

**Przykład 1.** Zastanówmy się jak wygląda rotacja wektora w układzie sferycznym.  $M = \mathbb{R}^3$ .

$$v \xrightarrow{\sharp} \Lambda^1(M) \xrightarrow{d} \Lambda^2(M) \xrightarrow{*} \Lambda^1(M) \rightarrow T_p M \rightarrow \left[ \right]_i$$

$$\text{rot} v = (* (dv^\sharp))^\flat$$

na początek dostajemy w smsie

$$\left[ \begin{array}{c} A^r \\ A^\theta \\ A^\varphi \end{array} \right]_{i_r, i_\theta, i_\varphi} = v = A^r \frac{\partial}{\partial r} + A^\theta \frac{1}{r} \frac{\partial}{\partial \theta} + A^\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

chcemy sobie zrobić jednoformę, która jest podniesionym wektorkiem:

$$\alpha = v^\sharp = g_{rr} A^r dr + g_{\theta\theta} \frac{1}{r} A^\theta d\theta + g_{\varphi\varphi} \frac{1}{r \sin \theta} A^\varphi d\varphi = A^r dr + r A^\theta d\theta + r \sin \theta A^\varphi d\varphi$$

$$d\alpha = (A^r_{,\theta} - (r A^\theta)_{,r}) d\theta \wedge dr + (A^r_{,\varphi} - (r \sin \theta A^\varphi)_{,r}) d\varphi \wedge dr + \\ + ((r A^\theta)_{,\varphi} - (r \sin \theta A^\varphi)_{,\theta}) d\varphi \wedge d\theta$$

$$*(dr \wedge d\theta) = \sin \theta d\varphi, \quad *(d\theta \wedge d\varphi) = \frac{1}{r^2} dr, \quad *(d\varphi \wedge dr) = \frac{1}{\sin \theta} d\theta$$

$$*d\alpha = ((r \sin \theta A^\varphi)_{,\theta} - (r A^\theta)_{,\varphi}) \frac{1}{r^2 \sin \theta} dr + (A^r_{,\varphi} - (r \sin \theta A^\varphi)_{,r}) \frac{1}{\sin \theta} d\theta + \\ + ((r A^\theta)_{,r} - A^r_{,\theta}) \sin \theta d\varphi.$$

notacja:  $\square, \heartsuit = \frac{\partial \square}{\partial \heartsuit}$ . Zostały nam jeszcze tylko dwie operacje.

$$(*d\alpha)^\flat = ((r \sin \theta A^\varphi)_{,\theta} - (r A^\theta)_{,\varphi}) \cdot 1 \cdot \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} + (A^r_{,\varphi} - (r \sin \theta A^\varphi)_{,r}) \frac{1}{\sin \theta} \frac{1}{r^2} \frac{\partial}{\partial \theta} + \\ + \left( (r A^\theta)_{,r} - A^r_{,\theta} \right) \sin \theta \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \varphi}.$$

Czyli

$$\text{rot} \left[ \begin{array}{c} A^r \\ A^\theta \\ A^\varphi \end{array} \right] = \left[ \begin{array}{c} \frac{1}{r^2 \sin \theta} ((r \sin \theta A^\varphi)_{,\theta} - (r A^\theta)_{,\varphi}) \\ \frac{1}{r \sin \theta} (A^r_{,\varphi} - (r \sin \theta A^\varphi)_{,r}) \\ \frac{1}{r} ((r A^\theta)_{,r} - A^r_{,\theta}) \end{array} \right].$$

**Przykład 2.** *To może teraz dywergencja rzutem na taśmę.*

$$\begin{aligned}
 \left[ \right] &= v \xrightarrow{\sharp} \Lambda^1(M) \xrightarrow{*} \Lambda^2(M) \xrightarrow{d} \Lambda^3(M) \xrightarrow{\flat} \Lambda^0(M) \\
 \text{div}(v) &= * (d(*v^{\sharp})) \\
 \begin{bmatrix} A^r \\ A^{\theta} \\ A^{\varphi} \end{bmatrix} &= v, \alpha = v^{\sharp} \\
 \alpha &= A^r dr + r A^{\theta} d\theta + A^{\varphi} r \sin \theta d\varphi \\
 * dr &= r^2 \sin \theta d\theta \wedge d\varphi \\
 * d\theta &= \sin \theta d\varphi \wedge dr \\
 * d\varphi &= \frac{1}{\sin \theta} dr \wedge d\theta \\
 * \alpha &= (A^r r^2 \sin \theta) d\theta \wedge d\varphi + (r \sin \theta A^{\theta}) d\varphi \wedge dr + (r A^{\varphi}) dr \wedge d\theta \\
 d(*\alpha) &= ((A^r r^2 \sin \theta)_{,r} + (r \sin \theta A^{\theta})_{,\theta} + (r A^{\varphi})_{,\varphi}) dr \wedge d\theta \wedge d\varphi
 \end{aligned}$$

$$\text{div} \begin{bmatrix} A^r \\ A^{\theta} \\ A^{\varphi} \end{bmatrix} = \frac{1}{r^2 \sin \theta} ((A^r r^2 \sin \theta)_{,r} + (r \sin \theta A^{\theta})_{,\theta} + (r A^{\varphi})_{,\varphi}).$$

$$\begin{aligned}
 f(r, \theta, \varphi) &\xrightarrow{d} \Lambda^1(M) \xrightarrow{*} \Lambda^2(M) \xrightarrow{d} \Lambda^3(M) \xrightarrow{*} \Lambda^0(M) \\
 \alpha = df &= \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \varphi} d\varphi \\
 * \alpha &= \left( \frac{\partial f}{\partial r} r^2 \sin \theta \right) d\theta \wedge d\varphi + \left( \frac{\partial f}{\partial \theta} \sin \theta \right) d\varphi \wedge dr + \left( \frac{\partial f}{\partial \varphi} \frac{1}{\sin \theta} \right) dr \wedge d\theta \\
 d(*\alpha) &= \left( \left( r^2 \sin \theta \frac{\partial f}{\partial r} \right)_{,r} + \sin \theta \frac{\partial f}{\partial \theta} \right)_{,\theta} + \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \varphi} \right)_{,\varphi} dr \wedge d\theta \wedge d\varphi \\
 *(d(*\alpha)) &= \frac{1}{r^2 \sin \theta} \left( \left( r^2 \sin \theta \frac{\partial f}{\partial r} \right)_{,r} + \left( \sin \theta \frac{\partial f}{\partial \theta} \right)_{,\theta} + \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \varphi} \right)_{,\varphi} \right).
 \end{aligned}$$

**Przykład 3.**  $M = \mathbb{R}^3, f \in \Lambda^0(M)$ .

$$\begin{aligned}
 ddf &= 0 \\
 ddf &= d \left( \left( (df)^{\flat} \right)^{\sharp} \right) \implies \text{rot}(\text{grad}(f)) = 0.
 \end{aligned}$$

Niech teraz  $v \in \Lambda^1(M)$ .

$$\begin{aligned}
 d \left( * \left( (* (dV^{\sharp}))^{\flat} \right)^{\sharp} \right) &= d(*(* (d(v^{\sharp})))) = dd(v^{\sharp}) = 0 \\
 \text{div}(\text{rot}(V)) &= 0.
 \end{aligned}$$

Weźmy sobie jakąś funkcję:  $f : (t, x, y, z) \rightarrow \mathbb{R}$ .

Zobaczmy jak  $*d(*df)$  wygląda w  $\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ .

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

$$*(dx^{i_1} \wedge \dots \wedge dx^{i_L}) = \frac{\sqrt{g}}{(n-L)!} g^{i_1 j_1} \dots g^{i_L j_L} \in_{j_1 \dots j_k k_1 \dots k_{n-L}} dx^{k_1} \wedge \dots \wedge dx^{k_{n-L}}$$

$$*(dx^0) = \frac{\sqrt{-(-1)}}{(4-1)!} g^{00} \in_{0k_1 k_2 k_3} dx^{k_1} \wedge dx^{k_2} \wedge dx^{k_3}, i, k = 0, \dots, 3$$

$$*(dx^0) = -\frac{1}{3!} 3! dx^1 \wedge dx^2 \wedge dx^3$$

$$*(dt) = -dx \wedge dy \wedge dz$$

$$*(dx^1) = \frac{\sqrt{-(-1)}}{(4-1)!} g^{11} \in_{1k_1 k_2 k_3} dx^{k_1} \wedge dx^{k_2} \wedge dx^{k_3}$$

$$*(dx) = 3! \frac{1}{3!} dy \wedge dt \wedge dz$$

$$*(dy) = dt \wedge dx \wedge dz$$

$$*(dz) = dx \wedge dt \wedge dy$$

$$*df = -\frac{\partial f}{\partial t} dx \wedge dy \wedge dz + \frac{\partial f}{\partial x} dy \wedge dt \wedge dz + \frac{\partial f}{\partial y} dt \wedge dx \wedge dz + \frac{\partial f}{\partial z} dx \wedge dt \wedge dy$$

$$d*df = \left( -\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dt \wedge dx \wedge dy \wedge dz.$$

Na koniec:

Mamy dwuformę pola elektromagnetycznego:

$$F = -E_x dt \wedge dx + E_y dt \wedge dy - E_z dt \wedge dz + B_x dy \wedge dz + B_y dz \wedge dy + B_z dy \wedge dx.$$

$dF = 0$  to jest pierwsza część równań Maxwella

$$\begin{bmatrix} \rho \\ \rho v^x \\ \rho v^y \\ \rho v^z \end{bmatrix} = \begin{bmatrix} \rho \\ j^x \\ j^y \\ j^z \end{bmatrix}$$

$$j = -g dt + j^x dx + j^y dy + j^z dz$$

$$d(*F) = *j \text{ a to druga.}$$