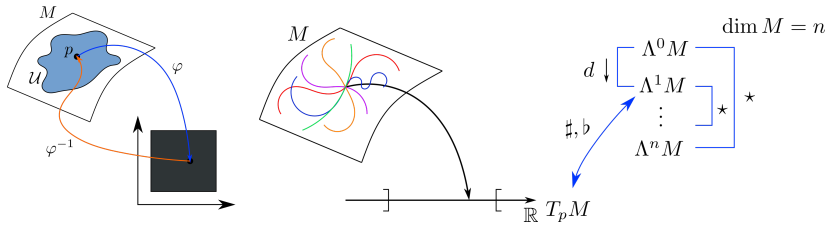
Przypomnienie



Rysunek 0.1: Przypomnienie

Niech $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Lambda^1(M)$, $v_1, v_2, \ldots, v_k \in T_pM$, to wtedy

$$\langle \alpha_1 \wedge \alpha_2 \wedge \ldots \alpha_k, v_1, v_2, \ldots, v_k \rangle = \left| \begin{bmatrix} \alpha_1(v_1) & \ldots & \alpha_k \\ \vdots & \ddots & \vdots \\ \alpha_1(v_k) & \ldots & \alpha_k(v_k) \end{bmatrix} \right|.$$

$$\langle v|w \rangle = [v]^T [g_{ij}] \begin{bmatrix} \\ w \\ \\ \end{bmatrix}.$$

$$A = A^1 \frac{\partial}{\partial x^1} + \ldots + A^n \frac{\partial}{\partial x^n}.$$

$$A^\sharp = A^1 g_{11} dx^1 + \ldots + A^n g_{nn} dx^n,$$

(gdy g_{ij} - diagonalna)

$$A^i g_{ij} dx^j.$$

To jak to było z tymi wektorami?

Niech $A \in T_pM$, $A = A^1 \frac{\partial}{\partial x^1} + \ldots + A^k \frac{\partial}{\partial x^k}$, $B = T_pM$, $B = B^1 \frac{\partial}{\partial x^1} + \ldots + \frac{\partial}{\partial x^k}$. Jaka jest interpretacja geometryczna wielkości

$$\left\langle A^\sharp, B \right\rangle, \quad (g_{ij} \text{ - diagonalna}).$$

$$A^\sharp = A^1 g_{11} dx^1 + \ldots + A^k g_{kk} dx^k.$$

$$\left\langle A^\sharp, B \right\rangle = \left\langle A^1 g_{11} dx^1 + \ldots + A^k g_{kk} dx^k, B^1 \frac{\partial}{\partial x^1} + \ldots + B^k \frac{\partial}{\partial x^k} \right\rangle =$$
$$= g_{11} A^1 B^1 + \ldots + g_{kk} A^k B^k = A \cdot B.$$

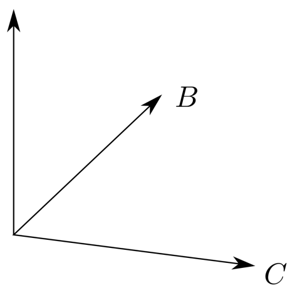
Czyli gdyby $\|B\| = 1$, to $\left\langle A^\sharp, B\right\rangle$ byłoby długością rzutu A na kierunek B.
Niech $\dim M = 3$, $\Lambda^2 M \ni A$,

$$A = A^1 dx^2 \wedge dx^3 + A^2 dx^3 \wedge dx^1 + A^3 dx^1 \wedge dx^2.$$

$$B = B^1 \frac{\partial}{\partial x^1} + B^2 \frac{\partial}{\partial x^2} + B^3 \frac{\partial}{\partial x^3}, \quad C = C^1 \frac{\partial}{\partial x^1} + \ldots + C^3 \frac{\partial}{\partial x^3} \in T_p M.$$

$$\begin{aligned}
\langle A, B, C\rangle &= A^1 \left\langle dx^2 \wedge dx^3, B, C\right\rangle + A^2 \left\langle dx^3 \wedge dx^1, B, C\right\rangle + A^3 \left\langle dx^1 \wedge dx^2, B, C\right\rangle = \\
&= A^1 \begin{bmatrix} \left\langle dx^2, B\right\rangle & \left\langle dx^3, B\right\rangle \\ \left\langle dx^2, C\right\rangle & \left\langle dx^3, C\right\rangle \end{bmatrix} + A^2 \begin{bmatrix} \left\langle dx^3, B\right\rangle & \left\langle dx^1, B\right\rangle \\ \left\langle dx^3, C\right\rangle & \left\langle dx^1, C\right\rangle \end{bmatrix} + \\
&+ A^3 \begin{bmatrix} \left\langle dx^1, B\right\rangle & \left\langle dx^2, B\right\rangle \\ \left\langle dx^1, C\right\rangle & \left\langle dx^2, C\right\rangle \end{bmatrix} = \\
&= A^1 \begin{bmatrix} B^2 & B^3 \\ C^2 & C^3 \end{bmatrix} + A^2 \begin{bmatrix} B^3 & B^1 \\ C^3 & C^1 \end{bmatrix} + A^3 \begin{bmatrix} B^1 & B^2 \\ C^1 & C^2 \end{bmatrix} = \\
&= A^1 \left(B^2 C^3 - B^3 C^2\right) + A^2 \left(B^3 C^1 - B^1 C^3\right) + A^3 \left(B^1 C^2 - B^2 C^1\right) = \\
&\text{"} = \text{"} A^1 (B \times C)_1 + A^2 (B \times C)_2 + A^3 (B \times C)_3 \text{"} = \text{"} A \cdot (B \times C) \\
&= \left| \begin{bmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{bmatrix} \right|.
\end{aligned}$$

Wychodzi tak jak na (rys 0.2)



Rysunek 0.2: Się okazuje, że wychodzi z tego coś jak iloczyn wektorowy

Problem

$\dim M = 3$, mamy

$$\Lambda^1 M \ni F = F^1 dx^1 + F^2 dx^2 + F^3 dx^3$$

oraz krzywą S w $\mathbb{R}^3$ (np. spiralę) (rys 0.3). Chcemy znaleźć pracę związaną z przemieszczeniem z punktu A do B.

1. sparametryzujmy kształt S, np.

$$S = \left\{ (x, y, z) \in \mathbb{R}^3, \begin{array}{c} x = \cos(t) \\ y = \sin(t) \\ z = t \end{array}, t \in [0, 4\pi] \right\}.$$

2. możemy na spirali wygenerować pole wektorów stycznych.
   Jeżeli $p = \left.\begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}\right|_{t=t_0}$, to

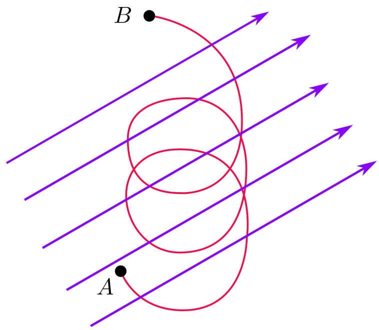
$$T_pM = \left.\left\langle \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{bmatrix} \right\rangle\right|_{t=t_0}.$$

   (rys 0.4)

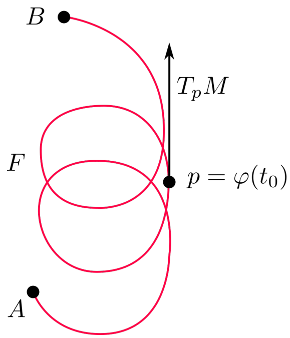
3. Niech $T_pM \ni v = -\sin(t)\frac{\partial}{\partial x} + \cos(t)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. (rys 0.5)
   Możemy policzyć np.

$$\int \langle F, v \rangle = \int_0^{4\pi} \left\langle F, -\sin(t)\frac{\partial}{\partial x} + \cos(t)\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right\rangle dt =$$

$$= \int_0^{4\pi} \left\langle F, \varphi_\star\left(\frac{\partial}{\partial t}\right) \right\rangle dt = \int_0^{4\pi} \left\langle \varphi^\star F, \frac{\partial}{\partial t} \right\rangle dt.$$



Rysunek 0.3: Mrówka (albo koralik) na spirali + jakieś pole wektorowe (grawitacyjne albo mocny wiatrak)



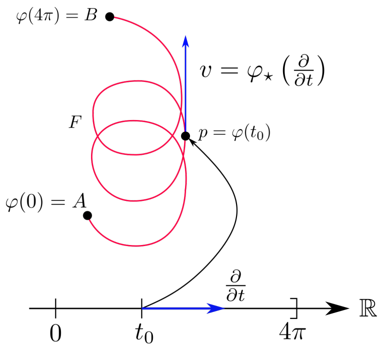
Rysunek 0.4: można jakoś to sparametryzować przez φ

Definicja 1. *Niech M - rozmaitość, L - krzywa na M, $w \in \Lambda^1 M$, $\varphi : [a, b] \to M$ - parametryzacja krzywej L, czyli*

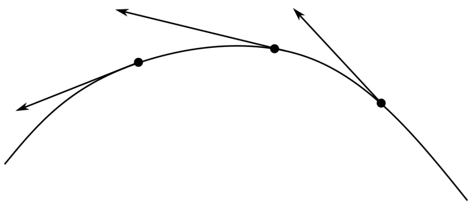
$$L = \{\varphi(t), t \in [a, b]\} .$$

Całką z jednoformy po krzywej nazywamy wielkość (rys 0.5)

$$\int_a^b \left\langle \varphi^\star \omega, \frac{\partial}{\partial t} \right\rangle dt.$$



Rysunek 0.5



Rysunek 0.6: Cała sztuka polega na takim kolekcjonowaniu wektorków stycznych

Przykład 1. *niech (rys 0.6)*

$$C_1 = \left\{(x,y) \in \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t-1 \\ 2t-1 \end{bmatrix}, 1 \leqslant t \leqslant 2\right\}$$

i

$$\omega = ydx = \left(y\frac{\partial}{\partial x}\right)^\sharp.$$

Wtedy mamy $\varphi(t) = \begin{bmatrix} t-1 \\ 2t-1 \end{bmatrix}$, $\varphi^\star\omega = \quad \begin{vmatrix} x = t-1 \\ dx = dt \end{vmatrix} = (2t-1)dt$

$$\left\langle \varphi^\star\omega, \frac{\partial}{\partial t}\right\rangle = \left\langle (2t-1)dt, \frac{\partial}{\partial t}\right\rangle = 2t-1$$

$$\int\limits_{C_1} \omega = \int\limits_1^2 \left\langle \varphi^\star\omega, \frac{\partial}{\partial t}\right\rangle dt = \int\limits_1^2 (2t-1)dt = \left[t^2 - t\right]_1^2 = 2$$

.

$$C_2 = \left\{(x,y) \in \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2-u \\ 5-2u \end{bmatrix}, 1 \leqslant u \leqslant 2\right\}, \varphi_1(u) = \begin{bmatrix} 2-u \\ 5-2u \end{bmatrix}.$$

$$\int\limits_{C_2} \omega = \int\limits_1^2 \left\langle \varphi_1^\star \omega, \frac{\partial}{\partial u} \right\rangle du,$$

ale $\begin{matrix} x = 2-u \\ dx = -u \end{matrix}$ *i mamy*

$$\varphi^\star \omega = (5-2u)(-du) = (2u-5)du.$$

Ostatecznie

$$\int\limits_{C_2} \omega = \int\limits_1^2 (2u-5)du = \left[u^2 - 5u\right]_1^2 = -6 + 4 = -2.$$