

Przykład 1 Zastanówmy się jak wygląda rotacja wektora w układzie sferycznym. $M = \mathbb{R}^3$.

$$v \xrightarrow{\sharp} \Lambda^1(M) \xrightarrow{d} \Lambda^2(M) \xrightarrow{*} \Lambda^1(M) \rightarrow T_p M \rightarrow \prod_i$$

$$\text{rot} v = (*dv^\sharp)^\flat$$

$$\text{na początek dostajemy w smsie } \begin{bmatrix} A^r \\ A^\theta \\ A^\varphi \end{bmatrix}_{i_r, i_\theta, i_\varphi} = v = A^r \frac{\partial}{\partial r} + A^\theta \frac{1}{r} \frac{\partial}{\partial \theta} + A^\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

chcemy sobie zrobić jednoformę, która jest podniesionym wektorkiem: $\alpha = v^\sharp =$

$$= g_{rr} A^r dr + g_{\theta\theta} \frac{1}{r} A^\theta d\theta + g_{\varphi\varphi} \frac{1}{r \sin \theta} A^\varphi d\varphi = A^r dr + r A^\theta d\theta + r \sin \theta A^\varphi d\varphi$$

$$d\alpha = (A^r_{,\theta} - (r A^\theta)_{,r}) d\theta \wedge dr + (A^r_{,\varphi} - (r \sin \theta A^\varphi)_{,r}) d\varphi \wedge dr + ((r A^\theta)_{,\varphi} - (r \sin \theta A^\varphi)_{,\theta}) d\varphi \wedge d\theta$$

$$*(dr \wedge d\theta) = \sin \theta d\varphi, \quad *(d\theta \wedge d\varphi) = \frac{1}{r^2} dr, \quad *(d\varphi \wedge dr) = \frac{1}{\sin \theta} d\theta$$

$$*d\alpha = ((r \sin \theta A^\varphi)_{,\theta} - (r A^\theta)_{,\varphi}) \frac{1}{r^2 \sin \theta} dr + (A^r_{,\varphi} - (r \sin \theta A^\varphi)_{,r}) \frac{1}{\sin \theta} d\theta +$$

$$+ ((r A^\theta)_{,r} - A^r_{,\theta}) \sin \theta d\varphi.$$

notacja: $\square, \heartsuit = \frac{\partial \square}{\partial \heartsuit}$. Zostały nam jeszcze tylko dwie operacje.

$$(*d\alpha)^\flat = ((r \sin \theta A^\varphi)_{,\theta} - (r A^\theta)_{,\varphi}) \cdot 1 \cdot \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} + (A^r_{,\varphi} - (r \sin \theta A^\varphi)_{,r}) \frac{1}{\sin \theta} \frac{1}{r^2} \frac{\partial}{\partial \theta} +$$

$$+ \left((r A^\theta)_{,r} - A^r_{,\theta} \right) \sin \theta \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \varphi}.$$

Czyli

$$\text{rot} \begin{bmatrix} A^r \\ A^\theta \\ A^\varphi \end{bmatrix} = \begin{bmatrix} \frac{1}{r^2 \sin \theta} ((r \sin \theta A^\varphi)_{,\theta} - (r A^\theta)_{,\varphi}) \\ \frac{1}{r \sin \theta} (A^r_{,\varphi} - (r \sin \theta A^\varphi)_{,r}) \\ \frac{1}{r} ((r A^\theta)_{,r} - A^r_{,\theta}) \end{bmatrix}.$$

Przykład 2 To może teraz dywergencja rzutem na taśmę.

$$\prod = v \xrightarrow{\sharp} \Lambda^1(M) \xrightarrow{*} \Lambda^2(M) \xrightarrow{d} \Lambda^3(M) \xrightarrow{\flat} \Lambda^0(M)$$

$$\text{div}(v) = *(d(*v^\sharp))$$

$$\begin{bmatrix} A^r \\ A^\theta \\ A^\varphi \end{bmatrix} = v, \alpha = v^\sharp$$

$$\alpha = A^r dr + r A^\theta d\theta + A^\varphi r \sin \theta d\varphi$$

$$*dr = r^2 \sin \theta d\theta \wedge d\varphi$$

$$*d\theta = \sin \theta d\varphi \wedge dr$$

$$*d\varphi = \frac{1}{\sin \theta} dr \wedge d\theta$$

$$*\alpha = (A^r r^2 \sin \theta) d\theta \wedge d\varphi + (r \sin \theta A^\theta) d\varphi \wedge dr + (r A^\varphi) dr \wedge d\theta$$

$$d(*\alpha) = ((A^r r^2 \sin \theta)_{,r} + (r \sin \theta A^\theta)_{,\theta} + (r A^\varphi)_{,\varphi}) dr \wedge d\theta \wedge d\varphi$$

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$$\operatorname{div} \begin{bmatrix} A^r \\ A^\theta \\ A^\varphi \end{bmatrix} = \frac{1}{r^2 \sin \theta} \left((A^r r^2 \sin \theta)_{,r} + (r \sin \theta A^\theta)_{,\theta} + (r A^\varphi)_{,\varphi} \right).$$

$$\begin{aligned} f(r, \theta, \varphi) &\xrightarrow{d} \Lambda^1(M) \xrightarrow{*} \Lambda^2(M) \xrightarrow{d} \Lambda^3(M) \xrightarrow{*} \Lambda^0(M) \\ \alpha = df &= \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \varphi} d\varphi \\ *\alpha &= \left(\frac{\partial f}{\partial r} r^2 \sin \theta \right) d\theta \wedge d\varphi + \left(\frac{\partial f}{\partial \theta} \sin \theta \right) d\varphi \wedge dr + \left(\frac{\partial f}{\partial \varphi} \frac{1}{\sin \theta} \right) dr \wedge d\theta \\ d(*\alpha) &= \left(\left(r^2 \sin \theta \frac{\partial f}{\partial r} \right)_{,r} + \sin \theta \frac{\partial f}{\partial \theta} + \left(\frac{1}{\sin \theta} \frac{\partial f}{\partial \varphi} \right)_{,\varphi} \right) dr \wedge d\theta \wedge d\varphi \\ *(d(*\alpha)) &= \frac{1}{r^2 \sin \theta} \left(\left(r^2 \sin \theta \frac{\partial f}{\partial r} \right)_{,r} + \left(\sin \theta \frac{\partial f}{\partial \theta} \right)_{,\theta} + \left(\frac{1}{\sin \theta} \frac{\partial f}{\partial \varphi} \right)_{,\varphi} \right). \end{aligned}$$

Przykład 3 $M = \mathbb{R}^3, f \in \Lambda^0(M)$.

$$\begin{aligned} ddf &= 0 \\ ddf &= d \left(\left((df)^\flat \right)^\sharp \right) \implies \operatorname{rot}(\operatorname{grad}(f)) = 0. \end{aligned}$$

Niech teraz $v \in \Lambda^1(M)$.

$$\begin{aligned} d \left(* \left((dV^\sharp)^\flat \right)^\sharp \right) &= d(*(*(*d(v^\sharp)))) = dd(v^\sharp) = 0 \\ \operatorname{div}(\operatorname{rot}(V)) &= 0. \end{aligned}$$

Weźmy sobie jakąś funkcję: $f : (t, x, y, z) \rightarrow \mathbb{R}$.

Zobaczmy jak $*d(*df)$ wygląda w $\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$.

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

$$*(dx^{i_1} \wedge \dots \wedge dx^{i_L}) = \frac{\sqrt{g}}{(n-L)!} g^{i_1 j_1} \dots g^{i_L j_L} \in_{j_1 \dots j_k k_1 \dots k_{n-L}} dx^{k_1} \wedge \dots \wedge dx^{k_{n-L}}$$

$$*(dx^0) = \frac{\sqrt{-(-1)}}{(4-1)!} g^{00} \in_{0k_1 k_2 k_3} dx^{k_1} \wedge dx^{k_2} \wedge dx^{k_3}, i, k = 0, \dots, 3$$

$$*(dx^0) = -\frac{1}{3!} 3! dx^1 \wedge dx^2 \wedge dx^3$$

$$*(dt) = -dx \wedge dy \wedge dz$$

$$*(dx^1) = \frac{\sqrt{-(-1)}}{(4-1)!} g^{11} \in_{1k_1 k_2 k_3} dx^{k_1} \wedge dx^{k_2} \wedge dx^{k_3}$$

$$*(dx) = 3! \frac{1}{3!} dy \wedge dt \wedge dz$$

$$*(dy) = dt \wedge dx \wedge dz$$

$$*(dz) = dx \wedge dt \wedge dy$$

$$*df = -\frac{\partial f}{\partial t} dx \wedge dy \wedge dz + \frac{\partial f}{\partial x} dy \wedge dt \wedge dz + \frac{\partial f}{\partial y} dt \wedge dx \wedge dz + \frac{\partial f}{\partial z} dx \wedge dt \wedge dy$$

$$d*df = \left(-\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dt \wedge dx \wedge dy \wedge dz.$$

Na koniec:

Mamy dwuformę pola elektromagnetycznego:

$$F = -E_x dt \wedge dx + E_y dt \wedge dy - E_z dt \wedge dz + B_x dy \wedge dz + B_y dz \wedge dy + B_z dy \wedge dx.$$

$dF = 0$ to jest pierwsza część równań Maxwella

$$\begin{bmatrix} \rho \\ \rho v^x \\ \rho v^y \\ \rho v^z \end{bmatrix} = \begin{bmatrix} \rho \\ j^x \\ j^y \\ j^z \end{bmatrix}$$

$$j = -gdt + j^x dx + j^y dy + j^z dz$$

$$d(*F) = *j \text{ a to druga.}$$