

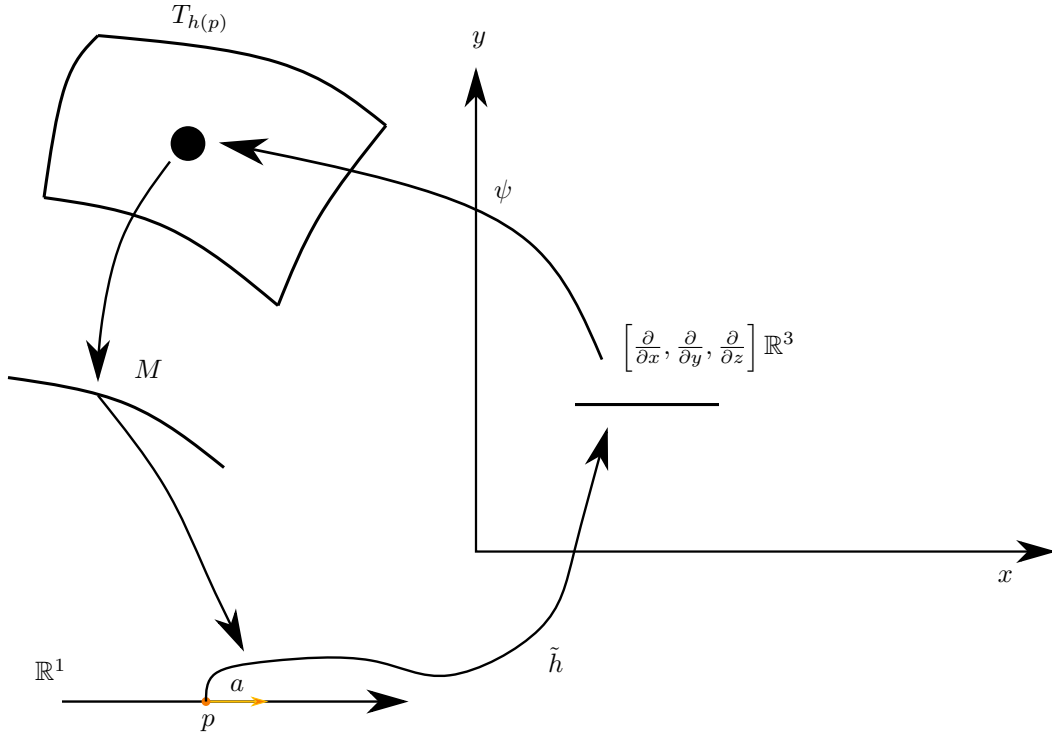
Przykład 1. (na pchnięcie wektora)

Niech $M = \mathbb{R}^1, N = \mathbb{R}^3, h(t) = \begin{bmatrix} f(t) \\ g(t) \\ r(t) \end{bmatrix}$

Niech $p \in \mathbb{R}^1$, niech $v \in T_p M, v = a \frac{\partial}{\partial t}$. $v = [\sigma], \tilde{\sigma}(t) = at + p, \sigma(c) = p, \frac{d\tilde{\sigma}(t)}{dt}|_{t=0} = a$.

$$h_x \sigma = \begin{bmatrix} f(at + p) \\ g(at + p) \\ r(at + p) \end{bmatrix}, h_x v = [h_x \sigma], \frac{d}{dt}(\tilde{h}_x \sigma)|_{t=0}.$$

$$h_x v = \begin{bmatrix} af'(p) \\ ag'(p) \\ ar'(p) \end{bmatrix} = af'(p) \frac{\partial}{\partial x} + ag'(p) \frac{\partial}{\partial y} + ar'(p) \frac{\partial}{\partial z}.$$



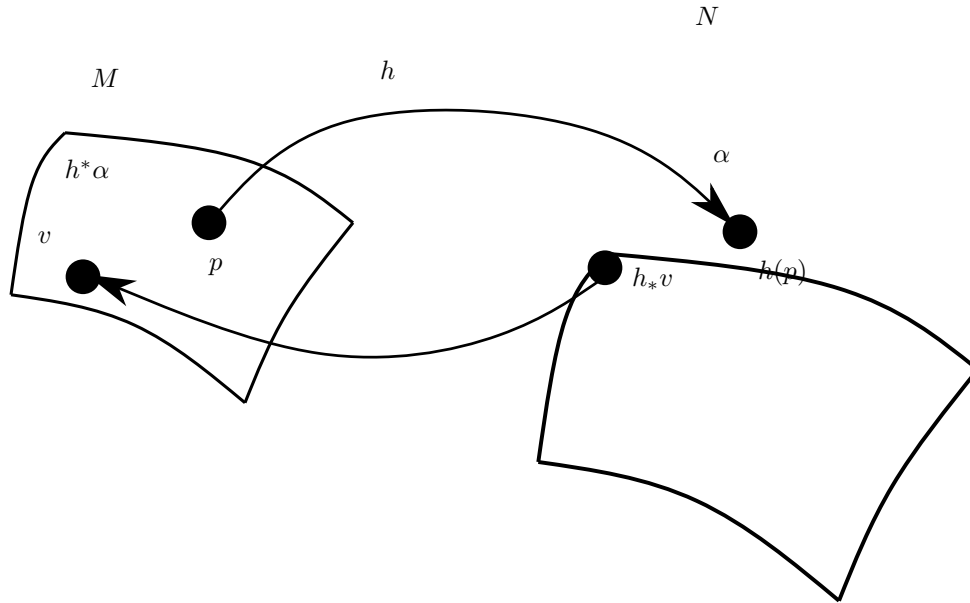
Definicja 1. Niech M, N - rozmaitości, $h : M \rightarrow N$ i niech $p \in M, \alpha \in T_{h(p)}^* N$.

Cofnięciem formy α w odwzorowaniu h nazywamy formę $h^* \alpha \in T_p^* M$, taką, że $\langle h^* \alpha, v \rangle = \langle \alpha, h_* v \rangle \quad \forall_{v \in T_p M}$ i ciągła. Jeżeli $\alpha_1, \alpha_2, \dots, \alpha_k \in \Lambda^1(N)$ i $v_1, \dots, v_k \in T_p(M)$, to

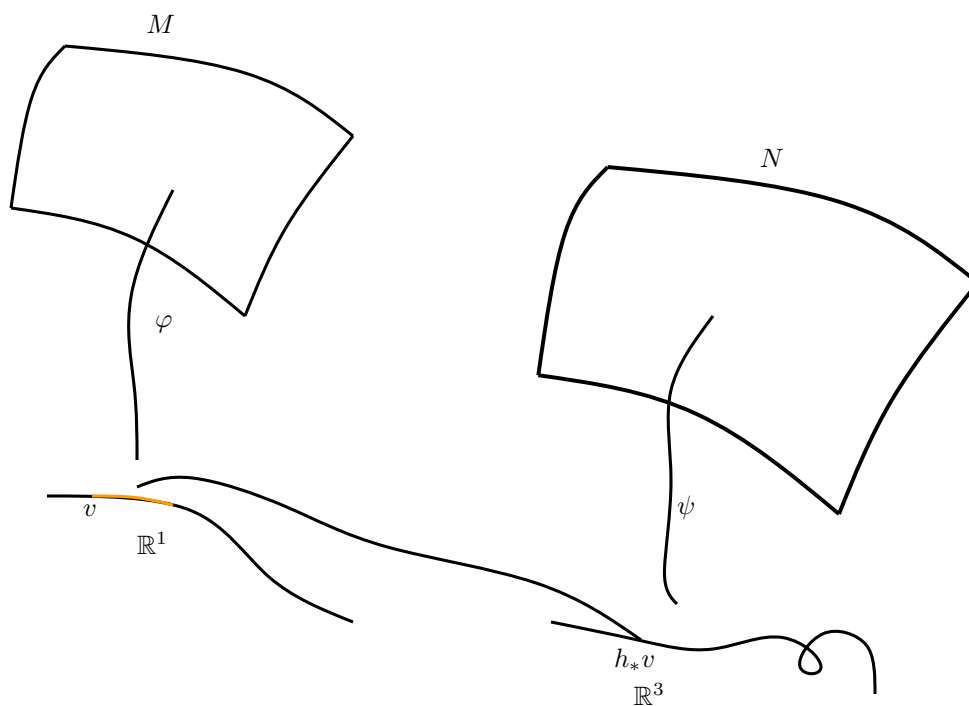
$$h^*(\alpha_1 \wedge \dots \wedge \alpha_k), v_1, \dots, v_k \stackrel{\text{def}}{=} \begin{bmatrix} \langle h^* \alpha_1, v_1 \rangle & \langle h^* \alpha_2, v_1 \rangle & \dots & \langle h^* \alpha_k, v_1 \rangle \\ \vdots & \vdots & & \vdots \\ \langle h^* \alpha_k, v_k \rangle & \langle h^* \alpha_k, v_k \rangle & \dots & \langle h^* \alpha_k, v_k \rangle \end{bmatrix}.$$

Czyli

$$h^*(\alpha_1 \wedge \dots \wedge \alpha_k) = (h^* \alpha_1) \wedge (h^* \alpha_2) \wedge \dots \wedge h^*(\alpha_k).$$



Rysunek 1: $\langle h^* \alpha, v \rangle \stackrel{\text{def}}{=} \langle \alpha, h_* v \rangle$



Przykład 2. (wstępny)

Niech $\alpha = 3(x^2 + y^2)dx - 2xdy + 2z^2dz, \alpha \in \Lambda^1(N)$ (jednoformy nad N , $\dim N = 3$, chociaż można dać więcej jak się chce).

$h(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \\ t \end{bmatrix}$. Czym jest $h^*\alpha$?

$$\langle h^*\alpha, v \rangle = \langle \alpha, h_*v \rangle.$$

Niech $v \in T_p M$ i $v = a \frac{\partial}{\partial t}$. Zatem $h_x v = a \cos(p) \frac{\partial}{\partial x} - a \sin(p) \frac{\partial}{\partial y} + a \cdot 1 \frac{\partial}{\partial t}$.

$$\begin{aligned} \langle \alpha, h_* v \rangle &= \langle 3 (\sin^2(t) + \cos^2(t)) dx - 2 (\sin(t)) dy + 2 (t^2) dz, h_x v \rangle = \\ &= \left\langle 3 dx - 2 \sin(t) dy + 2 t' dz, a \cos(t) \frac{\partial}{\partial x} - a \sin(t) \frac{\partial}{\partial y} + a \cdot 1 \frac{\partial}{\partial z} \right\rangle_{t=p} \\ &= 3a \cos(t) + 2a \sin^2(t) + at^2|_{t=p} = \\ &= \left\langle (3 \cos(t) dt + 2a \sin^2(t) + at^2)|_{t=p}, a \frac{\partial}{\partial t} \right\rangle = \\ &\text{czyli } h^* \alpha = (3 \cos(t) + 2 \sin^2(t) + t^2) dt \end{aligned}$$

Na skróty!

$$\begin{array}{ll} x = \sin(t) & dx = \cos(t) dt \\ y = \cos(t) & dy = -\sin(t) dt \\ z = t & dz = dt. \end{array}$$

Zatem

$$\begin{aligned} h^* \alpha &= 3 (\sin^2(t) + \cos^2(t)) \cos(t) dt - 2 \sin(t) (-\sin t dt) + 2t^2 dt \\ &= (3 \cos(t) + 2 \sin^2(t) + 2t^2) dt. \end{aligned}$$

Przykład 3. Niech $M = \mathbb{R}^4, N = \mathbb{R}^4$.

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-v^2}}, \\ c &= 1 \\ h: \quad t &= \gamma(t' - vx') \\ x &= \gamma(x' - vt') \\ y &= y' \\ z &= z'. \end{aligned}$$

Czyli

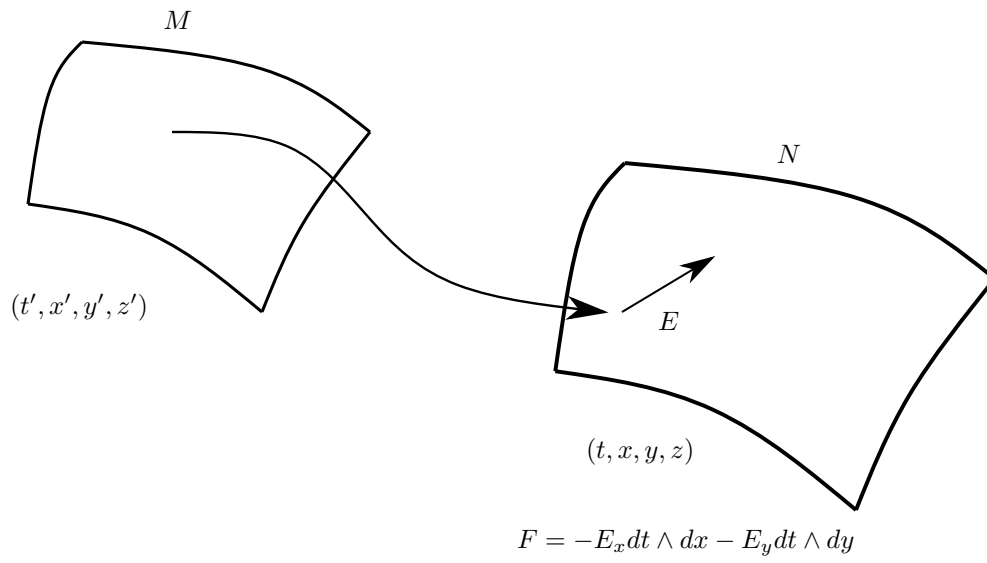
$$\begin{aligned} dt &= \gamma(dt' - vdx') \\ dx &= \gamma(dx' - vdt') \\ dy &= dy' \\ dz &= dz'. \end{aligned}$$

Chcemy cofnąć naszą formę. Na fizyce nie używamy słowa cofnięte.

$$\begin{aligned} F' &= -E_x (\gamma (dt' - vdx')) \wedge \gamma (dx' - vdt') - E_y \gamma (dt' - vdx') \wedge dy' = \\ &= -E_x \gamma^2 (1 - v^2) dt' \wedge dx' - E_y \gamma dt' \wedge dy' + E_y \gamma v dx' \wedge dy' = \\ &= -E_x \frac{1}{1-v^2} (1 - v^2) dt' \wedge dx' - E_y \gamma dt' \wedge dy' + \gamma v E_x dx' \wedge dy' \\ F' &= -E'_x dt' \wedge dx' - E'_y dt' \wedge dy' + B'_z dx' \wedge dy' \end{aligned}$$

Czyli

$$\begin{aligned}E'_x &= E_x \\E'_y &= \gamma E_y \\B'_z &= \gamma v E_y.\end{aligned}$$



Obserwacja: Niech $\alpha \in \Lambda^1(N)$, $\dim N = k$, niech M - rozmaitość, $\dim M = n$ i $h : M \rightarrow N$. Wówczas

$$h^*f \in \Lambda^0(M).$$

Oraz

$$d(h^*f) = h^*(df).$$

Dowód. Skoro $f \in \Lambda^0(N)$, to $f(x^1, x^2, \dots, x^k)$,
 $df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \dots + \frac{\partial f}{\partial x^k} dx^k$.

$$\langle h^*(df), v \rangle = \langle df, h_*v \rangle, v \in T_p M.$$

Niech $V \in T_p M$.

$$\tilde{h}(t_1, \dots, t_n) = \begin{bmatrix} h_1(t_1, \dots, t_n) \\ \vdots \\ h_k(t_1, \dots, t_n) \end{bmatrix}.$$

$$\text{Jeżeli } v = a_1 \frac{\partial}{\partial t^1} + a_2 \frac{\partial}{\partial t^2} + \dots + a_n \frac{\partial}{\partial t^n}, \text{ to } h_* v = \left([h'] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right)_{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}}.$$

$$\begin{aligned} h_x v &= \left(\begin{bmatrix} \frac{\partial h_1}{\partial t^1} & \dots & \frac{\partial h_k}{\partial t^k} \\ \vdots & & \vdots \\ \frac{\partial h_k}{\partial t^1} & \dots & \frac{\partial h_k}{\partial t^k} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right)_{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}} = \left(\frac{\partial h_1}{\partial t^1} a_1 + \dots + \frac{\partial h_1}{\partial t^n} a_n \right) \frac{\partial}{\partial x^1} + \\ &+ \dots + \left(\frac{\partial h_k}{\partial t^1} a_1 + \dots + \frac{\partial h_k}{\partial t^n} a_n \right) \frac{\partial}{\partial x^k}. \end{aligned}$$

Dalej

$$\begin{aligned} \langle df, h_* v \rangle &= \frac{\partial f}{\partial x^1} \left(\frac{\partial h_1}{\partial t^1} a_1 + \dots + \frac{\partial h_1}{\partial t^n} a_n \right) + \dots + \frac{\partial f}{\partial x^k} \left(\frac{\partial h_k}{\partial t^1} a_1 + \dots + \frac{\partial h_k}{\partial t^n} a_n \right) = \\ &= \left\langle df(h_1(t_1, \dots, t_n), h_2(t_1, \dots, t_n), \dots, h_k(t_1, \dots, t_n)), a_1 \frac{\partial}{\partial t^1} + \dots + a_n \frac{\partial}{\partial t^n} \right\rangle \end{aligned}$$

□