0.1 Konwencja z ćwiczeń z fizyki:

Przykład 1. Mamy funkcję $H(r,\varphi)=(f\circ\Psi)(r,\varphi)$

$$\begin{split} &\Psi_{1}(r,\varphi) = x(r,\varphi) \\ &\Psi_{2}(r,\varphi) = y(r,\varphi) \\ &\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} \\ &\frac{\partial f}{\partial \varphi} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \varphi} \end{split}$$

Przykład 2.

$$\begin{split} f(x,y): \mathbb{R}^2 &\to \mathbb{R}, \quad \begin{bmatrix} x = r\cos\varphi \\ y = r\sin\varphi \end{bmatrix} \\ \frac{\partial f}{\partial r} &= \cos\varphi \frac{\partial f}{\partial x} + \sin\varphi \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial \varphi} = -r\sin\varphi \frac{\partial f}{\partial x} + r\cos\varphi \frac{\partial f}{\partial y} \\ f(x,y): \mathbb{R}^2 &\to \mathbb{R}, \quad f' = \begin{bmatrix} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \end{bmatrix} \end{split}$$

0.2 Interpretacja geometryczna f'

Przykład 3. Rozważmy zbiór

$$P_c = \{(x,y) \in \mathbb{R}^2 : f(x,y) = c\} \text{ np. } f(x,y) = x^2 + y^2 : P_c = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = c\}.$$

Załóżmy, że f(x,y) - taka, że P_c można sparametryzować jako

$$\varphi(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, t \in D, \text{ to znaczy, } \dot{z}e \ P_c = \{(x(t), y(t)), t \in D\}$$

Przykład 4.

Niech
$$\varphi(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$
. Wtedy $P_c = \{(c \cdot \cos t, c \cdot \sin t) : t \in [0, 2\pi]\}$ $f(x(t), y(t)) = c \quad \forall \\ t \in D$ - powierzchnie ekwipotencjalne

$$\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right] \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = 0,$$

$$\left[2x, 2y \right] \left[\begin{matrix} -c \cdot \sin t \\ c \cdot \cos t \end{matrix} \right] = 0.$$

"../img/"fig_3.png

Rysunek 1: Trajektoria kluki

Definicja 1. Pochodna mieszana

$$f(x,y) = x^2 y^3, \quad \frac{\partial f}{\partial x} = 2xy^3, \frac{\partial f}{\partial y} = 3x^2 y^2, \quad \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x}\right) = 2y^3, \quad \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y}\right) = 6x^2 y$$
$$\frac{\partial^2 f}{\partial x \partial y} = 6xy^2 \quad \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

 $Przypadek \ref{eq:przypadek} \ref{eq:przypadek}$

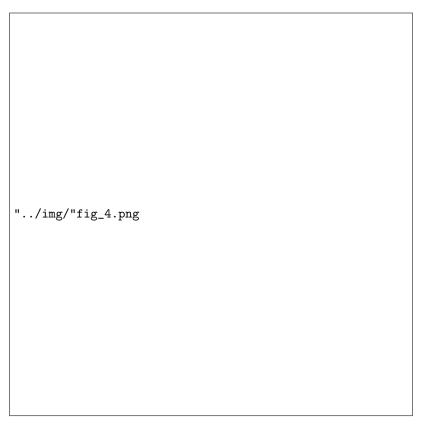
Twierdzenie 1. (Uogólnione twierdzenie Schwarza) Niech $f: \mathcal{O} \to \mathbb{R}$, $\mathcal{O} \subset \mathbb{R}^n$, otwarty i $f \in \mathcal{C}^2(\mathcal{O})$, wówczas

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}; i,j=1,\ldots,n$$

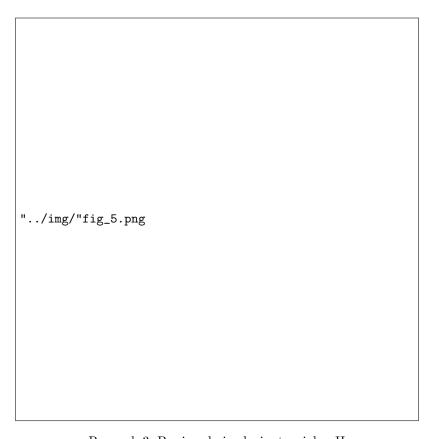
Dowód. Dowód dla n=2 Niech

$$w(x,y) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y),$$

$$\varphi(x) = f(x, y+k) - f(x, y)$$



Rysunek 2: Powierzchnia ekwipotencjalna I



Rysunek 3: Powierzchnia ekwipotencjalna II

wówczas

$$w = \varphi(x+h) - \varphi(x) = \frac{\partial \varphi}{\partial x}(\xi)h =$$

$$= \left[\frac{\partial f}{\partial x}(\xi, y+k) - \frac{\partial f}{\partial x}(\xi, y)\right]h =$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(\xi, \eta)\right)hk,$$
gdzie $x < \xi < x+h, \quad y < \eta < y+k$

Niech

$$\Psi(y) = f(x+h,y) - f(x,y)$$

$$\begin{split} w(x,y) &= \Psi(y+k) - \Psi(y) = \frac{\partial \Psi}{\partial y}(\eta_1)k = \\ &= \left[\frac{\partial f}{\partial y}(x+h,\eta_1) - \frac{\partial f}{\partial y}(x,\eta_1)\right]k = \\ &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y}(\xi,\eta)\right)kh, \end{split}$$

czyli

$$\exists_{\xi} \quad \xi \in]x, x+h[, \quad \xi_1 \in]x, x+h[, \quad \eta \in]y, y+k[, \quad \eta_1 \in]y, y+k[.$$

Jeżeli $h \to 0$,

$$\left(\frac{\partial^2 f}{\partial y \partial x}(\xi, \eta) = \frac{\partial^2 f}{\partial x \partial y}(\xi_1, \eta_1)\right),\,$$

to

$$\xi \to x, \xi_1 \to x, \eta \to y, \eta_1 \to y,$$

czyli:

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y),$$

jeżeli każda z tych wielkości jest ciągła.

0.3 Wzór Taylora (konstrukcja)

Niech $f: \mathcal{O} \to \mathbb{R}, \mathcal{O} \subset \mathbb{R}^n$ - otwarty $\varphi(t) = f(x_0 + th), h \in \mathbb{R}^n, t \in [0, 1].$ Dla

$$h = \begin{bmatrix} h^1 \\ \vdots \\ h^n \end{bmatrix}, x_0 = \begin{bmatrix} x_0^1 \\ \vdots \\ x_0^n \end{bmatrix}, \varphi(t) = f(x_0^1 + th^1, x_0^2 + th^2, \dots, x_0^n + th^n),$$

mamy

$$\frac{\partial \varphi}{\partial t} = \frac{\partial f}{\partial x^1} \bigg|_{x=x_0+th} h_1 + \frac{\partial f}{\partial x^2} \bigg|_{x=x_0+th} h_2 + \dots + \frac{\partial f}{\partial x^n} \bigg|_{x=x_0+th} h_n = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \bigg|_{x_0+th} h_i$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \sum_{j=1}^n \sum_{i=1}^n \frac{\partial f}{\partial x^i \partial x^j} \bigg|_{x_0+th} h_j h_i$$

$$\vdots$$

$$\frac{\partial^k \varphi}{\partial t^k} = \sum_{i_1, \dots, i^k}^n \frac{\partial^{(k)} f}{\partial x^{i_1} \dots \partial x^i} h_{i_1} \dots h_{i_k}$$

$$\varphi(t) = \varphi(0) = \varphi'(0)(t-0) + \frac{\varphi''(0)}{2!}(t-0)^2 + \dots + \frac{\varphi^k(0)}{k}(t-0)^k + r(\dots),$$

czyli:

$$\varphi(1) - \varphi(0) = \varphi'(0) + \frac{\varphi''(0)}{2!} + \frac{\varphi'''(0)}{3!} + \dots + \frac{\varphi^{k}(0)}{k!} + r(\dots)$$

$$f(x_0 + h) - f(x_0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x_0)h_i + \frac{1}{2!} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0)h_i h_j + \dots$$