

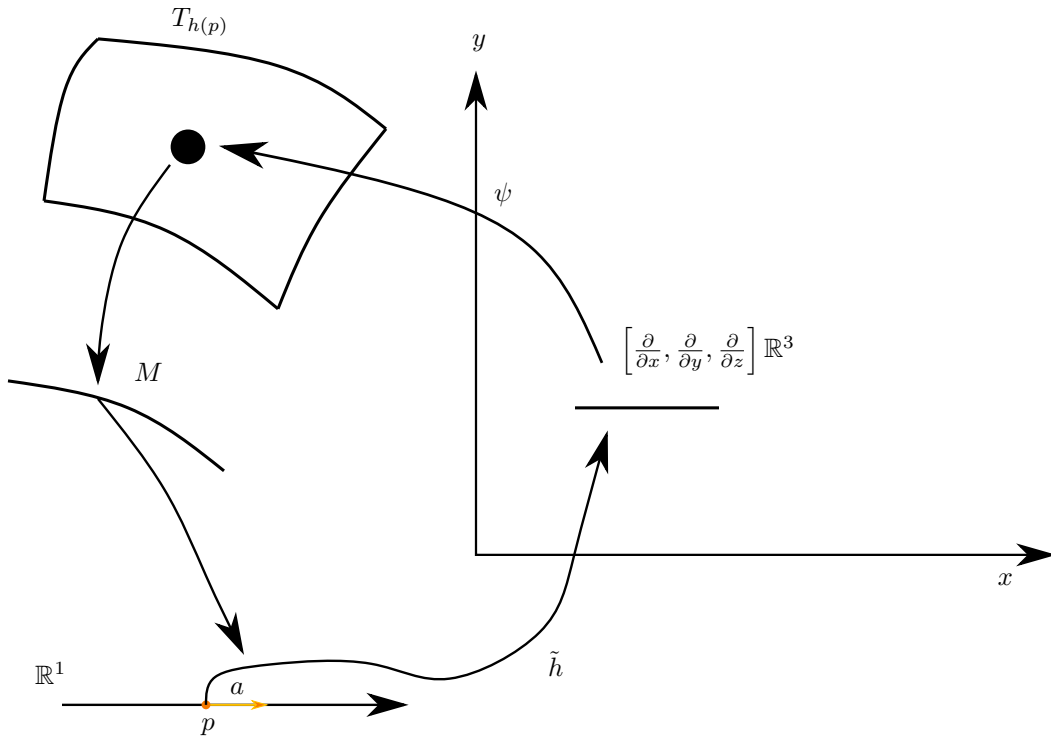
**Przykład 1** (na pchnięcie wektora)

Niech  $M = \mathbb{R}^1, N = \mathbb{R}^3, h(t) = \begin{bmatrix} f(t) \\ g(t) \\ r(t) \end{bmatrix}$

Niech  $p \in \mathbb{R}^1$ , niech  $v \in T_p M, v = a \frac{\partial}{\partial t}$ .  $v = [\sigma], \tilde{\sigma}(t) = at + p, \sigma(c) = p, \frac{d\tilde{\sigma}(t)}{dt}|_{t=0} = a$ .

$$h_x \sigma = \begin{bmatrix} f(at + p) \\ g(at + p) \\ r(at + p) \end{bmatrix}, h_x v = [h_x \sigma], \frac{d}{dt}(\tilde{h}_x \sigma)|_{t=0}.$$

$$h_x v = \begin{bmatrix} af'(p) \\ ag'(p) \\ ar'(p) \end{bmatrix} = af'(p) \frac{\partial}{\partial x} + ag'(p) \frac{\partial}{\partial y} + ar'(p) \frac{\partial}{\partial z}.$$



**Definicja 1** Niech  $M, N$  - rozmaitości,  $h : M \rightarrow N$  i niech  $p \in M, \alpha \in T_{h(p)}^* N$ .

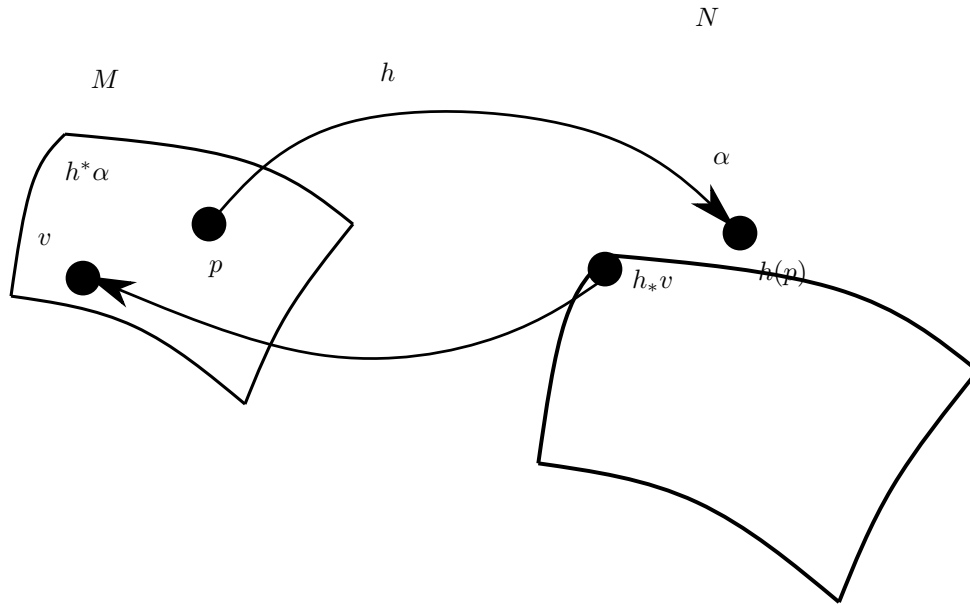
Cofnięciem formy  $\alpha$  w odwzorowaniu  $h$  nazywamy formę  $h^* \alpha \in T_p^* M$ , taką, że  $\langle h^* \alpha, v \rangle =$

$$\langle \alpha, hv \rangle_{v \in T_p M} \stackrel{\forall}{=} i \text{ caaa. Jeżeli } \alpha_1, \alpha_2, \dots, \alpha_k \in \Lambda^1(N) \text{ i } v_1, \dots, v_k \in T_p(M), \text{ to}$$

$$h^*(\alpha_1 \wedge \dots \wedge \alpha_k), v_1, \dots, v_k \stackrel{\text{def}}{=} \begin{bmatrix} \langle h^*\alpha_1, v_1 \rangle & \langle h^*\alpha_2, v_1 \rangle & \dots & \langle h^*\alpha_k, v_1 \rangle \\ \vdots & & & \\ \langle h^*\alpha_k, v_k \rangle & \langle h^*\alpha_k, v_k \rangle & \dots & \langle h^*\alpha_k, v_k \rangle \end{bmatrix}.$$

Czyli

$$h^*(\alpha_1 \wedge \dots \wedge \alpha_k) = (h^*\alpha_1) \wedge (h^*\alpha_2) \wedge \dots \wedge h^*(\alpha_k).$$

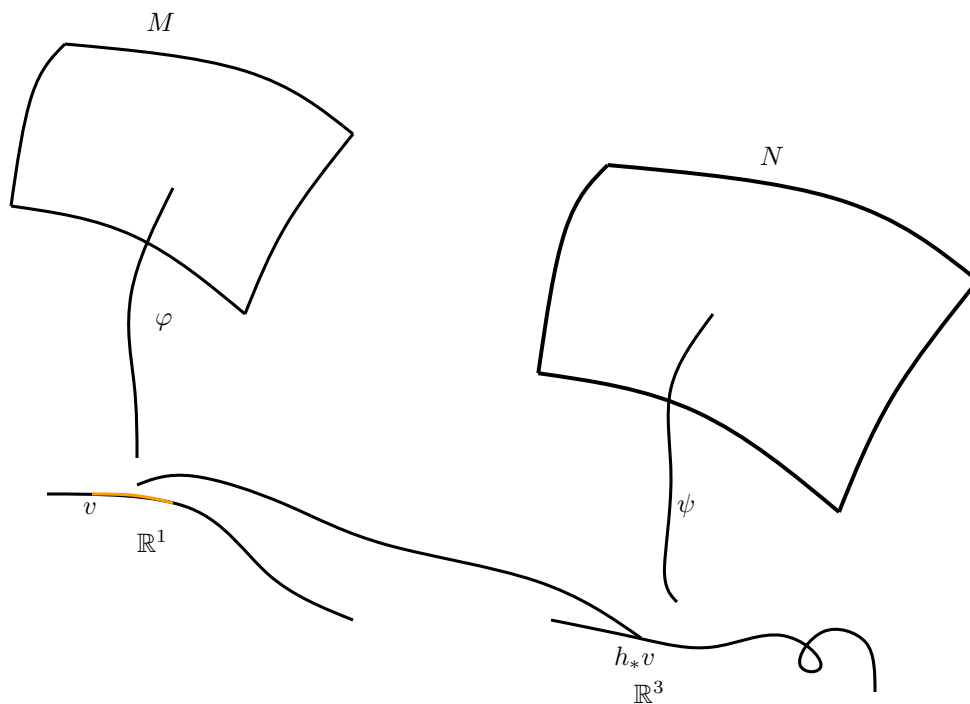


Rysunek 1:  $\langle h^*\alpha, v \rangle \stackrel{\text{def}}{=} \langle \alpha, h_*v \rangle$

**Przykład 2** (wstępny)

Niech  $\alpha = 3(x^2 + y^2)dx - 2xdy + 2z^2dz, \alpha \in \Lambda^1(N)$  (jednoformy nad  $N$ ,  $\dim N = 3$ , chociaż można dać więcej jak się chce).

$$h(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \\ t \end{bmatrix}. \text{ Czym jest } h^*\alpha?$$



$$\langle h^* \alpha, v \rangle = \langle \alpha, h_* v \rangle.$$

Niech  $v \in T_p M$  i  $v = a \frac{\partial}{\partial t}$ . Zatem  $h_* v = a \cos(p) \frac{\partial}{\partial x} - a \sin(p) \frac{\partial}{\partial y} + a \cdot 1 \frac{\partial}{\partial t}$ .

$$\begin{aligned} \langle \alpha, h_* v \rangle &= \langle 3 (\sin^2(t) + \cos^2(t)) dx - 2 (\sin(t)) dy + 2 (t^2) dz, h_* v \rangle = \\ &= \left\langle 3 dx - 2 \sin(t) dy + 2 t' dz, a \cos(t) \frac{\partial}{\partial x} - a \sin(t) \frac{\partial}{\partial y} + a \cdot 1 \frac{\partial}{\partial z} \right\rangle_{t=p} \\ &= 3a \cos(t) + 2a \sin^2(t) + at^2|_{t=p} = \\ &= \left\langle (3 \cos(t) dt + 2a \sin^2(t) + at^2) |_{t=p}, a \frac{\partial}{\partial t} \right\rangle = \\ &\text{czyli } h^* \alpha = (3 \cos(t) + 2 \sin^2(t) + t^2) dt \end{aligned}$$

Na skróty!

$$\begin{array}{ll}
x = \sin(t) & dx = \cos(t)dt \\
y = \cos(t) & dy = -\sin(t)dt \\
z = t & dz = dt.
\end{array}$$

Zatem

$$\begin{aligned}
h^*\alpha &= 3(\sin^2(t) + \cos^2(t))\cos(t)dt - 2\sin(t)(-\sin t dt) + 2t^2 dt \\
&= (3\cos(t) + 2\sin^2(t) + 2t^2) dt.
\end{aligned}$$

**Przykład 3** Niech  $M = \mathbb{R}^4, N = \mathbb{R}^4$ .

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-v^2}}, \\
c &= 1 \\
h: \quad t &= \gamma(t' - vx') \\
x &= \gamma(x' - vt') \\
y &= y' \\
z &= z'.
\end{aligned}$$

Czyli

$$\begin{aligned}
dt &= \gamma(dt' - vdx') \\
dx &= \gamma(dx' - vdt') \\
dy &= dy' \\
dz &= dz'.
\end{aligned}$$

Chcemy cofnąć naszą formę. Na fizyce nie używamy słowa cofnięte.

$$\begin{aligned}
F' &= -E_x(\gamma(dt' - vdx')) \wedge \gamma(dx' - vdt') - E_y\gamma(dt' - vdx') \wedge dy' = \\
&= -E_x\gamma^2(1-v^2)dt' \wedge dx' - E_y\gamma dt' \wedge dy' + E_y\gamma v dx' \wedge dy' = \\
&= -E_x\frac{1}{1-v^2}(1-v^2)dt' \wedge dx' - E_y\gamma dt' \wedge dy' + \gamma v E_x dx' \wedge dy' \\
F' &= -E'_x dt' \wedge dx' - E'_y dt' \wedge dy' + B'_z dx' \wedge dy'
\end{aligned}$$

Czyli

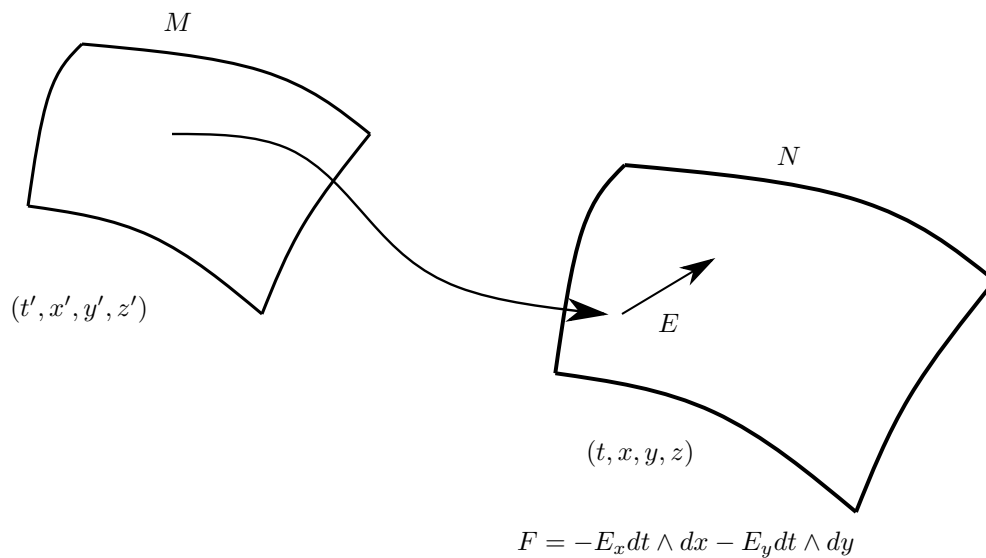
$$\begin{aligned}
E'_x &= E_x \\
E'_y &= \gamma E_y \\
B'_z &= \gamma v E_y.
\end{aligned}$$

Obserwacja: Niech  $\alpha \in \Lambda^1(N)$ ,  $\dim N = k$ , niech  $M$  - rozmaitość,  $\dim M = n$  i  $h: M \rightarrow N$ . Wówczas

$$h^*f \in \Lambda^0(M).$$

Oraz

$$d(h^*f) = h^*(df).$$



**Dowód 1** Skoro  $f \in \Lambda^0(N)$ , to  $f(x^1, x^2, \dots, x^k)$ ,  
 $df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \dots + \frac{\partial f}{\partial x^k} dx^k$ .

$$\langle h^*(df), v \rangle = \langle df, h_* v \rangle, v \in T^p M.$$

Niech  $V \in T_p M$ .

$$\tilde{h}(t_1, \dots, t_n) = \begin{bmatrix} h_1(t_1, \dots, t_n) \\ \vdots \\ h_k(t_1, \dots, t_n) \end{bmatrix}.$$

$$\text{Jeżeli } v = a_1 \frac{\partial}{\partial t^1} + a_2 \frac{\partial}{\partial t^2} + \dots + a_n \frac{\partial}{\partial t^n}, \text{ to } h_* v = \left( [h'] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right) \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}.$$

$$\begin{aligned}
h_x v &= \left( \begin{bmatrix} \frac{\partial h_1}{\partial t^1} & \dots & \frac{\partial h_k}{\partial t^k} \\ \vdots & & \vdots \\ \frac{\partial h_k}{\partial t^1} & \dots & \frac{\partial h_k}{\partial t^k} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right)_{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}} = \left( \frac{\partial h_1}{\partial t^1} a_1 + \dots + \frac{\partial h_1}{\partial t^n} a_n \right) \frac{\partial}{\partial x^1} + \\
&+ \dots + \left( \frac{\partial h_k}{\partial t^1} a_1 + \dots + \frac{\partial h_k}{\partial t^n} a_n \right) \frac{\partial}{\partial x^k}.
\end{aligned}$$

*Dalej*

$$\begin{aligned}
\langle df, h_* v \rangle &= \frac{\partial f}{\partial x^1} \left( \frac{\partial h_1}{\partial t^1} a_1 + \dots + \frac{\partial h_1}{\partial t^n} a_n \right) + \dots + \frac{\partial f}{\partial x^k} \left( \frac{\partial h_k}{\partial t^1} a_n + \dots + \frac{\partial h_k}{\partial t^n} a_n \right) = \\
&= \left\langle df(h_1(t_1, \dots, t_n), h_2(t_1, \dots, t_n), \dots, h_k(t_1, \dots, t_n)), a_1 \frac{\partial}{\partial t^1} + \dots + a_n \frac{\partial}{\partial t^n} \right\rangle
\end{aligned}$$