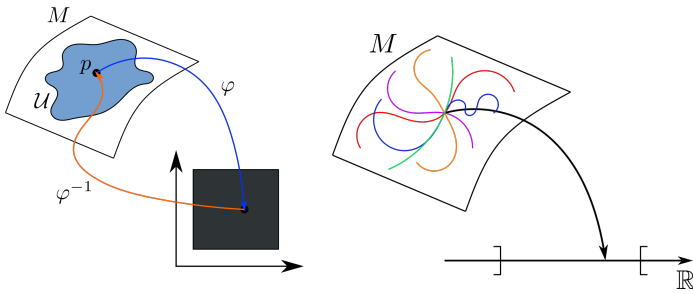


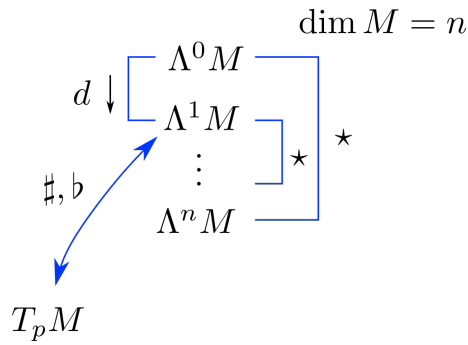
## Przypomnienie



Rysunek 0.1: Przypomnienie

Niech  $\alpha_1, \alpha_2, \dots, \alpha_k \in \Lambda^1(M)$ ,  $v_1, v_2, \dots, v_k \in T_p M$ , to wtedy

$$\langle \alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_k, v_1, v_2, \dots, v_k \rangle = \left| \begin{bmatrix} \alpha_1(v_1) & \dots & \alpha_k(v_1) \\ \vdots & \ddots & \vdots \\ \alpha_1(v_k) & \dots & \alpha_k(v_k) \end{bmatrix} \right|.$$



Rysunek 0.2: Przypomnienie c.d.

$$\langle v | w \rangle = [v]^T [g_{ij}] \begin{bmatrix} w \end{bmatrix}.$$

$$A = A^1 \frac{\partial}{\partial x^1} + \dots + A^n \frac{\partial}{\partial x^n}.$$

$$A^\sharp = A^1 g_{11} dx^1 + \dots + A^n g_{nn} dx^n,$$

(gdy  $g_{ij}$  - diagonalna)

$$A^i g_{ij} dx^j.$$

### To jak to było z tymi wektorami?

Niech  $A \in T_p M$ ,  $A = A^1 \frac{\partial}{\partial x^1} + \dots + A^k \frac{\partial}{\partial x^k}$ ,  $B \in T_p M$ ,  $B = B^1 \frac{\partial}{\partial x^1} + \dots + \frac{\partial}{\partial x^k}$ .  
Jaka jest interpretacja geometryczna wielkości

$$\langle A^\sharp, B \rangle, \quad (g_{ij} \text{ - diagonalna}).$$

$$A^\sharp = A^1 g_{11} dx^1 + \dots + A^k g_{kk} dx^k.$$

$$\begin{aligned} \langle A^\sharp, B \rangle &= \left\langle A^1 g_{11} dx^1 + \dots + A^k g_{kk} dx^k, B^1 \frac{\partial}{\partial x^1} + \dots + B^k \frac{\partial}{\partial x^k} \right\rangle = \\ &= g_{11} A^1 B^1 + \dots + g_{kk} A^k B^k = A \cdot B. \end{aligned}$$

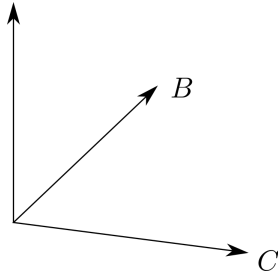
Czyli gdyby  $\|B\| = 1$ , to  $\langle A^\sharp, B \rangle$  byłoby długością rzutu  $A$  na kierunek  $B$ .  
Niech  $\dim M = 3$ ,  $\Lambda^2 M \ni A$ ,

$$A = A^1 dx^2 \wedge dx^3 + A^2 dx^3 \wedge dx^1 + A^3 dx^1 \wedge dx^2.$$

$$B = B^1 \frac{\partial}{\partial x^1} + B^2 \frac{\partial}{\partial x^2} + B^3 \frac{\partial}{\partial x^3}, \quad C = C^1 \frac{\partial}{\partial x^1} + \dots + C^3 \frac{\partial}{\partial x^3} \in T_p M.$$

$$\begin{aligned} \langle A, B, C \rangle &= A^1 \langle dx^2 \wedge dx^3, B, C \rangle + A^2 \langle dx^3 \wedge dx^1, B, C \rangle + A^3 \langle dx^1 \wedge dx^2, B, C \rangle = \\ &= A^1 \left[ \left\langle \frac{dx^2}{dx^2}, B \right\rangle \left\langle \frac{dx^3}{dx^3}, C \right\rangle \right] + A^2 \left[ \left\langle \frac{dx^3}{dx^3}, B \right\rangle \left\langle \frac{dx^1}{dx^1}, C \right\rangle \right] + \\ &+ A^3 \left[ \left\langle \frac{dx^1}{dx^1}, B \right\rangle \left\langle \frac{dx^2}{dx^2}, C \right\rangle \right] = \\ &= A^1 \begin{bmatrix} B^2 & B^3 \\ C^2 & C^3 \end{bmatrix} + A^2 \begin{bmatrix} B^3 & B^1 \\ C^3 & C^1 \end{bmatrix} + A^3 \begin{bmatrix} B^1 & B^2 \\ C^1 & C^2 \end{bmatrix} = \\ &= A^1 (B^2 C^3 - B^3 C^2) + A^2 (B^3 C^1 - B^1 C^3) + A^3 (B^1 C^2 - B^2 C^1) = \\ &= "A^1 (B \times C)_1 + A^2 (B \times C)_2 + A^3 (B \times C)_3" = "A \cdot (B \times C)" \\ &= \left| \begin{bmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{bmatrix} \right|. \end{aligned}$$

Wychodzi tak jak na (rys 0.3)



Rysunek 0.3: Się okazuje, że wychodzi z tego coś jak iloczyn wektorowy

### Problem

$\dim M = 3$ , mamy

$$\Lambda^1 M \ni F = F^1 dx^1 + F^2 dx^2 + F^3 dx^3$$

oraz krzywą  $S$  w  $\mathbb{R}^3$  (np. spiralę) (rys 0.3). Chcemy znaleźć pracę związaną z przemieszczeniem z punktu  $A$  do  $B$ .

1. sparametryzujemy kształt  $S$ , np.

$$S = \left\{ (x, y, z) \in \mathbb{R}^3, \begin{array}{l} x = \cos(t) \\ y = \sin(t), t \in [0, 4\pi] \\ z = t \end{array} \right\}.$$

2. możemy na spirali wygenerować pole wektorów stycznych.

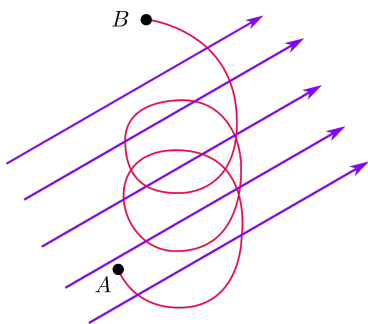
$$\text{Jeżeli } p = \left[ \begin{array}{c} \cos(t) \\ \sin(t) \\ t \end{array} \right] \Bigg|_{t=t_0}, \text{ to}$$

$$T_p M = \left\langle \left[ \begin{array}{c} -\sin(t) \\ \cos(t) \\ 1 \end{array} \right] \right\rangle \Bigg|_{t=t_0}.$$

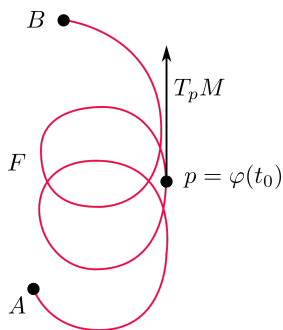
(rys 0.4)

3. Niech  $T_p M \ni v = -\sin(t) \frac{\partial}{\partial x} + \cos(t) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ . (rys 0.5)  
Możemy policzyć np.

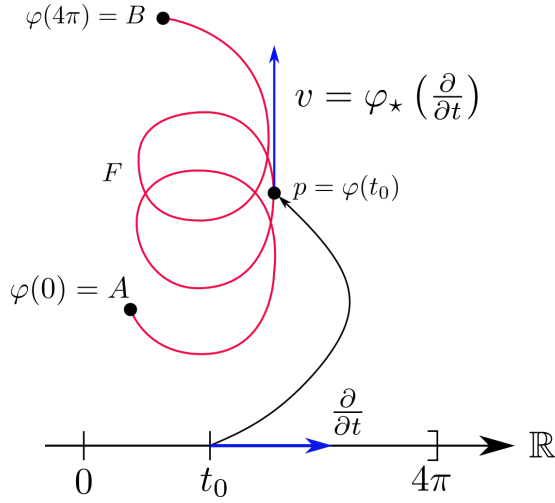
$$\begin{aligned} \int \langle F, v \rangle &= \int_0^{4\pi} \left\langle F, -\sin(t) \frac{\partial}{\partial x} + \cos(t) \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right\rangle dt = \\ &= \int_0^{4\pi} \left\langle F, \varphi_* \left( \frac{\partial}{\partial t} \right) \right\rangle dt = \int_0^{4\pi} \left\langle \varphi^* F, \frac{\partial}{\partial t} \right\rangle dt. \end{aligned}$$



Rysunek 0.4: Mrówka (albo koralik) na spirali + jakieś pole wektorowe (grawitacyjne albo mocny wiatrak)



Rysunek 0.5: można jakoś to sparametryzować przez  $\varphi$



Rysunek 0.6

**Definicja 1.** Niech  $M$  - rozmaitość,  $L$  - krzywa na  $M$ ,  $w \in \Lambda^1 M$ ,  $\varphi : [a, b] \rightarrow M$  - parametryzacja krzywej  $L$ , czyli

$$L = \{\varphi(t), t \in [a, b]\}.$$

Całkę z jednoformy po krzywej nazywamy wielkość (rys 0.6)

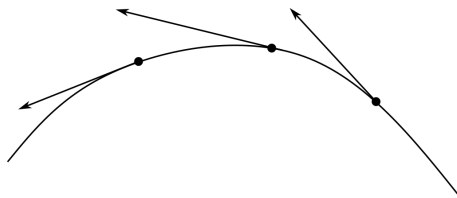
$$\int_a^b \left\langle \varphi^* \omega, \frac{\partial}{\partial t} \right\rangle dt.$$

**Przykład 1.** niech (rys 0.7)

$$C_1 = \left\{ (x, y) \in \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t-1 \\ 2t-1 \end{bmatrix}, 1 \leq t \leq 2 \right\}$$

$i$

$$\omega = ydx = \left( y \frac{\partial}{\partial x} \right)^\sharp.$$



Rysunek 0.7: Cała sztuka polega na takim kolekcjonowaniu wektorków stycznych

Wtedy mamy  $\varphi(t) = \begin{bmatrix} t-1 \\ 2t-1 \end{bmatrix}$ ,  $\varphi^*\omega = \begin{vmatrix} x=t-1 \\ dx=dt \end{vmatrix} = (2t-1)dt$

$$\left\langle \varphi^*\omega, \frac{\partial}{\partial t} \right\rangle = \left\langle (2t-1)dt, \frac{\partial}{\partial t} \right\rangle = 2t-1$$

$$\int_{C_1} \omega = \int_1^2 \left\langle \varphi^*\omega, \frac{\partial}{\partial t} \right\rangle dt = \int_1^2 (2t-1)dt = [t^2 - t]_1^2 = 2$$

$$C_2 = \left\{ (x, y) \in \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2-u \\ 5-2u \end{bmatrix}, 1 \leq u \leq 2 \right\}, \varphi_1(u) = \begin{bmatrix} 2-u \\ 5-2u \end{bmatrix}.$$

$$\int_{C_2} \omega = \int_1^2 \left\langle \varphi_1^*\omega, \frac{\partial}{\partial u} \right\rangle du,$$

ale  $\frac{x=2-u}{dx=-u}$  i mamy

$$\varphi^*\omega = (5-2u)(-du) = (2u-5)du.$$

Ostatecznie

$$\int_{C_2} \omega = \int_1^2 (2u-5)du = [u^2 - 5u]_1^2 = -6 + 4 = -2.$$