

Definicja 1. Jeżeli $\alpha \in \Lambda^k(M)$ taka, że $d\alpha = 0$, to mówimy, że α jest domknięta. Jeżeli $\exists \eta$ taka, że $d\eta = \alpha$, to mówimy, że α jest zupełna.

Przykład 1. $\mathbf{E} = -\nabla\varphi$, $\mathbf{B} = \text{rot}\mathbf{A}$, $\mathbf{B} = -\nabla f(x, y, z)$.

Dla $\omega = \frac{ydx - xdy}{x^2 + y^2}$, jest $d\omega = 0$. Było, że $\eta = \text{artctg}(\frac{x}{y})$, $d\eta = \omega$. Problem leży w punkcie (0,0) bo nie należy do dziedziny. (rys 7-1)

0.1 Zastosowania twierdzenia Stokesa (przypomnienie)

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

Dostaliśmy wektor $\begin{bmatrix} A^1 \\ A^2 \\ A^3 \end{bmatrix}$, który jest w koszarnej bazie $A^1 i_1 + A^2 i_2 + A^3 i_3$, ale można go zamienić na coś fajniejszego $A^1 \frac{1}{\sqrt{g_{11}}} \frac{\partial}{\partial x} + A^2 \sqrt{g^{22}} \frac{\partial}{\partial x^2} + A^3 \sqrt{g^{33}} \frac{\partial}{\partial x^3}$.

Dla trójki wektorów v_1, v_2, v_3 , ich $|v_1, v_2, v_3|$ to objętość. Paweł wprowadził taki napis

$$G(v_1, v_2, v_3) = \begin{bmatrix} \langle v_1 | v_1 \rangle & \langle v_1 | v_2 \rangle & \langle v_1 | v_3 \rangle \\ \langle v_2 | v_1 \rangle & \langle v_2 | v_2 \rangle & \langle v_2 | v_3 \rangle \\ \langle v_3 | v_1 \rangle & \langle v_3 | v_2 \rangle & \langle v_3 | v_3 \rangle \end{bmatrix}.$$

i zdefiniował objętość tak:

$$\text{vol}(v_1, v_2, v_3) = \sqrt{G(v_1, v_2, v_3)}.$$

$$A = \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) = \begin{bmatrix} v_1^1 & v_1^2 & v_1^3 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}.$$

Teraz

$$(\det A)^2 = (\det A)(\det A) = \det(A) \det(A^T) = \det(A^T A) = \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{bmatrix} \begin{bmatrix} v^1 & v^2 & v^3 \end{bmatrix} = G(v_1, v_2, v_3).$$

Definicja 2. Niech M - rozmaitość i γ krzywa na M .

$$\gamma = \{\gamma(t) \in M, t \in [a, b]\}.$$

Wówczas

$$\|\gamma\| \stackrel{\text{def}}{=} \int_a^b \left\| \frac{\partial}{\partial t} \right\| dt,$$

dla

$$\|v\| = \sqrt{\langle v | v \rangle}.$$

Przykład 2. (rys 7-2) M takie, że $\dim M = 2$

$$\gamma = \left\{ \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \in M, t \in [a, b] \right\}, \quad g_{ij} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}.$$

$$\frac{\partial}{\partial t} = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}, \quad \left\| \frac{\partial}{\partial t} \right\| = \sqrt{\left\langle \frac{\partial}{\partial t} \middle| \frac{\partial}{\partial t} \right\rangle} = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}.$$

$$\|\gamma\| = \int_a^b \sqrt{(x(t))^2 + (y(t))^2} dt.$$

dla zmiany parametryzacji na (rys 7-3) jest

$$\gamma = \int_A^B \left\| \frac{\partial}{\partial x} \right\| dx = \int_{x_0}^{x_1} \sqrt{1 + (f'(x))^2} dx.$$

$$\gamma = \left\{ \begin{bmatrix} x \\ f(x) \end{bmatrix} \in M, x_0 \leq x \leq x_1 \right\}.$$

$$\frac{\partial}{\partial x} = \begin{bmatrix} 1 \\ f'(x) \end{bmatrix}, \quad \left\| \frac{\partial}{\partial x} \right\| = \sqrt{\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle}.$$

I zmiana na biegunowe (rys 7-4)

$$\gamma = \left\{ \begin{bmatrix} r(\varphi) \\ \varphi \end{bmatrix} \in M, \varphi_0 \leq \varphi \leq \varphi_1 \right\}.$$

$$\gamma = \int_A^B \left\| \frac{\partial}{\partial \varphi} \right\| d\varphi, \quad g_{ij} = \begin{bmatrix} 1 & \\ & r^2 \end{bmatrix}.$$

Wektorek styczny jest taki

$$\frac{\partial}{\partial \varphi} = \begin{bmatrix} \frac{\partial}{\partial \varphi} r(\varphi) \\ 1 \end{bmatrix}, \quad \left\langle \frac{\partial}{\partial \varphi} \middle| \frac{\partial}{\partial \varphi} \right\rangle = \left(\begin{bmatrix} 1 & \\ & r^2 \end{bmatrix} \begin{bmatrix} r(\varphi) \\ 1 \end{bmatrix} \right)^T \begin{bmatrix} r'(\varphi) \\ 1 \end{bmatrix}.$$

Ale my wiemy, że $\langle v, w \rangle = g_{ij} v^i w^j$, dalej jest

$$\begin{bmatrix} \frac{\partial r(\varphi)}{\partial \varphi} & r^2 \end{bmatrix} \begin{bmatrix} \frac{\partial r(\varphi)}{\partial \varphi} \\ 1 \end{bmatrix} = r^2 + \left(\frac{\partial r(\varphi)}{\partial \varphi} \right)^2.$$

I w związku z tym możemy podać od razu

$$\|\gamma\| = \int_{\varphi_0}^{\varphi_1} \sqrt{r^2 + \left(\frac{\partial r}{\partial \varphi} \right)^2} d\varphi.$$

W powietrzu wisi **NIEZALEŻNOŚĆ OD WYBORU PARAMETRYZACJI**, ale to po przerwie.

Niech $M = \mathbb{R}^3$,

$$D = \left\{ \begin{bmatrix} D^1(t^1, t^2) \\ D^2(t^1, t^2) \\ D^3(t^1, t^2) \end{bmatrix} \mid a \leq t_1 \leq b, \quad c \leq t_2 \leq d \right\}.$$

$$\|D\| = \int \text{vol} \left(\frac{\partial}{\partial t^1}, \frac{\partial}{\partial t^2} \right) dt^1 dt^2.$$

Przykład 3. Niech

$$D = \left(\begin{bmatrix} x \\ y \\ f(x, y) \end{bmatrix}, \quad a \leq x \leq b, \quad c \leq y \leq d \right).$$

Liczymy $\text{vol}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$

$$\frac{\partial}{\partial x} = \begin{bmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{bmatrix}, \quad \frac{\partial}{\partial y} = \begin{bmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{bmatrix}.$$

$$\text{vol}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \sqrt{G\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)} = \sqrt{\left\| \begin{bmatrix} \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle & \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \\ \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right\rangle & \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right\rangle \end{bmatrix} \right\|}.$$

$$G\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \left\| \begin{bmatrix} 1 + (f_{,x})^2 & (f_{,x})(f_{,y}) \\ (f_{,x})(f_{,y}) & 1 + (f_{,y})^2 \end{bmatrix} \right\| = (1 + (f_{,x})^2)(1 + (f_{,y})^2) - (f_{,x})^2(f_{,y})^2.$$

$$\|D\| = \int_a^b \int_c^d \underbrace{\sqrt{1 + (f_x)^2 + (f_y)^2}}_{ds} dx dy.$$

Wracamy do napisu

$$\int_U d\omega = \int_{\partial U} \omega.$$

Niech A - wektor w bazie ortonormalnej. Dla $\dim M = 3$, $g = \begin{bmatrix} g_{11} & & \\ & g_{22} & \\ & & g_{33} \end{bmatrix}$,

$$A = A^1 \sqrt{g^{11}} \frac{\partial}{\partial x^1} + A^2 \sqrt{g^{22}} \frac{\partial}{\partial x^2} + A^3 \sqrt{g^{33}} \frac{\partial}{\partial x^3}.$$

niech $\alpha = A^\sharp \in \Lambda^1(M)$, γ - krzywa na M .

$$\alpha = g_{11} A^1 \sqrt{g^{11}} dx^1 + g_{22} A^2 \sqrt{g^{22}} dx^2 + g_{33} A^3 \sqrt{g^{33}} dx^3.$$

$$\int_{\gamma} \alpha = \int_{\gamma} A^{\sharp} = \int_{\gamma} \left\langle \varphi^{\star} \alpha, \frac{\partial}{\partial t} \right\rangle dt = \int_{\gamma} \left\langle \alpha, \varphi_{\star} \frac{\partial}{\partial t} \right\rangle dt = \int_{\gamma} \left\langle \alpha, \frac{\varphi_{\star} \frac{\partial}{\partial t}}{\left\| \varphi_{\star} \frac{\partial}{\partial t} \right\|} \right\rangle \left\| \varphi_{\star} \frac{\partial}{\partial t} \right\| dt.$$

Niech $v = v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} + v^3 \frac{\partial}{\partial x^3}$.

Pytanie: czym jest $\langle \alpha, v \rangle$?

$$\langle \alpha, v \rangle = A^1 \sqrt{g^{11}} g_{11} v^1 + A^2 \sqrt{g^{22}} g_{22} v^2 + A^3 \sqrt{g^{33}} g_{33} v^3.$$

czyli mamy

$$\int_{\gamma} A^{\sharp} = \int_{\gamma} \mathbf{A} \cdot \underbrace{\mathbf{t}_{st} dL}_{d\mathbf{L}}.$$

Znowu wracamy do Stokesa.

Niech $V \subset M$, $\dim M = 3$, $\dim V = 3$. Wtedy tw. Stokesa znaczy

$$\int_V d\omega = \int_{\partial V} \omega, \quad \omega \in \Lambda^2(M).$$

Niech $S \subset M$, $\dim M = 3$, $\dim S = 2$.

$$\int_S d\alpha = \int_{\partial S} \alpha, \quad \alpha \in \Lambda^1(M).$$

Pytanie 1. Niech $\alpha = A^{\sharp}$, czym jest $\int_S dA^{\sharp}$?

$$\begin{aligned} dA^{\sharp} &= \underbrace{\left(\left(g_{33} A^3 \sqrt{g^{33}} \right)_{,2} - \left(g_{22} A^2 \sqrt{g^{22}} \right)_{,3} \right)}_{D_1} dx^2 \wedge dx^3 + \underbrace{\left(\left(g_{11} A^1 \sqrt{g^{11}} \right)_{,3} - \left(g_{33} A^3 \sqrt{g^{33}} \right)_{,1} \right)}_{D_2} dx^3 \wedge dx^1 + \underbrace{(\dots)}_{D_3} dx^1 \wedge dx^2. \\ \int_S dA^{\sharp} &= \int \left\langle D^1 dx^2 \wedge dx^3, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\rangle + \left\langle D^2 dx^3 \wedge dx^1, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^1} \right\rangle + \left\langle D^3 dx^1 \wedge dx^2, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right\rangle. \\ &\quad \int \left\langle D^1 dx^2 \wedge dx^3, \frac{\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}}{\left\| \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\|} \right\rangle \underbrace{\left\| \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\|}_{ds} dx^2 dx^3 + \dots \end{aligned}$$

Pamiętamy, czym była $\text{rot}(A) = (\star dA^{\sharp})^{\flat} = \int (\text{rot}(A)) \mathbf{n} ds$