

**Definicja 1.** Niech  $\alpha_1, \alpha_2, \dots, \alpha_k \in T_p^*M \in \Lambda^1(M)$ , wówczas  $\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_k \in \Lambda^k(M)$  i dla  $v_1, v_2, \dots, v_k \in T_p^*M$ ,

$$\langle \alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_k; v_1, v_2, \dots, v_k \rangle \stackrel{def}{=} \begin{bmatrix} \alpha_1(v_1)\alpha_2(v_1) \dots \alpha_k(v_1) \\ \vdots \\ \alpha_1(v_k)\alpha_2(v_k) \dots \alpha_k(v_k) \end{bmatrix}.$$

Uwagi do operatora  $d$  ( $dd = 0$ ):  
Niech  $M = \mathbb{R}^3, f: \mathbb{R}^3 \rightarrow \mathbb{R}^1 \in \Lambda^0(M)$

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \\ ddf &= d\left(\frac{\partial f}{\partial x}\right) \wedge dx + d\left(\frac{\partial f}{\partial y}\right) \wedge dy + d\left(\frac{\partial f}{\partial z}\right) \wedge dz = \\ &= \left(\frac{\partial^2 f}{\partial x^2} dx + \frac{\partial^2 f}{\partial y \partial x} dy + \frac{\partial^2 f}{\partial z \partial x} dz\right) \wedge dx + \left(\frac{\partial^2 f}{\partial x \partial z} dx + \frac{\partial^2 f}{\partial y \partial z} dy\right) \wedge dy \\ &\quad + \left(\frac{\partial^2 f}{\partial x \partial z} dx + \frac{\partial^2 f}{\partial y \partial z} dz\right) \wedge dz \\ &= \left(\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y}\right) dy \wedge dx + \left(\frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial y \partial z}\right) dz \wedge dy + \\ &\quad + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right) dz \wedge dx = 0. \end{aligned}$$

Niech  $\alpha = A_x dx + A_y dy + A_z dz$

$$\begin{aligned} d\alpha &= \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) dy \wedge dx + \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) dz \wedge dy + \\ &\quad + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) dz \wedge dx \\ d d\alpha &= \left(\pm \left(\frac{\partial^2 A_x}{\partial z \partial y} - \frac{\partial^2 A_x}{\partial z \partial x}\right) \pm \left(\frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z}\right) \pm \left(\frac{\partial^2 A_z}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y \partial z}\right)\right) dx \wedge dy \wedge dz \\ &= 0. \end{aligned}$$

$$\begin{aligned} \beta &= A_x dy \wedge dz + A_y dx \wedge dz + A_z dy \wedge dx \\ d\beta &= 0 \\ dd\beta &= 0. \end{aligned}$$

Niech  $M = \mathbb{R}^4$ ,  $A = \phi dt + A_x dx + A_y dy + A_z dz$ .

$$dA = \underbrace{\left( \frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} \right)}_{E_x} dx \wedge dt + \underbrace{\left( \frac{\partial \phi}{\partial y} - \frac{\partial A_y}{\partial t} \right)}_{E_y} dy \wedge dt + \underbrace{\left( \frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} \right)}_{E_z} dz \wedge dt +$$

$$\underbrace{\left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)}_{B_z} dx \wedge dy + \underbrace{\left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right)}_{B_x} dy \wedge dz + \underbrace{\left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right)}_{B_y} dz \wedge dx$$

$$ddA = 0.$$

niech  $dA = F$

$$dF = 0.$$

Pytanie: niech  $M$  - rozmaitość wymiaru 3 (bo mamy bijekcję między  $\theta \in M$  i  $\mathbb{R}^3$ ). Czy istnieje  $\Lambda^4(M)$ ?

niech  $M = \mathbb{R}^3$

$$\Lambda^0(M) \quad f : \mathbb{R}^3 \rightarrow M$$

$$\dim \Lambda^0(M) = 1$$

$$\Lambda^1(M) \quad \alpha = A_x dx + A_y dy + A_z dz$$

$$\Lambda^1(\eta) = \underbrace{\langle dx, dy, dz \rangle}_3$$

$$\Lambda^2(M) \quad \beta = A_z dx \wedge dy + A_y dz \wedge dx + A_x dy \wedge dz$$

$$\Lambda^2(M) = \underbrace{\langle dx \wedge dy, dz \wedge dx, dy \wedge dz \rangle}_3$$

$$\Lambda^3(\eta) \quad \gamma = f dx \wedge dy \wedge dz$$

$$\Lambda^3(M) = \underbrace{\langle dx \wedge dy \wedge dz \rangle}_1$$

Niech  $M = \mathbb{R}^4$ .

$$\Lambda^0(M) \quad f(t, x, y, z) \rightarrow \mathbb{R}$$

$$\dim \Lambda^0(M) = 1$$

$$\Lambda^1(M) \quad \alpha = A_t dt + A_x dx + A_y dy + A_z dz$$

$$\dim \Lambda^1(M) = 4$$

$$\Lambda^2(M) \quad \beta = A_1 dt \wedge dx + A_2 dt \wedge dy + A_3 dt \wedge dz + B_1 dy \wedge dx + B_2 dz \wedge dx + C_1 dz \wedge dy$$

$$\dim \Lambda^2(M) = 6$$

$$\Lambda^3(M) : \quad \gamma = C_1 dy \wedge dt \wedge dx + C_2 dz \wedge dt \wedge dx + D_1 dz \wedge dt \wedge dy + E_1 dx \wedge dy \wedge dz$$

$$\dim \Lambda^3(M) = 4$$

$$\Lambda^4(M) \quad \delta = g dt \wedge dx \wedge dy \wedge dz$$

$$\dim \Lambda^4(M) = 1.$$

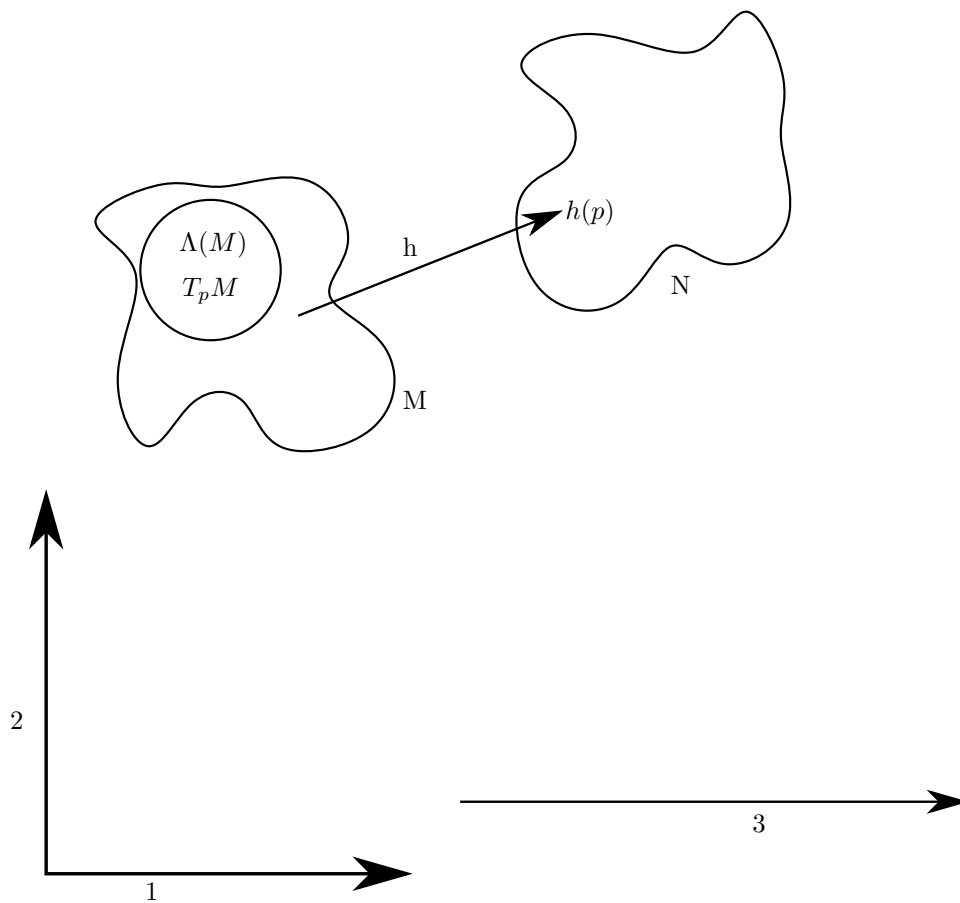
## 0.1 Pchnięcia i cofnięcia

**Definicja 2.** Niech  $M, N$  - rozmaitości  $\dim M = n, \dim N = k$  i niech  $h : M \rightarrow N$ . ( $h$  nie musi być bijekcją !!!)

Niech  $p \in M$ . Pchnięciem punktu  $p$  w odwzorowaniu  $h$  nazywamy punkt  $h_*(p) \stackrel{\text{def}}{=} h(p)$

**Przykład 1.** Niech  $M = \mathbb{R}^2$ ,  $N = \mathbb{R}$ ,  $h(x, y) = x + y, h : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

$$p = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, h_*(p) = 3$$



$$M = \mathbb{R}^1, N = \mathbb{R}^3, h(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}, p = \frac{\pi}{2}.$$

$$h_x(\frac{\pi}{2}) = \begin{bmatrix} \cos \frac{\pi}{2} \\ \sin \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$$

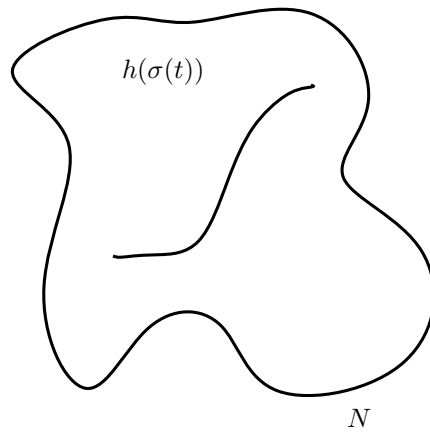
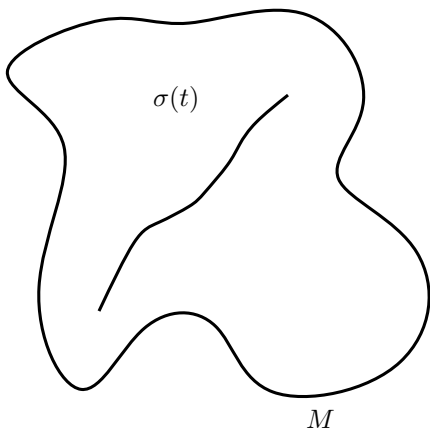
Niech  $\sigma(t)$  - krzywa na  $M$ . Pchnięciem krzywej  $\sigma$  w odwzorowaniu  $h$  nazywamy krzywą  $h_*(\sigma(t)) \stackrel{\text{def}}{=} h(\sigma(t))$

Niech  $f : N \rightarrow \mathbb{R}^2$ . Cofnięciem funkcji  $f$  w odwzorowaniu  $h$  nazywamy funkcję

$$h^* f(p) = f(h(p)).$$

**Przykład 2.**  $M = \mathbb{R}^2, N = \mathbb{R}, f : N \rightarrow \mathbb{R}^2, f(t) = \begin{bmatrix} 2t \\ t \end{bmatrix}, h(x, y) = x + y.$

$$h^* f(x, y) = f(h(x, y)) = \begin{bmatrix} 2(x + y) \\ x + y \end{bmatrix}.$$



**Definicja 3.** *Pchnięciem wektora  $V$  w odwzorowaniu  $h$  nazywamy wektor*

$$h_*V = [h(\sigma)], h_*v \in T_{h(p)}N.$$

**Przykład 3.** *Niech  $M = \mathbb{R}^2$ ,  $N = \mathbb{R}$ ,  $h(x, y) = x + 2y$ ,  $v = 2\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}$ . Co to jest  $h_*v$ ?*  
 $p = (1, 2) = (\varphi^1(p), \varphi^1(p))$

$$\sigma(t) : \frac{d}{dt}(\varphi(\sigma(t)))|_{t=0}$$

$$\varphi(\sigma(t)) = \begin{bmatrix} 2t + 1 \\ 3t + 2 \end{bmatrix}$$

$$h[\sigma(t)] = 2t + 1 + 2(3t + 2)$$

$$h[\sigma(t)] = 8t + 5$$

$$[h[\sigma(t)]] = 8\frac{\partial}{\partial t} \in t_s N.$$

$\dim M = n$ ,  $\varphi(\sigma(t)) = (\varphi^1(\sigma(t)), \varphi^2(\sigma(t)), \dots, \varphi^n(\sigma(t)))$ ,  $v \in T_p M$ .

$$v = \frac{\partial \varphi^1(\sigma(t))}{\partial t}|_{t=0} \frac{\partial}{\partial x^1} + \frac{\partial \varphi^2(\sigma(t))}{\partial t}|_{t=0} \frac{\partial}{\partial x^2} \dots \frac{\partial \varphi^n(\sigma(t))}{\partial t}|_{t=0} \frac{\partial}{\partial x^n}.$$

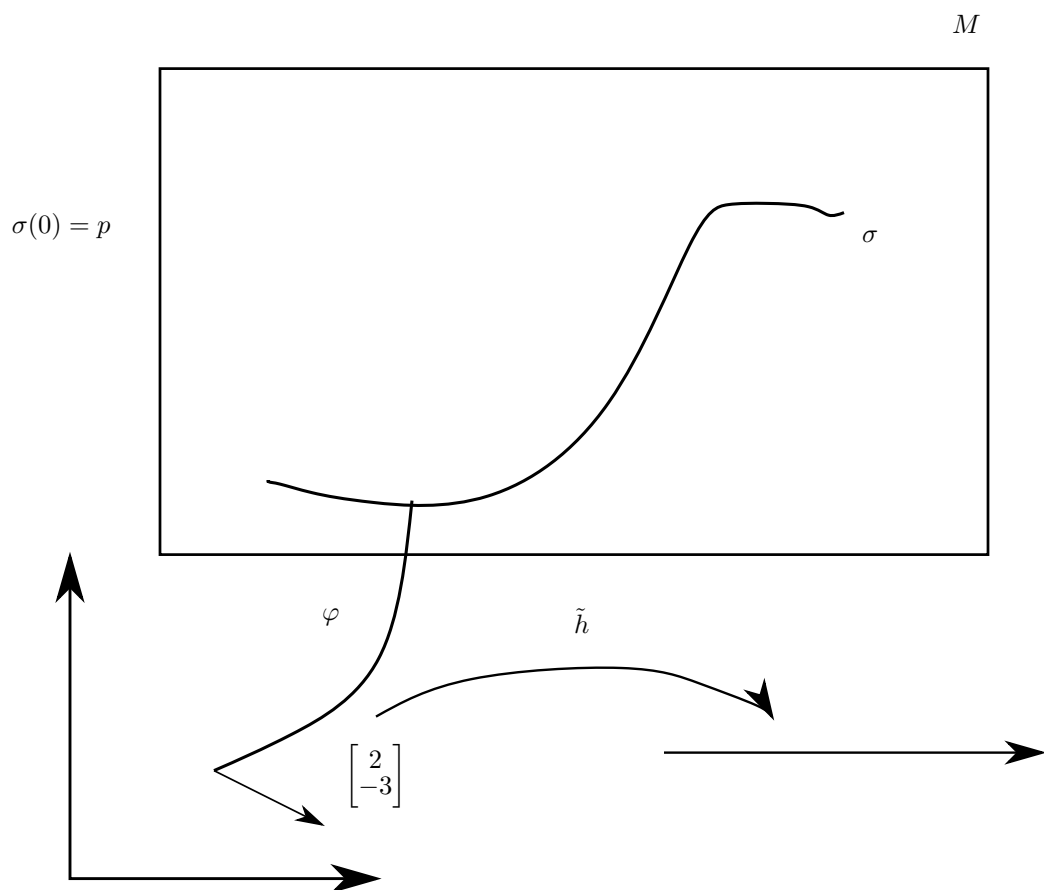
$$\frac{d(\varphi \circ h(\sigma(t)))}{dt}|_{t=0} = \frac{d}{dt}(\psi \circ h \circ \varphi^{-1}\sigma)|_{t=0} = \frac{d}{dt}(\tilde{h} \circ \tilde{\sigma}(t)).$$

$$= \frac{d}{dt}\tilde{h}(\tilde{\sigma}_1(t), \tilde{\sigma}_2(t), \dots, \tilde{\sigma}^n(t))|_{t=0} = \tilde{h}'_{\tilde{\sigma}(0)} \frac{d\tilde{\sigma}}{dt}|_{t=0} = \tilde{h}' \cdot v.$$

Czyli ostatecznie  $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}}$ ,  $\tilde{h}(x, y) = x + 2y \rightarrow \tilde{h}(x, y) = [1, 2]$ .

$$h_*v = [1, 2] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \cdot 1 + 6 = 8\frac{\partial}{\partial t}.$$

Niech  $\alpha \in \Lambda^1(?)$  - pytanie: czy formy się pcha, czy cofa?



Rysunek 1:  $\tilde{h} = \psi h \varphi^{-1}$