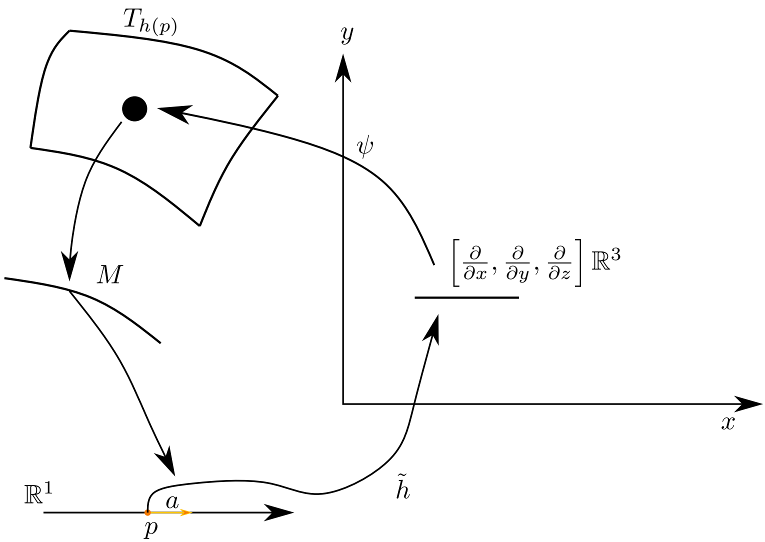
Przykład 1. *(na pchnięcie wektora)*

Niech $M = \mathbb{R}^1, N = \mathbb{R}^3, h(t) = \begin{bmatrix} f(t) \\ g(t) \\ r(t) \end{bmatrix}$

Niech $p \in \mathbb{R}^1$, *niech* $v \in T_pM, v = a\frac{\partial}{\partial t}$. $v = [\sigma], \tilde{\sigma}(t) = at + p$, $\sigma(c) = p$, $\frac{d\tilde{\sigma}(t)}{dt}|_{t=0} = a$.

$$h_x\sigma = \begin{bmatrix} f(at+p) \\ g(at+p) \\ r(at+p) \end{bmatrix}, h_xv = [h_x\sigma], \frac{d}{dt}(\tilde{h}_x\sigma)|_{t=0}.$$

$$h_xv = \begin{bmatrix} af'(p) \\ ag'(p) \\ ar'(p) \end{bmatrix} = af'(p)\frac{\partial}{\partial x} + ag'(p)\frac{\partial}{\partial y} + ar'(p)\frac{\partial}{\partial z}.$$

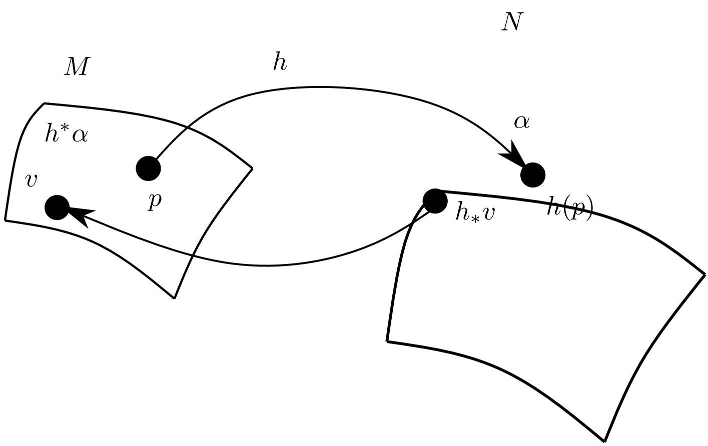


Definicja 1. *Niech M, N - rozmaitości, $h : M \to N$ i niech $p \in M, \alpha \in T^*_{h(p)}N$. Cofnięciem formy α w odwzorowaniu h nazywamy formę $h^*\alpha \in T_pM$, taką, że $\langle h^*\alpha, v\rangle = \langle \alpha, hv\rangle \underset{v\in T_pM}{\forall}$ i ciągła. Jeżeli $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Lambda^1(N)$ i $v_1, \ldots, v_k \in T_p(M)$, to*

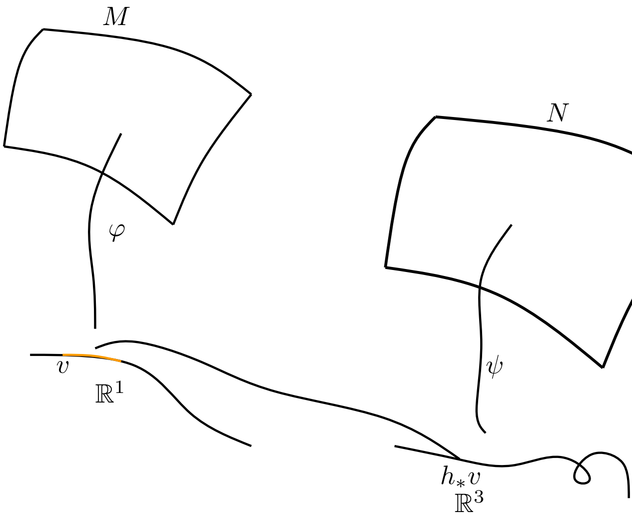
$$h^*(\alpha_1 \wedge \ldots \wedge \alpha_k), v_1, \ldots, v_k \overset{def}{=} \begin{bmatrix} \langle h^*\alpha_1, v_1\rangle & \langle h^*\alpha_2, v_1\rangle & \ldots & \langle h^*\alpha_k, v_1\rangle \\ \vdots & & & \\ \langle h^*\alpha_k, v_k\rangle & \langle h^*\alpha_k, v_k\rangle & \ldots & \langle h^*\alpha_k, v_k\rangle \end{bmatrix}.$$

Czyli

$$h^*(\alpha_1 \wedge \ldots \wedge \alpha_k) = (h^*\alpha_1) \wedge (h^*\alpha_2) \wedge \ldots \wedge h^*(\alpha_k).$$



Rysunek 1: $\langle h^*\alpha, v\rangle \overset{\text{def}}{=} \langle \alpha, h_*v\rangle$



Przykład 2. *(wstępny)*
Niech $\alpha = 3(x^2+y^2)dx - 2xdy + 2z^2dz, \alpha \in \Lambda^1(N)$ *(jednoformy nad* N*,* $\dim N = 3$*, chociaż można dać więcej jak się chce).*
$h(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \\ t \end{bmatrix}$*. Czym jest* $h^*\alpha$*?*

$$\langle h^*\alpha, v\rangle = \langle \alpha, h_x v\rangle .$$

Niech $v \in T_pM$ *i* $v = a\frac{\partial}{\partial t}$*. Zatem* $h_x v = a\cos(p)\frac{\partial}{\partial x} - a\sin(p)\frac{\partial}{\partial y} + a \cdot 1\frac{\partial}{\partial t}$*.*

$$\langle \alpha, h_* v\rangle = \left\langle 3\left(\sin^2(t) + \cos^2(t)\right)dx - 2\left(\sin(t)\right)dy + 2\left(t^2\right)dz, h_x v\right\rangle =$$
$$= \left\langle 3dx - 2\sin(t)dy + 2t'dz, a\cos(t)\frac{\partial}{\partial x} - a\sin(t)\frac{\partial}{\partial y} + a \cdot 1\frac{\partial}{\partial z}\right\rangle_{t=p}$$
$$= 3a\cos(t) + 2a\sin^2(t) + at^2|_{t=p} =$$
$$= \left\langle \left(3\cos(t)dt + 2a\sin^2(t) + at^2\right)|_{t=p}, a\frac{\partial}{\partial t}\right\rangle =$$
czyli $h^*\alpha = \left(3\cos(t) + 2\sin^2(t) + t^2\right)dt$

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Na skróty!

$$\begin{aligned} x &= \sin(t) & dx &= \cos(t)dt \\ y &= \cos(t) & dy &= -\sin(t)dt \\ z &= t & dz &= dt. \end{aligned}$$

Zatem

$$\begin{aligned} h^*\alpha &= 3\left(\sin^2(t)+\cos^2(t)\right)\cos(t)dt - 2\sin(t)\left(-\sin t dt\right) + 2t^2 dt \\ &= \left(3\cos(t) + 2\sin^2(t) + 2t^2\right) dt. \end{aligned}$$

Przykład 3. *Niech* $M = \mathbb{R}^4, N = \mathbb{R}^4$.

$$\begin{aligned} & \gamma = \frac{1}{\sqrt{1-v^2}}, \\ & c = 1 \\ h: \quad & t = \gamma(t' - vx') \\ & x = \gamma(x' - vt') \\ & y = y' \\ & z = z'. \end{aligned}$$

Czyli

$$\begin{aligned} dt &= \gamma(dt' - vdx') \\ dx &= \gamma(dx' - vdt') \\ dy &= dy' \\ dz &= dz'. \end{aligned}$$

Chcemy cofnąć naszą formę. Na fizyce nie używamy słowa cofnięte.

$$\begin{aligned} F' &= -E_x\left(\gamma\left(dt' - vdx'\right)\right) \wedge \gamma\left(dx' - vdt'\right) - E_y\gamma\left(dt' - vdx'\right) \wedge dy' = \\ &= -E_x\gamma^2\left(1-v^2\right)dt' \wedge dx' - E_y\gamma dt' \wedge dy' + E_y\gamma v dx' \wedge dy' = \\ &= -E_x\frac{1}{1-v^2}\left(1-v^2\right)dt' \wedge dx' - E_y\gamma dt' \wedge dy' + \gamma v E_x dx' \wedge dy' \\ F' &= -E'_x dt' \wedge dx' - E'_y dt' \wedge dy' + B'_z dx' \wedge dy' \end{aligned}$$

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Czyli

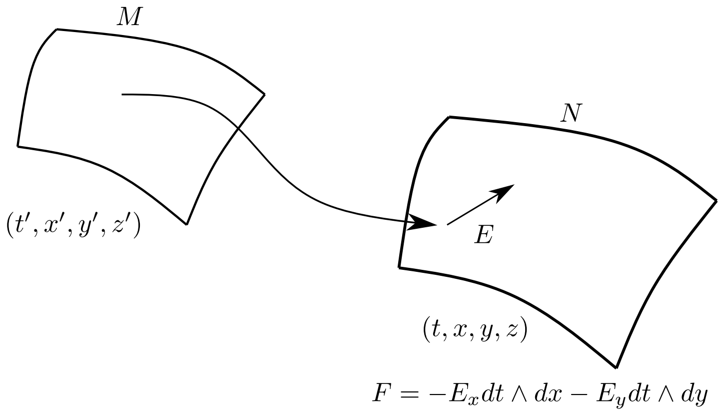
$$\begin{aligned} E'_x &= E_x \\ E'_y &= \gamma E_y \\ B'_z &= \gamma v E_y. \end{aligned}$$

Obserwacja: Niech $\alpha \in \Lambda^1(N)$, $\dim N = k$, niech M - rozmaitość, $\dim M = n$ i $h : M \to N$. Wówczas

$$h^* f \in \Lambda^0(M).$$

Oraz

$$d(h^* f) = h^*(df).$$



Dowód. Skoro $f \in \Lambda^0(N)$, to $f(x^1, x^2, \ldots, x^k)$,
$df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \ldots + \frac{\partial f}{\partial x^k} x^k$.

$$\langle h^*(df), v \rangle = \langle df, h_x v \rangle, v \in T^p M.$$

Niech $V \in T_p M$.

$$\tilde{h}(t_1, \ldots, t_n) = \begin{bmatrix} h_1(t_1, \ldots, t_n) \\ \vdots \\ h_k(t_1, \ldots, t_n) \end{bmatrix}.$$

Jeżeli $v = a_1 \frac{\partial}{\partial t^1} + a_2 \frac{\partial}{\partial t^2} + \ldots + a_n \frac{\partial}{\partial t_n}$, to $h_* v = \left(\begin{bmatrix} h' \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right)_{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}}$.

$$h_x v = \left(\begin{bmatrix} \frac{\partial h_1}{\partial t^1} & \cdots & \frac{\partial h_k}{\partial t^k} \\ \vdots & & \\ \frac{\partial h_k}{\partial t^1} & \cdots & \frac{\partial h_k}{\partial t^k} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right)_{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}} = \left(\frac{\partial h_1}{\partial t^1} a_1 + \ldots + \frac{\partial h_1}{\partial t^n} a_n \right) \frac{\partial}{\partial x^1} +$$
$$+ \ldots + \left(\frac{\partial h_k}{\partial t^1} a_1 + \ldots + \frac{\partial h_k}{\partial t^n} a_n \right) \frac{\partial}{\partial x^k}.$$

Dalej

$$\langle df, h_* v\rangle = \frac{\partial f_1}{\partial x^1}\left(\frac{\partial h_1}{\partial t^1}a_1 + \ldots + \frac{\partial h_1}{\partial t^n}a_n\right) + \ldots + \frac{\partial f}{\partial x^k}\left(\frac{\partial h_k}{\partial t^1}a_n + \ldots + \frac{\partial h_k}{\partial t^n}a_n\right) =$$

$$= \left\langle df(h_1(t_1,\ldots,t_n), h_2(t_1,\ldots,t_n),\ldots,h_k(t_1,\ldots,t_n)), a_1\frac{\partial}{\partial t^1} + \ldots + a_n\frac{\partial}{\partial t^n}\right\rangle$$

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$\square$