0.1 Końcówka dowodu (Stokesa na kostce)

Dowód. mamy definicję ścianki:

$$\partial I = \sum_{j=1}^{n} \sum_{\alpha=0,1} (-1)^{\alpha+j} I_{(j,\alpha)},$$

dla $I^n \subset \mathbb{R}^n$, $\omega \in \Lambda^{n-1}(M)$, $\omega = f(x^1, \ldots, x^n) = dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n$. Wtedy dla $x = (x^1, \ldots, x^n)$ i $d\tilde{x} = dx^1 \ldots dx^{i-1} dx^{i+1} \ldots dx^n$

$$\begin{aligned}
&\int_{I(j,\alpha)} \left\langle f(x)d\tilde{x}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{j-1}}, \frac{\partial}{\partial x^{j+1}}, \ldots, \frac{\partial}{\partial x^n} \right\rangle = \\
&= \delta_{ij} \int_{I(i,\alpha)} f(x^1, \ldots, x^{i-1}, \alpha, x^{i+1}, \ldots x^n) d\tilde{x} = \\
&= \int_0^1 dx^1 \ldots \int_0^1 dx^{i-1} \int_0^1 dx^{i+1} \ldots \int_0^1 dx^n f(x^1, \ldots, x^{i-1}, \alpha, x^{i+1}, \ldots, x^n) \overset{(\star)}{=} \\
&\overset{(\star)}{=} \int_0^1 dx^1 \ldots \int_0^1 dx^n f(x^1, \ldots, x^{i-1}, \alpha, x^{i+1}, \ldots, x^n) = \int_{I^n} f(x^1, \ldots, x^{i-1}, \alpha, x^{i+1}, \ldots, x^n).
\end{aligned}$$

Przechodzimy do sumy

$$\begin{aligned}
\int_{\partial I} \omega &= \sum_{j=1}^{n} \sum_{\alpha=0,1} (-1)^{\alpha+j} \int_{I(j,\alpha)} \omega = \\
&= \sum_{\alpha=0,1} (-1)^{\alpha+i} \int_{I^n} f(x^1, \ldots, x^{i-1}, \alpha, x^{j+1}, \ldots, x^n) = \\
&= (-1)^{i+0} \int_{I^n} f(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n) + (-1)^{i+1} \int_{I^n} f(x^1, \ldots, x^{i-1}, 1, x^{i+1}, \ldots, x^n).
\end{aligned}$$

$$\begin{aligned}
d\omega &= \frac{\partial f}{\partial x^i} dx^i \wedge dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n = \\
&= (-1)^{i+1} \frac{\partial f}{\partial x^i} dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^i \wedge dx^{i+1} \wedge \ldots \wedge dx^n.
\end{aligned}$$

Stąd

$$\begin{aligned}
&(-1)^{i+1} \int_{I^n} \left\langle \frac{\partial f}{\partial x^1} dx^1, \ldots, dx^n, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\rangle = (-1)^{i+1} \int_0^1 dx^1 \ldots \int_0^1 dx^i \ldots \int_0^1 dx^n \frac{\partial f}{\partial x^i}(x) = \\
&= (-1)^{i+1} \int_0^1 dx^1 \ldots \int_0^1 dx^{i-1} \int_0^1 dx^{i+1} \ldots \int_0^1 dx^n \cdot \\
&\cdot \left[f(x^1, \ldots, x^{i-1}, 1, x^{i+1}, \ldots, x^n) - f(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n) \right] \\
&= (-1)^{i+1} \int_0^1 dx^1 \ldots \int_0^1 dx^i \ldots \int_0^1 dx^n \cdot \\
&\cdot \left[f(x^1, \ldots, x^{i-1}, 1, x^{i+1}, \ldots, x^n) - f(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n) \right] = \\
&= (-1)^{i+1} \int_{I^n} \left[f(x^1, \ldots, x^{i-1}, 1, x^{i+1}, \ldots, x^n) - f(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n) \right].
\end{aligned}$$

$$LHS = RHS.$$

$\square$

Uwaga: Większą kostkę (w sensie długości krawędzi) możemy zawsze podzielić na sumę zorientowanych wspólnie kostek I^n. Całki na tych ścianach kostek, które się stykają dadzą w efekcie zero.

Przykład 1. *Niech* $[a,b] \in \mathbb{R}^1$ *i* $f \in \Lambda^0([a,b])$. *Wtedy twierdzenie Stokesa wygląda tak (xD):*

$$\int_{\partial[a,b]} f = \int_{[a,b]} df = \int_a^b \left\langle \frac{\partial f}{\partial x} dx, \frac{\partial}{\partial x} \right\rangle dx = \int_a^b \frac{\partial f}{\partial x} dx = f(b) - f(a).$$

Przykład 2. *Niech* γ *- krzywa na* M, $\dim M = 3$, $f \in \Lambda^0 M$.

$$\int_\gamma df = \int_{\partial\gamma} f = f(B) - f(A).$$

Przykład 3. $\dim M = 2$, *niech* $\alpha = xydx + x^2dy$. *Policzmy* $\int_{\partial S} \alpha$.

$$\int_{\partial S} \alpha = \int_{C_1} \alpha + \int_{C_2} \alpha + \int_{C_3} \alpha,$$

ale

$$\int_{C_1} \left\langle \varphi^\star \alpha, \frac{\partial}{\partial x} \right\rangle = 0,$$

φ *- parametryzacja* C_1. *Jeżeli weźmiemy sobie*

$$\int_{C_3} \left\langle \varphi_3^\star \alpha, -\frac{\partial}{\partial y} \right\rangle = 0,$$

φ_3 *- parametryzacja* C_3.

$$C_2 = \left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, 0 \leqslant \theta \leqslant \frac{\pi}{2} \right\},$$

zatem $\varphi_2^\star \alpha$ *przy* $x = \cos\theta \implies dx = -\sin\theta d\theta$, $y = \sin\theta \implies dy = \cos\theta d\theta$, *mamy*

$$\varphi_2^\star \alpha = \cos\theta \sin\theta(-\sin\theta d\theta) + (\cos^2\theta)\cos\theta d\theta = \cos\theta(\cos^2\theta - \sin^2\theta)d\theta.$$

$$\int_{\partial S} \alpha = \int_0^{\frac{\pi}{2}} d\theta \left\langle \cos\theta(\cos^2\theta - \sin^2\theta)d\theta, \frac{\partial}{\partial\theta} \right\rangle,$$

ale np. tw. Stokesa: $\int_{\partial S} \alpha = \int_S d\alpha$.

$$d\alpha = xdy \wedge dx + 2xdx \wedge dy = xdx \wedge dy.$$

$$\int_\square \left\langle xdx \wedge dy, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle = \int_0^1 dx \int_0^{\sqrt{1-x^2}} x = \int_0^1 dx \cdot x\sqrt{1-x^2} = \frac{2}{3}(1-x^2)^{\frac{3}{2}} \frac{(-1)}{2} \Big|_0^1 = \frac{1}{3}.$$

Przykład 4. *Niech* $\alpha = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \Lambda^1(M)$, $\partial K = \left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, 0 \leqslant \theta, 2\pi \right\}$

$$\int_{\partial K} \alpha = \int_0^{2\pi} \left\langle \varphi^\star \alpha, \frac{\partial}{\partial\theta} \right\rangle d\theta.$$

$$\varphi^\star \alpha = -\sin\theta(-\sin\theta)d\theta + \cos\theta\cos\theta d\theta = d\theta.$$

Czyli mamy

$$\int_{\partial K} \alpha = \int_0^{2\pi} d\theta = 2\pi.$$

Ale z drugiej strony dla

$$\begin{aligned} d\alpha &= \left[\left(-\frac{1}{x^2+y^2} + \frac{2y \cdot y}{(x^2+y^2)^2} \right) dy \wedge dx + \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} \right) dx \wedge dy \right] = \\ &= \left(\frac{2}{x^2+y^2} - \frac{2}{x^2+y^2} \right) dx \wedge dy = 0. \end{aligned}$$

wyjdzie, że twierdzenie Stokesa się złamało.

Wiemy, że

$$\int_\gamma df = \int_{\partial\gamma} f = f(B) - f(A).$$

Niech $\alpha = x^2dx + xydy + 2dz$. α jest potencjalna, jeżeli

$$\underset{\eta\in\Lambda^0 M}{\exists} d\eta = \alpha \implies d(d\eta) = 0,$$

(rotacja gradientu równa zero)

$$\int_\gamma \alpha = \int_\gamma d\eta = \eta(B) - \eta(A).$$

Definicja 1. *Niech M - rozmaitość,* $\dim M = n$,

$$i_v : T_pM \times \Lambda^k M \to \Lambda^{k-1}M$$

zdefiniowana następująco:

1. $i_v f = 0$, *jeżeli* $f \in \Lambda^0 M$
2. $i_v dx^i = v^i$, *jeżeli* $v = v^1 \frac{\partial}{\partial x^1} + \ldots + v^i \frac{\partial}{\partial x^i} + \ldots + v^n \frac{\partial}{\partial x^n}$
3. $i_v(\omega \wedge \theta) = i_v(\omega) \wedge \theta + (-1)^{st\omega} \omega \wedge i_v(\theta)$.

Operację i_v nazywamy iloczynem zewnętrznym i oznaczamy poprzez

$$i_v(\omega) \overset{ozn}{=} v \lrcorner \omega.$$

Obserwacja: $i_v(i_v\omega) = 0$ (w domu)

Przykład 5. *Niech* $v = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$,

$$\omega = dx \wedge dy + dz \wedge dx.$$

$$v \lrcorner \omega = \langle dx, v\rangle \wedge dy + (-1)^1 dx \langle dy, v\rangle + \langle dz, v\rangle \wedge dx + (-1)^1 dz \wedge \langle dx, v\rangle .$$

Przykład 6.

$$F = E^x dx \wedge dt + E^y dy \wedge dt + E^z dz \wedge dt + B^x dy \wedge dz + B^y dz \wedge dx + B^z dx \wedge dy.$$

$$j = e\frac{\partial}{\partial t} + ev^x \frac{\partial}{\partial x} + ev^y \frac{\partial}{\partial y} + ev^z \frac{\partial}{\partial z}.$$

$$j \lrcorner F = ?.$$