0.1 W ostatnim odcinku

$$\int_\gamma \alpha = \int_\gamma \vec{A}\cdot \underbrace{\vec{t}_{st} dL}_{d\vec{L}}.$$

$$dA^\sharp = \left(\overbrace{(.), -(.)}^{D_1} \right) dx^2 \wedge dx^3 + \ldots.$$

$$\int_S dA^\sharp = \int D^1 \left\langle dx^2 \wedge dx^3, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\rangle dx^2 dx^3 + \int D^2 dx^3 dx^1 + \int D^3 dx^1 dx^2.$$

Przypomnijmy sobie czym jest rotacja wektora (takiego fizycznego)

$$rot(\vec{A}) = \left(\star \left(d\vec{A}^\sharp \right) \right)^\flat,$$

ale

$$\star(dx^2 \wedge dx^3) = g^{22} g^{33} \sqrt{g} dx^1,$$
$$\star(dx^3 \wedge dx^1) = g^{11} g^{33} \sqrt{g} dx^2,$$
$$\star(dx^1 \wedge dx^2) = g^{11} g^{22} \sqrt{g} dx^3.$$

Więc

$$\star dA^\sharp = D^1 g^{22} g^{33} \sqrt{g} dx^1 + D^2 g^{33} g^{11} \sqrt{g} dx^2 + D^3 g^{11} g^{22} \sqrt{g} dx^3.$$

$$\begin{aligned}\left(\star dA^\sharp\right)^\flat &= D^1 g^{11} g^{22} g^{33} \sqrt{g} \frac{\partial}{\partial x^1} + D^2 g^{22} g^{33} g^{11} \sqrt{g} \frac{\partial}{\partial x^2} + D^3 g^{33} g^{11} g^{22} \sqrt{g} \frac{\partial}{\partial x^3} = \\ &= D^1 \sqrt{g^{22} g^{33}} \sqrt{g^{11}} \frac{\partial}{\partial x^1} + D^2 \sqrt{g^{11} g^{33}} \sqrt{g^{22}} \frac{\partial}{\partial x^2} + D^3 \sqrt{g^{11} g^{22}} \sqrt{g^{33}} \frac{\partial}{\partial x^3}.\end{aligned}$$

Czyli dla $\vec{A}$ - wektor w bazie ortonormalnej jest

$$rot\vec{A} = \begin{bmatrix} D^1 \frac{1}{\sqrt{g_{22} g_{33}}} \\ D^2 \frac{1}{\sqrt{g_{11} g_{33}}} \\ D^3 \frac{1}{g_{11} g_{22}} \end{bmatrix}.$$

ale $rot(\vec{A}) \cdot \vec{n} = D^1 \frac{1}{g_{22} g_{33}}$, ale

$$\left(rot\vec{A} \cdot \vec{n} \right) \cdot d\vec{s} = D^1 \frac{1}{g_{22} g_{33}} \sqrt{g_{22} g_{33}} dx^2 dx^3,$$

zatem

$$\int_S dA^\sharp = \int_S (rot\vec{A}) \cdot \vec{n} ds.$$

Czyli teraz mamy tak

$$\int_\gamma A^\sharp = \int_\gamma \vec{A} \cdot \vec{t}_{st} dL.$$

$$\int_S dA^\sharp = \int_{\partial S} A^\sharp.$$

$$\int_S \left(rot\vec{A} \right) \cdot \vec{n} ds = \int_{\partial S} \vec{A} \cdot \vec{t}_{st} dL.$$

Przykład 1. $\dim M = 3$, $V \subset M$, $\dim V = 3$

$$\int_{\partial V} \star A^\sharp = \int_V d \star A^\sharp.$$

Pytanie 1. *czym jest* $\int_{\partial V} \star A^\sharp$*?*

$$\star(dx^1) \sqrt{g} g^{11} dx^2 \wedge dx^3,$$
$$\star(dx^2) \sqrt{g} g^{22} dx^3 \wedge dx^1,$$
$$\star(dx^3) \sqrt{g} g^{33} dx^1 \wedge dx^2,$$

Odpowiedź:

$$\star A^\sharp = A^1 g_{11} \sqrt{g^{11}} \sqrt{g} g^{11} dx^2 \wedge dx^3 + A^2 g_{22} \sqrt{g^{22}} \sqrt{g} g^{22} dx^3 \wedge dx^1 + A^3 g_{33} \sqrt{g^{33}} \sqrt{g} g^{33} dx^1 \wedge dx^2,$$

następuje cudowne skrócenie i jest

$$A^1\sqrt{g_{22}g_{33}}\quad dx^2\wedge dx^3+A^2\sqrt{g_{11}g_{33}}\quad dx^3\wedge dx^1+A^3\sqrt{g_{11}g_{22}}\quad dx^1\wedge dx^2.$$

Całka z tego interesu:

$$\int_{\partial V}\star A^\sharp=\int A^1\sqrt{g_{22}g_{33}}\quad dx^2dx^3+\int A^2\sqrt{g_{11}g_{33}}\quad dx^3dx^1+\int A^3\sqrt{g_{11}g_{22}}\quad dx^1dx^2,$$

ale

$$\vec{A}\cdot\vec{n}\cdot ds=A^1\sqrt{g_{22}g_{33}}\quad dx^2dx^3.$$

Czyli ostatecznie

$$\int_{\partial V}\star A^\sharp=\int_{\partial V}\vec{A}\cdot\vec{n}ds.$$

Pytanie 2. *Jak wygląda $\int_V d\star A^\sharp$?*

$$\int_V d\star A^\sharp=\int_V\left\langle\left(A^1\sqrt{g_{22}g_{33}}\right)_{,1}+\left(A^2\sqrt{g_{11}g_{33}}\right)_{,2}+\left(A^3\sqrt{g_{11}g_{22}}\right)_{,3},dx^1\wedge dx^2\wedge dx^3,\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}\right\rangle dx^1dx^2dx^3.$$

Dywergencja to było coś takiego:

$$div\vec{A}=\star d\left(\star A^\sharp\right),$$

wiemy, że

$$\star\left(dx^1\wedge dx^2\wedge dx^3\right)=\sqrt{g}g^{11}g^{22}g^{33}=\sqrt{g^{11}g^{22}g^{33}},$$

więc

$$div\vec{A}\sqrt{g_{11}g_{22}g_{33}}\quad dx^1dx^2dx^3=div\vec{A}\quad dV.$$

Zatem ze zdania

$$\int_{\partial V}\star A^\sharp=\int_V d\star A^\sharp$$

wiemy, że

$$\int_{\partial V}\vec{A}\cdot\vec{n}ds=\int_V div\vec{A}\quad dV.$$