Przykład 1. *Zastanówmy się jak wygląda rotacja wektora w układzie sferycznym.* $M = \mathbb{R}^3$.

$$v \xrightarrow{\sharp} \Lambda^1(M) \xrightarrow{d} \Lambda^2(M) \xrightarrow{*} \Lambda^1(M) \to T_pM \xrightarrow{\flat} \big[\;\big]_i$$

$rot v = \left(*(dv^\sharp)\right)^\flat$
na początek dostajemy w smsie

$$\begin{bmatrix} A^r \\ A^\theta \\ A^\varphi \end{bmatrix}_{i_r, i_\theta, i_\varphi} = v = A^r \frac{\partial}{\partial r} + A^\theta \frac{1}{r} \frac{\partial}{\partial \theta} + A^\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi}$$

chcemy sobie zrobić jednoformę, która jest podniesionym wektorkiem:

$$\alpha = v^\sharp = g_{rr} A^r dr + g_{\theta\theta} \frac{1}{r} A^\theta d\theta + g_{\varphi\varphi} \frac{1}{r \sin\theta} A^\varphi d\varphi = A^r dr + r A^\theta d\theta + r \sin\theta A^\varphi d\varphi$$

$$\begin{aligned}
d\alpha &= \left(A^r_{,\theta} - (rA^\theta)_{,r}\right) d\theta \wedge dr + \left(A^r_{,\varphi} - (r\sin\theta A^\varphi)_{,r}\right) d\varphi \wedge dr + \\
&\quad + \left((rA^\theta)_{,\varphi} - (r\sin\theta A^\varphi)_{,\theta}\right) d\varphi \wedge d\theta \\
*(dr \wedge d\theta) &= \sin\theta d\varphi, \quad *(d\theta \wedge d\varphi) = \frac{1}{r^2} dr, \quad *(d\varphi \wedge dr) = \frac{1}{\sin\theta} d\theta \\
*d\alpha &= \left((r\sin\theta A^\varphi)_{,\theta} - (rA^\theta)_{,\varphi}\right) \frac{1}{r^2 \sin\theta} dr + \left(A^r_{,\varphi} - (r\sin\theta A^\varphi)_{,r}\right) \frac{1}{\sin\theta} d\theta + \\
&\quad + \left((rA^\theta)_{,r} - A^r_{,\theta}\right) \sin\theta d\varphi.
\end{aligned}$$

notacja: $\square_{,\heartsuit} = \frac{\partial \square}{\partial \heartsuit}$. *Zostały nam jeszcze tylko dwie operacje.*

$$\begin{aligned}
(*d\alpha)^\flat &= \left((r\sin\theta A^\varphi)_{,\theta} - (rA^\theta)_{,\varphi}\right) \cdot 1 \cdot \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial r} + \left(A^r_{,\varphi} - (r\sin\theta A^\varphi)_{,r}\right) \frac{1}{\sin\theta} \frac{1}{r^2} \frac{\partial}{\partial \theta} + \\
&\quad + \left((rA^\theta)_{,r} - A^r_{,\theta}) \sin\theta \frac{1}{r^2 \sin^2\theta}\right) \frac{\partial}{\partial \varphi}.
\end{aligned}$$

Czyli

$$rot \begin{bmatrix} A^r \\ A^\theta \\ A^\varphi \end{bmatrix} = \begin{bmatrix} \frac{1}{r^2 \sin\theta} \left((r\sin\theta A^\varphi)_{,\theta} - (rA^\theta)_{,\varphi})\right) \\ \frac{1}{r \sin\theta} \left(A^r_{,\varphi} - (r\sin\theta A^\varphi)_{,r}\right) \\ \frac{1}{r} \left((rA^\theta)_{,r} - A^r_{,\theta}\right) \end{bmatrix}.$$

Przykład 2. *To może teraz dywergencja rzutem na taśmę.*

$$\begin{aligned}
&\begin{bmatrix} \\ \\ \end{bmatrix} = v \xrightarrow{\sharp} \Lambda^1(M) \xrightarrow{*} \Lambda^2(M) \xrightarrow{d} \Lambda^3(M) \xrightarrow{\flat} \Lambda^0(M)\\
&div(v) = *\left(d(*v^\sharp)\right)\\
&\begin{bmatrix} A^r \\ A^\theta \\ A^\varphi \end{bmatrix} = v, \alpha = v^\sharp\\
&\alpha = A^r dr + rA^\theta d\theta + A^\varphi r \sin\theta d\varphi\\
&* dr = r^2 \sin\theta d\theta \wedge d\varphi\\
&* d\theta = \sin\theta d\varphi \wedge dr\\
&* d\varphi = \frac{1}{\sin\theta} dr \wedge d\theta\\
&* \alpha = \left(A^r r^2 \sin\theta\right) d\theta \wedge d\varphi + \left(r \sin\theta A^\theta\right) d\varphi \wedge dr + \left(rA^\varphi\right) dr \wedge d\theta\\
&d(*\alpha) = \left((A^r r^2 \sin\theta)_{,r} + (r\sin\theta A^\theta)_{,\theta} + (rA^\varphi)_{,\varphi}\right) dr \wedge d\theta \wedge d\varphi
\end{aligned}$$

.

$$div \begin{bmatrix} A^r \\ A^\theta \\ A^\varphi \end{bmatrix} = \frac{1}{r^2 \sin\theta} \left((A^r r^2 \sin\theta)_{,r} + (r\sin\theta A^\theta)_{,\theta} + (rA^\varphi)_{,\varphi}\right).$$

$$\begin{aligned}
&f(r,\theta,\varphi) \xrightarrow{d} \Lambda^1(M) \xrightarrow{*} \Lambda^2(M) \xrightarrow{d} \Lambda^3(M) \xrightarrow{*} \Lambda^0(M)\\
&\alpha = df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \varphi} d\varphi\\
&* \alpha = \left(\frac{\partial f}{\partial r} r^2 \sin\theta\right) d\theta \wedge d\varphi + \left(\frac{\partial f}{\partial \theta}\sin\theta\right) d\varphi \wedge dr + \left(\frac{\partial f}{\partial \varphi}\frac{1}{\sin\theta}\right) dr \wedge d\theta\\
&d(*\alpha) = \left(\left(r^2\sin\theta\frac{\partial f}{\partial r}\right)_{,r} + \sin\theta\frac{\partial f}{\partial \theta}_{,\theta} + \left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \varphi}_{,\varphi}\right)\right) dr \wedge d\theta \wedge d\varphi\\
&*\left(d(*\alpha)\right) = \frac{1}{r^2\sin\theta}\left(\left(r^2\sin\theta\frac{\partial f}{\partial r}\right)_{,r} + \left(\sin\theta\frac{\partial f}{\partial \theta}\right)_{,\theta} + \left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \varphi}\right)_{,\varphi}\right).
\end{aligned}$$

Przykład 3. $M = \mathbb{R}^3, f \in \Lambda^0(M)$.

$$\begin{aligned}
ddf &= 0\\
ddf &= d\left(\left((df)^\flat\right)^\sharp\right) \implies rot(grad(f)) = 0.
\end{aligned}$$

Niech teraz $v \in \Lambda^1(M)$.

$$\begin{aligned}
&d\left(*\left(\left(*(dV^\sharp)\right)^\flat\right)^\sharp\right) = d(*(\overset{**=\mathbb{I}?}{*}(d(v^\sharp)))) = dd(v^\sharp) = 0\\
&div(rot(V)) = 0.
\end{aligned}$$

Weźmy sobie jakąś funkcję: $f : (t, x, y, z) \to \mathbb{R}$.

Zobaczmy jak $*d(*df)$ wygląda w $\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$.

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

$$* \left(dx^{i_1} \wedge \ldots \wedge dx^{i_L}\right) = \frac{\sqrt{g}}{(n-L)!} g^{i_1 j_1} \ldots g^{i_L j_L} \in_{j_1 \ldots j_k k_1 \ldots k_{n-L}} dx^{k_1} \wedge \ldots \wedge dx^{k_{n-L}}$$

$$* \left(dx^0\right) = \frac{\sqrt{-(-1)}}{(4-1)!} g^{00} \in_{0 k_1 k_2 k_3} dx^{k_1} \wedge dx^{k_2} \wedge dx^{k_3}, i, k = 0, \ldots, 3$$

$$* \left(dx^0\right) = -\frac{1}{3!} 3! dx^1 \wedge dx^2 \wedge dx^3$$

$$* (dt) = -dx \wedge dy \wedge dz$$

$$* \left(dx^1\right) = \frac{\sqrt{-(-1)}}{(4-1)!} g^{11} \in_{1 k_1 k_2 k_3} dx^{k_1} \wedge dx^{k_2} \wedge dx^{k_3}$$

$$* (dx) = 3! \frac{1}{3!} dy \wedge dt \wedge dz$$

$$* (dy) = dt \wedge dx \wedge dz$$

$$* (dz) = dx \wedge dt \wedge dy$$

$$* df = -\frac{\partial f}{\partial t} dx \wedge dy \wedge dz + \frac{\partial f}{\partial x} dy \wedge dt \wedge dz + \frac{\partial f}{\partial y} dt \wedge dx \wedge dz + \frac{\partial f}{\partial z} dx \wedge dt \wedge dy$$

$$d * df = \left(-\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) dt \wedge dx \wedge dy \wedge dz.$$

Na koniec:
Mamy dwuformę pola elektromagnetycznego:

$$F = -E_x dt \wedge dx + E_y dt \wedge dy - E_2 dt \wedge dz + B_x dy \wedge dz + B_y dz \wedge dy + B_z dy \wedge dx.$$

$dF = 0$ to jest pierwsza część równań Maxwella

$$\begin{bmatrix} \rho \\ \rho v^x \\ \rho v^y \\ \rho v^z \end{bmatrix} = \begin{bmatrix} \rho \\ j^x \\ j^y \\ j^z \end{bmatrix}$$

$$j = -g dt + j^x dx + j^y dy + j^z dz$$

$d(*F) = *j$ a to druga.