Przykład 1. ***Uwaga:*** *jeżeli np.* $f:\mathbb{R}^2\to\mathbb{R}^2$, *to znaczy, że* $f(x,y)=\begin{bmatrix}f_1(x,y)\\f_2(x,y)\end{bmatrix}$, $f_1:\mathbb{R}^2\to\mathbb{R}^1$, $f_2:\mathbb{R}^2\to\mathbb{R}^1$, *wówczas*

$$\frac{\partial f}{\partial x}=\begin{bmatrix}\frac{\partial}{\partial x}f_1\\\frac{\partial}{\partial x}f_2\end{bmatrix},\frac{\partial}{\partial y}f=\begin{bmatrix}\frac{\partial}{\partial y}f_1\\\frac{\partial}{\partial y}f_2\end{bmatrix}$$

Przykład 2.

$$f(x,y)=\begin{bmatrix}2xy^2\\x^3y\end{bmatrix}$$

Wtedy pochodne cząkowe:

$$\frac{\partial f}{\partial x}=\begin{bmatrix}2y^2\\3x^2y\end{bmatrix},\frac{\partial f}{\partial y}=\begin{bmatrix}4xy\\x^3\end{bmatrix}$$

$$\begin{aligned}f(x+h)-f(x)&=\\&=\frac{\partial f}{\partial x}h^x+\frac{\partial f}{\partial y}h^y+r((x,y),h)=\\&=\begin{bmatrix}2y^2\\3x^2y\end{bmatrix}h^x+\begin{bmatrix}4xy\\x^3\end{bmatrix}h^y+r((x,y),h)\\&=\begin{bmatrix}2y^2&4xy\\3x^2y&x^3\end{bmatrix}\begin{bmatrix}h^x\\h^y\end{bmatrix}+r((x,y),h).\end{aligned}$$

Czyli

$$f'=\begin{bmatrix}\frac{\partial f_1}{\partial x}&\frac{\partial f_1}{\partial y}\\\frac{\partial f_2}{\partial x}&\frac{\partial f_2}{\partial y}\end{bmatrix}$$

i ogólniej: jeżeli $f:\mathbb{R}^n\to\mathbb{R}^k$, *to*

$$f'=\begin{bmatrix}\frac{\partial f_1}{\partial x^1}&\dots&\frac{\partial f_1}{\partial x^n}\\\vdots&\ddots&\vdots\\\frac{\partial f_k}{\partial x^1}&\dots&\frac{\partial f_k}{\partial x^n}\end{bmatrix}$$

0.1 Uzupełnienie:

Stwierdzenie 1. *Niech* V *- przestrzeń wektorowa z normą* $||.||$ *i* $x_0\in V$, *wówczas*

$$f(x)=||x||,f:V\to\mathbb{R}^1 \text{ - ciągła w } x_0.$$

Dowód. Chcemy pokazać, że

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta}{\exists}\ \underset{x}{\forall}\ d_x(x,x_0)<\delta\implies d_{\mathbb{R}}(f(x),f(x_0))<\varepsilon$$

ale

$$d_x(x,y)=||x-y||,d_{\mathbb{R}^1}(x,y)=|x-y|.$$

Czyli pokażemy, że

$$\underset{\varepsilon>0}{\forall}\,\underset{\delta}{\exists}\,\underset{x}{\forall} \quad ||x - x_0|| < \delta \implies \Big|||x|| - ||x_0||\Big| < \varepsilon.$$

Ale wiemy, że

$$||x|| = ||x - y + y|| \leqslant ||x - y|| + ||y||, ||x|| - ||y|| \leqslant ||x - y||,$$

$$||y|| = ||y - x + x|| \leqslant ||y - x|| + ||x||,$$

$$||y|| - ||x|| \leqslant ||x - y||,$$

czyli $|\|x\| - \|y\|| \leqslant \|x - y\|$. Niech $\delta = \frac{\varepsilon}{2}$, otrzymujemy $\varepsilon > \frac{\varepsilon}{2} > ||x-y|| \geqslant \Big|||x||-||y||\Big| \geqslant 0$

$\square$

Pytanie 1. *Niech* $f(x, y) = 7x+6y^2$ *i* $g(t) = \begin{bmatrix} cos(t) \\ sin(t) \end{bmatrix}$. *Wówczas* $h(t) = (f \circ g)(t) : \mathbb{R} \to \mathbb{R}$. *Ile wynosi pochodna?*

$$f' = [7, 12y], g' = \begin{bmatrix} -sin(t) \\ cos(t) \end{bmatrix}$$

Twierdzenie 1. *Niech* $G : U \to Y, U \subset X, U$ *- otwarte,*
X *- przestrzeń wektorowa unormowana,*
$F : G(U) \to Z, G(U) \subset V$
G *- różniczkowalna w* $x_0 \in U$*,*
F *- różniczkowalna w* $G(x_0) \in U$*.*
Wówczas: $(F \circ G)$ *- różniczkowalna w* x_0 *oraz*

$$(F \circ G)'(x_0) = F'(x)|_{x=G(x_0)}\, G'(x_0).$$

Dowód.

$$G(x_0 + h_1) - G(x_0) = G'(x_0)h_1 + r_1(x_0, h_1), \text{ gdy } \frac{r(x_0, h_1)}{||h_1||_x} \to 0$$

$$F(y_0 + h_2) - F(y_0) = F'(y_0)h_2 + r_2(y_0, h_2), \text{ gdy } \frac{r(y_0, h_2)}{||h_2||_y} \to 0$$

$$\begin{aligned} F(G(x_0 + h)) - F(G(x_0)) = & \\ &= F(G(x_0) + G'(x_0)h_1 + r_1(x_0, h_1)) - F(G(x_0)) = \\ &= F(G(x_0)) + F'(G(x_0)) \cdot (G'(x_0)h_1 + r_1(x_0, h_1)) + \\ &= r_2(G(x_0), G'(x_0)h_1 + r_1(x_0, h_1)) - F(G(x_0)). \end{aligned}$$

zatem:

$$\begin{aligned} F(G(x_0 + h)) - F(G(x_0)) = & \\ &= F'(G(x_0)) \cdot G'(x_0)h_1 + F'(G(x_0)) \cdot r_1(x_0, h_1) + \\ &= r_2 \cdot (G(x_0), G'(x_0)h_1 + r_1(x_0, h_1)). \end{aligned}$$

Wystarczy pokazać, że

$$\frac{r_3}{||h_1||} \to 0,$$

ale

$$\frac{r_3}{||h_1||} = F'(G(x_0))\frac{r_1(x_0,h_1)}{||h_1||} +$$
$$+ \underbrace{\frac{r_2(G(x_0), G'(x_0)h_1 + r_1(x_0,h_1))}{||G'(x_0)h_1 + r_1(x_0,h_1)||}}_{\to 0 \text{ kiedy } h_1 \to 0} \cdot \underbrace{\frac{||G'(x_0)h_1 + r_1(x_0,h_1)||}{||h_1||}}_{\text{jest ograniczony}},$$

ale jeżeli $h_1 \to 0$, to $h_2 = G'(x_0)h_1 + r_1(x_0,h_1)$, zatem $F(G(x))$ - różniczkowalna w x_0 $\square$

Przykład 3. $f(x,y) = \begin{bmatrix} 2xy^2 \\ x^3y \end{bmatrix}, \varphi(t) = \begin{bmatrix} 2t^2 \\ t^3 \end{bmatrix}, h(t) = (f \circ \varphi)(t), h : \mathbb{R} \to \mathbb{R}^2.$

Policzmy H'. $f' = \begin{bmatrix} 2y^2 & 4xy \\ 3x^2y & x^3 \end{bmatrix}, \varphi'(t) = \begin{bmatrix} 4t \\ 3t^2 \end{bmatrix}$, *tzn.*

$$H' = \begin{bmatrix} 2y^2 & 4xy \\ 3x^2y & x^3 \end{bmatrix}\Bigg|_{x=2t^2, y=t^3} \cdot \begin{bmatrix} 4t \\ 3t^2 \end{bmatrix} = \begin{bmatrix} 2(2t^2)^2 4t + 4(2t^2)(t^3)3t^2 \\ 3(2t^2)^2 t^3 4 + (2t^3)^3 3t^2 \end{bmatrix}$$

Przykład 4. *Niech* $f : \mathbb{R}^2 \to \mathbb{R}$,
$\Psi : \mathbb{R}^2 \to \mathbb{R}^2$,
$\Psi(r,\varphi) = \begin{bmatrix} \Psi_1(r,\varphi) \\ \Psi_2(r,\varphi) \end{bmatrix}$ $\Psi_1 : \mathbb{R}^2 \to \mathbb{R}$ $\Psi_2 : \mathbb{R}^2 \to \mathbb{R}$

Niech $H(r,\varphi) = (f \circ \Psi)(r,\varphi)$, *czyli* $H : \mathbb{R}^2 \to \mathbb{R}$.
Szukamy pochodnej H, *ale*

$$f' = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}], \Psi' = \begin{bmatrix} \frac{\partial \Psi_1}{\partial r} & \frac{\partial \Psi_1}{\partial \varphi} \\ \frac{\partial \Psi_2}{\partial r} & \frac{\partial \Psi_2}{\partial \varphi} \end{bmatrix}$$

Czyli

$$H' = \Big[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\Big]\Bigg|_{x=\Psi_1(r,\varphi), y=\Psi_1(r,\varphi)} \begin{bmatrix} \frac{\partial \Psi_1}{\partial r} & \frac{\partial \Psi_1}{\partial \varphi} \\ \frac{\partial \Psi_2}{\partial r} & \frac{\partial \Psi_2}{\partial \varphi} \end{bmatrix}$$

Co daje:

$$\Big[\frac{\partial H}{\partial r}, \frac{\partial H}{\partial \varphi}\Big] = \Big[\frac{\partial f}{\partial x}\frac{\partial \Psi_1}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial \Psi_2}{\partial r}, \frac{\partial f}{\partial x}\frac{\partial \Psi_1}{\partial \varphi} + \frac{\partial f}{\partial y}\frac{\partial \Psi_2}{\partial \varphi}\Big]\Bigg|_{x=\Psi_1(r,\varphi), y=\Psi_2(r,\varphi)}$$