Definicja 1 *Niech* $\alpha_1, \alpha_2, \ldots, \alpha_k \in T_p^*M \in \Lambda'(M)$, *wówczas* $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k \in \Lambda^k(M)$ *i dla* $v_1, v_2, \ldots, v_k \in T_p^*M$,

$$\langle \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k ; v_1, v_2, \ldots, v_k \rangle \stackrel{def}{=} \begin{bmatrix} \alpha_1(v_1)\alpha_2(v_1)\ldots\alpha_k(v_1) \\ \vdots \\ \alpha_1(v_k)\alpha_2(v_k)\ldots\alpha_k(v_k) \end{bmatrix}.$$

Uwagi do operatora d ($dd = 0$):
Niech $M = \mathbb{R}^3, f : \mathbb{R}^3 \to \mathbb{R}^1 \in \Lambda^0(M)$

$$\begin{aligned}
&df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\
&ddf = d\left(\frac{\partial f}{\partial x}\right) \wedge dx + d\left(\frac{\partial f}{\partial y}\right) \wedge dy + d\left(\frac{\partial f}{\partial z}\right) \wedge dz = \\
&= \left(\frac{\partial^2 f}{\partial x^2}dx + \frac{\partial^2 f}{\partial y \partial x}dy + \frac{\partial^2 f}{\partial z \partial x}dz\right) \wedge dx \left(\frac{\partial^2 f}{\partial x \partial z}dx + \frac{\partial^2 f}{\partial y \partial z}dy\right) \wedge dy \\
&\left(\frac{\partial^2 f}{\partial x \partial z}dx + \frac{\partial^2 f}{\partial y \partial z}dz\right) \wedge dz \\
&= \left(\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y}\right) dy \wedge dx + \left(\frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial y \partial z}\right) dz \wedge dy + \\
&= \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right) dz \wedge dx = 0.
\end{aligned}$$

Niech $\alpha = A_x dx + A_y dy + A_z dz$

$$\begin{aligned}
&d\alpha = \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) dy \wedge dx + \left(\frac{\partial A_z}{\partial y} - \frac{\partial y}{\partial z}\right) dz \wedge dy + \\
&+ \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) dz \wedge dx \\
&dd\alpha = \left(\pm\left(\frac{\partial^2 A_x}{\partial z \partial y} - \frac{\partial^2 A_x}{\partial z \partial x}\right) \pm \left(\frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z}\right) \pm \left(\frac{\partial^2 A_z}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y \partial z}\right)\right) dx \wedge dy \wedge dz \\
&.
\end{aligned}$$

$$\begin{aligned}
&\beta = A_x dy \wedge dz + A_y dx \wedge dz + A_z dy \wedge dz \\
&d\beta = ()\, dx \wedge dy \wedge dz \\
&dd\beta = 0.
\end{aligned}$$

Niech $M = \mathbb{R}^4$, $A = \phi dt + A_x dx + A_y dy + A_z dz$.

$$dA = \Bigg(\underbrace{\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}}_{E_x}\Bigg) dx \wedge dt + \Bigg(\underbrace{\frac{\partial \phi}{\partial y} - \frac{\partial A_y}{\partial t}}_{E_y}\Bigg) dy \wedge dt + \Bigg(\underbrace{\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t}}_{E_z}\Bigg) dz \wedge dt +$$
$$\Bigg(\underbrace{\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}}_{B_z}\Bigg) dx \wedge dy + \Bigg(\underbrace{\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}}_{B_x}\Bigg) dy \wedge dz + \Bigg(\underbrace{\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}}_{B_y}\Bigg) dz \wedge dx$$
$$ddA = 0.$$

niech $dA = F$

$$dF = 0.$$

Pytanie: niech M - rozmaitość wymiaru 3 (bo mamy bijekcję między $\theta \in M$ i $\mathbb{R}^3$). Czy istnieje $\Lambda^4(M)$?

niech $M = \mathbb{R}^3$

$\Lambda^0(M)$ $\quad f : \mathbb{R}^3 \to M$ $\quad \dim \Lambda^1(M) = 3$

$\Lambda^1(M)$ $\quad \alpha = A_x dx + A_y dy + A_z dz$ $\quad \Lambda^1(\eta) = \langle dx, dy, dz \rangle$

$\Lambda^2(M)$ $\quad \beta = A_z dx \wedge dy + A_y dz \wedge dx + A_z dy \wedge dz$ $\quad \Lambda^2(M) = \langle dx \wedge dy, dz \wedge dx, dy \wedge dz \rangle$

$\dim(\Lambda^2(M)) = 3$

$\Lambda^3(\eta)$ $\quad \gamma = f dx \wedge dy \wedge dz$ $\quad \Lambda^3(M) = \langle dx \wedge dy \wedge dz \rangle$

$\dim(\Lambda^3(M)) = 1.$

Niech $M = \mathbb{R}^4$.

$\Lambda^0(M)$ $\quad f(t, x, y, z) \to \mathbb{R}$ $\quad \dim \Lambda^0(M) = 1$

$\Lambda^1(M)$ $\quad \alpha = A_t dt + A_x dx + A_y dy + A_z dz$ $\quad \dim \Lambda^1(M) = 4$

$\Lambda^2(M)$ $\quad \beta = A_1 dt \wedge dx + A_2 dt \wedge dy + A_3 dt \wedge dz + B_1 dy \wedge dx + B_2 dz \wedge dx + C_1 dz \wedge dy$ $\quad \dim \Lambda^2(M) = 6$

$\Lambda^3(M) :$ $\quad \gamma = C_1 dy \wedge dt \wedge dx + C_2 dz \wedge dt \wedge dx + D_1 dz \wedge dt \wedge dy + E_1 dx \wedge dy \wedge dz$ $\quad \dim \Lambda^3(M) = 4$

$\Lambda^4(M)$ $\quad \delta = g dt \wedge dx \wedge dy \wedge dz$ $\quad \dim \Lambda^4(M) = 1.$

0.1 Pchnięcia i cofnięcia

Niech M, N - rozmaitości $\dim M = n, \dim N = k$ i niech $h : M \to N$. (h nie musi być bijekcją !!!)

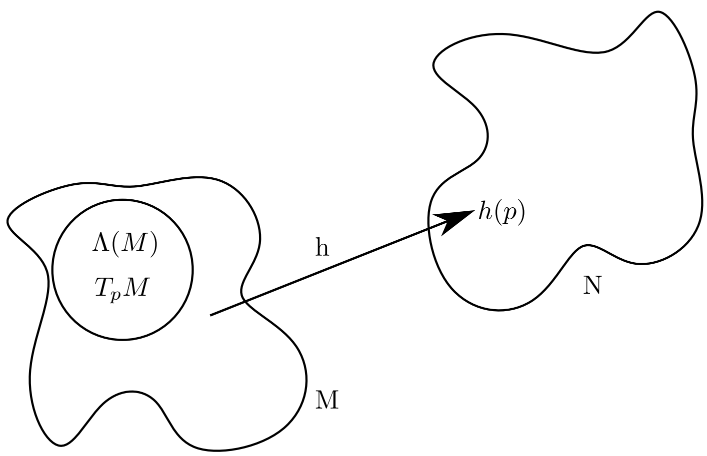
Niech $p \in M$. Pchnięciem punktu p w odwzorowaniu h nazywamy punkt $h_*(p) \overset{\text{def}}{=} h(p)$

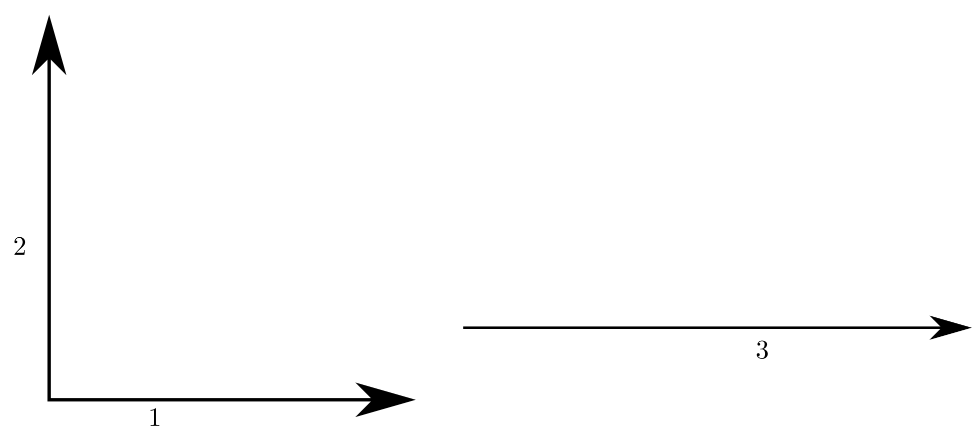
Przykład 1 *Niech* $M = \mathbb{R}^2$, $N = \mathbb{R}$, $h(x, y) = x + y, h : \mathbb{R}^2 \to \mathbb{R}$.

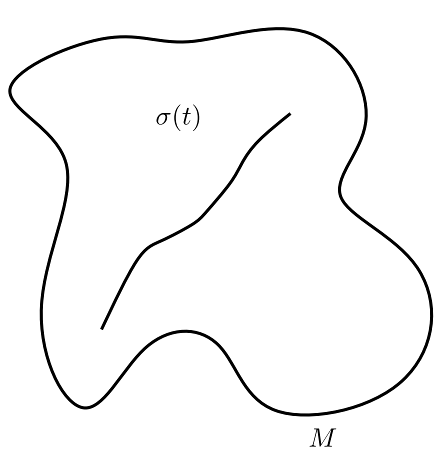
$p = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, h_*(p) = 3$

$M = \mathbb{R}^1$, $N = \mathbb{R}^3$, $h(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}, p = \frac{\pi}{2}$.

$h_x(\frac{\pi}{2}) = \begin{bmatrix} \cos \frac{\pi}{2} \\ \sin \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$


$\Lambda(M)$
T_pM
h
$h(p)$
N
M


2
3
1



Niech $\sigma(t)$ - krzywa na M. Pchnięciem krzywej σ w odwzorowaniu h nazywamy krzywą $h_*(\sigma(t)) \stackrel{\text{def}}{=} h(\sigma(t))$

Niech $f : N \to \mathbb{R}^2$. Cofnięciem funkcji f w odwzorowaniu h nazywamy funkcję

$$h^* f(p) = f(h(p)).$$

Przykład 2 *$M = \mathbb{R}^2, N = \mathbb{R}, f : N \to \mathbb{R}^2, f(t) = \begin{bmatrix} 2t \\ t \end{bmatrix}, h(x, y) = x + y.$*

$$h^* f(x, y) = f(h(x, y)) = \begin{bmatrix} 2(x + y) \\ x + y \end{bmatrix}.$$

Pchnięciem wektora V w odwzorowaniu h nazywamy wektor

$$h_* V = [h(\sigma)], h_* v \in T_{h(p)} N.$$

Przykład 3 *Niech $M = \mathbb{R}^2, N = \mathbb{R}, h(x, y) = x + 2y, v = 2\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}$. Co to jest $h_* v$?*

$p = (1, 2) = (\varphi^1(p), \varphi^1(p))$

$$
\begin{aligned}
&\sigma(t) : \frac{d}{dt}(\varphi(\sigma(t)))|_{t=0} \\
&\varphi(\sigma(t)) = \begin{bmatrix} 2t+1 \\ 3t+2 \end{bmatrix} \\
&h[\sigma(t)] = 2t+1+2(3t+2) \\
&h[\sigma(t)] = 8t+5 \\
&[h[\sigma(t)]] = 8\frac{\partial}{\partial t} \in t_s N.
\end{aligned}
$$

$\dim M = n$, $\varphi(\sigma(t)) = \left(\varphi^1(\sigma(t)), \varphi^2(\sigma(t)), \ldots, \varphi^n(\sigma(t))\right), v \in T_pM$.

$$
v = \frac{\partial \varphi^1(\sigma(t))}{\partial t}|_{t=0}\frac{\partial}{\partial x^1} + \frac{\partial \varphi^2(\sigma(t))}{\partial t}|_{t=0}\frac{\partial}{\partial x^2} \cdots \frac{\partial \varphi^n(\sigma(t))}{\partial t}|_{t=0}\frac{\partial}{\partial x^n}.
$$

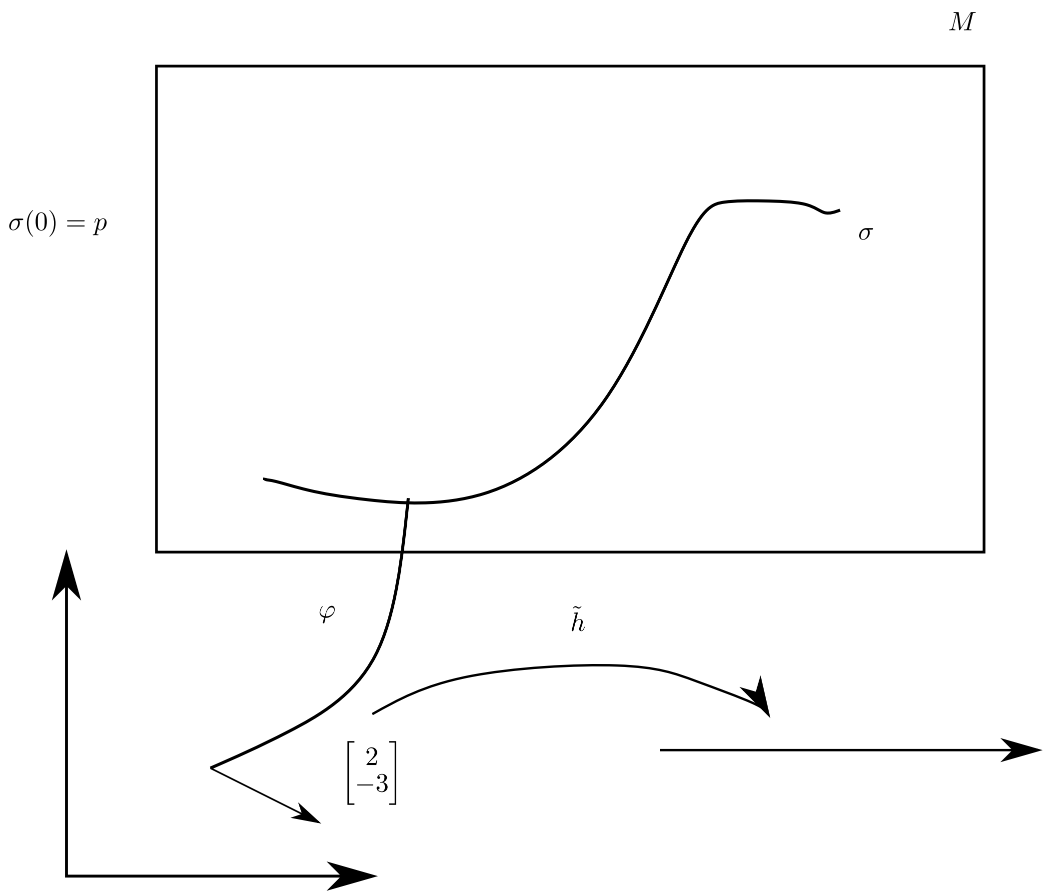
$$
\frac{d(\varphi \circ h(\sigma(t)))}{dt}|_{t=0} = \frac{d}{dt}\left(\psi \circ h \circ \varphi^{-1}\sigma\right)_{t=0} = \frac{d}{dt}\left(\tilde{h} \circ \tilde{\sigma}(t)\right).
$$

$$
= \frac{d}{dt}\tilde{h}\left(\tilde{\sigma}_1(t), \tilde{\sigma}_2(t), \ldots, \tilde{\sigma}^n(t)\right)_{t=0} = \tilde{h}'_{\tilde{\sigma}(0)}\frac{d\tilde{\sigma}}{dt}_{t=0} = \tilde{h}' \cdot v.
$$

Czyli ostatecznie $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}}, \tilde{h}(x,y) = x + 2y \to \tilde{h}(x,y) = [1,2].$

$$
h_* v = [1,2]\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \cdot 1 + 6 = 8\frac{\partial}{\partial t}.
$$

Niech $\alpha \in \Lambda^1(?)$ - pytanie: czy formy się pcha, czy cofa?



Rysunek 1: $\tilde{h} = \psi h \varphi^{-1}$