Introduction. MIM 1

Def: Mathematics (OXFORD ENG. DICTIONARY):
The abstract science which investigates deductively the conclusions implicit in the elementary conceptions of spatial and numerical relations.
This science can be divided in 6 main topics:
- Foundations: logic, Set theory, Proof theory, etc
- Algebra: numbers, arithmetical operations, order theory
- Analysis: differentiation, integration, measure, etc
- Geometry and topology: properties of space, shape, position of figures.
- Combinatorics: graph theory, partition theory, etc
- Applied Mathematics: computational sciences, probability
The range of applications of Mathematics is wide, such as:
* Banking and Finance: Black and Schole equation
* Aeronautical Engineering: Fluid mechanics, shape design
* Chemistry: Models for protein folding, thermodynamics
* Informatic: Cryptography, computational algebra, parallel programming, etc
Summary of the program: 1st Semester
- Real numbers
- Sequence and Series of numbers
- Continuity and limit
- Differentiation (1D)
- Integration (1D)

Chap 1: Real Numbers and some basic concepts

1. Set of points:

We recall here some basical concepts.

- **Def:** A Set is a collection of distinct objects.

Ex: 2, 5, 7 are different objects (numbers). They can compose the set $\{2,5,7\}$, where $\{\dots\}$ denotes the set composed by the objects... □

- **Not:** If an object x is a member of a set Θ, we denote:

$$x \in \Theta \quad \text{else we denote } x \notin \Theta.$$

Ex: $\Theta = \{0,5,7\}$, $x=5$ and $y=9$:

$$x \in \Theta \quad \text{and} \quad y \notin \Theta \;\square$$

- **Def:** Considering two sets A and B. If every elements of A is a member of B, A is said to be a subset of B, and we denote:

$$A \subseteq B, \text{ else we denote } A \nsubseteq B \; (\text{or } A \not\subset B)$$

Furthermore, if it exists at least one element of B which is not a member of A, A is said to be a proper subset of B, and we denote

$$A \subset B.$$

Ex: $A=\{1,2,3\}$ $\quad B=\{0,1,2,3,4\}$ $\quad C=\{0,1,2\}$

$$A \subseteq A \qquad A \subset B \qquad A \nsubseteq C \; (\text{or } A \not\subset C)$$

- **Def:** Some set operators: Let A and B being two sets.

* $\cup$: union : The union of A and B is the set

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$$A \cup B = \{x \,/\, x \in A \text{ or } x \in B\}$$

$/$: such that

*** $\cap$: intersection:** The intersection of A and B is the set

$$A \cap B = \{x \,/\, x \in A \text{ and } x \in B\}.$$

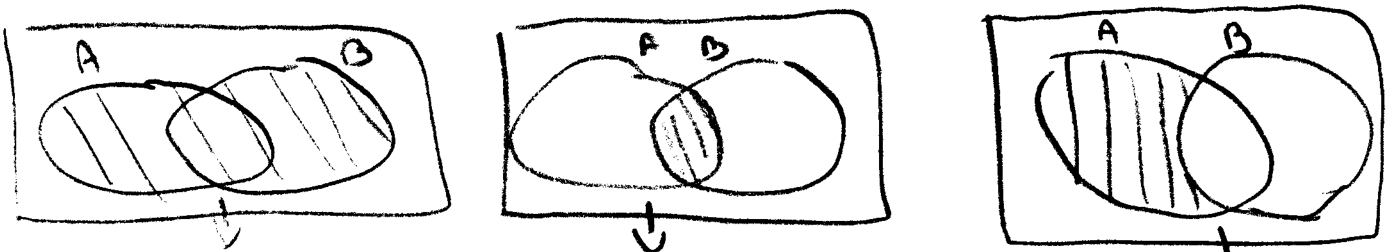
*** $\setminus$: complement:**

$$A \setminus B = \{x \,/\, x \in A \text{ and } x \notin B\}.$$

Ex: $A = \{3, 5, 7\}$ $\quad B = \{5, 7, 10\}$

- $A \cup B = \{3, 5, 7, 10\}$ • $A \cap B = \{5, 7\}$
- $A \setminus B = \{3\}$ • $B \setminus A = \{10\}$ □

Geometrical representation (GR):



$A \cup B$ $\qquad$ $A \cap B$ $\qquad$ $A \setminus B$

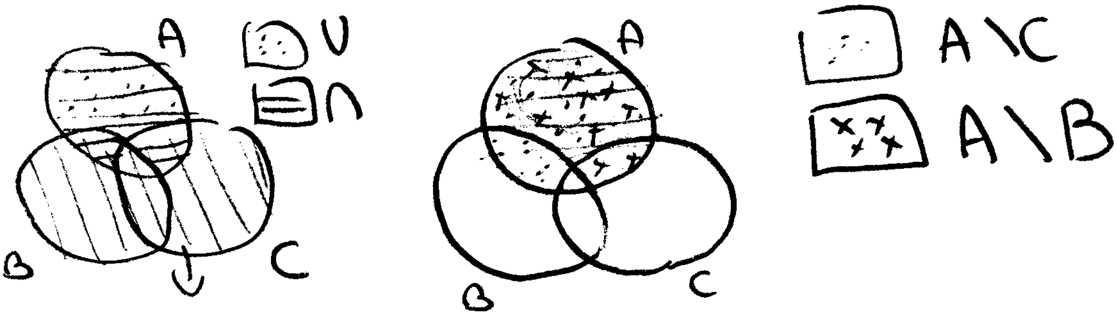
□

- Properties:

a) $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$

b) $A \setminus (B \cap C) = (A \setminus B) \cap (A \setminus C)$

GR:



$B \cup C$

- Not: In the case of various $^{(n)}$ set denoted by $A_1, A_2, \ldots, A_n$, instead of writing: MIM 4

$A_1 \cup A_2 \cup \ldots \cup A_n$ we write $\bigcup_{k=1}^{n} A_k$

$A_1 \cap A_2 \cap \ldots \cap A_n$ or $\bigcap_{k=1}^{n} A_k$

Ex: $A_1 = \{1,2,3\}$ $A_2 = \{5,6,7\}$ $A_3 = \{1,5,9\}$

$$\bigcup_{k=1}^{3} A_k = A_1 \cup A_2 \cup A_3 = \{1,2,3,5,7,9\}$$

- Observation: We can apply the same notation in case of infinite $^{(\infty)}$ numbers of set $\{A_1, \ldots, A_{100}, \ldots\}$:

infinity

$$\bigcup_{k=1}^{\infty} A_k \quad \text{and} \quad \bigcap_{k=1}^{\infty} A_k$$

some examples and the concept of infinity will be defined in next Sections. □

- Def: The cartesian product of two sets A and B is denoted by $A \times B$ and defined as:

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

where (a,b) is called ordered pair.

Ex: $A = \{1,2,3\}$ $B = \{7,9\}$

$$A \times B = \{(1,7),(1,9),(2,7),(2,9),(3,7),3,9)\}$$ □

More properties will be introduced later in this chapter.

II. Some common set of sets of real points:

Here we only introduce the set of points used in next Chapters:

Def:

* $\mathbb{R} = \{\ldots, -10, \ldots, -7, \ldots, 0, \ldots, 4, \ldots 1000, \ldots\}$ is called the set of real numbers which contains all positive an negative numbers.

* $\mathbb{N} = \{1, 2, 3, \ldots\}$ is called the set of natural numbers which contains all the strictly positive integer numbers

* $\mathbb{Z} = \{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3 \ldots\}$ is called the set of integer numbers and contains the positive and negative integers.

* $\phi = \{\}$ the empty set represents the sets without any element. Ej: $A = \{1, 4\}$, $B = \{3, 4\}$ $A \cap B = \phi$ (no coincidence btw A and B)

* $\mathbb{Q} = \{x \in \mathbb{R} / x = \frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{Z}$ and $n \neq 0\}$ is called the set of rational numbers and contains the real numbers that can be written as a quotient of integer numbers.

* $\text{Odd} = \{x \in \mathbb{R} / \underbrace{\exists}_{\substack{\text{Not: it} \\ \text{exists}}} k \in \mathbb{N} \underbrace{\text{s.t}}_{\substack{\text{Not:} \\ \text{such that}}} x = 2k+1\}$ is the set of the odd integer numbers.

* $\text{Even} = \{x \in \mathbb{R} / \exists k \in \mathbb{N} \text{ s.t. } x = 2k\}$ is the set of the even integer numbers

* $\mathbb{C} = \{x + iy \mid x \in \mathbb{R}$ and $y \in \mathbb{R}\}$ is the set of complex numbers (cultural, not studied here). Useful, e.g. electricity

Not: i denotes the imaginary number that verify $i^2 = -1$. $\square$

Obs: $\phi \subset \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

Def: A relation, denoted by $\leq$, is a total order on a set S if it verifies:

1. Reflexivity: $\underbrace{\forall}_{\text{Not: forall}} a \in S$, $a \leq a$
2. Antisymmetry: $a \leq b$ and $b \leq a \overset{\text{Not: implies}}{\Rightarrow} a = b$
3. Transitivity: $a \leq b$ and $b \leq c \Rightarrow a \leq c$

If only verifies ⇒ partial order

4. Comparability: $\forall a, b \in S$, either $a \leq b$ or $b \leq a$

Ej: * The relation $\leq$ applied to $\mathbb{R}$ is a total order.
* The relation $\subset$ applied to sub-set of $\mathbb{R}$ is <u>not</u> a total order: $\{1,2\}$ and $\{2,4\}$ cannot be compared □

Def: A set plus a total order relation is called total ordered set.

Ej: $(\mathbb{R}, \leq)$: rules
1. $a = b$: a equal b
2. $a < b$ $(a > b)$: a strictly inferior (superior) to b (not equal)
3. $a \leq b$ $(a \geq b)$: a inferior or equal (superior) to b. □

Def: Infinity, denoted by ∞, is an abstract concept describing a limitless quantity (e.g. number).

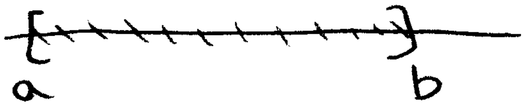
Prop: * $\forall x \in \mathbb{R}$, $-\infty \leq x$ and $x \leq +\infty$.
* $-\infty$ and $+\infty \notin \mathbb{R}$ □

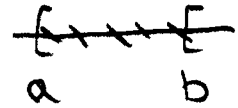
Def: An interval is a real subset containing all the values between two given points, included or not. It can be of the type:

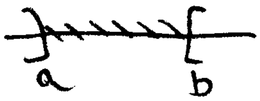
$a, b \in \mathbb{R}$:
* Open interval: $\overset{\text{Not}}{(a, b)} = \{x \in \mathbb{R} / a < x < b\}$
* closed interval: $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
* left closed interval: $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
* left open interval: $(a, b] = \{x \in \mathbb{R} / a < x \leq b\}$ □

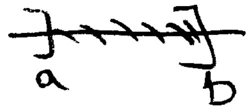
Prop: $\mathbb{R} = (-\infty, +\infty)$. $\square$

Graphical representation (GR):

$[a,b]$: 

$[a,b)$: 

(a,b): 

$(a,b]$:  $\square$

Def: Axiomatic definition of $\mathbb{R}$

The real number system $(\mathbb{R}, +, \cdot, <)$ is a set where the following rules are defined.

a) addition $(+)$ a function $\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$

$(x,y) \longmapsto x+y$

with the following properties

N: first this operation

* Associativity: $\forall x,y,z \in \mathbb{R}$ $\quad (x+y)+z = x+(y+z)$
* Commutativity: $\forall x,y \in \mathbb{R}$ $\quad x+y = y+x$
* Identity element: $\exists\, 0 \in \mathbb{R} \,/\, 0+x = x+0 = x$
* Opposite element: $\forall x \in \mathbb{R}$, $\exists !\, -x \in \mathbb{R} \,/\, x+(-x) = (-x)+x$

N. it exists an unique

b) Multiplication $(\cdot)$: a function $\mathbb{R}\times\mathbb{R} \to \mathbb{R}$

$(x,y) \mapsto x.y$

with the following properties

* Associativity: $\forall x,y,z \in \mathbb{R}$ $\quad (xy)z = x(yz)$
* Commutativity: $\forall x,y \in \mathbb{R}$ $\quad xy = yx$
* Identity element: $\exists\, 1 \in \mathbb{R} \,/\, 1.x = x.1 = x$
* Inverse element: $\forall x \in \mathbb{R}\setminus\{0\}$, $\exists !\, \frac{1}{x} \in \mathbb{R} \,/\, x.\frac{1}{x} = \frac{1}{x}.x =$
* Distributivity: $\forall x,y,z \in \mathbb{R}$

$x.(y+z) = x.y + x.z$

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c) The field $(\mathbb{R}, +, \cdot)$ is ordered: * $\geq$ is a total order.

* $\forall x, y, z \in \mathbb{R}$, if $x \geq y$ then $x+z \geq y+z$
* $\forall x, y \geq 0$ $\quad xy \geq 0$

d) The order is Dedekind-complete:

If $A \subseteq \mathbb{R}$, $A \neq \emptyset$ and $\exists k \in \mathbb{R} / \forall a \in A, a \leq k$ (k is called upper bound) then $\exists \alpha$, denoted $\sup A$ and called least upper bound, such that $\forall a \in A$ $a \leq \alpha$ and $\forall k \in \mathbb{R}$ upper bound of A $\alpha \leq k$. $\square$

Obs: - $\mathbb{N}$ cannot be defined axiomatically (e.g. $0 \notin \mathbb{N}$) $\square$

In previous definition we have used the concept of function, in next section we present some definitions and properties of functions used in next chapter.

III. Mathematical Functions:

Def: Let A and B being two sets. A function from A to B, is a relation between A and B, denoted by $f: A \to B$, such that $\forall a \in A$, $\exists! b \in B / f(a) = b$.

The elements of A are called arguments of f.

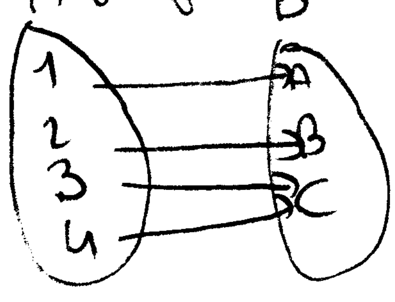
The element $b \in B$ s.t $f(a) = b$, with $a \in A$, is called value at a or image of a under f.

A is called domain of f, $D(f)$, and $R(f) = \{b \in B / \exists a \in A$ s.t $f(a) = b\}$ is the codomain. $\square$

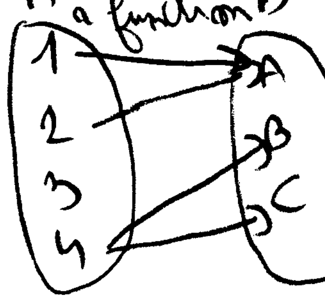
Not: * $f: A \to B$
$\quad a \mapsto f(a)$

* we can write f maps A to B.

Ex: A f is function B



A f is not a function B



$f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2 + 1$

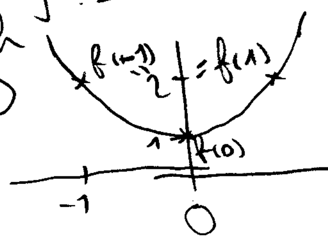
$f(1) = 1^2 + 1 = 2$

$f(2) = 2^2 + 1 = 5$

Not: x^n, $n \in$

Def: The graph of a function is the set of ordered pairs $F=\{(a,f(a)), \forall a\in A\}$. □

G.I: $f: \mathbb{R}\to\mathbb{R}$
$x\mapsto x^2+1$ $\overset{\text{graph}}{\Longrightarrow}$



□

Def: a) If $E\subseteq A$, the image of E under f is:

$$f(E)=\{f(x) \,/\, x\in E\}$$

b) If $H\subseteq B$, the preimage of H under f is:

Not

$$f^{(-1)}(H)=\{x\in A \,/\, f(x)\in H\}. \;□$$

Ej: $A=B=\mathbb{R}$ $\quad f(x)=x^2$

* $E=[0,2]\subset\mathbb{R}$ $\quad f(E)=[0,4]$
* $H=\{4,9\}\subset\mathbb{R}$ $\quad f^{-1}(H)=\{-3,-2,2,3\}$ □

Def: $f: A\to B$

a) f is called injective if, $\forall a_1,a_2\in A$, $f(a_1)=f(a_2)\Rightarrow a_1=a$

b) f is called surjective if $\forall b\in B$, $\exists a\in A \,/\, f(a)=b$.

c) f is called bijective if f is injective and surjective. □

Ej: a) $f: \mathbb{R}\to\mathbb{R}$, $x\mapsto x^2$ is not injective: $f(1)=f(-1)$
is not surjective: $-1\in\mathbb{R}$ and $\underbrace{\nexists}_{\text{Not: does not exist}} a\in A \,/\, f(a)$

b) $f:[0,1]\to[1,2]$, $x\mapsto x+1$ is injective: $f(x_1)=f(x_2)\Rightarrow x_1+1=x_2+1$
$\Rightarrow x_1=x_2$
is surjective: $\forall x_1\in[1,2]$, $x_2=x_1-1\in[0$
and $f(x_2)=x_1$

Thus f is bijective □

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Def: Let $f: A \to B$ and $g: B \to C$, the composition of f with g is a function denoted as $g \circ f$ and defined by:

$$g \circ f : A \to C$$
$$a \longmapsto g(\underbrace{f(a)}_{\in B}) \quad \square$$

Eg: $g: \mathbb{R} \to \mathbb{R}$, $x \mapsto \cos(x)$ and $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$

$$g \circ f(x) = \cos(x^2) \neq f \circ g(x) = (\cos(x))^2 \quad \square$$

Def: The identity function f on A, is the function

$$id_A : A \to A$$
$$x \longmapsto x \quad \square$$

Def: Let $f: A \to B$ bijective. The inverse function of f, denoted by f^{-1}, is the function $f^{-1}: B \to A$ such that:

$$\circledast \ b = f(a) \iff a = f^{-1}(b)$$

Nota: if and only if $\square$

Eg: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x+1$ $\qquad f^{-1}: \mathbb{R} \to \mathbb{R}$, $x \mapsto x-1$

$$x = f(y) = y+1 \iff y = x-1 = f^{-1}(x) \quad \square$$

Obs: $\circledast$ can be replaced by

$$f \circ f^{-1}(b) = id_B$$

and

$$f^{-1} \circ f(a) = id_A$$

Eg: $f \circ f^{-1}(x) = f(x-1) = (x-1)+1 = x$ and $f^{-1} \circ f(x) = (x+1)-1 = x$ $\square$

Def: we call real function with real variable a function of the type $f: \mathbb{R} \to \mathbb{R}$ □ | MIM1

Note: More properties and definitions about function will be seen in next chapters. □

IV. Some Properties of particular real sub-sets.

a) Odd and Even sets:

Prop: Let $o_1, o_2 \in$ Odd and $e_1, e_2 \in$ Even:

a) $O_1 + O_2$ is even
b) $e_1 + e_2$ is even
c) $e_1 + o_1$ is odd
d) $e_1 \times e_2$ is even
e) $O_1 \times O_2$ is odd
f) $e_1 \times O_1$ is even □

Proof: let $e_1 = 2k_1$, $e_2 = 2k_2$, $O_1 = 2k_3 + 1$ and $O_2 = 2k_4 + 1$ with k_1, k_2, k_3, k_4 in $\mathbb{Z}$

a) $O_1 + O_2 = 2k_3 + 1 + 2k_4 + 1 = 2(k_3 + k_4) + 2 = \overbrace{2(\underbrace{k_3 + k_4 + 1}_{\in \mathbb{Z}})}^{\text{even}}$

b) $e_1 + O_1 = 2k_1 + 2k_3 + 1 = \overbrace{2(\underbrace{k_1 + k_3}_{\in \mathbb{Z}})}^{\text{even}} + 1$

d) $e_1 \times e_2 = 2k_1 \times 2k_2 = \underbrace{2(\underbrace{2k_1k_2}_{\in \mathbb{Z}})}_{\text{even}}$ → see next sub-sections.

f) $e_1 \times O_1 = 2k_1 \times (2k_3 + 1) = 2k_1k_3 + 2k_1$
$= \underbrace{2(\underbrace{k_1k_3 + h_1}_{\in \mathbb{Z}})}_{\text{even}}$ □

Exercice 1.1: Prop:

a) <u>true</u>: $\Leftarrow$ n is even, from previous proposition $n^2 = n \times n$ is even.

$\Rightarrow$] n^2 is even. Proof by reduction to the absurd.

If n is odd $n^2 = n \times n$ is odd, which is absurd (due to the assumption.

b) true $(n+p)^2$ is even $\Rightarrow (n+p)$ even $\Rightarrow$ n, p even or n, p odd

if n, p even $\Rightarrow n-p$ is even $\Rightarrow (n-p)^2$ is even

if n, p odd $\Rightarrow n-p$ is even $\Rightarrow (n-p)^2$ is even

$\Leftarrow$] same idea.

c) true np odd $\Rightarrow n$ and p odd $\Rightarrow n+p$ is even.

d) true $\Rightarrow$] $n^2+np+p^2 = \underbrace{(n+p)^2}-np$ even $\Rightarrow$ absurd

if np odd $\Rightarrow (n+p)^2$ is odd $\Rightarrow (n+p)$ is odd $\Rightarrow$ contrapositive n even, p odd $\Rightarrow np$ even (absurd).

note: false
$\Leftarrow$] np even $\Rightarrow n$ and p even $\Rightarrow n^2+np+p^2$ even

or n odd and p even $\Rightarrow n^2$ odd $+ p^2$ even is odd

n^2+np+p^2 is odd $\Rightarrow$ false ej: $n=1$ $p=2$ $np=2$

$1+2+4=7$ odd

e) true: absurd

$n^2+np+p^2 = (n+p)^2 - \underbrace{np}_{\text{even}}$

if n and p odd $\Rightarrow np$ odd and $(n+p)^2$ even $\Rightarrow n^2+np+p^2$ odd

if n even and p odd $\Rightarrow np$ even and $(n+p)^2$ odd

absurd

b) $\mathbb{N}$ and $\mathbb{Z}$

Prop: $n_1, n_2 \in \mathbb{N}$ and $z_1, z_2 \in \mathbb{Z}$

a) $n_1+n_2 \in \mathbb{N}$ b) $n_1 \times n_2 \in \mathbb{N}$

c) $z_1+z_2 \in \mathbb{Z}$ d) $z_1 \times z_2 \in \mathbb{Z}$

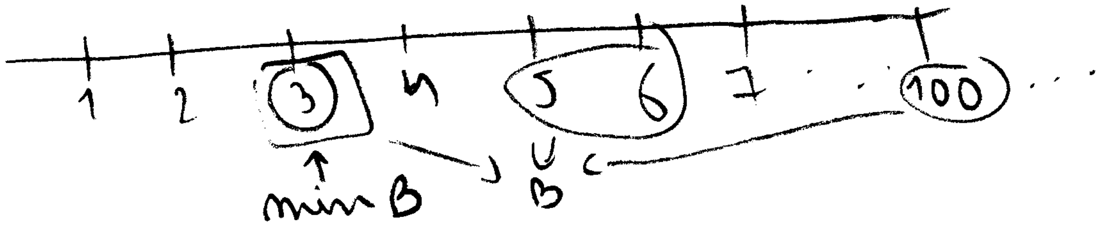
e) $n_1 \geq n_2$ or $n_2 \geq n_1$

f) $z_1 \geq z_2$ or $z_2 \geq z_1$

$\square$

Prop: (well-ordering principle): Let $B \subseteq \mathbb{N}$ and $B \neq \emptyset$.
It always exists $n_0 \in B$ such that $\forall m \in B$, $n_0 \leq m$.
Such n_0 is called minimum of B and denoted $\min B$ (see next subsection).

G.1



Def: Mathematical induction:
We want to demonstrate a statement P_n involving $n \in \mathbb{N}$ for all values of n.
We follow the step:
1- We prove that the statement holds for the first value of n.
2- We prove that, if the statement holds for n, then it holds for $n+1$. $\square$

Ex:

Proof: W.O.P. Assume it is false.
$J = \mathbb{N} \setminus B$.
$P_n = $ "$\{1,..,n\} \in J$". We start with:
1- $P_1 = $ "$1 \in J$": true else $\min B = 1$.
2- $P_n = \{1,..,n \in J\}$, then $n+1 \in J$ else $\min B = n+1$ (as $1,..,n \notin B$).
$\Rightarrow \forall n \in \mathbb{N} \in J \Rightarrow B = \emptyset \Rightarrow$ ABSURD.

Ex 1.2 a) $\displaystyle\overset{\text{Not}}{\sum_{k=1}^{n} k} = \frac{n(n+1)}{2} = P_n$

Not: Sum: (see later) $\displaystyle\sum_{k=1}^{n} k = 1+2+3+\dots+n$

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$* P_1 = 1 = \frac{1(1+1)}{2} = 1$

$(* P_2 = 1+2 = 3 = \frac{2(2+1)}{2} = 3)$ OPTIONAL

$* P_n$ true: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \Rightarrow \sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + n+1$

$= \frac{n(n+1)}{2} + n+1 = \frac{n(n+1)+2n+2}{2} = \frac{(n+1)(n+2)}{2}$

$\Rightarrow \sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$ P_{n+1} true.

5) $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$ $\quad \sum_{k=0}^{n} r^k = \underbrace{r^0}_{=1} + \underbrace{r^1}_{=r} + r^2 + r^3 + \dots + r^n$

$* P_1$: $r^0 + r = 1 + r = \frac{1-r^2}{1-r} = \frac{(1+r)\cancel{(1-r)}}{\cancel{1-r}}$ true

$* P_n$ true: $\sum_{k=0}^{n+1} r^k = \sum_{k=0}^{n} r^k + r^{n+1} = \frac{1-r^{n+1}}{1-r} + r^{n+1}$

$= \frac{1 - \cancel{r^{n+1}} + \cancel{r^{n+1}} - r.r^{n+1}}{1-r}$, $r.r^{n+1} = r^{n+2}$ $= \frac{1-r^{n+2}}{1-r}$ P_{n+1} true $\square$

c) $\mathbb{Q}$:

Prop: $Q_1, Q_2 \in \mathbb{Q}$

a) $Q_1 + Q_2 \in \mathbb{Q}$ $\quad$ b) $Q_1 \times Q_2 \in \mathbb{Q}$

c) $\frac{1}{Q_1} \in \mathbb{Q}$ (if $Q_1 \neq 0$) d) $Q_1 \leq Q_2$ or $Q_2 \leq Q_1$

Ex: 1.3 a) $p \subset \mathbb{Q}$ $p \neq 0 \Rightarrow p = \frac{a}{b}$ $\quad a, b \in \mathbb{Z} \setminus \{0\}$

$x \notin \mathbb{Q}$

If $p+x \in \mathbb{Q}$, $\exists\, c, d \in \mathbb{Z}$ t.q.:

$$p + x = \frac{c}{d} \Rightarrow \frac{a}{b} + x = \frac{c}{d} \Rightarrow x = \frac{c}{d} - \frac{a}{b} = \frac{\overset{\in \mathbb{Z}}{(bc-ad)}}{\underset{\in \mathbb{Z}\setminus\{0\}}{(db)}}$$

$\Rightarrow x \in \mathbb{Q}$ absurd.

b) same idea $\square$

Ex $\sqrt{2}$ is irrational. Proof:

Absurd: $\sqrt{2} = \frac{p}{q} \Rightarrow \frac{p^2}{q^2} = 2 \Rightarrow p^2 = 2q^2$

$\Rightarrow p$ even $\Rightarrow p^2$ divisible by 4 (i.e. $p^2 = 4m$, $m \in \mathbb{N}$)

We assume p and q have no common factors (i.e. $\nexists\, n \in \mathbb{N}\setminus\{1\} / p = kn$ and $q = ln$ $k, l \in \mathbb{Z}$)

Thus, $q^2 = 2m \Rightarrow q$ even (i.e. $q = 2s$, $s \in \mathbb{N}$)

Then, p and q have a common factor (2).

Absurd. $\square$

Prop: $\sqrt{n}$, with $n \in \mathbb{N}$ and such that n is not a square number (i.e., $\nexists\, p \in \mathbb{N} / p^2 = n$), is irrational

Def: An algebraic number is a real number that is a root of a non-zero polinomial with rational coefficients.

i.e.: n is algebraic, it exists a polinom

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k = \sum_{i=0}^{k} a_i x^i$$

such that $p(n) = 0$.

Prop: If a_1 and a_2 algebraic, thus $a_1 + a_2$, $a_1 a_2$, $-a_1$ and $\frac{1}{a_1}$ ($a_1 \neq 0$) are algebraic

Ex: a) If $\sqrt{2} - \sqrt{3}$ is rational $\Rightarrow (\sqrt{2} - \sqrt{3})^2$ is rational

$\Rightarrow 5 - 2\underbrace{\sqrt{2}\sqrt{3}}_{=\sqrt{6}}$ is rational $\Rightarrow \sqrt{6}$ rational $\Rightarrow$ absurd

MIM1(

$1-\sqrt[3]{2+\sqrt{5}}$ is algebraic:

2 is algebraic and $\sqrt{5}$ is algebraic (x^2-5)

$\Rightarrow 2+\sqrt{5}$ is algebraic and root ... of $P(x)$.

$\Rightarrow \sqrt[3]{2+\sqrt{5}}$ is root of $P(x^3)$ (i.e all exponents of x are increased by 3, e.g: $P(x)=x^2+x-1 \Rightarrow P(x^3)=x^6+x^3-1$

$\Rightarrow$ 1 is algebraic $\Rightarrow 1-\sqrt[3]{2+\sqrt{5}}$ is algebraic.

<u>Def</u>: a decimal representation of a real number r is an expression of the form:

$$r = a_0, a_1 a_2 a_3 a_4 \dots \quad \text{with } a_i \in \text{[illegible]}, i \in \mathbb{N}$$

or (see Serie chapter)

$$r = \sum_{i=1} \frac{a_i}{10^i} \quad \square$$

<u>Ex</u>: $\frac{1}{3} = 0,3333\dots$ $\quad \sqrt{2} = 1,41421\dots$

<u>Prop</u>: The decimal representation of a real number r either terminates (i.e, $\exists n_0 \in \mathbb{N} / a_i = 0 \ \forall i \geq n_0$) or begins to repeat a same finite sequence over and over if and only if r is rational.

<u>Ex</u>: $0,127\,841\,841\,841\dots$ is rational

$\parallel$ not

$0,127\overline{841}$

$\pi = 3.14159265\dots$ is irrational $\square$

d) $\mathbb{R}$:

<u>Prop</u>: $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in $\mathbb{R}$: $\forall x_1, x_2 \in \mathbb{R}, x_1 < x_2 \ \exists q \in \mathbb{Q}$ and $m \in \mathbb{R}\setminus\mathbb{Q} / x_1 < q < x_2$ and $x_1 < m < x_2$.

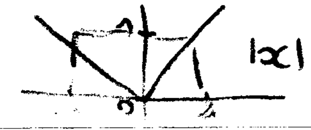
* <u>Distance in $\mathbb{R}$</u>:

<u>Def</u>: The absolute value or modulus of a real number x is denoted $|x|$ and defined by:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad \square$$

<u>G.R</u>: 1.1: $\mathbb{R} \to \mathbb{R}$

$x \mapsto |x|$



a) $|x| = \sqrt{x^2}$

b) $|xy| = |x||y|$

c) $|x+y| \leq |x| + |y|$ (Sub additivity) $\square$

Proof: c) $x \leq |x|$ and $y \leq |y|$
$-x \leq |x|$ and $-y \leq |y|$ Nota: $\pm x \leq |x|$ and $\pm y \leq |y|$

$\Rightarrow$ $x+y \leq |x| + |y|$
$-(x+y) \leq |x|+|y|$ $\Rightarrow |x+y| \leq |x|+|y|$ $\square$

Def: $\forall x, y \in \mathbb{R}$, we call $|x-y|$ the distance from x to y.

Prop: $\forall x, y, z \in \mathbb{R}$:

a) $|x-y| = 0 \Leftrightarrow x = y$

b) $|x-y| = |y-x|$

c) $|x-y| \leq |x-z| + |z-y|$ (triangle inequality). $\square$

Ex: 1.5 a) $|2x+3| - 1 < |x|$

we determine all the cases of signs.

$x < -\frac{3}{2}$, $0 > x \geq -\frac{3}{2}$ and $x \geq 0$

$x \geq 0 \Rightarrow$ $|2x+3| = 2x+3 \geq 0$ and $|x| = x \geq 0$

the equation leads to:

$2x+3-1 < x \Rightarrow x < -2 \Rightarrow$ Absurd $\square$

$x < -\frac{3}{2}$: $|2x+3| = -2x-3$ and $|x| = -x$

$-2x-3-1 < -x \Rightarrow -2x-4 < -x$

$\Rightarrow$ $2x+4 > x \Rightarrow x > -4 \Rightarrow x \in \left(-4, -\frac{3}{2}\right)$

rule: $a < b \Leftrightarrow -a > -b$

MIM1

$x \in [-\frac{3}{2}, 0)$: $|2x+3| = 2x+3$ $\quad |x| = -x$

$$2x+3-1 < -x \Rightarrow 3x < -2 \Rightarrow x < -\frac{2}{3}$$

$$\Rightarrow x \in [-\frac{3}{2}, -\frac{2}{3})$$

The solutions are $x \in (-4, -\frac{2}{3})$ $\square$

b) Same idea: $x \geq 4$; $4 > x \geq 0$; $x < 0$

1.6 $\quad |2-|x|| = 2+|x|$

$x \geq 2$: $|x| = x \quad |2-|x|| = -2+x$

$\Rightarrow -2+x = 2+x \Rightarrow 2 = -2$ absurd

$2 > x \geq 0$: $|x| = x \quad |2-|x|| = 2-x$

$\Rightarrow 2-x = 2+x \Rightarrow x = -x \Rightarrow \boxed{x=0}$

$0 > x \geq -2$: $|x| = -x \quad |2-|x|| = 2+x$

$\Rightarrow 2-x = 2+x \Rightarrow x = 0$

$x < -2$: $|x| = -x \quad |2-|x|| = -2+x$

$-2+x = 2-x \Rightarrow 2x = 4 \Rightarrow x = 2$ absurd

The solution is $x = 0$.

* Some applications of the axiomatic definition of $\mathbb{R}$:

1.7 a) $ax = a$ and $a \neq 0 \Rightarrow x = \frac{a}{a} = 1$

b) $(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$

c) $(x+y)(x-y) = x^2 + xy - xy - y^2 = x^2 - y^2$

d) $x^2 = y^2 \Rightarrow x = \pm\sqrt{y^2} \Rightarrow \begin{matrix} x = +|y| \\ x = -|y| \end{matrix} \Rightarrow x = \pm y$

MIM19

e) $(x-y)(x^2+xy+y^2) = x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3$
$= x^3 - y^3$

f) $(x-y)\left(\sum_{i=0}^{n-1} x^{n-(i+1)} y^{i}\right) = \sum_{i=0}^{n-1} x^{n-i} y^{i} - \sum_{i=0}^{n-1} x^{n-(i+1)} y^{i+1}$

$= x^n + \sum_{i=1}^{n-1} x^{n-i} y^{i} - \underbrace{\sum_{i=0}^{n-2} x^{n-(i+1)} y^{i+1}}_{=-\sum_{i=1}^{n-1} x^{n-i} y^{i}} - y^n = x^n - y^n$ □

rext 1

add 1

1.8 $0 < a < b$

* $\underbrace{\frac{a+b}{2}}_{0} < \underbrace{\sqrt{\frac{a^2+b^2}{2}}}_{0} \Rightarrow \frac{a^2+b^2+2ab}{4} < \frac{a^2+b^2}{2}$

$\Rightarrow 2ab < a^2+b^2 \Rightarrow 0 < a^2+b^2-2ab \Rightarrow 0 < \underbrace{(a-b)^2}_{<0}$

$\Rightarrow$ true

* $\sqrt{ab} < \frac{a+b}{2} \Rightarrow 2\underset{0}{\underset{\vee}{\sqrt{ab}}} < \underset{0}{\underset{\vee}{a+b}} \Rightarrow 4ab < a^2+b^2+2ab$

$\Rightarrow 2ab < a^2+b^2$ true

* $\frac{2ab}{a+b} < \sqrt{ab} \Rightarrow \underset{0}{\underset{\vee}{2ab}} < \underset{0}{\underset{\vee}{\sqrt{ab}}}(a+b) \Rightarrow 4a^2b^2 < ab(a+b)^2$

$\Rightarrow 4ab < a^2+b^2+2ab$ true.

1.9

a) $a \leq b$ and $\forall \varepsilon > 0$ $a \leq b \leq a+\varepsilon$:

Absurd: if $a < b \Rightarrow 0 < b-a \Rightarrow \exists \varepsilon_b > 0 / \varepsilon_b < b-a$

$\Rightarrow a+\varepsilon_b < b$ with $\varepsilon_b > 0 \Rightarrow$ ABSURD $\Rightarrow a = b$

b) $a \leq b$ and $\forall \varepsilon > 0$ $b - \varepsilon \leq a \leq b$:

ABSURD: if $a<b \Rightarrow$ $a-b<0 \Rightarrow \exists \varepsilon_a>0$ / $a-b<-\varepsilon_a$ (MIM2

$\Rightarrow a<b-\varepsilon_a$, $\varepsilon_a>0$ absurd $\Rightarrow a=b$. □

* Boundaries of real subsets:

Def: Let $A \subseteq \mathbb{R}$. A is called bounded from above if $\exists k \in \mathbb{R} / \forall a \in A$ $a \leq k$. In this case, k is called upper bound of S.

A is called bounded from below if $\exists k \in \mathbb{R} / \forall a \in A$ $a \geq k$. k is called lower bound.

If A has both upper and lower bounds, then A is called bounded set. □

Ex: $(0,+\infty)$ is bounded from below by any $k \leq 0$ but not bounded from above. □

Def: $A \subseteq \mathbb{R}$ $\begin{cases}\alpha \\ \beta\end{cases}$ is called $\begin{cases}\text{supremum} \\ \text{infimum}\end{cases}$ of A (or $\begin{cases}\text{least upper} \\ \text{greatest lower}\end{cases}$ bound) if:

a) $\begin{cases}\alpha \\ \beta\end{cases}$ is an $\begin{cases}\text{upper} \\ \text{lower}\end{cases}$ bound of A

b) $\forall k \in \mathbb{R}$ $\begin{cases}\text{upper} \\ \text{lower}\end{cases}$ bound of A, $\begin{cases}\alpha \leq k \\ \beta \geq k\end{cases}$

We denote $\alpha = \sup A$ and $\beta = \inf A$. □

Ex: $A=[3,5]$ $\sup A = 5$ and $\inf A = 3$.

Prop: * If $A \subseteq \mathbb{R}$ is bounded from above, then A admits a supremum.

* If $A \subseteq \mathbb{R}$ is bounded from below, then A admits an infimum

Def: Let $A \subseteq \mathbb{R}$.

* If $\sup A$ exists and $\sup A \in A$, then $\sup A$ is called maximum of A and is denoted $\max A$.

* — $\inf A$ ———— $\inf A \in A$ — $\inf A$ —— minimum — $\min A$

Ex: $A = [3, 4)$ $\quad \min A = \inf A = 3$

$\sup A = 4 \notin A \Rightarrow \max A$ does not exist. □

1.10: A bounded and $A_0 \subseteq A$

* from above: If A_0 is not bounded from above (b. f. a)

$\forall k \in \mathbb{R}, \exists a_0 \in A_0 \ / \ a_0 > k \Rightarrow \forall k \in \mathbb{R}, \exists a_0 \in A \ /$
$a_0 > k \Rightarrow A$ is not b. f. a $\Rightarrow$ absurd.

Furthermore, $\forall a \in A_0$, $a \in A$ and $a \leq \sup A \Rightarrow$
$\sup A$ is an upper bound of $A_0 \Rightarrow \sup A \geq \sup A_0$.

* from below: same idea □

1.11: $A, B \subset \mathbb{R}$ bounded:

$A + B = \{ x \in \mathbb{R} / \ x = a + b$ with $a \in A$ and $b \in B \}$

i) * $\forall x \in A + B$, $\quad x = a + b \leq \sup A + \sup B$ as $a \leq \sup A$, $b \leq \sup B$

$\Rightarrow \sup A + \sup B$ is an upper bound of $A + B$.

* Let $k \in \mathbb{R} \ / \ \forall x \in A + B \quad x \leq k$

$\Rightarrow \forall a \in A, b \in B \quad a + b \leq k \Rightarrow \forall a \in A \quad a \leq k - b$ $\quad \forall b \in B$ is u.b of A

$\Rightarrow \sup A \leq k - b$, $\forall b \in B$

$\Rightarrow \forall b \in B$, $b \leq \underbrace{k - \sup A}_{\text{u.b of } B} \Rightarrow \sup B \leq k - \sup A$

$\Rightarrow \sup A + \sup B \leq k$.

Thus $\sup A + B = \sup A + \sup B$.

(i) same idea

(ii) $\alpha > 0$: $\quad \alpha A = \{ x \in \mathbb{R} / \ x = \alpha a$ with $a \in A \}$

* $\forall x \in A$, $\quad x = \alpha a \underset{\text{as } \alpha > 0}{\leq} \alpha \sup A$ as $a \leq \sup A$

* Let $k \in \mathbb{R} \ / \ \forall x \in \alpha A \quad x \leq k \Rightarrow \forall a \in A \ \alpha a \leq k$

since $\alpha>0$ α

Thus $\alpha \sup A = \sup \alpha A$.

$\inf \alpha A$ = same idea.

(iv) same idea but $\alpha<0$ (changes in inequalities).

* $\forall x \in A,\ x = \alpha A \geq \alpha \sup A$
* $\forall a \in A$ $\alpha a \geq k \Rightarrow \forall a \in A$ $a \leq \frac{k}{\alpha} \Rightarrow \sup A \leq \frac{k}{\alpha} \Rightarrow \alpha \sup A \geq k$.

Ex 1.12:

1) $A = \{2; 2{,}2; 2{,}22; 2{,}222; \dots\}$

* $\forall a \in A$, $\forall x \in (-\infty, 2]$ $x \leq a \Rightarrow (-\infty, 2]$ is the set of l.b.
* $\inf A = 2$ (the greatest l.b).
* $\inf A \in A \Rightarrow \min A = 2$
* $\forall a \in A$, $\forall x \in [2+\frac{10}{45} = 2{,}\overline{2}, +\infty)$, $x \geq a \Rightarrow [2+\frac{10}{45}, +\infty)$ set of U.b
* $\sup A = 2 + \frac{10}{45}$ (least U.b)
* $\sup A \notin A$, max does not exists.

2) $\forall r \in \mathbb{R}$, $z = \text{floor}(r)+1 \in \mathbb{Z}$ and $z > r$

$z = \text{floor}(r) - 1 \in \mathbb{Z}$ and $z < r \Rightarrow \mathbb{Z}$ does not admit u... l. bounds.

Not: floor(r) corresponds to the largest integer that does n... exceed r; : Ex: round$(3.32) = 3$

3,6) study the roots.

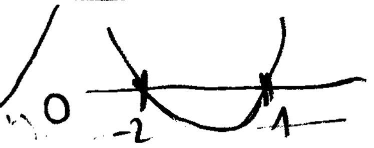
Note: $\forall r \in (-2,1)$ $\exists m, s \in \mathbb{R}\setminus\mathbb{Q}$ s.t: $-2 < m < r$, $r < s < 1$.

4) $2r^3 - 1 < 15 \Rightarrow r^3 < 8 \Rightarrow r < 2$

* $[2, +\infty)$ is the set of upper bound
* $\sup A = 2$ * $\sup A \notin A$
* no lower bound.

5) $x^2 - x - 2 < 0$ roots $\Rightarrow -2, 1$.

density of $\mathbb{Q}$ in $\mathbb{R}$



* $(-\infty, -2]$ set of l.b and $[1, +\infty)$ set of U.b
* $\sup A = 1 \in \mathbb{Q}$ $\inf A = -2 \in \mathbb{Q}$ * no max and min