

Rysunek 1: (a)

Zabawki działające dzięki wnioskowi z Tw. wyżej

**Definicja 1** *Funkcje uwikłane*

$$x + y = 1 \quad (\text{a}).$$

$$x^2 + y^2 = 1 \quad (\text{b}).$$

$$H(x, y) = \sin x e^{xy} + \operatorname{tg} y - x = 0.$$

**Przykład 1** *Równanie gazowe*

$$H(p, V, T) = 0, H : \mathbb{R}^3 \rightarrow \mathbb{R}^1.$$

$$p(V, T) = 0, \mathbb{R}^2 \rightarrow \mathbb{R}^1.$$

$$V(p, T) = 0, \mathbb{R}^2 \rightarrow \mathbb{R}^1.$$

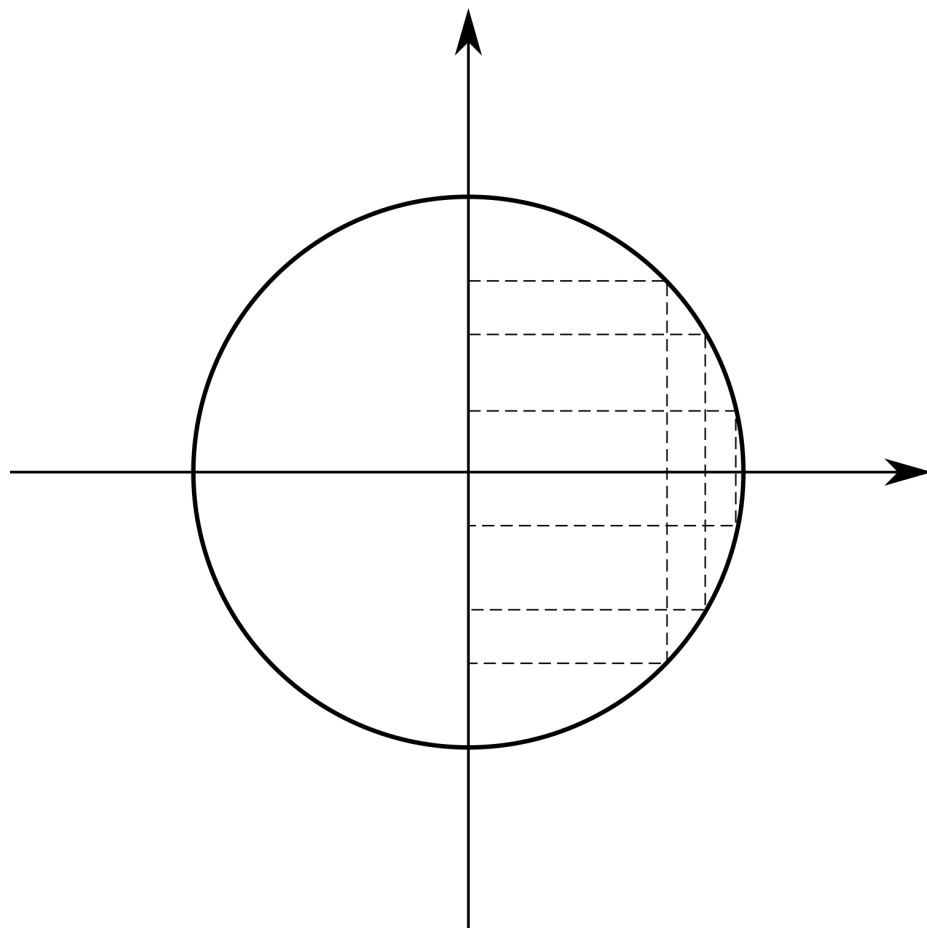
$$T(p, V) = 0, \mathbb{R}^2 \rightarrow \mathbb{R}^1.$$

*istnienie przedziałów, w których funkcja uwikłana zadaje inne funkcje*

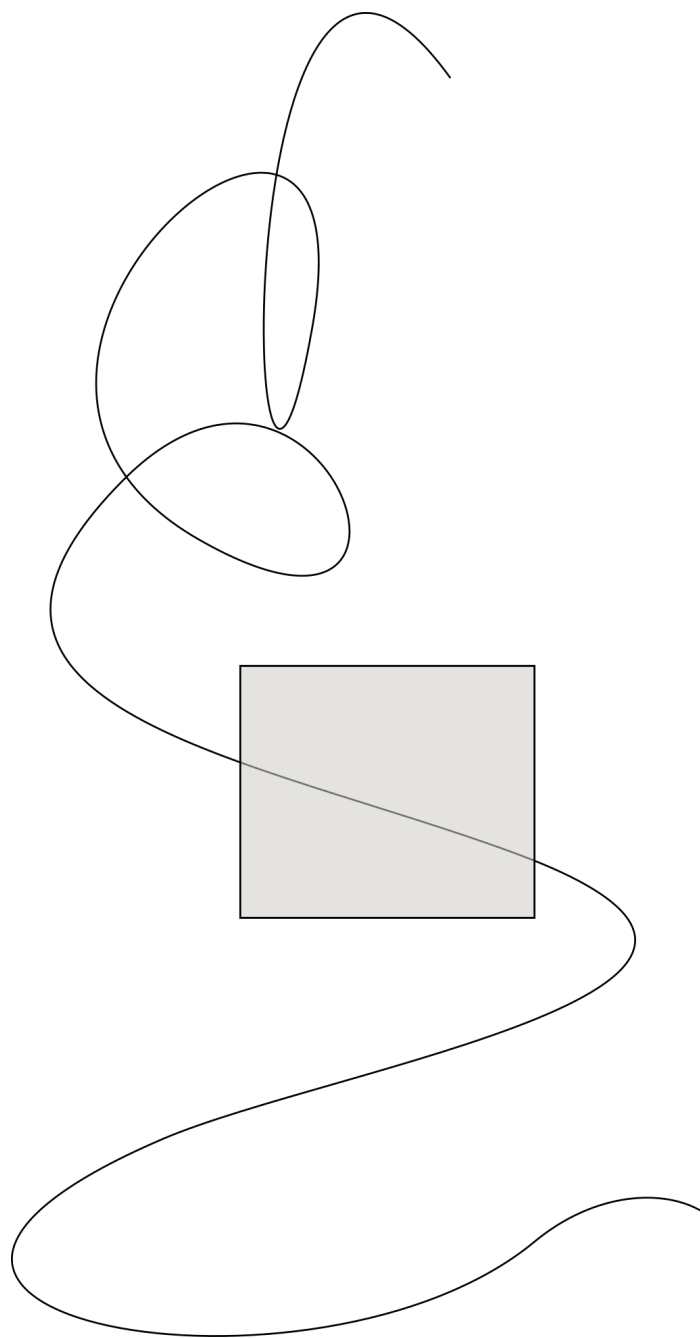
**Przykład 2**

$$H(x, y) : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^1.$$

**Pytanie 1** *Czy istnieje  $y(x) : H(x, y(x)) = 0$ , dla  $x \in V$ ?*



Rysunek 2: (b)



Rysunek 3: (c)

$$\frac{dH}{dx}(x, y(x)) = \frac{d}{dx}(H(x, y) \circ g(x)).$$

$$H' = \left[ \frac{\partial H}{\partial x}, \frac{\partial H}{\partial y} \right].$$

$$g(x) : \mathbb{R}^1 \rightarrow \mathbb{R}^2, g(x) = \begin{bmatrix} x \\ y(x) \end{bmatrix}, g'(x) = \begin{bmatrix} 1 \\ y'(x) \end{bmatrix}.$$

$$H'(x, y)g'(x) = 0 \implies \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} \frac{\partial y}{\partial x} = 0 \implies \frac{\partial y}{\partial x} = -\frac{\frac{\partial H}{\partial x}}{\frac{\partial H}{\partial y}}.$$

Więc

$$\frac{\partial y}{\partial x} = \frac{-\cos y + ye^{xy} - 1}{xe^y + \frac{1}{\cos^2 y}}.$$

### Przykład 3

$$H(x_1, x_2, x_3, x_4, x_5) = \begin{bmatrix} 2e^{x_1} + x_2x_3 - 4x_3 + 3 \\ x^2 \cos x_1 - 6x_1 + 2x_3 - x_5, H : \mathbb{R}^5 \rightarrow \mathbb{R}^3 \end{bmatrix}.$$

$$H(x_1, \dots, x_5) = 0 \text{ może zadać funkcję } g : \mathbb{R}^3 \rightarrow \mathbb{R}^2.$$

$$x_4(x_1, x_2, x_3), x_5(x_1, x_2, x_3).$$

$$g(x_1, g_2, g_3) = \begin{bmatrix} g_1(x_1, x_2, x_3) \\ g_2(x_1, x_2, x_3) \end{bmatrix}.$$

**Obserwacja 1**  $H(0, 1, 3, 2, 7) = 0$

$$H : \mathbb{R}^5 \rightarrow \mathbb{R}^2, H(x_1, x_2, y_1, y_2, y_3) = 0.$$

$$H(x_1, x_2, y_1, y_2, y_3) = \begin{bmatrix} H_1(x_1, x_2, y_1, y_2, y_3) \\ H_2(x_1, x_2, y_1, y_2, y_3) \end{bmatrix}.$$

**Pytanie 2** Czy  $H(x_1, x_2, y_1, y_2, y_3) = 0$  zadaje nam

$$g_1(y_1, y_2, y_3).$$

$$g_2(y_1, y_2, y_3)?$$

$$\text{czyli } g(y_1, y_2, y_3) = \begin{bmatrix} g_1(y_1, y_2, y_3) \\ g_2(y_1, y_2, y_3) \end{bmatrix}, g : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$H_1(g_1(y_1, y_2, y_3), g_2(y_1, y_2, y_3), y_1, y_2, y_3) = 0.$$

$$H_2(g_1(y_1, y_2, y_3), g_2(y_1, y_2, y_3), y_1, y_2, y_3) = 0.$$

Szukamy  $g'$ .

$$g' = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \frac{\partial g_2}{\partial y_3} \end{bmatrix}.$$

$$\frac{\partial H_1}{\partial x_1} \frac{\partial g_1}{\partial y_2} + \frac{\partial H_1}{\partial x_2} \frac{\partial g_2}{\partial y_1} + \frac{\partial H_1}{\partial y_1} = 0.$$

$$\frac{\partial H_1}{\partial x_1} \frac{\partial g_1}{\partial y_3} + \frac{\partial H_1}{\partial x_2} \frac{\partial g_2}{\partial y_1} + \frac{\partial H_1}{\partial y_1} = 0.$$

$$\frac{\partial H_1}{\partial x_1} \frac{\partial g_1}{\partial y_3} + \frac{\partial H_1}{\partial x_2} \frac{\partial g_2}{\partial y_3} + \frac{\partial H_1}{\partial y_3} = 0.$$

$$\frac{\partial H_2}{\partial x_1} \frac{\partial g_1}{\partial y_1} + \frac{\partial H_2}{\partial x_2} \frac{\partial g_2}{\partial y_1} + \frac{\partial H_2}{\partial y_1} = 0.$$

$$\frac{\partial H_2}{\partial x_1} \frac{\partial g_1}{\partial y_2} + \frac{\partial H_2}{\partial x_2} \frac{\partial g_2}{\partial y_2} + \frac{\partial H_2}{\partial y_2} = 0.$$

$$\frac{\partial H_2}{\partial x_1} \frac{\partial g_1}{\partial y_3} + \frac{\partial H_2}{\partial x_2} \frac{\partial g_2}{\partial y_3} + \frac{\partial H_2}{\partial y_3} = 0.$$

napięcie rośnie (6 równań oho)

$$\begin{matrix} \begin{bmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} \end{bmatrix} & \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \frac{\partial g_2}{\partial y_3} \end{bmatrix} & = - \begin{bmatrix} \frac{\partial H_1}{\partial y_1} & \frac{\partial H_1}{\partial y_2} & \frac{\partial H_1}{\partial y_3} \\ \frac{\partial H_2}{\partial y_1} & \frac{\partial H_2}{\partial y_2} & \frac{\partial H_2}{\partial y_3} \end{bmatrix} \\ H'_x & g' & H'_y \end{matrix}$$

$$H'_x g' = -H'_y \implies g' = -(H'_x)^{-1} H'_y.$$

**Twierdzenie 1** (o funkcji uwikłanej)

Niech  $H : E \subset \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$ ,  $H \in \mathcal{C}^1$  na  $E$ .  $(x_0, y_0) \in E$ ,  $H(x_0, y_0) = 0$ ,  $(x_0, y_0) = (x_0^1, \dots, x_0^n, y_0^1, \dots, y_0^m)$ ,  $H$  - odwracalna.

Wówczas istnieje  $U \subset E$  takie, że  $(x_0, y_0) \in U$ ,  $\exists_{W \subset \mathbb{R}^n}$ , że  $x_0 \in W$ ,  $\forall_{x \in W} \exists! y$   $H(x, y) = 0$ ,  $(x, y) \in U$ .

Jeżeli  $y = \varphi(x)$ , to  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^m$  i  $\varphi \in \mathcal{C}^1$  na  $W$ .  $\varphi'(x) = -(H'_y)^{-1} H'_x$

**Dowód 1** Oznaczenia:

$$H(x^1, \dots, x^n, y^1, \dots, y^m) = \begin{bmatrix} H^1(x^1, \dots, x^n, y^1, \dots, y^m) \\ \vdots \\ H^m(x^1, \dots, x^n, y^1, \dots, y^m) \end{bmatrix}.$$

$$H'_y = \begin{bmatrix} \frac{\partial H^1}{\partial y^1} & \dots & \frac{\partial H^1}{\partial y^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial H^m}{\partial y^1} & \dots & \frac{\partial H^m}{\partial y^m} \end{bmatrix}, H'_x = \begin{bmatrix} \frac{\partial H^1}{\partial x^1} & \dots & \frac{\partial H^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial H^m}{\partial x^1} & \dots & \frac{\partial H^m}{\partial x^n} \end{bmatrix}.$$

Wprowadźmy funkcję  $F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m}$

$$F(x^1, \dots, x^n, y^1, \dots, y^m) = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \\ H^1(x^1, \dots, x^n, y^1, \dots, y^m) \\ \vdots \\ H^m(x^1, \dots, x^n, y^1, \dots, y^m) \end{bmatrix}.$$

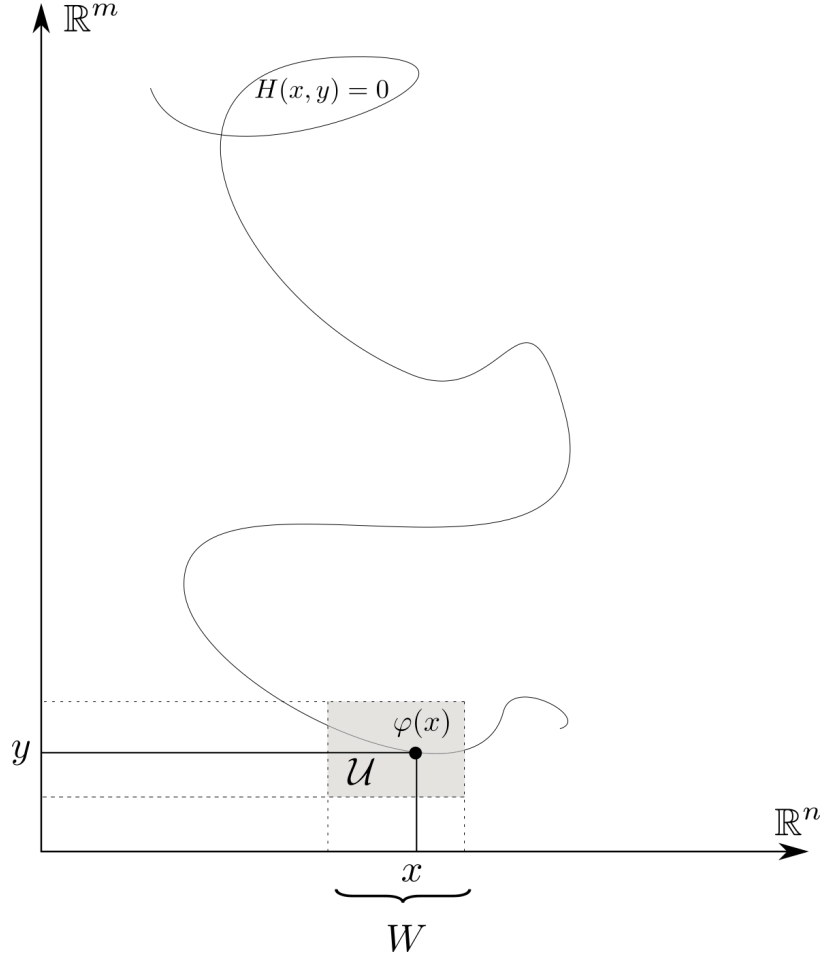
Jakie własności ma  $F$ ?

$$F(x_0, y_0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

Ale

$$F' = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \\ H'_x & & H'_y \end{bmatrix}, \det F' = \det H'_y.$$

Jeżeli  $H'_y(x_0, y_0)$  - odwracalna, to  $F'(x_0, y_0)$  - też. Oznacza to (na podstawie tw. o lokalnej odwra-



Rysunek 4

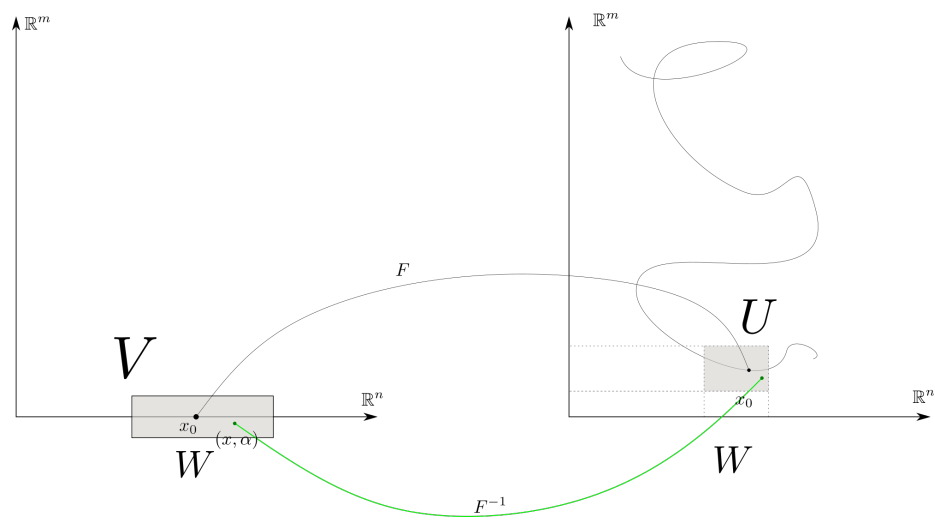
calności), że

$$\exists_{U \subset \mathbb{R}^{n+m}}, (x_0, y_0) \in U, \exists_{V \subset \mathbb{R}^{n+m}}, (x_0, 0) \in V,$$

że  $F$  jest bijekcją między  $U$  i  $V$  oraz  $\exists F^{-1} : V \rightarrow U$ ,  $F^{-1}$  - różniczkowalna taka, że

$$F^{-1}(x, \alpha) = (a(x, \alpha), b(x, \alpha)), x, \alpha \in V,$$

gdzie  $a(x, \alpha) : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n$ ,  $b(x, \alpha) : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m$



Rysunek 5