

Uwaga: Jeżeli np. $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, to znaczy, że
 $f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}$, $f_1: \mathbb{R}^2 \rightarrow \mathbb{R}^1$, $f_2: \mathbb{R}^2 \rightarrow \mathbb{R}^1$, wówczas

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial}{\partial x} f_1 \\ \frac{\partial}{\partial x} f_2 \end{bmatrix}$$

$$\frac{\partial}{\partial y} f = \begin{bmatrix} \frac{\partial}{\partial y} f_1 \\ \frac{\partial}{\partial y} f_2 \end{bmatrix}$$

Przykład 1

$$f(x, y) = \begin{bmatrix} 2xy^2 \\ x^3y \end{bmatrix}$$

Wtedy pochodne czątkowe: $\frac{\partial f}{\partial x} = \begin{bmatrix} 2y^2 \\ 3x^2y \end{bmatrix}$, $\frac{\partial f}{\partial y} = \begin{bmatrix} 4xy \\ x^3 \end{bmatrix}$

$$f(x+h) - f(x) = \frac{\partial f}{\partial x} h^x + \frac{\partial f}{\partial y} h^y + r((x, y), h) = \begin{bmatrix} 2y^2 \\ 3x^2y \end{bmatrix} h^x + \begin{bmatrix} 4xy \\ x^3 \end{bmatrix} h^y + r((x, y), h) = \begin{bmatrix} 2y^2 & 4xy \\ 3x^2y & x^3 \end{bmatrix} \begin{bmatrix} h^x \\ h^y \end{bmatrix} + r((x, y), h)$$

$$\text{Czyli } f' = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

i ogólniej: jeżeli $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$, to

$$f' = \begin{bmatrix} \frac{\partial f_1}{\partial x^1} & \cdots & \frac{\partial f_1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x^1} & \cdots & \frac{\partial f_k}{\partial x^n} \end{bmatrix}$$

0.1 Uzupełnienie:

Niech V - przestrzeń wektorowa z normą $||\cdot||$ i $x_0 \in V$,
 wówczas $f(x) = ||x||$, $f: V \rightarrow \mathbb{R}^1$ - ciągła w x_0 .

Dowód 1

Chcemy pokazać, że $\forall \epsilon > 0 \exists \delta > 0 \forall x \quad d_x(x, x_0) < \delta \implies d_{\mathbb{R}}(f(x), f(x_0)) < \epsilon$
 ale $d_x(x, y) = ||x - y||$, $d_{\mathbb{R}^1}(x, y) = |x - y|$.

Chcemy pokazać, że $\forall \epsilon > 0 \exists \delta > 0 \forall x \quad ||x - x_0|| < \delta \implies |||x| - |x_0||| < \epsilon$
 ale $||x|| = ||x - y + y|| \leq ||x - y|| + ||y||$, $||x|| - ||y|| \leq ||x - y||$,
 $||y|| = ||y - x + x|| \leq ||y - x|| + ||x||$,
 $||y|| - ||x|| \leq ||x - y||$, czyli $|||x| - |y||| \leq ||x - y||$. Niech $\delta = \frac{\epsilon}{2}$, otrzymujemy $\epsilon > \frac{\epsilon}{2} > ||x - y|| \leq |||x| - |y||| \geq 0 \square$

Pytanie 1 Niech $f(x, y) = 7x + 6y^2$ i $g(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$. Wówczas $h(t) = (f \circ g)(t): \mathbb{R} \rightarrow \mathbb{R}$. Ile wynosi pochodna?

$$f' = [7, 12y], g' = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

Twierdzenie 1 Niech $G : U \rightarrow Y, U \subset X, U$ - otwarte, X - przestrzeń wektorowa unormowana, $F : G(U) \rightarrow Z, G(U) \subset V$
 G - różniczkowalna w $x_0 \in U$, F - różniczkowalna w $G(x_0) \in U$.

$$G(x_0 + h_1) - G(x_0) = G'(x_0)h_1 + r_1(x_0, h_1), \text{ gdy } \frac{r(x_0, h_1)}{\|h_1\|_x} \rightarrow 0$$

$$F(y_0 + h_2) - F(y_0) = F'(y_0)h_2 + r_2(y_0, h_2), \text{ gdy } \frac{r(y_0, h_2)}{\|h_2\|_y} \rightarrow 0$$

Wówczas: $(F \circ G)$ - różniczkowalna w x_0

$$\text{oraz } (F \circ G)'(x_0) = F'(x)|_{x=G(x_0)} G'(x_0)$$

Dowód 2

$$\begin{aligned} F(G(x_0 + h)) - F(G(x_0)) &= \\ F(G(x_0) + G'(x_0)h_1 + r_1(x_0, h_1)) - F(G(x_0)) &= \\ F(G(x_0)) + F'(G(x_0))(G'(x_0)h_1 + r_1(x_0, h_1)) + r_2(G(x_0)) &= \\ G'(x_0)h_1 + r_1(x_0, h_1)) - F(G(x_0)) &= \end{aligned}$$

zatem:

$$F(G(x_0)) + F(G(x_0 + h)) = F'(G(x_0))G'(x_0)h_1 + F'(G(x_0))r_1(x_0, h_1) + r_2(G(x_0), G'(x_0)h_1 + r_1(x_0, h_1))$$

Wystarczy pokazać, że $\frac{r_3}{\|h_1\|} \rightarrow 0$, ale $\frac{r_3}{\|h_1\|} = F'(G(x_0)) \frac{r_1(x_0, h_1)}{\|h_1\|} + \underbrace{\frac{r_2(G(x_0), G'(x_0)h_1 + r_1(x_0, h_1))}{\|G'(x_0)h_1 + r_1(x_0, h_1)\|}}_{\rightarrow 0 \text{ kiedy } h_1 \rightarrow 0}$

$$\underbrace{\frac{\|G'(x_0)h_1 + r_1(x_0, h_1)\|}{\|h_1\|}}_{\text{jest ograniczony}}, \text{ ale jeżeli } h_1 \rightarrow 0, \text{ to } h_2 = G'(x_0)h_1 + r_1(x_0, h_1), \text{ zatem } F(G(x)) - \text{różniczkowalna w } x_0 \quad \square$$

Przykład 2

$$f(x, y) = \begin{bmatrix} 2xy^2 \\ x^3y \end{bmatrix}, \varphi(t) = \begin{bmatrix} 2t^2 \\ t^3 \end{bmatrix}, h(t) = (f \circ \varphi)(t), h : \mathbb{R} \rightarrow \mathbb{R}^2.$$

$$\text{Policzmy } h'. f' = \begin{bmatrix} 2y^2 & 4xy \\ 3x^2y & x^3 \end{bmatrix}, \varphi'(t) = \begin{bmatrix} 4t \\ 3t^2 \end{bmatrix}, \text{ tzn. } H' = \begin{bmatrix} 2y^2 & 4xy \\ 3x^2y & x^3 \end{bmatrix} \Big|_{x=2t^2, y=t^3} \begin{bmatrix} 4t \\ 3t^2 \end{bmatrix} =$$

$$\begin{bmatrix} 2(2t^2)^2 4t + 4(2t^2)(t^3) 3t^2 \\ 3(2t^2)^2 t^3 4 + (2t^3)^3 3t^2 \end{bmatrix}$$

$$\text{Weźmy przykład: Niech } f : \mathbb{R}^2 \rightarrow \mathbb{R}, \Psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \Psi(r, \varphi) = \begin{bmatrix} \Psi_1(r, \varphi) \\ \Psi_2(r, \varphi) \end{bmatrix}$$

$$\Psi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\Psi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\text{Niech } H(r, \varphi) = (f \circ \Psi)(r, \varphi), \text{ czyli } H : \mathbb{R}^2 \rightarrow \mathbb{R}.$$

$$\text{Szukamy pochodnej } H, \text{ ale } f' = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right], \Psi' = \begin{bmatrix} \frac{\partial \Psi_1}{\partial r} & \frac{\partial \Psi_1}{\partial \varphi} \\ \frac{\partial \Psi_2}{\partial r} & \frac{\partial \Psi_2}{\partial \varphi} \end{bmatrix}$$

$$\text{Czyli } H' = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right] \bigg|_{x=\Psi_1(r,\varphi), y=\Psi_1(r,\varphi)} \begin{bmatrix} \frac{\partial \Psi_1}{\partial r} & \frac{\partial \Psi_1}{\partial \varphi} \\ \frac{\partial \Psi_2}{\partial r} & \frac{\partial \Psi_2}{\partial \varphi} \end{bmatrix}$$

$$\text{Co daje: } \left[\frac{\partial H}{\partial r}, \frac{\partial H}{\partial \varphi} \right] = \left[\frac{\partial f}{\partial x} \frac{\partial \Psi_1}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial \Psi_2}{\partial r}, \frac{\partial f}{\partial x} \frac{\partial \Psi_1}{\partial \varphi} + \frac{\partial f}{\partial y} \frac{\partial \Psi_2}{\partial \varphi} \right] \bigg|_{x=\Psi_1(r,\varphi), y=\Psi_2(r,\varphi)}$$