Ćwiczenie 1. *Mając dane 1-formy ω_1, ω_2*

$$\omega_1 = 2x^2 dx + (x+y)dy, \quad \omega_2 = -xdx + (x-2y)dy,$$

oblicz $\omega_1 \wedge \omega_2$, pochodne zewnętrzne $d\omega_i$ i sprawdź, czy $dd\omega_i = 0$.

$$\begin{aligned}\omega_1 \wedge \omega_2 &= 2x^2(x-2y)dx \wedge dy - x(x+y)dy \wedge dx = \\ &= \left(2x^2(x-2y) + x(x+y)\right) dx \wedge dy = \\ &= \left(2x^3 - 4x^2y + x^2 + xy\right) dx \wedge dy.\end{aligned}$$

$$\begin{aligned}d\omega_1 &= d\left(2x^2dx + (x+y)dy\right) = \\ &= d(2x^2dx) + d\left((x+y)dy\right) = \\ &= 4xdx \wedge dx + dx \wedge dy + dy \wedge dy = dx \wedge dy.\end{aligned}$$

Ogólnie mamy

$$d\left(f(x,y)dx + g(x,y)dy\right) = \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial g}{\partial x} dx \wedge dy.$$

$$\begin{aligned}d\omega_2 = d(-xdx) + d\left((x-2y)dy\right) = -dx \wedge dx + \\ + dx \wedge dy - 2dy \wedge dy = dx \wedge dy.\end{aligned}$$

$$d\left(dx \wedge dy\right) = 0.$$

Ćwiczenie 2. *Jeszcze przykład iloczynu zewnętrznego*

$$\begin{aligned}\omega_1 &= \left(x^2 + y + 2z^3\right) dx \wedge dy + xyzdy \wedge dz \\ \omega_2 &= x^2ydx + z^2xdy + xdz \\ \omega_1 \wedge \omega_2 &= \left(x^2 + y + 2z^3\right) xdx \wedge dy \wedge dz + \\ &\quad + x^3y^2zdy \wedge ddz \wedge dx = \\ &= \left(x^3 + xy + 2z^3x + x^3y^2z\right) dx \wedge dy \wedge dz.\end{aligned}$$

$$\begin{aligned}d\omega_1 &= 6zdz \wedge dx \wedge dy + yzdx \wedge dy \wedge dz = \\ &= (6z + yz)\, dx \wedge dy \wedge dz.\end{aligned}$$

$$\begin{aligned}d\omega_2 &= -x^2dx \wedge dy - z^2dy \wedge dx - 2zxdy \wedge dz - dz \wedge dx = \\ &= (z^2 - x^2)dx \wedge dy - 2zxdy \wedge dz + dx \wedge dz.\end{aligned}$$

$$dd\omega_2 = 2zdz \wedge dx \wedge dy - 2zdx \wedge dy \wedge dz = 0.$$

Ćwiczenie 3. *Niech $f = x^2 + y^2 - 3z^4$. Znaleźć gradient f we współrzędnych kartezjańskich i walcowych, korzystając z odpowiedniości 0-form.*

$$\nabla f = (df)^\flat, \quad g_{ij} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}.$$

$$\begin{aligned}df &= 2xdx + 2ydy - 12z^3dz \\ \nabla f &= 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - 12z^3\frac{\partial}{\partial z}.\end{aligned}$$

$$
\begin{aligned}
g_{ij} &= \begin{bmatrix} 1 & & \\ & \rho^2 & \\ & & 1 \end{bmatrix} \quad (\rho, \varphi, z) \\
f &= (\rho\cos\varphi)^2 + (\rho\sin\varphi)^2 - 3z^4 \\
df &= 2\rho d\rho - 12z^3 dz \\
g^{ij} &= \begin{bmatrix} 1 & & \\ & \frac{1}{\rho^2} & \\ & & 1 \end{bmatrix} \\
(df)^\flat &= g^{\rho\rho}\frac{\partial f}{\partial\rho}\frac{\partial}{\partial\rho} + g^{\varphi\varphi}\frac{\partial f}{\partial\varphi}\frac{\partial}{\partial\varphi} + g^{zz}\frac{\partial f}{\partial z}\frac{\partial}{\partial z} = \\
&= 2\rho\frac{\partial}{\partial\rho} - 12z^3\frac{\partial}{\partial z}\underbrace{\left(+\frac{1}{\rho^2}\frac{\partial f}{\partial\varphi}\frac{\partial}{\partial\varphi}\right)}_{\text{znika}}.
\end{aligned}
$$

Ćwiczenie 4. *Znaleźć rotację pól wektorowych korzystając z odpowiedniości pól wektorowych i form różniczkowych*

$$
rot(v) = \left(\star\left(d\left(v^\sharp\right)\right)\right)^\flat .
$$

$$
A = \left(2x^2, x+y, 0\right), \quad B = \left(\rho^2\sin\varphi, \rho\cos\varphi, 3z\right).
$$

$$
\begin{aligned}
A^\sharp &= 2x^2 dx + (x+y)dy \\
d(A^\sharp) &= dx \wedge dy \\
\star(d(A^\sharp)) &= dz \\
\left(\star\left(d(A^\sharp)\right)\right)^\flat &= \frac{\partial}{\partial z}.
\end{aligned}
$$

$$
\nabla \times A = (0,0,1).
$$

W walcowych g_{ij} jest postaci

$$
g_{ij} = \begin{bmatrix} 1 & & \\ & \rho^2 & \\ & & 1 \end{bmatrix}.
$$

$$
\begin{aligned}
B^\sharp &= \rho^2\sin\varphi d\rho + \rho^3\cos\varphi d\varphi + 3z dz \\
d(B^\sharp) &= \rho^2\cos\varphi d\varphi \wedge d\rho + 3\rho^2\cos\varphi d\rho\wedge d\varphi = \\
&= 2\rho^2\cos\varphi d\rho\wedge d\varphi \star\left(d(B^\sharp)\right) = \frac{\rho}{(3-2)!}2\rho^2\cos\varphi g^{\rho\rho}g^{\varphi\varphi}\epsilon_{123}dz = \\
&= 2\rho^3\cos\varphi\frac{1}{\rho^2}dz = 2\rho\cos\varphi dz \\
\left(\star\left(d(B^\sharp)\right)\right)^\flat &= 2\rho\cos\varphi\frac{\partial}{\partial z}.
\end{aligned}
$$

Ćwiczenie 5. *Znaleźć dywergencję pól*

$$
A = (0,0,x^2-y^2), \quad B = (r^2, \cos\varphi, 3\cos\theta).
$$

$$
\begin{aligned}
A &= (0,0,x^2-y^2) = (x^2-y^2)\frac{\partial}{\partial z} \\
A^\sharp &= (x^2-y^2)dz \\
\star A^\sharp &= (x^2-y^2)dx\wedge dy \\
d\star A^\sharp &= 0 dx\wedge dy\wedge dz = 0.
\end{aligned}
$$

$$
\begin{aligned}
g_{ij} &= \begin{bmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin\theta \end{bmatrix} \\
B^\sharp &= r^2 dr + r^2 \cos\varphi d\theta + 3r^2 \cos\theta \sin^2\theta d\varphi \\
\star d\theta &= \frac{r^2 \sin\theta}{(3-1)!} g^{\theta j_1} \epsilon_{j_1 k_1 k_2} dx^{k_1} \wedge dx^{k_2} = \\
&= -\sin\theta dr \wedge d\varphi \\
\star dr &= r^2 \sin\theta d\theta \wedge d\varphi \\
\star d\varphi &= r^2 \sin\theta \frac{1}{r^2 \sin^2\theta} dr \wedge d\theta = \\
&= \frac{1}{\sin\theta} dr \wedge d\theta \\
\star B^\sharp &= r^2 \cdot r^2 \sin\theta d\theta \wedge d\varphi + \ldots \\
d(\star B^\sharp) &= 4r^3 \sin\theta dr \wedge d\theta \wedge d\varphi \\
\star d \star B^\sharp &= \frac{4r^3 \sin\theta}{r^2 \sin\theta} = 4r.
\end{aligned}
$$

Ćwiczenie 6. *Obliczyć (wyprowadzić) gradient, rotację i dywergencję we współrzędnych bisferycznych* (σ, τ, φ)*:*

$$
g_{ij} = \frac{\sigma^2}{\cosh\tau - \cos\sigma} \begin{bmatrix} 1 & & \\ & 1 & \\ & & \sin^2\sigma \end{bmatrix}.
$$