

### Końcówka dowodu (Stokesa na kostce)

Dowód. mamy definicję ścianki:

$$\partial I = \sum_{j=1}^n \sum_{\alpha=0,1} (-1)^{\alpha+j} I_{(j,\alpha)},$$

dla  $I^n \subset \mathbb{R}^n$ ,  $\omega \in \Lambda^{n-1}(M)$ ,  $\omega = f(x^1, \dots, x^n) = dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n$ . Wtedy dla  $x = (x^1, \dots, x^n)$  i  $d\tilde{x} = dx^1 \dots dx^{i-1} dx^{i+1} \dots dx^n$

$$\begin{aligned} & \int_{I(j,\alpha)} \left\langle f(x) d\tilde{x}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{j-1}}, \frac{\partial}{\partial x^{j+1}}, \dots, \frac{\partial}{\partial x^n} \right\rangle = \\ &= \delta_{ij} \int_{I(i,\alpha)} f(x^1, \dots, x^{i-1}, \alpha, x^{i+1}, \dots, x^n) d\tilde{x} = \\ &= \int_0^1 dx^1 \dots \int_0^1 dx^{i-1} \int_0^1 dx^{i+1} \dots \int_0^1 dx^n f(x^1, \dots, x^{i-1}, \alpha, x^{i+1}, \dots, x^n) \stackrel{(*)}{=} \\ &\stackrel{(*)}{=} \int_0^1 dx^1 \dots \int_0^1 dx^n f(x^1, \dots, x^{i-1}, \alpha, x^{i+1}, \dots, x^n) = \int_{I^n} f(x^1, \dots, x^{i-1}, \alpha, x^{i+1}, \dots, x^n). \end{aligned}$$

Przechodzimy do sumy

$$\begin{aligned} \int_{\partial I} \omega &= \sum_{j=1}^n \sum_{\alpha=0,1} (-1)^{\alpha+j} \int_{I(j,\alpha)} \omega = \\ &= \sum_{\alpha=0,1} (-1)^{\alpha+i} \int_{I^n} f(x^1, \dots, x^{i-1}, \alpha, x^{i+1}, \dots, x^n) = \\ &= (-1)^{i+0} \int_{I^n} f(x^1, \dots, x^{i-1}, 0, x^{i+1}, \dots, x^n) + \\ &+ (-1)^{i+1} \int_{I^n} f(x^1, \dots, x^{i-1}, 1, x^{i+1}, \dots, x^n). \end{aligned}$$

$$\begin{aligned} d\omega &= \frac{\partial f}{\partial x^i} dx^i \wedge dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n = \\ &= (-1)^{i+1} \frac{\partial f}{\partial x^i} dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^i \wedge dx^{i+1} \wedge \dots \wedge dx^n. \end{aligned}$$

Stąd

$$\begin{aligned}
& (-1)^{i+1} \int_{I^n} \left\langle \frac{\partial f}{\partial x^1} dx^1, \dots, dx^n, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\rangle = \\
& = (-1)^{i+1} \int_0^1 dx^1 \dots \int_0^1 dx^i \dots \int_0^1 dx^n \frac{\partial f}{\partial x^i}(x) = \\
& = (-1)^{i+1} \int_0^1 dx^1 \dots \int_0^1 dx^{i-1} \int_0^1 dx^{i+1} \dots \int_0^1 dx^n \cdot \\
& \cdot [f(x^1, \dots, x^{i-1}, 1, x^{i+1}, \dots, x^n) - f(x^1, \dots, x^{i-1}, 0, x^{i+1}, \dots, x^n)] \\
& = (-1)^{i+1} \int_0^1 dx^1 \dots \int_0^1 dx^i \dots \int_0^1 dx^n \cdot \\
& \cdot [f(x^1, \dots, x^{i-1}, 1, x^{i+1}, \dots, x^n) - f(x^1, \dots, x^{i-1}, 0, x^{i+1}, \dots, x^n)] = \\
& = (-1)^{i+1} \int_{I^n} [f(x^1, \dots, x^{i-1}, 1, x^{i+1}, \dots, x^n) - f(x^1, \dots, x^{i-1}, 0, x^{i+1}, \dots, x^n)].
\end{aligned}$$

$$LHS = RHS.$$

□

**Uwaga:** Większą kostkę (w sensie długości krawędzi) możemy zawsze podzielić na sumę zorientowanych wspólnie kostek  $I^n$ . Całki na tych ścianach kostek, które się stykają dadzą w efekcie zero.

**Przykład 1.** Niech  $[a, b] \in \mathbb{R}^1$  i  $f \in \Lambda^0([a, b])$ . Wtedy twierdzenie Stokesa wygląda tak ( $xD$ ):

$$\int_{\partial[a, b]} f = \int_{[a, b]} df = \int_a^b \left\langle \frac{\partial f}{\partial x} dx, \frac{\partial}{\partial x} \right\rangle dx = \int_a^b \frac{\partial f}{\partial x} dx = f(b) - f(a).$$

**Przykład 2.** Niech  $\gamma$  - krzywa na  $M$ ,  $\dim M = 3$ ,  $f \in \Lambda^0 M$ .

$$\int_{\gamma} df = \int_{\partial\gamma} f = f(B) - f(A).$$

**Przykład 3.**  $\dim M = 2$ , niech  $\alpha = xydx + x^2dy$ . Policzmy  $\int_{\partial S} \alpha$ .

$$\int_{\partial S} \alpha = \int_{C_1} \alpha + \int_{C_2} \alpha + \int_{C_3} \alpha,$$

ale

$$\int_{C_1} \left\langle \varphi^* \alpha, \frac{\partial}{\partial x} \right\rangle = 0,$$

$\varphi$  - parametryzacja  $C_1$ . Jeżeli weźmiemy sobie

$$\int_{C_3} \left\langle \varphi_3^* \alpha, -\frac{\partial}{\partial y} \right\rangle = 0,$$

$\varphi_3$  - parametryzacja  $C_3$ .

$$C_2 = \left\{ \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, 0 \leq \theta \leq \frac{\pi}{2} \right\},$$

zatem  $\varphi_2^* \alpha$  przy  $x = \cos \theta \implies dx = -\sin \theta d\theta$ ,  $y = \sin \theta \implies dy = \cos \theta d\theta$ , mamy

$$\varphi_2^* \alpha = \cos \theta \sin \theta (-\sin \theta d\theta) + (\cos^2 \theta) \cos \theta d\theta = \cos \theta (\cos^2 \theta - \sin^2 \theta) d\theta.$$

$$\int_{\partial S} \alpha = \int_0^{\frac{\pi}{2}} d\theta \left\langle \cos \theta (\cos^2 \theta - \sin^2 \theta) d\theta, \frac{\partial}{\partial \theta} \right\rangle,$$

ale np. tw. Stokesa:  $\int_{\partial S} \alpha = \int_S d\alpha$ .

$$d\alpha = x dy \wedge dx + 2x dx \wedge dy = x dx \wedge dy.$$

$$\int_{\square} \left\langle x dx \wedge dy, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle = \int_0^1 dx \int_0^{\sqrt{1-x^2}} x = \int_0^1 dx \cdot x \sqrt{1-x^2} = \frac{2}{3} (1-x^2)^{\frac{3}{2}} \frac{(-1)}{2} \Big|_0^1 = \frac{1}{3}.$$

**Przykład 4.** Niech  $\alpha = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \in \Lambda^1(M)$ ,  $\partial K = \left\{ \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, 0 \leq \theta, 2\pi \right\}$

$$\int_{\partial K} \alpha = \int_0^{2\pi} \left\langle \varphi^* \alpha, \frac{\partial}{\partial \theta} \right\rangle d\theta.$$

$$\varphi^* \alpha = -\sin \theta (-\sin \theta) d\theta + \cos \theta \cos \theta d\theta = d\theta.$$

Czyli mamy

$$\int_{\partial K} \alpha = \int_0^{2\pi} d\theta = 2\pi.$$

*Ale z drugiej strony dla*

$$\begin{aligned} d\alpha &= \left[ \left( -\frac{1}{x^2 + y^2} + \frac{2y \cdot y}{(x^2 + y^2)^2} \right) dy \wedge dx + \left( \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} \right) dx \wedge dy \right] = \\ &= \left( \frac{2}{x^2 + y^2} - \frac{2}{x^2 + y^2} \right) dx \wedge dy = 0. \end{aligned}$$

*wyjdzie, że twierdzenie Stokesa się złamało.*

Wiemy, że

$$\int_{\gamma} df = \int_{\partial\gamma} f = f(B) - f(A).$$

Niech  $\alpha = x^2 dx + xy dy + 2dz$ .  $\alpha$  jest potencjalna, jeżeli

$$\exists_{\eta \in \Lambda^0 M} d\eta = \alpha \implies d(d\eta) = 0,$$

(rotacja gradientu równa zero)

$$\int_{\gamma} \alpha = \int_{\gamma} d\eta = \eta(B) - \eta(A).$$

**Definicja 1.** Niech  $M$  - rozmaitość,  $\dim M = n$ ,

$$i_v : T_p M \times \Lambda^k M \rightarrow \Lambda^{k-1} M$$

zdefiniowana następująco:

1.  $i_v f = 0$ , jeżeli  $f \in \Lambda^0 M$
2.  $i_v dx^i = v^i$ , jeżeli  $v = v^1 \frac{\partial}{\partial x^1} + \dots + v^i \frac{\partial}{\partial x^i} + \dots + v^n \frac{\partial}{\partial x^n}$
3.  $i_v(\omega \wedge \theta) = i_v(\omega) \wedge \theta + (-1)^{st\omega} \omega \wedge i_v(\theta)$ .

Operację  $i_v$  nazywamy iloczynem zewnętrznym i oznaczamy poprzez

$$i_v(\omega) \stackrel{ozn}{=} v \lrcorner \omega.$$

**Obserwacja:**  $i_v(i_v \omega) = 0$  (w domu)

**Przykład 5.** Niech  $v = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$ ,

$$\omega = dx \wedge dy + dz \wedge dx.$$

$$v \lrcorner \omega = \langle dx, v \rangle \wedge dy + (-1)^1 dx \langle dy, v \rangle + \langle dz, v \rangle \wedge dx + (-1)^1 dz \wedge \langle dx, v \rangle.$$

**Przykład 6.**

$$F = E^x dx \wedge dt + E^y dy \wedge dt + E^z dz \wedge dt + B^x dy \wedge dz + B^y dz \wedge dx + B^z dx \wedge dy.$$

$$j = e \frac{\partial}{\partial t} + ev^x \frac{\partial}{\partial x} + ev^y \frac{\partial}{\partial y} + ev^z \frac{\partial}{\partial z}.$$

$$j \lrcorner F = ?.$$