Discrete Mathematics and Mathematical Logic I

Grado en Ingeniería Informática (UCM)

Homework 1 September 2020

Exercises on logic and proof methods

1 Training exercises

Exercise 1 Determine whether these implications are true or false:

1. If $1 + 1 = 2$, then $2 + 2 = 5$.
2. If $1 + 1 = 3$, then $2 + 2 = 4$.
3. If $1 + 1 = 3$, then $2 + 2 = 5$.
4. If $1 + 1 = 3$, then God exists.

Exercise 2 Prove that if the universe of discourse U is $\{a_1, \dots, a_n\}$, the following propositions are equivalent:

1. $\forall x P(x)$ and $P(a_1) \wedge \dots \wedge P(a_n)$.
2. $\exists x P(x)$ and $P(a_1) \vee \dots \vee P(a_n)$.

Exercise 3 Determine whether $\forall x(P(x) \to Q(x))$ and $\forall x P(x) \to \forall x Q(x)$ have the same truth value. If $P(x)$ stands for "x lives in the ocean", $Q(x)$ for "x is a fish" and the universe for x consists of all animals, what is the meaning of those assertions in English?

Exercise 4 Study whether the proposition $\forall x(x + 2 < x)$ is true when the universe is $\mathbb{R}$ and when the universe consists of the real solutions to the equation $(x^2 + 2)^2 = 1$.

Exercise 5 Study whether the proposition $\exists x(x + 2 > x)$ is true when the universe is $\mathbb{R}$ and when the universe consists of the real solutions to the equation $(x^2 + 2)^2 = 1$.

Exercise 6 Study whether the following propositions are true or false when the universe is $\mathbb{N}$ and when the universe is $\mathbb{R}$; in case they are false, show a counterexample.

1. $\forall x \forall y(x^2 = y^2 \to x = y)$.
2. $\forall x \exists y(y^2 = x)$.
3. $\forall x \forall y(xy \geq x)$.

Exercise 7 Prove that the following assertions are true:

1. For all integers a and b, if both are odd then so is ab.
2. There exist two integers a and b such that $ab = 60$ and $a + b < 20$.

Exercise 8 Prove by cases that $\max(x, y) + \min(x, y) = x + y$, where x, y are real numbers.

2 Additional exercises

Exercise 1 Prove that if $17n + 2$ is odd then n is odd.

Exercise 2 Prove that $\neg \exists x Q(x)$ is equivalent to $\forall x \neg Q(x)$.

Exercise 3 Prove or provide a counterexample:

1. If c and d are perfect squares, then cd is a perfect square.
2. If cd is a perfect square and $c \neq d$, then c and d are perfect squares.

Exercise 4 Prove by contradiction that if m and n are integers such that $n + n^2 + n^3 = m + m^2$, then n is even.