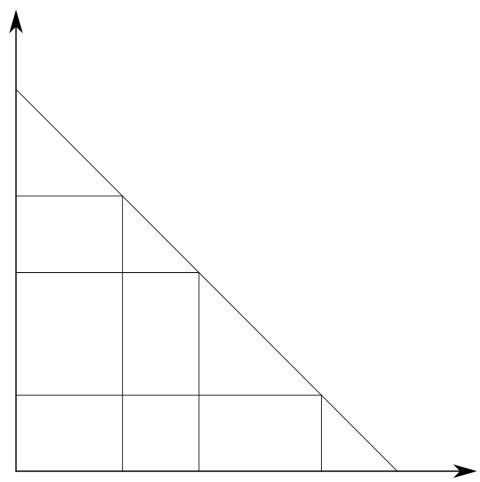


Rysunek 1: (a)

Zabawki działające dzięki wnioskom z Tw. wyżej

Definicja 1. *Funkcje uwikłane*

$$x + y = 1 \quad \text{(a)}.$$

$$x^2 + y^2 = 1 \quad \text{(b)}.$$

$$H(x, y) = \sin x e^{xy} + \operatorname{tg} y - x = 0.$$

Przykład 1. *Równanie gazowe*

$$H(p, V, T) = 0, H : \mathbb{R}^3 \to \mathbb{R}^1.$$

$$p(V, T) = 0, \mathbb{R}^2 \to \mathbb{R}^1.$$

$$V(p, T) = 0, \mathbb{R}^2 \to \mathbb{R}^1.$$

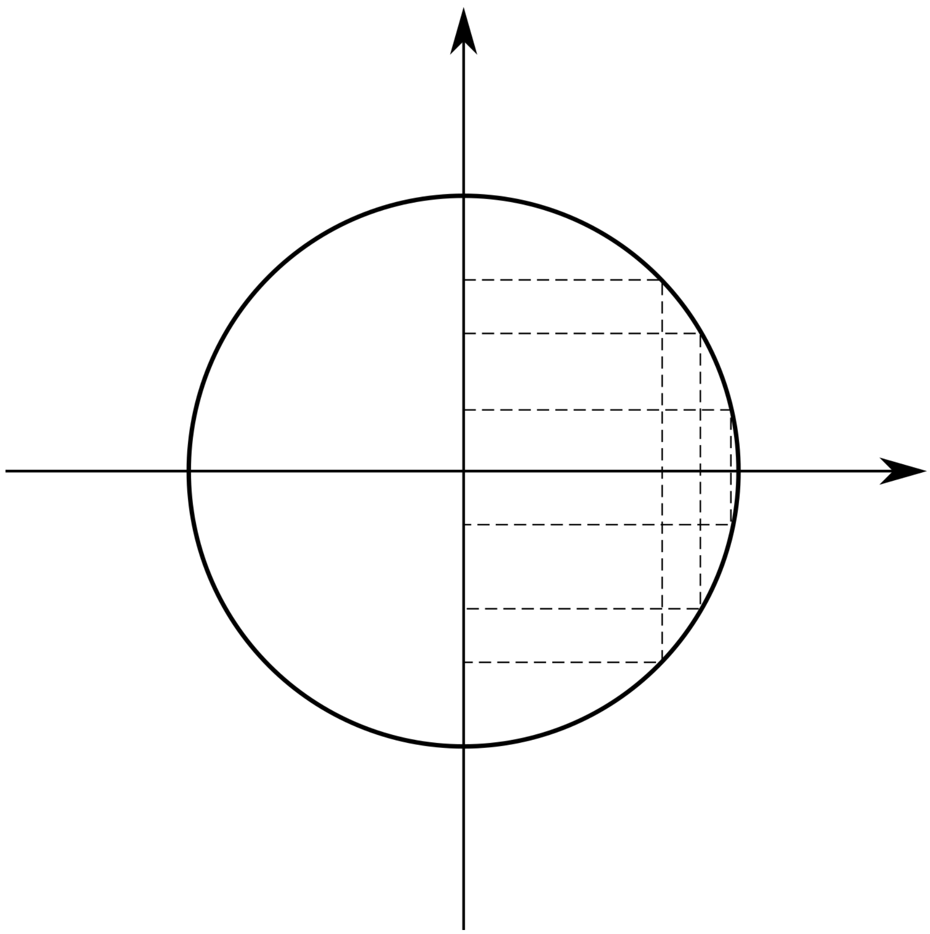
$$T(p, V) = 0, \mathbb{R}^2 \to \mathbb{R}^1.$$

istnienie przedziałów, w których funkcja uwikłana zadaje inne funkcje

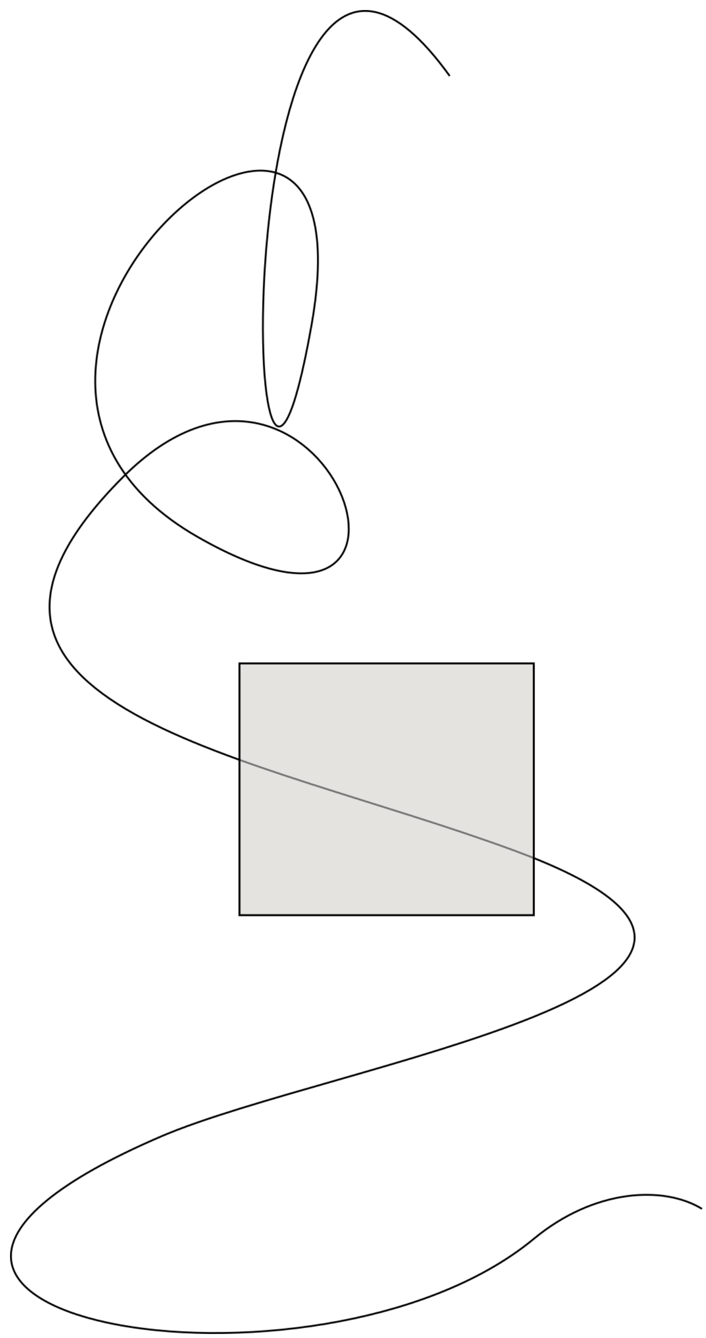
Przykład 2.

$$H(x, y) : U \subset \mathbb{R}^2 \to \mathbb{R}^1.$$

Pytanie 1. *Czy istnieje $y(x) : H(x, y(x)) = 0$, dla $x \in V$?*



Rysunek 2: (b)



Rysunek 3: (c)

$$\frac{dH}{dx}(x,y(x)) = \frac{d}{dx}(H(x,y)\circ g(x)).$$

$$H' = \left[\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\right].$$

$$g(x) : \mathbb{R}^1 \to \mathbb{R}^2, g(x) = \begin{bmatrix} x \\ y(x) \end{bmatrix}, g'(x) = \begin{bmatrix} 1 \\ y'(x) \end{bmatrix}.$$

$$H'(x,y)g'(x) = 0 \implies \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial y}{\partial x} = 0 \implies \frac{\partial y}{\partial x} = -\frac{\frac{\partial H}{\partial x}}{\frac{\partial H}{\partial y}}.$$

Więc

$$\frac{\partial y}{\partial x} = \frac{-\cos y + ye^{xy} - 1}{xe^y + \frac{1}{\cos^2 y}}.$$

Przykład 3.

$$H(x_1,x_2,x_3,x_4,x_5) = \begin{bmatrix} 2e^{x_1} + x_2x_3 - 4x_3 + 3 \\ x^2\cos x_1 - 6x_1 + 2x_3 - x_5, H : \mathbb{R}^5 \to \mathbb{R}^3 \end{bmatrix}.$$

$$H(x_1,\dots,x_5) = 0 \text{ może zadać funkcję } g : \mathbb{R}^3 \to \mathbb{R}^2.$$

$$x_4(x_1,x_2,x_3), x_5(x_1,x_2,x_3).$$

$$g(x_1,g_2,g_3) = \begin{bmatrix} g_1(x_1,x_2,x_3) \\ g_2(x_1,x_2,x_3 \end{bmatrix}.$$

Obserwacja 1. $H(0,1,3,2,7) = 0$

$$H : \mathbb{R}^5 \to \mathbb{R}^2, H(x_1,x_2,y_1,y_2,y_3) = 0.$$

$$H(x_1,x_2,y_1,y_2,y_3) = \begin{bmatrix} H_1(x_1,x_2,y_1,y_2,y_3) \\ H_2(x_1,x_2,y_1,y_2,y_2) \end{bmatrix}.$$

Pytanie 2. *Czy* $H(x_1,x_2,y_1,y_2,y_3) = 0$ *zadaje nam*

$$g_1(y_1,y_2,y_3).$$

$$g_2(y_1,y_2,y_3)?$$

czyli $g(y_1,y_2,y_3) = \begin{bmatrix} g_1(y_1,y_2,y_3) \\ g_2(y_1,y_2,y_3) \end{bmatrix}, g : \mathbb{R}^3 \to \mathbb{R}^2$

$$H_1(g_1(y_1,y_2,y_3), g_2(y_1,y_2,y_3), y_1,y_2,y_3) = 0.$$

$$H_2(g_1(y_1,y_2,y_3), g_2(y_1,y_2,y_3), y_1,y_2,y_3) = 0.$$

Szukamy g'.

$$g' = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \frac{\partial g_3}{\partial y_3} \end{bmatrix}.$$

$$\frac{\partial H_1}{\partial x_1}\frac{\partial g_1}{\partial y_2} + \frac{\partial H_1}{\partial x_2}\frac{\partial g_2}{\partial y_1} + \frac{\partial H_1}{\partial y_1} = 0.$$

$$\frac{\partial H_1}{\partial x_1}\frac{\partial g_1}{\partial y_3} + \frac{\partial H_1}{\partial x_2}\frac{\partial g_2}{\partial y_1} + \frac{\partial H_1}{\partial y_1} = 0.$$

$$\frac{\partial H_1}{\partial x_1}\frac{\partial g_1}{\partial y_3} + \frac{\partial H_1}{\partial x_2}\frac{\partial g_2}{\partial y_3} + \frac{\partial H_1}{\partial y_3} = 0.$$

$$\frac{\partial H_2}{\partial x_1}\frac{\partial g_1}{\partial y_1} + \frac{\partial H_2}{\partial x_2}\frac{\partial g_2}{\partial y_1} + \frac{\partial H_2}{\partial y_1} = 0.$$

$$\frac{\partial H_2}{\partial x_1}\frac{\partial g_1}{\partial y_2} + \frac{\partial H_2}{\partial x_2}\frac{\partial g_2}{\partial y_2} + \frac{\partial H_2}{\partial y_2} = 0.$$

$$\frac{\partial H_2}{\partial x_1}\frac{\partial g_1}{\partial y_3} + \frac{\partial H_2}{\partial x_2}\frac{\partial g_2}{\partial y_3} + \frac{\partial H_2}{\partial y_3} = 0.$$

napięcie rośnie (6 równań oho)

$$\underbrace{\begin{bmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x)2} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} \end{bmatrix}}_{H'_x} \underbrace{\begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \frac{\partial g_2}{\partial y_3} \end{bmatrix}}_{g'} = -\underbrace{\begin{bmatrix} \frac{\partial H_1}{\partial y_1} & \frac{\partial H_1}{\partial y_2} & \frac{\partial H_1}{\partial y_3} \\ \frac{\partial H_2}{\partial y_1} & \frac{\partial H_2}{\partial y_2} & \frac{\partial H_2}{\partial p_3} \end{bmatrix}}_{H'_y}.$$

$$H'_x g' = -H'_y \implies g' = -(H'_x)^{-1} H'_y.$$

Twierdzenie 1. *(o funkcji uwikłanej)*
Niech $H : E \subset \mathbb{R}^{n+m} \to \mathbb{R}^m, H \in \mathcal{C}^1$ *na* E. $(x_0, y_0) \in E, H(x_0, y_0) = 0, (x_0, y_0) = (x_0^1, \ldots, x_0^n, y_0^1, \ldots, y_0^m), H$ *- odwracalna.*
Wówczas istnieje $U \subset E$ *takie, że* $(x_0, y_0) \in U, \underset{W \subset \mathbb{R}^n}{\exists}$, *że* $x_0 \in W, \underset{x \in W}{\forall} \underset{y}{\exists!} H(x, y) = 0, (x, y) \in U$.
Jeżeli $y = \varphi(x)$, *to* $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ *i* $\varphi \in \mathcal{C}^1$ *na* W. $\varphi'(x) = -(H'_y)^{-1} H'_x$

Dowód 1. *Oznaczenia:*

$$H(x^1, \ldots, x^n, y^1, \ldots, y^m) = \begin{bmatrix} H^1(x^1, \ldots, x^n, y^1, \ldots, y^m) \\ \vdots \\ H^2(x^1, \ldots, x^n, y^1, \ldots, y^m) \end{bmatrix}.$$

$$H'_y = \begin{bmatrix} \frac{\partial H^1}{\partial y^1} & \cdots & \frac{\partial H^1}{\partial y^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial H^m}{\partial y^1} & \cdots & \frac{\partial H^m}{\partial y^n} \end{bmatrix}, H'_x = \begin{bmatrix} \frac{\partial H^1}{\partial x^1} & \cdots & \frac{\partial H^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial H^m}{\partial x^1} & \cdots & \frac{\partial H^m}{\partial x^n} \end{bmatrix}.$$

Wprowadźmy funkcję $F : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$

$$F(x^1, \ldots, x^n, y^1, \ldots, y^m) = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \\ H^1(x^1, \ldots, x^n, y^1, \ldots, y^m) \\ \vdots \\ H^m(x^1, \ldots, x^n, y^1, \ldots, y^m) \end{bmatrix}.$$

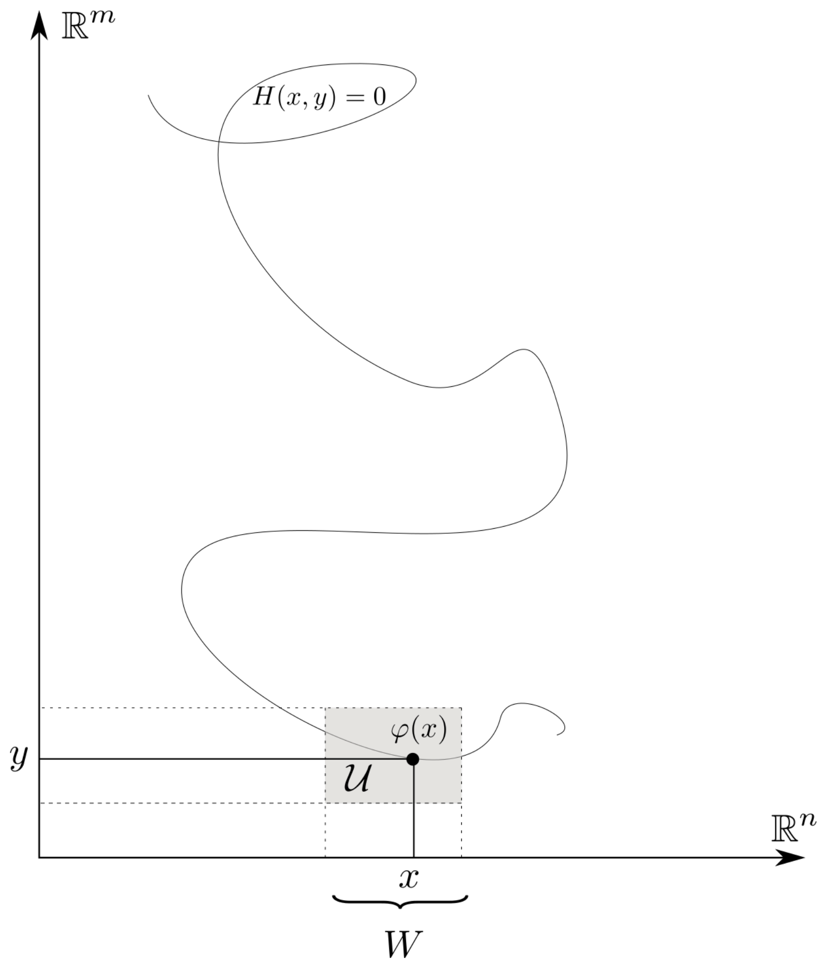
Jakie własności ma F*?*

$$F(x_0, y_0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

Ale

$$F' = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ & H'_x & & H'_y \end{bmatrix}, \det F' = \det H'_y.$$

Jeżeli $H'_y(x_0, y_0)$ - odwracalna, to $F'(x_0, y_0)$ - też. Oznacza to (na podstawie tw. o lokalnej odwra-



Rysunek 4

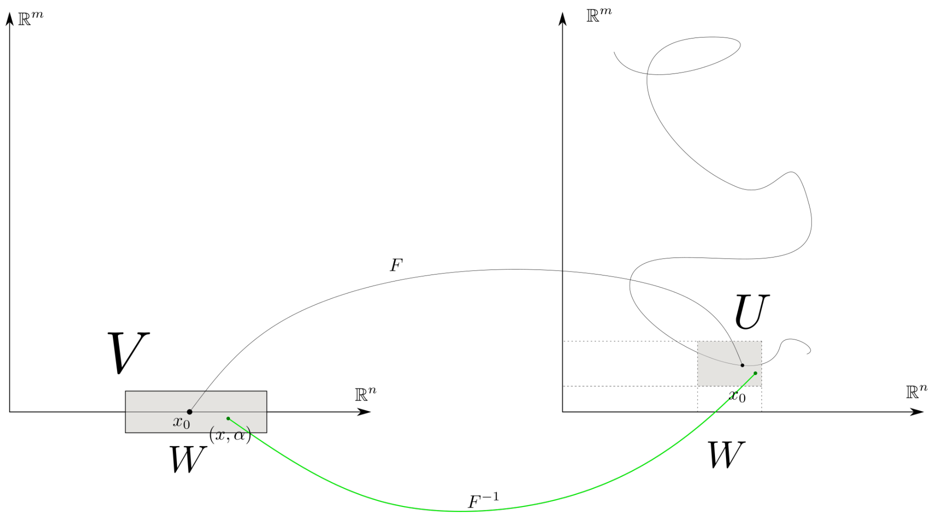
calności), że

$$\underset{U \subset \mathbb{R}^{n+m}}{\exists}, (x_0, y_0) \in U, \underset{V \subset \mathbb{R}^{n+m}}{\exists}, (x_0, 0) \in V.,$$

że F jest bijekcją między U i V oraz $\exists F^{-1} : V \to U, F^{-1}$ - różniczkowalna taka, że

$$F^{-1}(x, \alpha) = (a(x, \alpha), b(x, \alpha)), x, \alpha \in V.,$$

gdzie $a(x, \alpha) : \mathbb{R}^{m+n} \to \mathbb{R}^n, \quad b(x, \alpha) : \mathbb{R}^{m+n} \to \mathbb{R}^m$



Rysunek 5