

$$\frac{\partial H}{\partial r} = \frac{\partial f}{\partial x} \Big|_{\substack{x=\Psi_1(r,\varphi), \\ y=\Psi_2(r,\varphi)}} \frac{\partial \Psi_1}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial \Psi_2}{\partial r}$$

$$\frac{\partial H}{\partial \varphi} = \frac{\partial f}{\partial x} \frac{\partial \Psi_1}{\partial \varphi} + \frac{\partial f}{\partial y} \frac{\partial \Psi_2}{\partial \varphi}$$

Konwencja z ćwiczeń z fizyki:

$$H(r, \varphi) = (f \circ \Psi)(r, \varphi)$$

$$H(r, \varphi) = f(r, \varphi)$$

$$\Psi_1(r, \varphi) = x(r, \varphi)$$

$$\Psi_2(r, \varphi) = y(r, \varphi)$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial \varphi} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \varphi}$$

Przykład 1

$$f(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \end{cases}$$

$$\frac{\partial f}{\partial r} = \cos \varphi \frac{\partial f}{\partial x} + \sin \varphi \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial \varphi} = -r \sin \varphi \frac{\partial f}{\partial x} + r \cos \varphi \frac{\partial f}{\partial y}$$

$$f(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f' = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]$$

Interpretacja geometryczna Rozważmy zbiór

$$P_c = \{(x, y) \in \mathbb{R}^2 : f(x, y) = c\} \text{ np. } f(x, y) = x^2 + y^2 : P_c = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = c\}$$

Założmy, że $f(x, y)$ - taka, że P_c - można sparametryzować jako

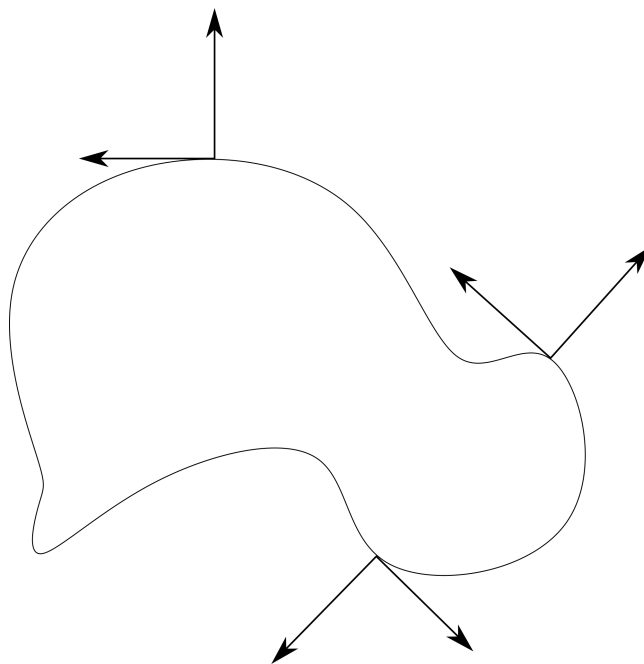
$$\varphi(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, t \in D, \text{ to znaczy, że } P_c = \{(x(t), y(t)), t \in D\}$$

Przykład 2

$$\text{Niech } \varphi(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}. \text{ Wtedy } P_c = \{(c \cos t, c \sin t); t \in [0, 2\pi]\}$$

$$f(x(t), y(t)) = c \quad \forall_{t \in D} \text{ - powierzchnie ekwipotencjalne}$$

$$\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right] \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = 0, \quad \begin{bmatrix} 2x, 2y \end{bmatrix} \begin{bmatrix} -c \sin t \\ c \cos t \end{bmatrix} = 0$$



Rysunek 1: Trajektoria kluki

Definicja 1 *Pochodna mieszana*

$$f(x, y) = x^2 y^3, \quad \frac{\partial f}{\partial x} = 2xy^3, \quad \frac{\partial f}{\partial y} = 3x^2 y^2, \quad \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = 2y^3, \quad \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = 6x^2 y$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6xy^2 \quad \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

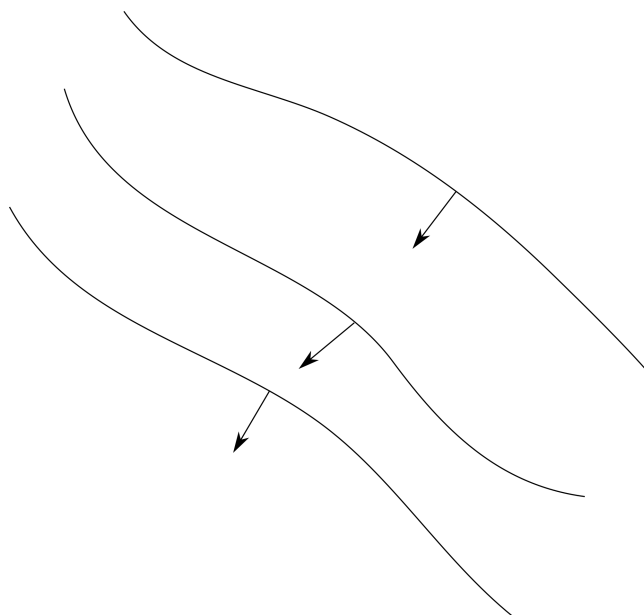
Przypadek???

Twierdzenie 1 *Niech $f : \mathcal{O} \rightarrow \mathbb{R}$, $\mathcal{O} \subset \mathbb{R}^n$, otwarty i $f \in \mathcal{C}^2(\mathcal{O})$, wówczas*

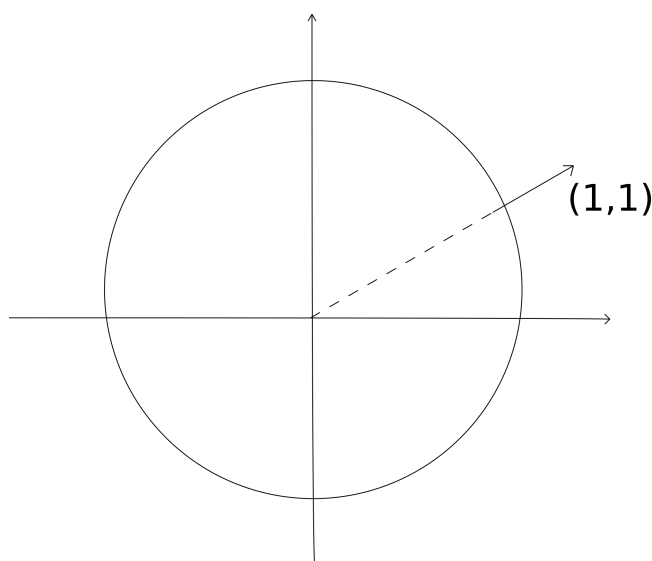
$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}; i, j = 1, \dots, n$$

Dowód 1 *Dowód dla $n = 2$*

$$\begin{aligned} \text{Niech } w(x, y) &= f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y) \\ \varphi(x) &= f(x, y+k) - f(x, y) \end{aligned}$$



Rysunek 2: Powierzchnia ekwipotencjalna I



Rysunek 3: Powierzchnia ekwipotencjalna II

wówczas

$$\begin{aligned} w &= \varphi(x+h) - \varphi(x) = \frac{\partial \varphi}{\partial x}(\xi)h = \\ &= \left[\frac{\partial f}{\partial x}(\xi, y+k) - \frac{\partial f}{\partial x}(\xi, y) \right] h = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x}(\xi, \eta) \right) hk, \\ &\text{gdzie } x < \xi < x+h, \quad y < \eta < y+k \end{aligned}$$

Niech $\Psi(y) = f(x+h, y) - f(x, y)$

$$\begin{aligned} w(x, y) &= \Psi(y+k) - \Psi(y) = \frac{\partial \Psi}{\partial y}(\eta_1)k = \left[\frac{\partial f}{\partial y}(x+h, \eta_1) - \frac{\partial f}{\partial y}(x, \eta_1) \right] k = \\ &\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y}(\xi, \eta) \right) kh, \text{ czyli } \exists_{\xi \in]x, x+h[}, \quad \xi_1 \in]x, x+h[, \quad \eta \in]y, y+k[, \quad \eta_1 \in]y, y+k[\\ &\quad (y < \eta_1 < y+k) \\ &\text{Jeżeli } h \rightarrow 0 \\ &\left(\frac{\partial^2 f}{\partial y \partial x}(\xi, \eta) = \frac{\partial^2 f}{\partial x \partial y}(\xi_1, \eta_1) \right) \\ &\text{to } \xi \rightarrow x, \xi_1 \rightarrow x, \eta \rightarrow y, \eta_1 \rightarrow y, \text{ czyli:} \end{aligned}$$

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{\partial^2 f}{\partial x \partial y}(x, y)$$

Jeżeli każda z tych wielkości jest ciągła \square

Wzór Taylora Niech $f: \mathcal{O} \rightarrow \mathbb{R}, \mathcal{O} \subset \mathbb{R}^n$ - otwarty
 $\varphi(t) = f(x_0 + th), h \in \mathbb{R}^n, t \in [0, 1]$

$$\text{Dla } h = \begin{bmatrix} h^1 \\ \vdots \\ h^n \end{bmatrix}, x_0 = \begin{bmatrix} x_0^1 \\ \vdots \\ x_0^n \end{bmatrix}, \varphi(t) = f(x_0^1 + th^1, x_0^2 + th^2, \dots, x_0^n + th^n)$$

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \frac{\partial f}{\partial x^1} \Big|_{x=x_0+th} h_1 + \frac{\partial f}{\partial x^2} \Big|_{x=x_0+th} h_2 + \dots + \frac{\partial f}{\partial x^n} \Big|_{x=x_0+th} h_n = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \Big|_{x_0+th} h_i \\ \frac{\partial^2 \varphi}{\partial t^2} &= \sum_{j=1}^n \sum_{i=1}^n \frac{\partial^2 f}{\partial x^i \partial x^j} \Big|_{x_0+th} h_j h_i \\ &\vdots \end{aligned}$$

$$\begin{aligned} \frac{\partial^k \varphi}{\partial t^k} &= \sum_{i^1, \dots, i^k} \frac{\partial^{(k)} f}{\partial x^{i^1} \dots \partial x^{i^k}} h_{i_1} \dots h_{i_k} \\ \varphi(t) &= \varphi(0) = \varphi'(0)(t-0) + \frac{\varphi''(0)}{2!}(t-0)^2 + \dots + \frac{\varphi^{(k)}(0)}{k!}(t-0)^k + r(\dots) \\ &\text{Czyli:} \end{aligned}$$

$$\varphi(1) - \varphi(0) = \varphi'(0) + \frac{\varphi''(0)}{2!} + \frac{\varphi'''(0)}{3!} + \dots + \frac{\varphi^{(k)}(0)}{k!} + r(\dots)$$

$$f(x_0 + h) - f(x_0) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(x_0) h_i + \frac{1}{2!} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0) h_i h_j + \dots \square$$