$\frac{\partial H}{\partial r} = \frac{\partial f}{\partial x}\Big|_{\substack{x=\Psi_1(r,\varphi),\\ y=\Psi_2(r,\varphi)}} \frac{\partial \Psi_1}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial \Psi_2}{\partial r}$
$\frac{\partial H}{\partial \varphi} = \frac{\partial f}{\partial x}\frac{\partial \Psi_1}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial \Psi_2}{\partial \varphi}$

Konwencja z ćwiczeń z fizyki:

$H(r,\varphi) = (f \circ \Psi)(r,\varphi)$
$H(r,\varphi) = f(r,\varphi)$

$$\Psi_1(r,\varphi) = x(r,\varphi)$$
$$\Psi_2(r,\varphi) = y(r,\varphi)$$

$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$
$\frac{\partial f}{\partial \varphi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varphi}$

Przykład 1

$$f(x,y) : \mathbb{R}^2 \to \mathbb{R}, \quad \begin{bmatrix} x = r\cos\varphi \\ y = r\sin\varphi \end{bmatrix}$$
$$\frac{\partial f}{\partial r} = \cos\varphi \frac{\partial f}{\partial x} + \sin\varphi \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial \varphi} = -r\sin\varphi \frac{\partial f}{\partial x} + r\cos\varphi \frac{\partial f}{\partial y}$$
$$f(x,y) : \mathbb{R}^2 \to \mathbb{R}, \quad f' = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

Interpretacja geometryczna Rozważmy zbiór

$$P_c = \{(x,y) \in \mathbb{R}^2 : f(x,y) = c\} \text{ np. } f(x,y) = x^2 + y^2 : P_c = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = c\}$$

Załóżmy, że $f(x,y)$ - taka, że P_c - można sparametryzować jako

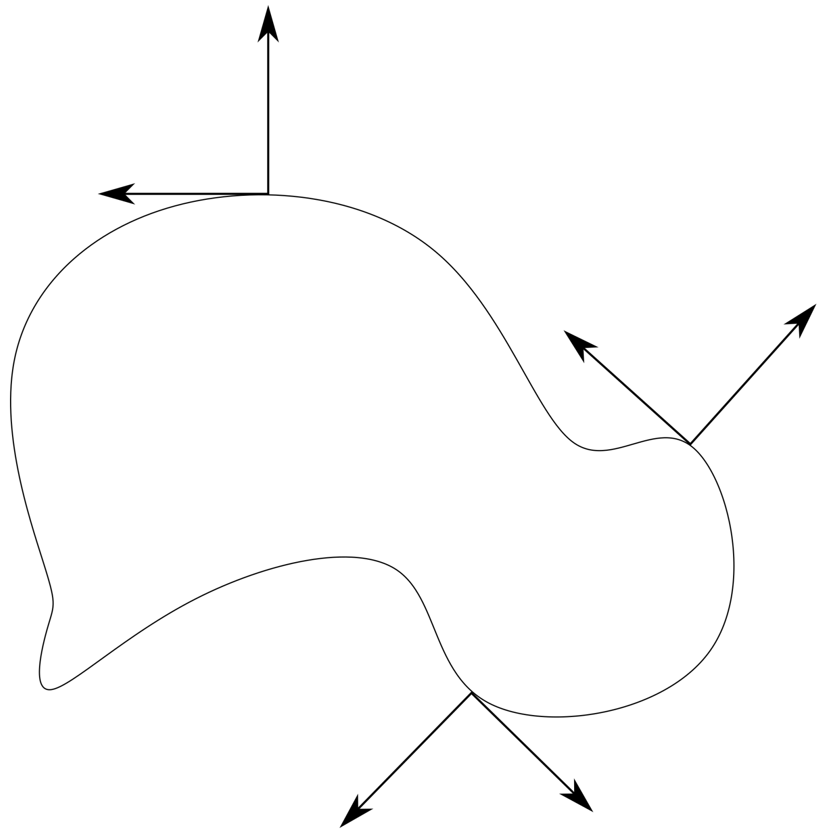
$$\varphi(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, t \in D, \text{ to znaczy, że } P_c = \{(x(t), y(t)), t \in D\}$$

Przykład 2

Niech $\varphi(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$. Wtedy $P_c = \{(c\cos t, c\sin t); t \in [0, 2\pi]\}$

$f(x(t), y(t)) = c \quad \underset{t \in D}{\forall}$ - powierzchnie ekwipotencjalne

$$\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right] \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = 0, \begin{bmatrix} 2x, 2y \end{bmatrix} \begin{bmatrix} -c\sin t \\ c\cos t \end{bmatrix} = 0$$



Rysunek 1: Trajektoria kluki

Definicja 1 *Pochodna mieszana*

$$f(x,y) = x^2y^3, \quad \frac{\partial f}{\partial x} = 2xy^3, \frac{\partial f}{\partial y} = 3x^2y^2, \quad \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = 2y^3, \quad \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6xy^2 \quad \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

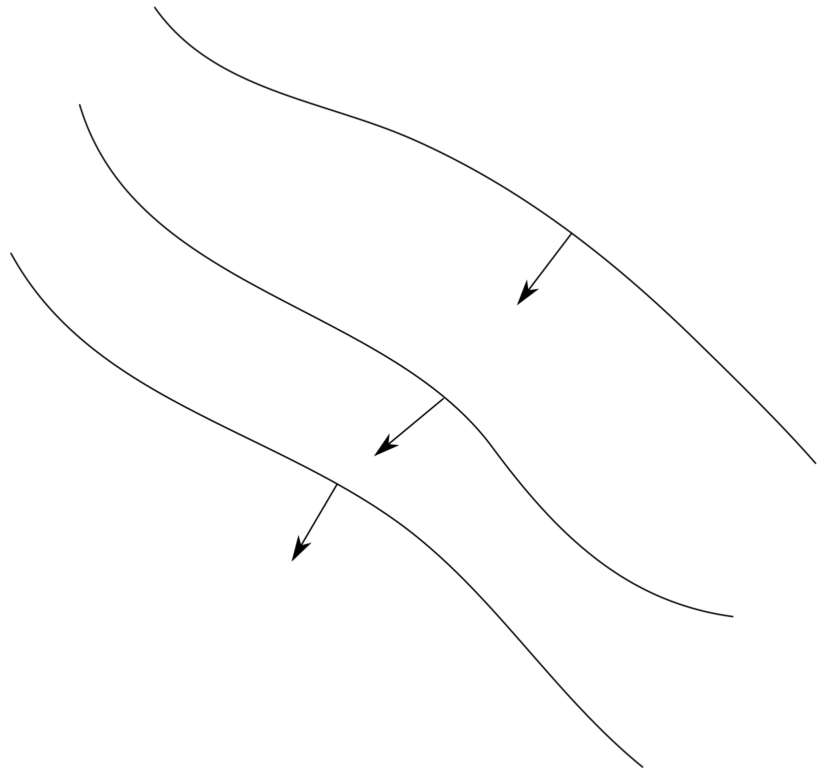
Przypadek???

Twierdzenie 1 *Niech $f : \mathcal{O} \to \mathbb{R}, \mathcal{O} \subset \mathbb{R}^n$, otwarty i $f \in \mathcal{C}^2(\mathcal{O})$, wówczas*

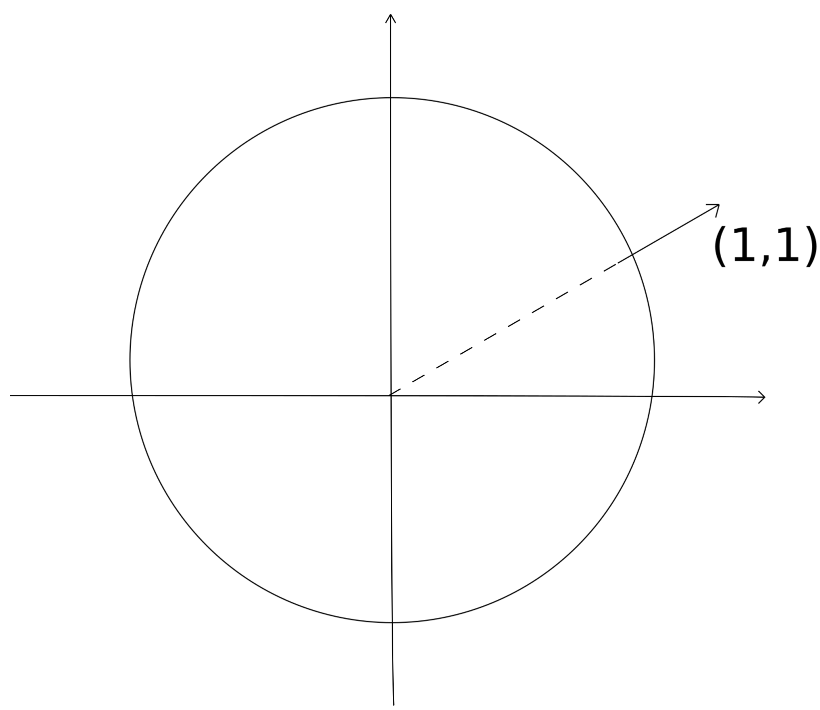
$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}; i, j = 1, \ldots, n$$

Dowód 1 *Dowód dla $n = 2$*

Niech $w(x,y) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x,y)$
$\varphi(x) = f(x, y+k) - f(x,y)$



Rysunek 2: Powierzchnia ekwipotencjalna I



Rysunek 3: Powierzchnia ekwipotencjalna II

wówczas

$$
\begin{aligned}
&w = \varphi(x+h) - \varphi(x) = \frac{\partial \varphi}{\partial x}(\xi)h = \\
&= \Big[\frac{\partial f}{\partial x}(\xi, y+k) - \frac{\partial f}{\partial x}(\xi, y)\Big]h = \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}(\xi,\eta)\Big)hk, \\
&\text{gdzie } x < \xi < x+h, \quad y < \eta < y+k
\end{aligned}
$$

Niech $\Psi(y) = f(x+h,y) - f(x,y)$
$w(x,y) = \Psi(y+k) - \Psi(y) = \frac{\partial \Psi}{\partial y}(\eta_1)k = \Big[\frac{\partial f}{\partial y}(x+h,\eta_1) - \frac{\partial f}{\partial y}(x,\eta_1)\Big]k = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}(\xi,\eta)\Big)kh$, czyli
$\underset{\xi\in]x,x+h[}{\exists}, \quad \xi_1 \in]x,x+h[, \quad \eta \in]y,y+k[, \quad \eta_1 \in]y,y+k[\quad (y < \eta_1 < y+k)$
Jeżeli $h \to 0$
$\left(\frac{\partial^2 f}{\partial y \partial x}(\xi,\eta) = \frac{\partial^2 f}{\partial x \partial y}(\xi_1,\eta_1)\right)$
to $\xi \to x, \xi_1 \to x, \eta \to y, \eta_1 \to y$, czyli:

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$$

Jeżeli każda z tych wielkości jest ciągła $\square$
Wzór Taylora Niech $f : \mathcal{O} \to \mathbb{R}, \mathcal{O} \subset \mathbb{R}^n$ - otwarty
$\varphi(t) = f(x_0 + th), h \in \mathbb{R}^n, t \in [0,1]$

Dla $h = \begin{bmatrix} h^1 \\ \vdots \\ h^n \end{bmatrix}, x_0 = \begin{bmatrix} x_0^1 \\ \vdots \\ x_0^n \end{bmatrix}, \varphi(t) = f(x_0^1 + th^1, x_0^2 + th^2, \dots, x_0^n + th^n)$

$\frac{\partial \varphi}{\partial t} = \frac{\partial f}{\partial x^1}\Big|_{x=x_0+th} h_1 + \frac{\partial f}{\partial x^2}\Big|_{x=x_0+th} h_2 + \dots + \frac{\partial f}{\partial x^n}\Big|_{x=x_0+th} h_n = \sum_{i=1}^n \frac{\partial f}{\partial x^i}\Big|_{x_0+th} h_i$
$\frac{\partial^2 \varphi}{\partial t^2} = \sum_{j=1}^n \sum_{i=1}^n \frac{\partial f}{\partial x^i \partial x^j}\Big|_{x_0+th} h_j h_i$
$\vdots$
$\frac{\partial^k \varphi}{\partial t^k} = \sum_{i^1,\dots,i^k}^n \frac{\partial^{(k)} f}{\partial x^{i_1} \dots \partial x^i} h_{i_1} \dots h_{i_k}$
$\varphi(t) = \varphi(0) = \varphi'(0)(t-0) + \frac{\varphi''(0)}{2!}(t-0)^2 + \dots + \frac{\varphi^k(0)}{k}(t-0)^k + r(\dots)$
Czyli:

$$\varphi(1) - \varphi(0) = \varphi'(0) + \frac{\varphi''(0)}{2!} + \frac{\varphi'''(0)}{3!} + \dots + \frac{\varphi^k(0)}{k!} + r(\dots)$$

$$f(x_0+h) - f(x_0) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(x_0)h_i + \frac{1}{2!}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0)h_i h_j + \dots\dots \square$$