

W ostatnim odcinku:  
 $M, N$  - rozmaitości,  $\dim M = n, \dim N = k, h : M \rightarrow N$ ,  $\langle h^* \alpha, v \rangle = \langle \alpha, h_* v \rangle$  i ogólnie, jeżeli  $\alpha_1, \dots, \alpha_k \in \Lambda^1(N)$  to  $\langle h^*(\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_k), v_1, \dots, v_n \rangle = \langle \alpha_1 \wedge \dots \wedge \alpha_k, h_* v_1, \dots, h_* v_n \rangle$ .

**Przykład 1** Niech  $N = \mathbb{R}^2$  i  $M = \mathbb{R}^1$ ,  $\alpha = 7dx \wedge dy \in \Lambda^2(N)$ ,

$$h(t) = \begin{bmatrix} 2t \\ 3t \end{bmatrix} \rightarrow (x = 2t, y = 3t \implies dx = 2dt, dy = 3dt).$$

$$h^* \alpha = 7 \cdot 2dt \wedge 3dt = h^* \alpha = 0.$$

Ostatnio chcieliśmy pokazać, że  $d(h^* f) = h^*(df)$ . To jest istotne w kontekście tej dwuformy przekształcenia transformacji Lorentza co była ostatnio. ( $d(h^* F) = 0 \implies dF = 0, h^* F \xrightarrow{h} F$ ).

Wzięliśmy sobie  $f : N \rightarrow \mathbb{R} : f(x_1, \dots, x_k)$ . Potem mieliśmy  $h : M \rightarrow N : h(t_1, \dots, t_n) = \begin{bmatrix} h^1(t_1, \dots, t_n) \\ \vdots \\ h^k(t_1, \dots, t_n) \end{bmatrix}$  i chcieliśmy pokazać, że  $h^*(df) = d(h^* f)$ .

Wiemy, że  $\langle h^*(df), v \rangle = \langle df, h_* v \rangle$  ( $v \in T_p M : v = a_1 \frac{\partial}{\partial t^1} + \dots + a_n \frac{\partial}{\partial t^n}$ ). Przepchnięcie wektorka  $h_* v = \begin{pmatrix} [h'] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \end{pmatrix} \begin{matrix} \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k} \end{matrix} = \begin{bmatrix} \frac{\partial h^1}{\partial t^1} & \dots & \frac{\partial h^1}{\partial t^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h^k}{\partial t^1} & \dots & \frac{\partial h^k}{\partial t^n} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \left( a_1 \frac{\partial h^1}{\partial t^1} + \dots + a_n \frac{\partial h^1}{\partial t^n} \right) \frac{\partial}{\partial x^1} + \dots + \left( a_1 \frac{\partial h^k}{\partial t^1} + \dots + a_n \frac{\partial h^k}{\partial t^n} \right) \frac{\partial}{\partial x^k}.$

$$df = \frac{\partial f}{\partial x^1} dx^1 + \dots + \frac{\partial f}{\partial x^k} dx^k.$$

$$\begin{aligned} \langle df, h_* v \rangle &= \frac{\partial f}{\partial x^1} \frac{\partial h^1}{\partial t^1} a_1 + \frac{\partial f}{\partial x^1} \frac{\partial h^1}{\partial t^2} a_2 + \dots + \frac{\partial f}{\partial x^1} \frac{\partial h^1}{\partial t^n} a_n + \dots + \frac{\partial f}{\partial x^1} \frac{\partial h^k}{\partial t^1} a_1 + \dots + \frac{\partial f}{\partial x^k} \frac{\partial h^k}{\partial t^n} a_n = \\ &= a_1 \left( \frac{\partial f}{\partial x^1} \frac{\partial h^1}{\partial t^1} + \frac{\partial f}{\partial x^2} \frac{\partial h^2}{\partial t^1} + \dots + \frac{\partial f}{\partial x^k} \frac{\partial h^k}{\partial t^1} \right) + \dots + a_n \left( \frac{\partial f}{\partial x^1} \frac{\partial h^1}{\partial t^n} + \frac{\partial f}{\partial x^2} \frac{\partial h^2}{\partial t^n} + \dots + \frac{\partial f}{\partial x^k} \frac{\partial h^k}{\partial t^n} \right) = \\ &= \left\langle ?, a_1 \frac{\partial}{\partial t^1} + a_2 \frac{\partial}{\partial t^2} + \dots + a_n \frac{\partial}{\partial t^n} \right\rangle \\ &= \left\langle \left( \frac{\partial f}{\partial x^1} \frac{\partial h^1}{\partial t^1} + \frac{\partial f}{\partial x^2} \frac{\partial h^2}{\partial t^1} + \dots + \frac{\partial f}{\partial x^k} \frac{\partial h^k}{\partial t^1} \right) dt^1 + \dots + \left( \frac{\partial f}{\partial x^1} \frac{\partial h^1}{\partial t^n} + \dots + \frac{\partial f}{\partial x^k} \frac{\partial h^k}{\partial t^n} \right) dt^n, a_1 \frac{\partial}{\partial t^1}, \dots, a_n \frac{\partial}{\partial t^n} \right\rangle = \\ &= \left\langle \underbrace{f(h^1(t^1, \dots, t^n), h^2(t^1, \dots, t^n), \dots, h^k(t^1, \dots, t^n))}_{h^* f}, \frac{\partial}{\partial t^1}, \dots, a_n \frac{\partial}{\partial t^n} \right\rangle = \langle d(h^* f), v \rangle \end{aligned}$$

co daje

$$d(h^*(\alpha_1 \wedge \dots \wedge \alpha_k)) = h^*(d(\alpha_1 \wedge \dots \wedge \alpha_k)) \quad \square.$$

## 0.1 Bazy w $T_p M$

Obserwacja: Niech  $M$  - rozmaitość i  $\langle | \rangle$  - iloczyn skalarny. Niech  $e_1, \dots, e_n$  - baza  $T_p M$ . Wówczas, jeżeli  $v = a_1 e_1 + \dots + a_n e_n$  i  $w = b_1 e_1 + \dots + b_n e_n$  ( $a_i, b_i \in \mathbb{R}, i = 1, \dots, n$ ).

$$\begin{aligned} \langle v | w \rangle &= \langle a_1 e_1 + \dots + a_n e_n, b_1 e_1 + \dots + b_n e_n \rangle = \\ &= a_1 b_1 \langle e_1 | e_1 \rangle + a_1 b_2 \langle e_1 | e_2 \rangle + \dots + a_1 b_n \langle e_1 | e_n \rangle + \dots + a_n b_n \langle e_n | e_n \rangle = \\ &= \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}^T \begin{bmatrix} \langle e_1 | e_1 \rangle & \langle e_1 | e_2 \rangle & \dots & \langle e_1 | e_n \rangle \\ \vdots & \ddots & & \\ \langle e_n | e_1 \rangle & \dots & \dots & \langle e_n | e_n \rangle \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}. \end{aligned}$$

Macierz  $[g_{ij}]$  nazywamy tensorem metrycznym  $\det [g_{ij}] \stackrel{\text{ozn}}{=} g \cdot [g_{ij}]^{-1} \stackrel{\text{ozn}}{=} [g^{ij}]$  - macierz odwrotna.

W zwykłym  $\mathbb{R}^4$ :  $[g_{ij}] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ , p. Minkowskiego:  $g_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \mu, \nu = 0, \dots, 3$

Bazy w  $\mathbb{R}$

$$\begin{array}{ccc} M = \mathbb{R}^2, & & N = \mathbb{R}^2 \\ \begin{bmatrix} x, y \\ e_x, e_y \\ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \end{bmatrix} & \xrightarrow{x=r \cos \varphi, y=r \sin \varphi} & \begin{bmatrix} r, \varphi \\ e_r, e_\varphi \\ \frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi} \end{bmatrix} \\ g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & & [?] \end{array}$$

$$h^*(e_r) = \left( [h'] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)_{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}}, h^*(e_\varphi)$$

$$h(r, \varphi) = \begin{bmatrix} r \cos \varphi \\ r \sin \varphi \end{bmatrix}, h' = \begin{bmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{bmatrix}$$

$$h^*(e_r) = \begin{bmatrix} \cos \varphi \\ \sin \varphi \end{bmatrix}_{e_x, e_y}, e_r = \cos \varphi e_x + \sin \varphi e_y$$

$$z = \cos \varphi e_x + \sin \varphi e_y$$

$$h^*(e_\varphi) = [h'] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -r \sin \varphi \\ r \cos \varphi \end{bmatrix}, e_\varphi = -r \sin \varphi e_x + r \cos \varphi e_y$$

$$\frac{\partial}{\partial \varphi} = -r \sin \varphi \frac{\partial}{\partial x} + r \cos \varphi \frac{\partial}{\partial y}$$

$$g_{ij} = \begin{bmatrix} \langle e_1 | e_1 \rangle & \langle e_1 | e_2 \rangle \\ \langle e_2 | e_1 \rangle & \langle e_2 | e_2 \rangle \end{bmatrix}, [g_{ij}]_{x,y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \langle e_x | e_x \rangle = 1, \langle e_x | e_y \rangle = 0$$

$$\langle e_r | e_r \rangle = \langle \cos \varphi e_x + \sin \varphi e_y | \cos \varphi e_x + \sin \varphi e_y \rangle = \cos^2 \varphi \langle e_x | e_x \rangle + \sin^2 \varphi \langle e_y | e_y \rangle$$

$$\langle e_r | e_\varphi \rangle = \langle \cos \varphi e_x + \sin \varphi e_y | -r \sin \varphi e_x + r \cos \varphi e_y \rangle = 0$$

$$\left\| \frac{\partial}{\partial \varphi} \right\|^2 = \langle e_\varphi | e_\varphi \rangle = \langle -r \sin \varphi e_x + r \cos \varphi e_y | -r \sin \varphi e_x + r \cos \varphi e_y \rangle = r^2.$$

$$\left\| \frac{\partial}{\partial \varphi} \right\| = r, [g_{ij}]_{r,\varphi} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}.$$

baza  $\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi} \right\rangle$  nie jest bazą ortonormalną!!!

$$e_x, e_y, e_z \rightarrow g_{ij} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \text{ - jest fajnie.}$$

$$e_r, e_\theta, e_\varphi \rightarrow \begin{bmatrix} 1 & & \\ & r^2 & \\ & r^2 & \sin^2 \theta \end{bmatrix}, \|e_\theta\| = r, \|e_\varphi\| = r \sin \theta$$

**Przykład 2** Dostałem wektorek  $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$  w sferycznych. Ale w jakiej konkretnie bazie?

W fizyce mierzone wielkości np. wektorowe podajemy zawsze we współrzędnych ortonormalnych.

We współrzędnych sferycznych mamy dwie bazy: - ortogonalną:  $e_r, e_\theta, e_\varphi : \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi} \right)$

- ortonormalną:  $i_r, i_\theta, i_\varphi : \left( \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right)$ . Więc jeżeli ktoś powiedział, że dostał  $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$  to znaczy,

że ma  $2 \frac{\partial}{\partial r} + 3 \frac{1}{r} \frac{\partial}{\partial \theta} + 4 \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$ .

Obserwacja: niech  $v = a_1 e_1 + a_2 e_2 + a_3 e_3$  i niech  $w = b_1 e_1 + b_2 e_2 + b_3 e_3$  i niech  $g_{ij} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$  - tensor metryczny. Wówczas wiemy, że  $\langle v|w \rangle = [v]^T [g_{ij}] [w] = \underbrace{\left[ a_1 g_{11} + a_2 g_{21} + a_3 g_{31}, \sum_{i=1}^3 a_i g_{i2}, \sum_{i=1}^3 a_i g_{i3} \right]}_{\langle v|}$

Ale w sumie to mogę wziąć coś takiego  $\langle v|$ .

$$\left( \sum_{i=1}^3 a^i g_{i1} \right) dx^1 + \left( \sum_{i=1}^3 a^i g_{i2} \right) dx^2 + \left( \sum_{i=1}^3 a^i g_{i3} \right) dx^3 = .$$

$$= \sum_{i=1}^3 \sum_{j=1}^3 a^i g_{ij} dx^j = a^i g_{ij} dx^j.$$

Zapomniałem o sumach, bo  $a^i b_i \stackrel{\text{ozn}}{=} a^1 b_1 + a^2 b_2 + a^3 b_3$ , w odróżnieniu od  $a^\mu b_\mu = a^0 b_0 + a^1 b_1 + \dots$  (Konwencja sumacyjna Einsteina).

Ozn.  $\sum_{i=1}^3 a^i g_{ik} \stackrel{\text{ozn}}{=} a^i g_{ik} = a_k$

**Definicja 1** niech  $M$  - rozmaitość wymiaru  $n$ ,  $g_{ij}$  - tensor metryczny na  $M$ , operację  $\sharp : T_p M \rightarrow T_p^* M$  taką, że dla  $v = a^1 \frac{\partial}{\partial x^1} + \dots + a^n \frac{\partial}{\partial x^n}$ ,

$$v^\sharp = a^i g_{i1} dx^1 + a^i g_{i2} dx^2 + \dots + a^i g_{in} dx^n, i = 1, \dots, n.$$

zadaje izomorfizm między  $T_p M$  a  $T_p^* M$ .

**Przykład 3**  $v = 7 \frac{\partial}{\partial r} + 8 \frac{\partial}{\partial \theta} + 9 \frac{\partial}{\partial \varphi}$ .

$$\alpha \in T_p^* M = v^\sharp = 7q_{11} dr + 8q_{22} d\theta + 9q_{33} d\varphi = 7dr + 8r^2 d\theta + 9r^2 \sin^2 \theta d\varphi.$$