 - 2u)\, v_m t\right).$$

Solving for u, we obtain the previous formula. In summary:

$$u(x,t) = \begin{cases} \dfrac{1}{3} & x < \min\left\{\dfrac{1}{3}v_m t, \dfrac{1}{2} - \dfrac{1}{12}v_m t\right\} \\ \dfrac{4 + 5x - 5v_m t}{2(6 - 5v_m t)} & \dfrac{1}{3}v_m t \le x \le 1 - \dfrac{1}{2}v_m t \\ \dfrac{3}{4} & x > \max\left\{1 - \dfrac{1}{2}v_m t, \dfrac{1}{2} - \dfrac{1}{12}v_m t\right\}. \end{cases}$$

* **Problem 3.2.6** (Traffic in a tunnel). *A realistic model for the velocity inside a very long tunnel is*

$$v(\rho) = \begin{cases} v_m & 0 \le \rho \le \rho_c \\ \lambda \log\left(\rho_m/\rho\right) & \rho_c \le \rho \le \rho_m \end{cases}$$

where ρ is the vehicles density and $\lambda = \frac{v_m}{\log(\rho_m/\rho_c)}$. Note v is continuous also at the point $\rho_c = \rho_m e^{-v_m/\lambda}$, which represents a critical density, below which drivers are free to cruise at the maximum speed allowed. Practical values are $\rho_c = 7$ cars/Km, $v_m = 90$ Km/h, $\rho_m = 110$ cars/Km, $v_m/\lambda = 2.75$.

Suppose the tunnel entrance is placed at $x = 0$, and that prior to the tunnel opening (at time $t = 0$) a queue has formed. The initial datum is

$$\rho(x,0) = g(x) = \begin{cases} \rho_m & x < 0 \\ 0 & x > 0. \end{cases}$$

a) *Determine the traffic density and velocity, and sketch the graphs of these functions.*

b) *Determine and sketch on the xt-plane the path of a car initially at $x = x_0 < 0$, then compute how long it takes it to enter the tunnel.*

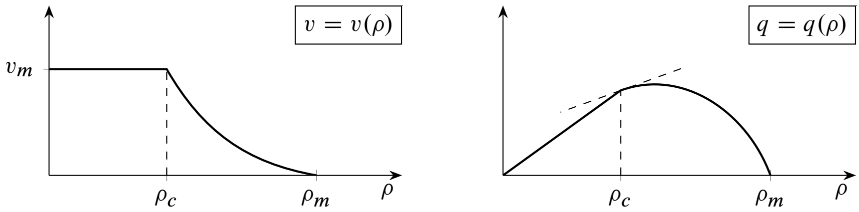

Fig. 3.10 Velocity and flux function for the traffic in a tunnel

Solution. a) By using the usual convective model the problem to solve reads

$$\begin{cases} \rho_t + q'(\rho)\rho_x = 0 & x \in \mathbb{R},\ t > 0 \\ \rho(x,0) = g\,(x) = \begin{cases} \rho_m & x < 0 \\ 0 & x > 0, \end{cases} \end{cases}$$

where

$$q(\rho) = \rho v(\rho)$$

and hence ($e^{-v_m/\lambda} = \rho_c/\rho_m$)

$$q'(\rho) = \begin{cases} v_m & 0 \le \rho < \rho_c \\ \lambda\,[\log\,(\rho_m/\rho) - 1] & \rho_c < \rho \le \rho_m. \end{cases}$$

The graphs of v and q in terms of the density ρ are shown in Fig. 3.10. Notice how q' jumps at $\rho = \rho_c$:

$$q'(\rho_c^-) = v_m \quad \text{and} \quad q'(\rho_c^+) = v_m - \lambda.$$

The characteristic from $(\xi, 0)$, i.e. the line $x = \xi + q'(g(\xi))t$, is

$$x = \xi - \lambda t \quad \text{for } \xi < 0, \qquad \text{and } x = \xi + v_m t \quad \text{for } \xi > 0.$$

Therefore we obtain immediately the solution is certain regions:

$$\rho(x,t) = \rho_m \quad \text{for} \quad x < -\lambda t.$$

It remains to find ρ in the sector

$$S = \{(x,t) : -\lambda t \le x \le v_m t\}.$$

For this we recall that q' is discontinuous at $\rho = \rho_c$:

$$q'(\rho_c^-) = v_m \quad \text{and} \quad q'(\rho_c^+) = v_m - \lambda.$$

This suggests writing $S = S_1 \cup S_2$, with

$$S_1 = \{(x,t) : -\lambda t \le x \le (v_m - \lambda)t\},$$

where $\rho_c < \rho \le \rho_m$, and

$$S_2 = \{(x,t) : (v_m - \lambda)t \le x \le v_m t\},$$

where $0 < \rho \le \rho_c$.

In S_1 we proceed as follows. When $\rho_c < \rho \le \rho_m$ we have

$$q''(\rho) = -\lambda/\rho < 0,$$

so that q is strictly concave. Since the initial datum is decreasing we seek a solution in the form of a rarefaction wave, centred at the origin, that attains continuously the value ρ_m on the line $x = -\lambda t$. The wave is given by $\rho(x,t) = R(x/t)$ where $R = (q')^{-1}$. To find R we solve for ρ the equation

$$q'(\rho) = \lambda \left[\log\left(\frac{\rho_m}{\rho}\right) - 1 \right] = y.$$

This gives

$$R(y) = \rho_m \exp\left(-1 - \frac{y}{\lambda}\right)$$

and hence we find

$$\rho(x,t) = \rho_m \exp\left(-1 - \frac{x}{\lambda t}\right)$$

in the region

$$-\lambda \le \frac{x}{t} \le v_m - \lambda.$$

Notice that $\rho = \rho_c$ on the straight line $x = (v_m - \lambda)t$. In S_2, where $\rho \le \rho_c$, we have $q'(\rho) = v_m$. Thus, q is not strictly convex of concave, and there is no possibility to construct a solution via a rarefaction wave. Changing perspective, we construct the entropic solution by solving the equation in the "quadrant" $\{x > (v_m - \lambda)t, t > 0\}$, prescribing the values $\rho = \rho_c$ on $x = (v_m - \lambda)t$ and 0 on $t = 0$. We have already found $\rho = 0$ when $x > v_m t$ (Fig. 3.11). In the sector S_2 ρ is constant along the characteristics

$$x = v_m t + k,$$

that carry the value $\rho = \rho_c = c^{-v_m/\lambda}$.

To sum up,

$$\rho(x,t) = \begin{cases} \rho_m & x \ge -\lambda t \\ \rho_m e^{-(1+x/(\lambda t))} & -\lambda t \le x \le (v_m - \lambda)t \\ \rho_m e^{-v_m/\lambda} & (v_m - \lambda)t \le x < v_m t \\ 0 & x > v_m t. \end{cases}$$

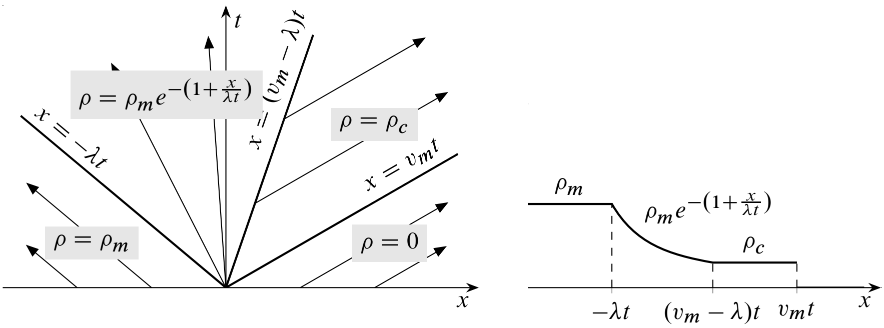

Fig. 3.11 Problem 3.2.6 b): characteristics (*left*); solution at time t (*right*)

In Fig. 3.11 (on the right) we see the density behaviour at a given time: it decreases from its maximum value (at zero speed) to reach the critical density (maximum speed). Note that the solution is discontinuous only along $x = v_m t$. This type of discontinuity is called *contact discontinuity*

b) Consider the vehicle initially placed at $x_0 < 0$. We want to describe its trajectory on the xt-plane. Observe first that the car will not move until time $t_0 = |x_0|/\lambda$ (Fig. 3.12). At that moment it enters the region S where the velocity is

$$v(\rho(x,t)) = \lambda \log (e^{1+x/\lambda t}) = \lambda + \frac{x}{t}.$$

If $x = x(t)$ denotes the vehicle path, we have

$$\begin{cases} x'(t) = \lambda + \dfrac{x(t)}{t} \\ x(t_0) = x_0. \end{cases}$$

The equation is linear, and integrating gives

$$x(t) = \lambda t \left(\log \frac{\lambda t}{|x_0|} - 1 \right).$$

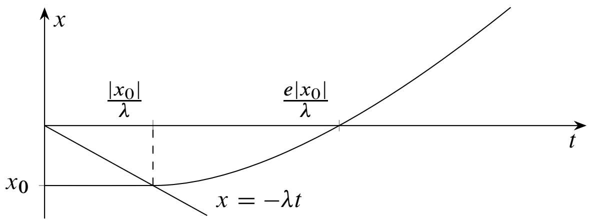

Fig. 3.12 Trajectory of a car in Problem 3.2.6

The car enters the tunnel at the time T such that $x(T) = 0$. The required lapse is then

$$T = \frac{e|x_0|}{\lambda}.$$

Problem 3.2.7 (Shock formation in a traffic model)**.** *Let u, $0 \le u \le 1$ be the normalised density that solves the following traffic problem:*

$$\begin{cases} u_t + v_m(1-2u)u_x = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R}. \end{cases}$$

Assume $g \in C^1(\mathbb{R})$, that g' has a unique maximum point x_1 and that

$$g'(x_1) = \max_{\mathbb{R}} g'(x) > 0.$$

a) *Study the qualitative behaviour of the characteristics and deduce that the solution develops a shock.*

b) *Verify that for small times u is defined implicitly by*

$$u = g(x - v_m t(1-2u)).$$

Deduce that the first instant t_s at which the shock forms (critical time) is the first time for which

$$1 - 2v_m t g'(x - v_m t(1-2u)) = 0.$$

c) *Show that the initial point (x_s, t_s) of the shock belongs to the characteristic Γ_{x_1} emanating from $(x_1, 0)$, and*

$$t_s = \frac{1}{2v_m g'(x_1)}.$$

In case $v_m = 1$, $g(x) = \frac{3}{4}\left[\frac{2}{\pi}\arctan x + 1\right]$, analyse numerically the graph of u at various times and interpret the results.

Solution. a) The characteristic Γ_ξ from the point $(\xi, 0)$ has equation

$$x = \xi + (1 - 2g(\xi))v_m t. \tag{3.10}$$

Under the given hypotheses g is strictly increasing in a neighbourhood of x_1, thus the characteristics starting in the neighbourhood meet, generating a shock.

b) On Γ_ξ we know that $u(x,t) = g(\xi)$, and from (3.10) we find

$$\xi = x - (1 - 2g(\xi))v_m t.$$

Hence

$$u(x,t) = g(x - (1 - 2u(x,t))v_m t).$$

Now we verify when the equation

$$h(x,t,u) = u - g(x - (1 - 2u)v_m t) = 0, \tag{3.11}$$

really defines an implicit function u of x and t. The sufficient conditions provided by the implicit function theorem are the following:

1. h is C^1, true because g is C^1.
2. (3.11) can be solved at some point, in fact

   $$h(x,0,g(x)) = g(x) - g(x) = 0$$

   at all points on the x-axis.
3. Finally,

   $$h_u(x,t,u) = 1 - 2v_m t g'(x - (1 - 2u)v_m t) \neq 0. \tag{3.12}$$

   As g' is either negative, or bounded when positive, equation (3.12) is always true for small times.

As long as (3.12) holds, by the implicit function theorem, equation (3.11) defines *a unique function* $u = u(x,t)$ in $C^1(\mathbb{R})$. This solution cannot develop (shock) discontinuities. On the other hand the same inverse function theorem gives a formula for u_x:

$$u_x(x,t) = -\frac{h_x(x,t,u)}{h_u(x,t,u)} = \frac{g'(x - (1 - 2u)v_m t)}{1 - 2v_m t g'(x - (1 - 2u)v_m t)}. \tag{3.13}$$

So if $t_s > 0$ is the first instant for which h_u is zero (for some $x = x_s$), necessarily

$$u_x(x,t) \to \infty \quad \text{as} \quad (x,t) \to (x_s, t_s)$$

since the numerator of (3.13) does not vanish at (x_s, t_s) (it goes to $(2v_m t_s)^{-1}$). Therefore t_s must be the *critical time*, i.e. when the shock starts.

c) Let us find t_s. Consider the characteristic Γ_ξ. For any $(x,t) \in \Gamma_\xi$

$$x - (1 - 2u(x,t)v_m t) = \xi,$$

so that (3.12) fails when

$$h_u(x,t,u(x,t)) = 1 - 2v_m t g'(\xi) = 0, \qquad \text{i.e.} \qquad t = \frac{1}{2v_m g'(\xi)}.$$

From part a) we know that t_s is the smallest (positive) t for which the previous equation

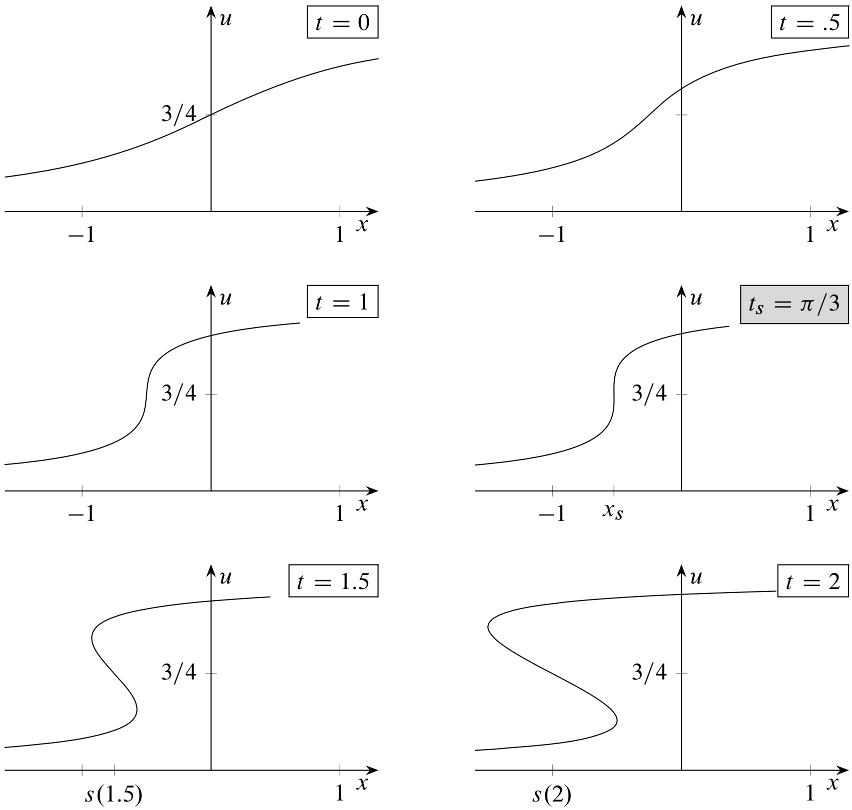

Fig. 3.13 Curve implicitly defined by the equation $u - \frac{3}{4}\left[\frac{2}{\pi}\arctan(x - (1-2u)v_m t) + 1\right] = 0$ at various times. The abscissas $s\,(1.5)$ and $s\,(2)$ denote the shock positions obtained by the *equal-area rule* [18, Chap. 4, Sect. 4]

holds. By assumption $g'(x_1) \geq g'(\xi)$ for any ξ, therefore (x_s, t_s) belongs to Γ_{x_1}, and moreover

$$t_s = \frac{1}{2v_m g'(x_1)},$$

$$x_s = x_1 + \frac{1}{2g'(x_1)}\,(1 - 2g\,(x_1)).$$

In case

$$g\,(x) = \frac{3}{4}\left[\frac{2}{\pi}\arctan x + 1\right],$$

the curve defined implicitly by (3.11) evolves as in Fig. 3.13.

Problem 3.2.8 (Envelope of characteristics and shock formation)**.** *Consider the Cauchy problem:*

$$\begin{cases} u_t + q(u)_x = 0 & x \in \mathbb{R},\, t > 0 \\ u(x,0) = g(x) & x \in \mathbb{R}. \end{cases}$$

Suppose $q \in C^2(\mathbb{R})$, $q'' < 0$ *and* $g \in C^1(\mathbb{R})$, *with*

$$g(x) = \begin{cases} g(x) = 0 & x \leq 0 \\ g'(x) > 0 & 0 < x < 1 \\ g(x) = 1 & x \geq 1. \end{cases}$$

a) *Show that the family of characteristics*

$$x = q'(u)t + \xi = q'(g(\xi))t + \xi, \quad \xi \in (0,1)$$

admit an envelope.

b) *Determine the point* (x_s, t_s) *of the envelope with smallest time coordinate, and show that this is the point where the shock originates from. Recover the result of Problem 3.2.7.*

c) *Show that* (x_s, t_s) *is a singular point for the envelope, meaning that the tangent vector at* (x_s, t_s) *is zero (assume* q *and* g *are regular enough.)*

Solution. a) Figure 3.14 shows the envelope of the characteristics

$$x = q'(g(\xi))t + \xi,$$

$\xi \in (0,1)$, in two particular cases.

To check the existence of an envelope, we consider the system

$$\begin{cases} x = q'(g(\xi))t + \xi \\ 0 = q''(g(\xi))g'(\xi)t + 1 = 0 \end{cases}$$

where the second equation is the derivative of the first with respect to ξ. As $q'' < 0$ and $g' > 0$ for $\xi \in (0,1)$, we have $q''(g(\xi))g'(\xi) < 0$ and the envelope is given by the parametric equations

$$x_{inv}(\xi) = \xi - \frac{q'(g(\xi))}{q''(g(\xi))g'(\xi)}, \qquad t_{inv}(\xi) = -\frac{1}{q''(g(\xi))g'(\xi)},$$

obtained by solving for ξ the system in the variables x and t.

b) The shock forms in correspondence to the point (x_s, t_s) of the envelope with smallest time coordinate, because that is the first point where two characteristics meet. As

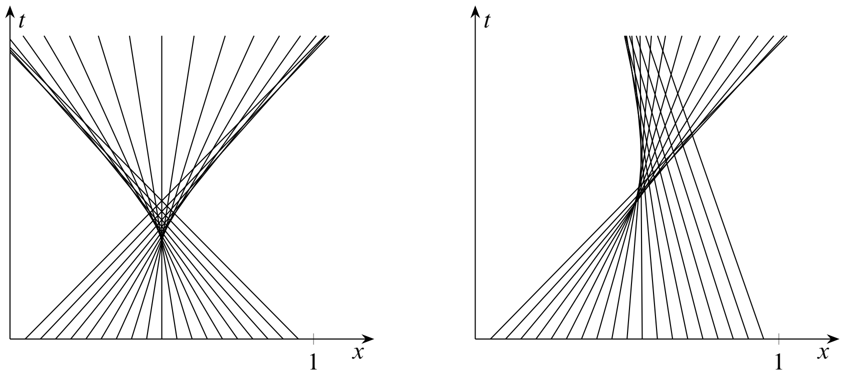

Fig. 3.14 Problem 3.2.8, envelope of characteristics with a cusp, in the case $q(u) = u - u^2$ and: $g(\xi) = (1 - \cos(\pi\xi))/2$ (*left*); $g(\xi) = 5\xi^2 e^{-2\xi}$ (*right*)

$g'(0) = g'(1) = 0$ and $q''(g(\xi))g'(\xi) < 0$ for $0 < \xi < 1$, the function

$$z(\xi) = -q''(g(\xi))g'(\xi)$$

has a positive maximum at some $\xi_M \in (0, 1)$. From the second equation

$$t_s = \min_{\xi \in (0,1)} \frac{1}{z(\xi)} = \frac{1}{z(\xi_M)}.$$

For Problem 3.2.7 (page 168) we have

$$\begin{aligned} q(u) &= v_m(u - u^2) \\ q'(u) &= v_m(1 - 2u) \\ q'' &= -2v_m < 0. \end{aligned}$$

In a neighbourhood of x_1, the positive maximum of g', we have $g' > 0$, so that the characteristics starting there have an envelope. From

$$z(\xi) = -q''(g(\xi))g'(\xi) = 2v_m g'(\xi)$$

we deduce $\xi_M = x_1$, and the solution has a shock starting at time

$$t_s = \frac{1}{2v_m g'(x_1)},$$

confirming the result in Problem 3.2.7.

c) To check that (x_s, t_s), origin of the shock and "origin " of the envelope, is singular, we need to show that

$$\frac{dx}{d\xi} \quad \text{and} \quad \frac{dt}{d\xi}$$

vanish at $\xi = \xi_M$. Assume q has three derivatives and g two. Then

$$\frac{dx}{d\xi} = \frac{-q'(g(\xi))}{z^2(\xi)} z'(\xi) = -\frac{dt}{d\xi}.$$

Since z has a (positive) maximum at $\xi = \xi_M$, we have

$$z'(\xi_M) = 0$$

and the derivatives vanish. The shock starts at the singular points of the envelope (cusps).

Problem 3.2.9 (Non-homogeneous conservation laws). *Consider the problem*

$$\begin{cases} u_t + q(u)_x = f(u, x, t) & x \in \mathbb{R},\ t > 0 \\ u(x, 0) = g(x) & x \in \mathbb{R}. \end{cases}$$

a) *Let $x = x(t)$ be a characteristic for the homogeneous equation ($f = 0$) and set*

$$z(t) = u(x(t), t).$$

Which Cauchy problems do $x(t)$ and $z(t)$ solve?

b) *Supposing f and g bounded, define weak solutions for the problem.*

c) *Deduce the Rankine-Hugoniot conditions for a shock curve $x = s(t)$.*

Solution. a) Set $z = u(x(t), t)$. We have

$$z'(t) = u_t(x(t), t) + u_x(x(t), t)\, x'(t),$$

and by the conservation law

$$u_t(x(t), t) + u_x(x(t), t)\, q'(z(t)) = f(z(t), x(t), t).$$

The characteristic from $(\xi, 0)$ solves

$$x'(t) = q'(z(t)), \qquad x(0) = \xi$$

while z satisfies the Cauchy problem

$$z'(t) = f(z(t), x(t), t), \qquad z(0) = g(\xi),$$

which uniquely determines u along the characteristic, under the usual smoothness assumption on f.

b) We mimic the procedure for homogeneous equations. Let us multiply the equation by a test function $\varphi \in C_0^1(D)$, which is C^1 with compact support K contained in

$$D = \{(x,t) \in \mathbb{R}^2 : t \geq 0\},$$

and integrate over D, obtaining

$$\int_D (q(u)_x + u_t)\varphi \, dxdt = \int_D f(u,x,t)\varphi \, dxdt.$$

The integrals are finite because the support of φ is bounded. The notion of weak solution is found essentially by integrating by parts. Interpreting $q(u)_x + u_t$ as the divergence of the vector field $(q(u), u)$ we may apply Green's theorem:

$$\begin{aligned}\int_K (q(u)_x + u_t)\varphi \, dxdt = &-\int_K [q(u)\varphi_x + u\varphi_t] \, dxdt - \int_{\mathbb{R}} u(x,0)\varphi(x,0) \, dx \\ &+ \int_{\partial K \cap \{t>0\}} [q(u)\varphi n_1 + u\varphi n_2] \, ds\end{aligned}$$

where (n_1, n_2) is the outer unit normal to ∂K and ds the infinitesimal length element. The last integral is zero (φ is continuous, hence null on $\partial K \cap \{t > 0\}$). So we define *weak solution* a *locally bounded function* u such that

$$\int_D [q(u)\varphi_x + u\varphi_t] \, dxdt + \int_{\mathbb{R}} u\varphi \, dx + \int_D f(u,x,t)\varphi \, dxdt \qquad \text{for any } \varphi \in C_0^1(D).$$

As for the homogeneous situation, a weak solution which is C^1 in $\mathbb{R} \times \{t \geq 0\}$ is a classical solution as well.

c) Suppose a curve Γ, $x = s(t)$, splits an open set $V \subset \{t > 0\}$ into two disjoint subdomains

$$V^- = \{(x,t) : x < s(t)\} \text{ and } V^+ = \{(x,t) \in V : x > s(t)\}.$$

Assume that u is a weak solution, which is C^1 in the closures $\overline{V^-}$ and $\overline{V^+}$ separately, with a jump discontinuity along Γ. In particular, this implies

$$u_t + q(u)_x = f(u,x,t)$$

in V^- and V^+. If $(x,t) \in \Gamma$, write $u^+(x,t)$ for the limit of u when approaching Γ on the right, $u^-(x,t)$ for the limit from the left. Pick a test function φ, with support in V that intersects Γ. From part b)

$$-\int_{V^-} [q(u)\varphi_x + u\varphi_t] \, dxdt - \int_{V^+} [q(u)\varphi_x + u\varphi_t] \, dxdt = \int_V f(u,x,t)\varphi \, dxdt.$$

Since u is regular on V^-, V^+, we can invoke Green's theorem on the integrals on the left. Recalling that $\varphi = 0$ on $\partial V^\pm \setminus \Gamma$:

$$-\int_{V^\pm} [q(u)\varphi_x + u\varphi_t]\, dxdt = \int_{V^\pm} (q(u)_x + u_t)\varphi\, dxdt \mp \int_\Gamma \left[q(u^\pm)n_1 + u^\pm n_2\right] \varphi\, ds$$
$$= \int_V f(u,x,t)\varphi\, dxdt \mp \int_\Gamma \left[q(u^\pm)n_1 + u^\pm n_2\right] \varphi\, ds$$

where (n_1, n_2) is the outward unit normal to Γ with respect to V^+ (we used the fact that $u_t + q(u)_x = f(u,x,t)$ on $V^\pm$). Substituting into the definition of weak solution, we find

$$\int_\Gamma \left[(q(u^+) - q(u^-))n_1 + (u^+ - u^-)n_2\right] \varphi\, ds = 0.$$

Since φ is arbitrary, and the jumps $q(u^+) - q(u^-)$, $u^+ - u^-$ are continuous along Γ, we deduce

$$(q(u^+) - q(u^-))n_1 + (u^+ - u^-)n_2 = 0 \ \text{ along } \Gamma.$$

On the other hand, if $s \in C^1$ we have

$$(n_1, n_2) = \frac{1}{\sqrt{1 + s'(t)^2}}(-1, s'(t)),$$

so

$$s' = \frac{q(u^+) - q(u^-)}{u^+ - u^-}.$$

The Rankine-Hugoniot condition coincides with the one for the non-homogeneous case.

Problem 3.2.10 (Fluid in a porous tube)**.** *Consider a cylindrical tube, infinitely long, placed along the x-axis, containing a fluid moving to the right. Denote by $\rho = \rho(x,t)$ the fluid density, and suppose that the speed at each point depends on the density by $v = \frac{1}{2}\rho$. Assume, further, that the tube wall is made of a porous material that leaks at the rate $H = k\rho^2$ (mass per unit length, per unit time).*

a) *Deduce that if ρ is smooth, it satisfies*

$$\rho_t + \left(\frac{1}{2}\rho^2\right)_x = -k\rho^2.$$

b) *Compute the solution with $\rho(x,0) = 1$ and the corresponding characteristics.*

Solution. a) We are dealing with a transport model. The leaking rate H leads to write the conservation law

$$\rho_t + q(\rho)_x = -H = -k\rho^2.$$

Due to the convective nature of motion the flow is described by

$$q(\rho) = v(\rho)\rho = \frac{1}{2}\rho^2,$$

yielding the required equation.

b) From Problem 3.2.9 a), indicating by $x = x(t)$ the characteristic from $(0, \xi)$ and setting $z = \rho(x(t), t)$, we have

$$\begin{cases} x'(t) = z(t) & x(0) = \xi \\ z'(t) = -kz^2(t) & z(0) = 1. \end{cases}$$

From the second equation we get

$$z(t) = 1/(kt + 1);$$

as the latter does not depend on ξ, we may write

$$\rho(x, t) = \frac{1}{1 + kt}.$$

The characteristics are parallel logarithms:

$$x(t) = \frac{1}{k} \ln(1 + kt) + \xi.$$

** **Problem 3.2.11** (A saturation problem)**.** *Suppose a certain substance is poured into a semi-infinite container (aligned along the axis $x \geq 0$) with a solvent; the substance concentration $u = u(x, t)$ is governed by the equation*

$$u_x + (1 + f'(u))u_t = 0 \qquad \text{with } u(x, 0) = 0,\ x > 0,\ t > 0.$$

At the entrance ($x = 0$) the substance is maintained at the concentration

$$g(t) = \begin{cases} \dfrac{c_0}{\alpha} t & 0 \leq t \leq \alpha \\ c_0 & t \geq \alpha. \end{cases} \qquad (c_0, \alpha > 0)$$

Study the evolution of u if one takes

$$f(u) = \frac{\gamma u}{1 + u} \qquad (\text{Langmuir isothermal } (\gamma > 0))$$

and discuss the case where α tends to zero[a]*.*

[a] See [28, Vol. 1, Chap. 6.4], also for the physical-chemical interpretation of the model.

Solution. First of all let us remark that, compared to the conservation laws seen so far, the roles of x and t are exchanged. We have

$$q'(u) = 1 + f'(u) = 1 + \frac{\gamma}{(1 + u)^2},$$

and since q is concave and g increasing, we expect a shock. The characteristics are the lines

$$t = (1 + f'(u))x + k = \left(1 + \frac{\gamma}{(1 + u)^2}\right) x + k \qquad k \in \mathbb{R}.$$