

0.1 Konwencja z ćwiczeń z fizyki:

Przykład 1. Mamy funkcję $H(r, \varphi) = (f \circ \Psi)(r, \varphi)$

$$\begin{aligned}\Psi_1(r, \varphi) &= x(r, \varphi) \\ \Psi_2(r, \varphi) &= y(r, \varphi) \\ \frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} \\ \frac{\partial f}{\partial \varphi} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \varphi}\end{aligned}$$

Przykład 2.

$$\begin{aligned}f(x, y) : \mathbb{R}^2 &\rightarrow \mathbb{R}, \quad \begin{bmatrix} x = r \cos \varphi \\ y = r \sin \varphi \end{bmatrix} \\ \frac{\partial f}{\partial r} &= \cos \varphi \frac{\partial f}{\partial x} + \sin \varphi \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial \varphi} = -r \sin \varphi \frac{\partial f}{\partial x} + r \cos \varphi \frac{\partial f}{\partial y} \\ f(x, y) : \mathbb{R}^2 &\rightarrow \mathbb{R}, \quad f' = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right]\end{aligned}$$

0.2 Interpretacja geometryczna f'

Przykład 3. Rozważmy zbiór

$$P_c = \{(x, y) \in \mathbb{R}^2 : f(x, y) = c\} \text{ np. } f(x, y) = x^2 + y^2 : P_c = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = c\}.$$

Załóżmy, że $f(x, y)$ - taka, że P_c można sparametryzować jako

$$\varphi(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, t \in D, \text{ to znaczy, że } P_c = \{(x(t), y(t)), t \in D\}$$

Przykład 4.

Niech $\varphi(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$. Wtedy $P_c = \{(c \cdot \cos t, c \cdot \sin t) : t \in [0, 2\pi]\}$

$f(x(t), y(t)) = c \quad \forall_{t \in D}$ - powierzchnie ekwipotencjalne

$$\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right] \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = 0,$$

$$\begin{bmatrix} 2x, 2y \end{bmatrix} \begin{bmatrix} -c \cdot \sin t \\ c \cdot \cos t \end{bmatrix} = 0.$$

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Rysunek 1: Trajektoria kluki

Definicja 1. *Pochodna mieszana*

$$f(x, y) = x^2y^3, \quad \frac{\partial f}{\partial x} = 2xy^3, \quad \frac{\partial f}{\partial y} = 3x^2y^2, \quad \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = 2y^3, \quad \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6xy^2 \quad \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

Przypadek???

Twierdzenie 1. *(Uogólnione twierdzenie Schwarz'a)*

Niech $f : \mathcal{O} \rightarrow \mathbb{R}$, $\mathcal{O} \subset \mathbb{R}^n$, otwarty i $f \in \mathcal{C}^2(\mathcal{O})$, wówczas

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}; i, j = 1, \dots, n$$

Dowód. Dowód dla $n = 2$ Niech

$$w(x, y) = f(x + h, y + k) - f(x + h, y) - f(x, y + k) + f(x, y),$$

$$\varphi(x) = f(x, y + k) - f(x, y)$$

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Rysunek 2: Powierzchnia ekwipotencjalna I

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Rysunek 3: Powierzchnia ekwipotencjalna II

wówczas

$$\begin{aligned}w &= \varphi(x+h) - \varphi(x) = \frac{\partial \varphi}{\partial x}(\xi)h = \\&= \left[\frac{\partial f}{\partial x}(\xi, y+k) - \frac{\partial f}{\partial x}(\xi, y) \right] h = \\&= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x}(\xi, \eta) \right) hk, \\&\text{gdzie } x < \xi < x+h, \quad y < \eta < y+k\end{aligned}$$

Niech

$$\Psi(y) = f(x+h, y) - f(x, y)$$

$$\begin{aligned}w(x, y) &= \Psi(y+k) - \Psi(y) = \frac{\partial \Psi}{\partial y}(\eta_1)k = \\&= \left[\frac{\partial f}{\partial y}(x+h, \eta_1) - \frac{\partial f}{\partial y}(x, \eta_1) \right] k = \\&= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y}(\xi, \eta) \right) kh,\end{aligned}$$

czyli

$$\exists_{\xi} \quad \xi \in]x, x+h[, \quad \xi_1 \in]x, x+h[, \quad \eta \in]y, y+k[, \quad \eta_1 \in]y, y+k[.$$

Jeżeli $h \rightarrow 0$,

$$\left(\frac{\partial^2 f}{\partial y \partial x}(\xi, \eta) = \frac{\partial^2 f}{\partial x \partial y}(\xi_1, \eta_1) \right),$$

to

$$\xi \rightarrow x, \xi_1 \rightarrow x, \eta \rightarrow y, \eta_1 \rightarrow y,$$

czyli:

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{\partial^2 f}{\partial x \partial y}(x, y),$$

jeżeli każda z tych wielkości jest ciągła. □

0.3 Wzór Taylora (konstrukcja)

Niech $f: \mathcal{O} \rightarrow \mathbb{R}$, $\mathcal{O} \subset \mathbb{R}^n$ - otwarty

$\varphi(t) = f(x_0 + th)$, $h \in \mathbb{R}^n$, $t \in [0, 1]$.

Dla

$$h = \begin{bmatrix} h^1 \\ \vdots \\ h^n \end{bmatrix}, x_0 = \begin{bmatrix} x_0^1 \\ \vdots \\ x_0^n \end{bmatrix}, \varphi(t) = f(x_0^1 + th^1, x_0^2 + th^2, \dots, x_0^n + th^n),$$

mamy

$$\begin{aligned}\frac{\partial \varphi}{\partial t} &= \left. \frac{\partial f}{\partial x^1} \right|_{x=x_0+th} h_1 + \left. \frac{\partial f}{\partial x^2} \right|_{x=x_0+th} h_2 + \dots + \left. \frac{\partial f}{\partial x^n} \right|_{x=x_0+th} h_n = \sum_{i=1}^n \left. \frac{\partial f}{\partial x^i} \right|_{x_0+th} h_i \\ \frac{\partial^2 \varphi}{\partial t^2} &= \sum_{j=1}^n \sum_{i=1}^n \left. \frac{\partial^2 f}{\partial x^i \partial x^j} \right|_{x_0+th} h_j h_i \\ &\vdots \\ \frac{\partial^k \varphi}{\partial t^k} &= \sum_{i^1, \dots, i^k}^n \frac{\partial^{(k)} f}{\partial x^{i_1} \dots \partial x^{i_k}} h_{i_1} \dots h_{i_k}\end{aligned}$$

$$\varphi(t) = \varphi(0) = \varphi'(0)(t-0) + \frac{\varphi''(0)}{2!}(t-0)^2 + \dots + \frac{\varphi^k(0)}{k}(t-0)^k + r(\dots),$$

czyli:

$$\begin{aligned}\varphi(1) - \varphi(0) &= \varphi'(0) + \frac{\varphi''(0)}{2!} + \frac{\varphi'''(0)}{3!} + \dots + \frac{\varphi^k(0)}{k!} + r(\dots) \\ f(x_0 + h) - f(x_0) &= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(x_0) h_i + \frac{1}{2!} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0) h_i h_j + \dots\end{aligned}$$