0.1 Konwencja z ćwiczeń z fizyki:

Przykład 1. *Mamy funkcję* $H(r,\varphi) = (f \circ \Psi)(r,\varphi)$

$$\begin{aligned}
&\Psi_1(r,\varphi) = x(r,\varphi)\\
&\Psi_2(r,\varphi) = y(r,\varphi)\\
&\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\\
&\frac{\partial f}{\partial \varphi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varphi}
\end{aligned}$$

Przykład 2.

$$f(x,y) : \mathbb{R}^2 \to \mathbb{R}, \quad \begin{bmatrix} x = r\cos\varphi \\ y = r\sin\varphi \end{bmatrix}$$

$$\frac{\partial f}{\partial r} = \cos\varphi\frac{\partial f}{\partial x} + \sin\varphi\frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial \varphi} = -r\sin\varphi\frac{\partial f}{\partial x} + r\cos\varphi\frac{\partial f}{\partial y}$$

$$f(x,y) : \mathbb{R}^2 \to \mathbb{R}, \quad f' = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

0.2 Interpretacja geometryczna f'

Przykład 3. *Rozważmy zbiór*

$P_c = \{(x,y) \in \mathbb{R}^2 : f(x,y) = c\}$ *np.* $f(x,y) = x^2 + y^2 : P_c = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = c\}$.

Załóżmy, że $f(x,y)$ *- taka, że* P_c *można sparametryzować jako*

$$\varphi(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, t \in D, \text{ to znaczy, że } P_c = \{(x(t), y(t)), t \in D\}$$

Przykład 4.

Niech $\varphi(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$. Wtedy $P_c = \{(c\cdot\cos t, c\cdot\sin t) : t \in [0, 2\pi]\}$
$f(x(t), y(t)) = c \quad \underset{t\in D}{\forall}$ - powierzchnie ekwipotencjalne

$$\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = 0,$$

$$\begin{bmatrix} 2x, 2y \end{bmatrix}\begin{bmatrix} -c\cdot\sin t \\ c\cdot\cos t \end{bmatrix} = 0.$$

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Rysunek 1: Trajektoria kluki

Definicja 1. *Pochodna mieszana*

$$f(x,y) = x^2y^3, \quad \frac{\partial f}{\partial x} = 2xy^3, \frac{\partial f}{\partial y} = 3x^2y^2, \quad \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = 2y^3, \quad \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6xy^2 \quad \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

Przypadek???

Twierdzenie 1. *(Uogólnione twierdzenie Schwarza)*
Niech $f : \mathcal{O} \to \mathbb{R}, \mathcal{O} \subset \mathbb{R}^n$, otwarty i $f \in \mathcal{C}^2(\mathcal{O})$, wówczas

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}; i, j = 1, \ldots, n$$

Dowód. Dowód dla $n = 2$ Niech

$$w(x,y) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x,y),$$

$$\varphi(x) = f(x, y+k) - f(x,y)$$

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Rysunek 2: Powierzchnia ekwipotencjalna I

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Rysunek 3: Powierzchnia ekwipotencjalna II wówczas

$$\begin{aligned} w = \varphi(x+h) - \varphi(x) &= \frac{\partial \varphi}{\partial x}(\xi)h = \\ &= \left[\frac{\partial f}{\partial x}(\xi, y+k) - \frac{\partial f}{\partial x}(\xi, y)\right] h = \\ &= \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}(\xi, \eta)\Big) hk, \\ &\text{gdzie } x < \xi < x+h, \quad y < \eta < y+k \end{aligned}$$

Niech

$$\Psi(y) = f(x+h, y) - f(x, y)$$

$$\begin{aligned} w(x,y) = \Psi(y+k) - \Psi(y) &= \frac{\partial \Psi}{\partial y}(\eta_1)k = \\ &= \left[\frac{\partial f}{\partial y}(x+h, \eta_1) - \frac{\partial f}{\partial y}(x, \eta_1)\right] k = \\ &= \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}(\xi, \eta)\Big) kh, \end{aligned}$$

czyli

$$\underset{\xi}{\exists} \quad \xi \in]x, x+h[, \quad \xi_1 \in]x, x+h[, \quad \eta \in]y, y+k[, \quad \eta_1 \in]y, y+k[.$$

Jeżeli $h \to 0$,

$$\left(\frac{\partial^2 f}{\partial y \partial x}(\xi, \eta) = \frac{\partial^2 f}{\partial x \partial y}(\xi_1, \eta_1)\right),$$

to

$$\xi \to x, \xi_1 \to x, \eta \to y, \eta_1 \to y,$$

czyli:

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{\partial^2 f}{\partial x \partial y}(x, y),$$

jeżeli każda z tych wielkości jest ciągła. □

0.3 Wzór Taylora (konstrukcja)

Niech $f : \mathcal{O} \to \mathbb{R}, \mathcal{O} \subset \mathbb{R}^n$ - otwarty
$\varphi(t) = f(x_0 + th), h \in \mathbb{R}^n, t \in [0, 1]$.
Dla

$$h = \begin{bmatrix} h^1 \\ \vdots \\ h^n \end{bmatrix}, x_0 = \begin{bmatrix} x_0^1 \\ \vdots \\ x_0^n \end{bmatrix}, \varphi(t) = f(x_0^1 + th^1, x_0^2 + th^2, \ldots, x_0^n + th^n),$$

mamy

$$\frac{\partial\varphi}{\partial t} = \left.\frac{\partial f}{\partial x^1}\right|_{x=x_0+th} h_1 + \left.\frac{\partial f}{\partial x^2}\right|_{x=x_0+th} h_2 + \dots + \left.\frac{\partial f}{\partial x^n}\right|_{x=x_0+th} h_n = \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x^i}\right|_{x_0+th} h_i$$

$$\frac{\partial^2\varphi}{\partial t^2} = \sum_{j=1}^{n}\sum_{i=1}^{n} \left.\frac{\partial f}{\partial x^i \partial x^j}\right|_{x_0+th} h_j h_i$$

$$\vdots$$

$$\frac{\partial^k\varphi}{\partial t^k} = \sum_{i^1,\dots,i^k}^{n} \frac{\partial^{(k)} f}{\partial x^{i_1}\dots\partial x^{i}} h_{i_1}\dots h_{i_k}$$

$$\varphi(t) = \varphi(0) = \varphi'(0)(t-0) + \frac{\varphi''(0)}{2!}(t-0)^2 + \dots + \frac{\varphi^k(0)}{k}(t-0)^k + r(\dots),$$

czyli:

$$\varphi(1) - \varphi(0) = \varphi'(0) + \frac{\varphi''(0)}{2!} + \frac{\varphi'''(0)}{3!} + \dots + \frac{\varphi^k(0)}{k!} + r(\dots)$$

$$f(x_0+h) - f(x_0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x_0) h_i + \frac{1}{2!}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0) h_i h_j + \dots\dots$$