Calculus

1. Real Numbers.

1.1.

a) Find the least and greatest numbers of the following sets: 1) $A = \{2n : n \geq 5\}$ 2) $\{2k^2 + 7 \ : 8 \geq k \geq 2\}$

b)We consider $\{\frac{1}{n} : n \in \mathbb{N}\backslash\{0\}\}$. Does this set admits a least or greatest element?

1.2. Prove by induction:

(a) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(b) $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$

(c) $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$, si $r \neq 1$

(d) $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2^n} \geq 1 + \frac{n}{2}$

Tip: Note that the number of elements of $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2^n}$ is 2^n and for $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2^{n+1}}$ it is 2^{n+1}.

5) Let $0 < x_0 < 1$ and we define $x_n = x_{n-1}\frac{n}{n+1}$ for $n \geq 1$. Prove that $0 < x_n < 1$ for all $n \in \mathbb{N}$.

1.3. The following properties are tue and can be deduced from the axioms de **complete ordered field**. Prove that:

Using the properties of the sum: a) if $x + y = x + z$, then $y = z$ b) if $x + y = x$, then $y = 0$ c) if $x + y = 0$, then $y = -x$ d) $-(-x) = x$.

Using the properties of the product: we assume that $x \neq 0$, a) if $xy = xz$, then $y = z$ b) if $xy = x$, then $y = 1$ c) if $xy = 1$, then $y = \frac{1}{x}$ d) $\frac{1}{\frac{1}{x}} = x$.

Using the properties of distributivity: a) $0x = 0$ b) if $x \neq 0$ and $y \neq 0$, than $xy \neq 0$. c) $(-x)y = -(xy) = x(-y)$ d) $(-x)(-y) = xy$.

Using the properties of order: a) if $x > 0$, then $(-x) < 0$ b) if $x < 0$ and $y < z$, then $xy > xz$. c) if $x \neq 0$, then $x^2 > 0$ and $1 > 0$. d) if $0 < x < y$, then $0 < 1/y < 1/x$.

1.4. A machine works only with three-digit numbers and is capable of inserting a comma between the digits. Prove that the machine **does not** respect the **associative** property of the sum (**Tip:** Consider $12,2$; $3,19$ y $4,12$. and use the fact that rounding may eliminates the last decimal).

1.5. Prove that:

1) $(x + y)^2 = x^2 + 2xy + y^2$
2) $x^2 - y^2 = (x + y)(x - y)$
3) if $x^2 = y^2$, then $x = y$ or $x = -y$.
4) if $ax^2 + bx + c = 0$ and $a \neq 0$, then $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ Is it always true?

1.6. Simplify:

1) $\frac{x^2-a^2}{x-a}$ 2) $\frac{x^2+2ax+a^2}{x+a}$ 3) $\frac{x^3-a^3}{x-a}$.

1.7. If $0 < a < b$ are real numbers, prove that:

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}}.$$

1.8. Draw the following sets in in $\mathbb{R}$. Compute the upper and lower boundaries, the supremum and infimum, and the maximum and minimum number of each set (if they exist).

(a) $\{1-\frac{1}{n} : n \in \mathbb{N}\}$, (b) $[1, 3)\cup(2, \pi]$, (c) $[3, \frac{25}{3}]\cap(\frac{5}{4}, 8]$, (d) $\{(-1)^n+\frac{n}{n+1}, n \in \mathbb{N}\}$.

1.9. Compute the upper and lower boundaries, the supremum and infimum, and the maximum and minimum (if they exist) of the following sets:s:
(a) $\{3, 3'3, 3'33, 3'333, ...\}$ (b)$\{x \in \mathbb{R} : x = 1 - \frac{1}{r},$ con $r > 0\}$
(c) $A \subset \mathbb{R}$ such that if $x \in A$ and the decimal form of x is $x = 0, a_1a_2a_3....a_n...$ then $a_{2k} = 1$ for all $k \in \mathbb{N}$.

1.10. Fidn all real numbers x such that:

(a) $-5(2 - x) < 15$

(b) $x^2 - 1 < 0$

(c) $\frac{x+3}{2x+5} \geq 3$

(d) $\frac{x-1}{x+1} > 0$

(e) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$

(f) $x^2 - 4 \geq |2x + 4|$

(g) $\frac{1-2x}{x+2} \leq 3$

(h) $\sqrt{1 + x} < 1 + \frac{1}{x}$

(i) $x^3(x^6 - 62)(x + 3)^2 < 0$

(j) $\frac{1-2x}{x+2} \leq 3$

1.11. Solve:

(a) $|x - 3| + |x - 7| = 2$

(b) $|x - 3| + |x - 7| = 4$

(c) $||3 - x| - |x|| = |x| + 1$

(d) $|2 - |x|| = 2 + |x|$

1.12. Prove that:
(a) $\{x \in \mathbb{R} \ : \ |2x + 3| < 6\} = \{x \in \mathbb{R} \ : \ -9/2 < x < 3/2\} = (-9/2, 3/2)$.
(b) $\{x \in \mathbb{R} \ : \ |x - 1| < |x|\} = \{x \in \mathbb{R} \ : \ x > 1/2\} = (1/2, \infty)$.
(c) $\{x \in \mathbb{R} \ : \ |x(x - 4)| < |x - 4| - |x| \} = (2 - \sqrt{8}, 3 - \sqrt{5})$.

1.13. Let $y = 2x + |2 - x|$, write x in function of y.