Calculus I

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October 6, 2020

Chapter 1

Introduction

Definition 1.1. Mathematics (according to Oxford Eng. Dictionary)

The abstract science which investigates deductively the conclusions implicit in the elementary conception of spacial and numerical relations. This science can be divided in 6 main topics:

1. **Foundations:** logic, set theory, proof theorems, etc.
2. **Algebra:** numbers, arithmetical operations, order theorems.
3. **Analysis:** differentiation, integration, measure, etc.
4. **Geometry and topology:** proporties of space, shape, position of figures.
5. **Combinatories:** graph theory, partition theory, etc.
6. **Applied Mathematics:** computational sciences, probability, the range of applications of Mathematics is wide, such as:
   (a) **Banking and Finance:** Black-Scholes equation.
   (b) **Aeronautical engineering:** Fluid mechanics, shape design.
   (c) **Chemistry:** Models for protein folding, thermodynamics.
   (d) **Informatic:** Cryptography, computational algebra, parallel programming, etc.

Summary of the program: 1st Semester

- Real numbers
- Sequence and series of numbers
- Continuity and limit
- Differentiation
- Integration

Chapter 2

Real numbers and some basic concepts

2.1 Set of points

We recall here some basic concepts:

Definition 2.1. A **set** is a **collection of distinct objects.**

Example 2.2. 2, 5, 7 are different objects (numbers). They can compose the set $\{2,\ 5,\ 7\}$, where $\{\ldots\}$ denotes the set composed by the objects ....

Note. If an object x is a member of a set θ, we denote:

$$x \in \theta,\ \text{else we denote } x \notin \theta$$

Example 2.3.

$$\theta = \{0,\ 5,\ 7\}, \quad \text{if } x = 5 \text{ and } y = 9:$$
$$x \in \theta \text{ and } y \notin \theta$$

Remark. A set cannot have two times the same object.

Definition 2.4. Considering two sets A and B. If every element of A is a member of B, A is said to be a **subset** of B, and we denote:

$$\boldsymbol{A \subseteq B}$$

, else we denote

$$\boldsymbol{A \nsubseteq B}$$

Furthermore, if it exists at least one element of B which is not a member of A (A is strictly in B), A is said to be a **proper subset** of B, and we denote

$$\boldsymbol{A \subset B}$$

.

Example 2.5.

$$A = \{1,\ 2,\ 3\}$$
$$B = \{0,\ 1,\ 2,\ 3,\ 4\}$$
$$C = \{0,\ 1,\ 2\}$$
$$\therefore A \subseteq A, \quad A \subset B, \quad A \nsubseteq C$$

Definition 2.6. Set operators Let A and B be two sets.

Union: $\cup$

The **union** of $\boldsymbol{A}$ and $\boldsymbol{B}$ is the set

$$\boldsymbol{A \cup B = \{x | x \in A \vee x \in B\}}$$

Intersection: $\cap$

The **intersection** of $\boldsymbol{A}$ and $\boldsymbol{B}$ is the set

$$\boldsymbol{A \cap B = \{x | x \in A \wedge x \in B\}}$$

Complement: $\setminus$

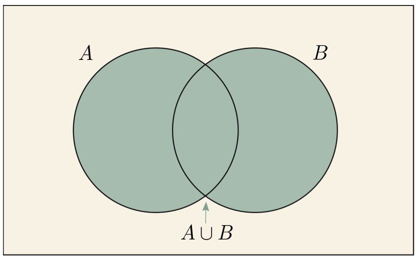
$$\boldsymbol{A \setminus B = \{x | x \in A \wedge x \notin B\}}$$

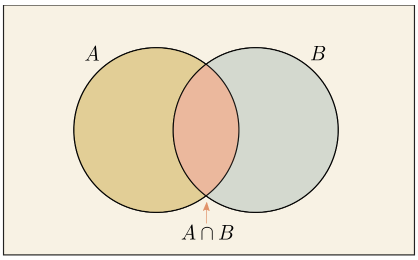
Example 2.7. $A = \{3, 5, 7\}, B = \{5, 7, 10\}$

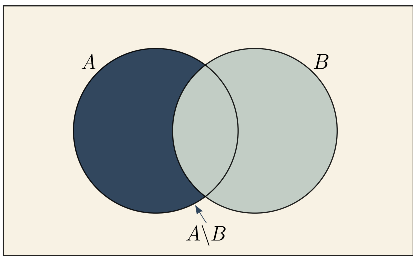
- $A \cup B = \{3, 5, 7, 10\}$
- $A \cap B = \{5, 7\}$
- $A \backslash B = \{3\}$
- $B \backslash A = \{10\}$

Remark. A set of one element is called a **singleton**.

Geometrical representation



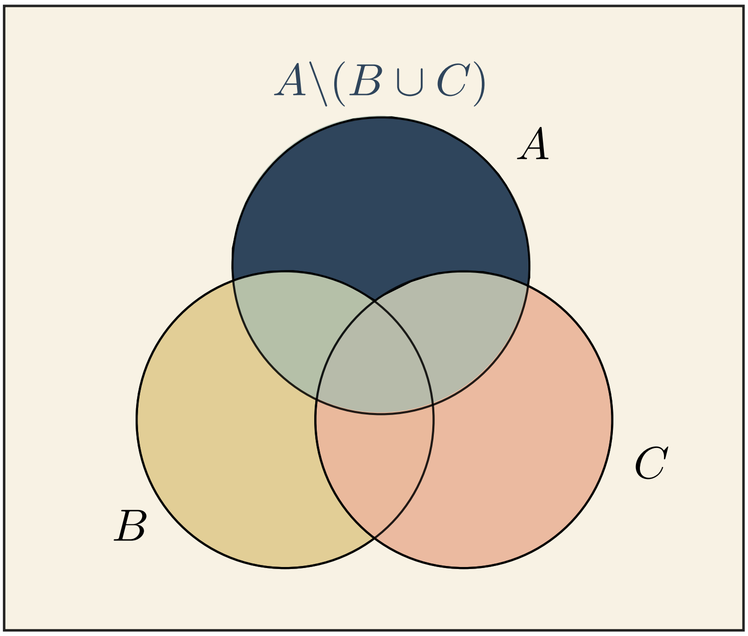


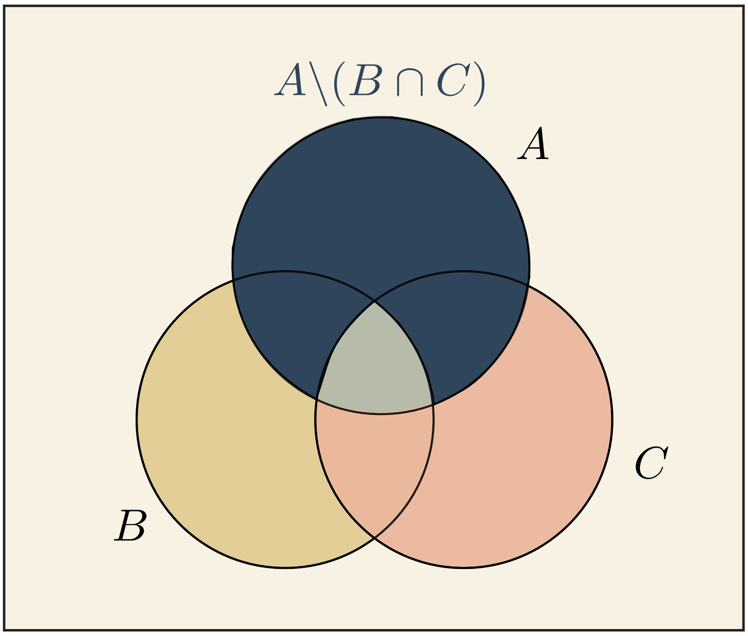


Properties (Morgan for sets):

- $A \backslash (B \cup C) = (A \backslash B) \cup (A \backslash C)$
- $A \backslash (B \cap C) = (A \backslash B) \cap (A \backslash C)$

Graphically (Venn diagrams):





Note. In the case of various n sets denoted by $A_1,\ A_2,\ A_n$, instead of writing:

$$A_1 \cup A_2 \cup \ldots \cup A_n \text{ we write } \bigcup_{k=1}^{n} A_k$$

or

$$A_1 \cap A_2 \cap \ldots \cap A_n \text{ we write } \bigcap_{k=1}^{n} A_k$$

Example 2.8.

$$A_1 = \{1,\ 2,\ 3\}, \quad A_2 = \{5,\ 6,\ 7\}, \quad A_3 = \{1,\ 5,\ 9\}$$

$$\bigcup_{k=1}^{3} A_k = A_1 \cup A_2 \cup A_3 = \{1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 9\}$$

Remark. We can apply the same notation in case of infinite (∞) numbers of a set $\{A_1, \ldots, A_{100}, \ldots\}$.

$$\bigcup_{k=1}^{\infty} A_k \quad \text{and} \quad \bigcap_{k=1}^{\infty} A_k$$

some examples and the concept of infinity will be defined in the next sections.

Definition 2.9. The cartesian product of two sets A and B is denoted by $\boldsymbol{A \times B}$ and defined as:

$$\boldsymbol{A \times B = \{(a,\ b) \mid a \in A \text{ and } b \in B\}}$$

where (a, b) is called **ordered pair.**

Example 2.10.

$$A = \{1,\ 2,\ 3\}, \quad B = \{7,\ 9\}$$

$$A \times B = \{(1,7),\ (1,9),\ (2,7),\ (2,9),\ (3,7),\ (3,9)\}$$

The order is very important, it always goes first the elements of the first named set and then the ones of the second one.More properties of sets will be introduced later in this chapter.

Some common sets of real points

Here we only introduce the set of points used in next chapters.

Definition 2.11.

- $\mathbb{R} = \boldsymbol{\{\ldots,\ \ldots,\ -10,\ \ldots,\ -7,\ \ldots,\ 0,\ \ldots,\ 4,\ \ldots, 1000,\ \ldots\}}$ is called the set of **real numbers** which contains **all positive and negative numbers**.
- $\mathbb{N} = \boldsymbol{\{1,\ 2,\ 3, \ldots\}}$ is called the set of **natural numbers** which contains **all the strictly positive integer numbers.**

  Remark. $\mathbb{N}^*$ includes the 0.
- $\mathbb{Z} = \boldsymbol{\{\ldots, -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3, \ldots\}}$ is called the set of **integer numbers** and contains the **positive and negative integers.**
- $\varnothing = \boldsymbol{\{\}}$ the **empty set** represents the sets without any elements.

  Example 2.12.

  $$\boldsymbol{A = \{1,\ 4\},\ \ B = \{3,\ 4\}} \quad \boldsymbol{A \cap B} = \varnothing, \text{ ie no coincidences between A and B}$$

- $\mathbb{Q} = \boldsymbol{\{x \in} \mathbb{R} \mid \boldsymbol{x = \frac{m}{n},\ m \in} \mathbb{Z}\boldsymbol{,\ n \in} \mathbb{Z}$ and $\boldsymbol{n \neq 0\}}$ is called the set of **rational numbers** and contains the **real numbers that can be written as a quotient of integer numbers** Numbers that don't belong in this set, ie $\boldsymbol{\sqrt{2}}$ or $\boldsymbol{\pi}$ are part of the **irrationals**, preferably noted as $\notin \mathbb{Q}$.
- **Odd**= $\boldsymbol{\{x \in} \mathbb{R} \mid \exists\ \boldsymbol{k \in} \mathbb{N}$ st $\boldsymbol{x = 2k+1\}}$ is the set of the **odd integer numbers**.
- **Even**= $\boldsymbol{\{x \in} \mathbb{R} \mid \exists\ \boldsymbol{k} \in \mathbb{N}$ st $\boldsymbol{x = 2k\}}$ is the set of the **even integer numbers**.
- $\mathbb{C} = \boldsymbol{\{x + iy \mid x \in} \mathbb{R}$ and $\boldsymbol{y \in} \mathbb{R}\boldsymbol{\}}$ is the set of **complex numbers**. Note: i denotes the imaginary number that verifies $\boldsymbol{i^2 = -1}$.

Remark.

$$\varnothing \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

Definition 2.13. Let A and B be two sets, and their cartesian product $A \times B$, any subset $S \subseteq A \times B$ is called a **relation** between A and B.

Example 2.14. $A = \{1, 2, 3\}, B = \{a, b\}$. A relation could be $R = \{(1, a), (2, b), (3, b)\}$.

Definition 2.15. Some properties of relation S are:

- **Reflexive:** $\forall a \in S,\ (a, a) \in S$
- **Symetric:** $(a, b) \in S$ and $(b, a) \in S \Rightarrow\ a = b$
- **Antisymmetric:** $(a, b) \in S$ and $(b, a) \in S \Rightarrow\ a = b$
- **Transitive:** $(a, b) \in S$ and $(b, c) \in S \Rightarrow (a, c) \in S$
- **Comparable (or connex):** $\forall\ a, b \in S$, either $(a, b) \in S$ or $(b, a) \in S$

Important note: antisymmetric **IS NOT** the negation of symetric. The negation of symetric would be asymetric.

Definition 2.16. A **relation**, noted by $\leq$, is a **total order** on a set S if it verifies:

1. **Reflexivity:** $\forall \boldsymbol{a} \in \boldsymbol{S},\ \boldsymbol{a} \leq \boldsymbol{a}$
2. **Antisymmetry:** $\boldsymbol{a} \leq \boldsymbol{b}$ and $\boldsymbol{b} \leq \boldsymbol{a} \Rightarrow\ \boldsymbol{a} = \boldsymbol{b}$
3. **Transitivity:** $\boldsymbol{a} \leq \boldsymbol{b}$ and $\boldsymbol{b} \leq \boldsymbol{c} \Rightarrow \boldsymbol{a} \leq \boldsymbol{c}$
4. **Comparability:** $\forall\ \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{S}$, either $\boldsymbol{a} \leq \boldsymbol{b}$ or $\boldsymbol{b} \leq \boldsymbol{a}$

   When Reflexivity, Antisymmetry and Transitivity occurs but no Comparability, then we have a **partial order**. For a **total order**, we would need Comparability.

Example 2.17. The divisibility relation, denoted by "$|$", on the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ is a classic example of a partial order relation.

- The relation "$|$" is reflexive, because any $a \in \mathbb{N}$ divides itself.
- The relation "$|$" is antisymmetric. Indeed, if $a|b$, then $ak = b$, where k is an integer. Similarly, if $b|a$, then $bl = a$, where l is an integer. Hence, $akl = l \Rightarrow kl = 1$. This last equation only holds if $k = l = 1$, which means that $a = b$.
- The relation "$|$" is transitive. Suppose $a|b$ and $b|c$. Then $ak = b$ and $bl = c$, where k, l are certain integers. Hence $akl = c$, and kl is an integer. That means $a|c$.

Example 2.18.

- The relation $\leq$ applied to $\mathbb{R}$ is a total order.
- The relation $\subset$ applied to a subset of $\mathbb{R}$ is **not** a total order. For example, $\{1,\ 2\}$ and $\{2,\ 4\}$ cannot be compared.

Definition 2.19. A set plus a total order relation is called a **total ordered set**.

Example 2.20.

$$(\mathbb{R}, \leq)$$

Rules:

1. $a = b$
2. $a < b$ or $a > b$ a strictly inferior (or superior) to b (not equal).

Definition 2.21. Infinity: denoted by ∞, is an abstract concept ??? a limitless quantity (e.g. number).

Properties:

- $\forall x \in \mathbb{R}, \quad -\infty \leq x$ and $x \leq +\infty \ \therefore \ \mathbb{R} = (-\infty, +\infty)$.
- $-\infty$ and $+\infty \notin \mathbb{R}$

Definition 2.22. An **interval** is a real subset containing all the values between two given points, included or not. It can be of the type:

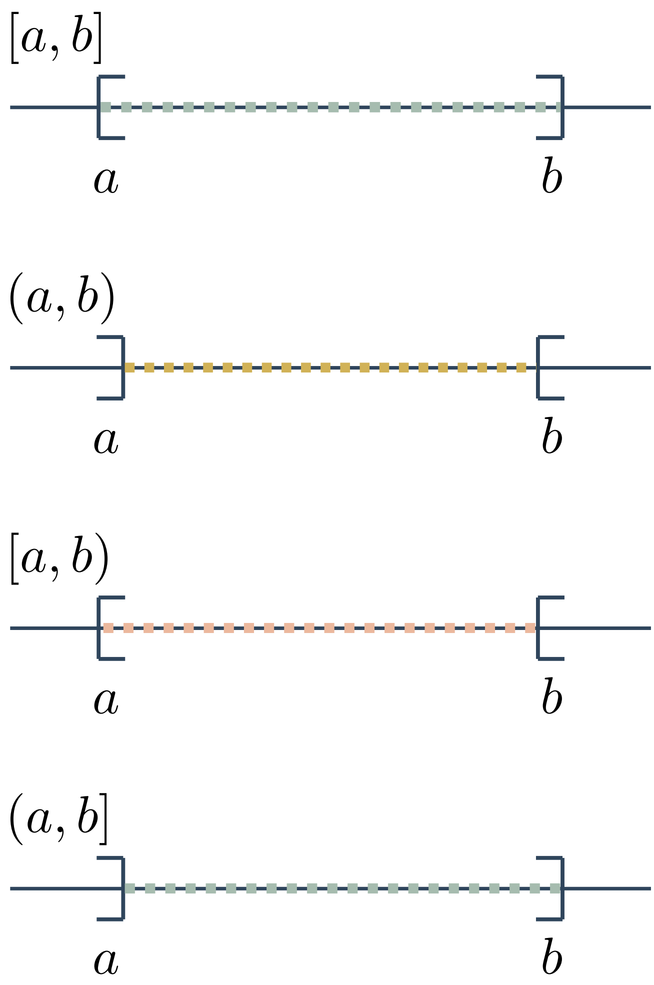
Let $a, \ b \in \mathbb{R}$:

- **Open interval**: $(a, b) = \{x \in \mathbb{R} | a < x < b\}$
- **Closed interval**: $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$
- **Left closed interval**: $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$
- **Left open interval**: $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$

Properties:

$\mathbb{R} = (-\infty, +\infty)$

Graphical representation:



Definition 2.23. Axiomatic definition of $\mathbb{R}$.

The real number system $(\mathbb{R}, +, \cdot, <)$ is a set where the following rules are defined.

- **Addition** $(+)$: a function

$$\begin{aligned}\mathbb{R}\times\mathbb{R} &\mapsto \mathbb{R}\\ (x,y) &\mapsto x+y\end{aligned}$$

  with the following properties:

  - **Associativity**: $\forall x,y,z\in\mathbb{R},\ (x+y)+z=x+(y+z)$
  - **Commutativity**: $\forall x,y\in\mathbb{R},\ x+y=y+x$
  - **Identity element**: $\exists 0\in\mathbb{R}|\ 0+x=x+0=x$
  - **Opposite element**: $\forall x\in\mathbb{R},\exists!-x\in\mathbb{R}|\ x+(-x)=(-x)+x=0$

- **Multiplication** $(\cdot)$: a function

$$\begin{aligned}\mathbb{R}\times\mathbb{R} &\longrightarrow \mathbb{R}\\ (x,y) &\longrightarrow x\cdot y\end{aligned}$$

  with the following properties:

  - **Associativity**: $\forall x,y,z\in\mathbb{R},\ (xy)z=x(yz)$
  - **Commutativity**: $\forall x,y\in\mathbb{R},\ xy=yx$
  - **Identity element**: $\exists 1\in\mathbb{R}|\ 1\cdot x=x\cdot 1=x$
  - **Inverse element**: $\forall x\in\mathbb{R},\exists!\frac{1}{x}\in\mathbb{R}|\ x\cdot\frac{1}{x}=\frac{1}{x}\cdot x=1$
  - **Distributivity**: $\forall x,y,z\in\mathbb{R}\backslash\{0\},\ x(y+z)=xy+xz$

- The field $(\mathbb{R},+,\cdot)$ is ordered:

  - $\geq$ is a total order.
  - $\forall x,y,z\in\mathbb{R},\ x\geq y\Rightarrow x+y\geq y+z$
  - $\forall x,y\geq 0,\ xy\geq 0$

- The order is **Dedekind complete** (the supremum property):

  $A\neq\emptyset,\ A\subseteq\mathbb{R}\wedge\exists k\in\mathbb{R}|\forall a\in A, a\leq k$ (where k is called *upper bound*) $\Rightarrow\exists\alpha$ denoted $\sup A$ and called least upper bound, such that $\forall a\in\mathbb{A}, a\leq\alpha$ and $\forall k\in\mathbb{R}$ upper bound of $A,\ \alpha\leq k$.

Remark. $\mathbb{N}$ cannot be defined axiomatically (e.g. $0\notin\mathbb{N}$)

2.2 Mathematical Functions

Definition 2.24. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ being two sets. A function from $\boldsymbol{A}$ to $\boldsymbol{B}$, is a relation between $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{f:A\rightarrow B}$, such that $\boldsymbol{\forall a\in A,\ \exists!\ b\in B/\ f(a)=b}$.

The elements of A are called **arguments of** f. The element $b\in B$ such that $f(a)=b$, with $a\in A$ is called **value** at a or **image** of a under f.

A is called **domain** of f, $D(f)$, and $R(f)=\{b\in B|\ \exists a\in A|\ f(a)=b\}$ is the **range**.

Notation.

$$\begin{aligned} f : A &\longrightarrow B \\ a &\longrightarrow f(a) \end{aligned}$$

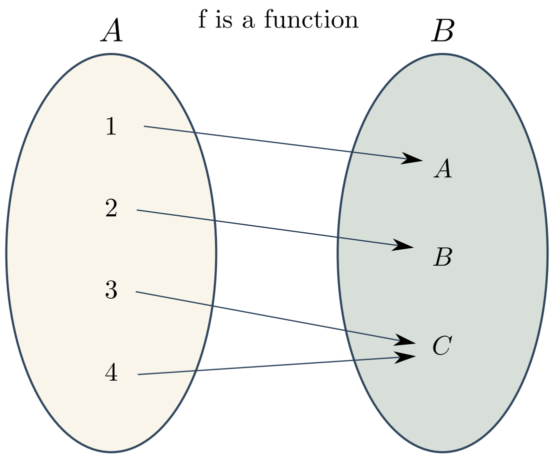
We can write $\boldsymbol{f}$ **maps** $\boldsymbol{A}$ to $\boldsymbol{B}$.

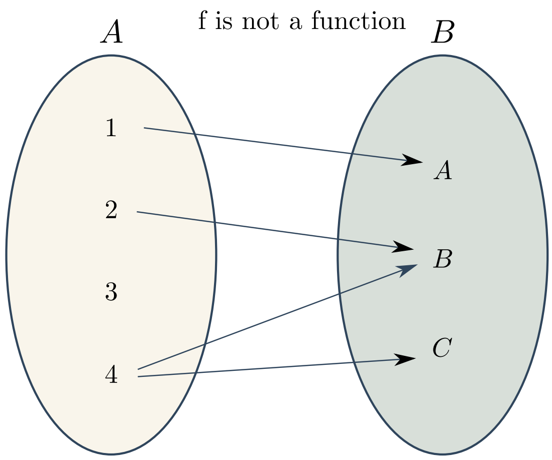
Example 2.25.

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longrightarrow x^2 + 1 \end{aligned}$$

$f(1) = 1^2 + 1 = 2$

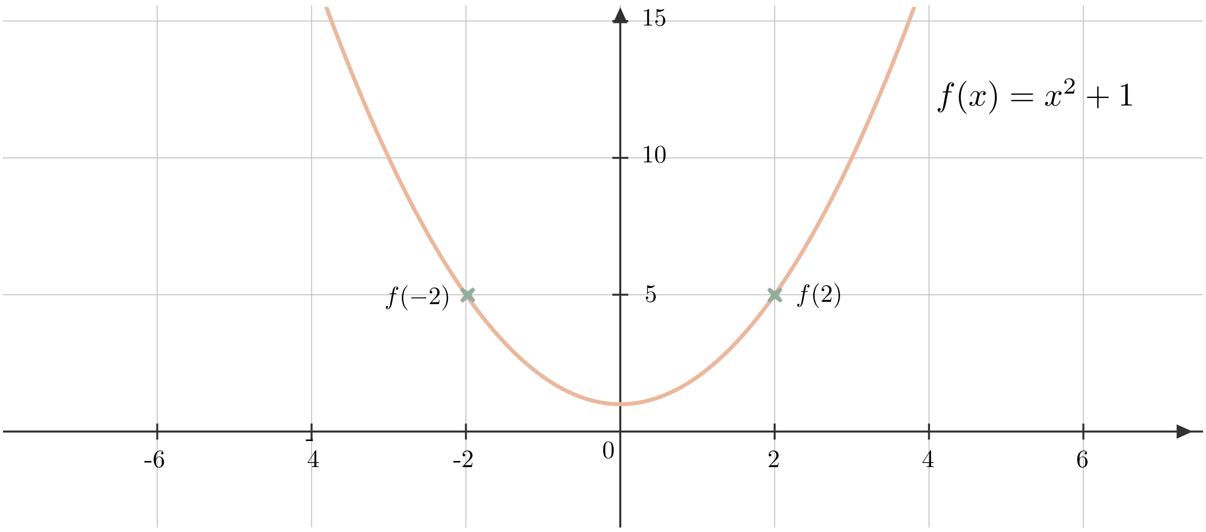
$f(2) = 2^2 + 1 = 5$





Definition 2.26. The graph of a function is **its set of ordered pairs** $\boldsymbol{F = \{(a, f(a)),\ \forall a \in A\}}$

$$\begin{aligned} \boldsymbol{f :} \mathbb{R} &\longrightarrow \mathbb{R} \\ \boldsymbol{x} &\longrightarrow \boldsymbol{x^2 + 1} \end{aligned}$$



Definition 2.27.
- If $E \subseteq A$, the image of E under f is $f(E) = \{f(x)|\ x \in E\}$.
- If $H \subseteq B$, the preimage of H under f is $f^{-1}(H) = \{x \in A|\ f(x) \in H\}$

Example 2.28. $A = B = \mathbb{R},\ f(x) = x^2$.
- $E = [0,2] \subset \mathbb{R} \quad f(E) = [0,4]$
- $H = \{4,9\} \subset \mathbb{R} \quad f^{-1}(H) = \{-3,-2,2,3\}$

Definition 2.29. $\boldsymbol{f : A \longrightarrow B}$
- $\boldsymbol{f}$ is called **inyective** if $\boldsymbol{\forall a_1, a_2 \in A, f(a_1) = f(a_2) \Rightarrow a_1 = a_2}$
- $\boldsymbol{f}$ is called **suryective** if $\boldsymbol{\forall b \in B,\ \exists a \in A|\ f(a) = b}$
- $\boldsymbol{f}$ is called **biyective** if $\boldsymbol{f}$ is **inyective and suryective.**

Remark. If a function is biyective you can obtain its inverse.

Example 2.30.
- $$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longrightarrow x^2 \end{aligned}$$

  It's not inyective, since $f(1) = f(-1)$, and it's not suryective since $-1 \in \mathbb{R} \wedge \nexists a \in \mathbb{R}|\ f(a) = -1$
- $$\begin{aligned} f : [0,1] &\longrightarrow [1,2] \\ x &\longrightarrow x+1 \end{aligned}$$

  It is inyective, since $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$. It is also suryective, since $\forall x_1 \in [1,2],\ x_2 = x_1 - 1 \in [0,1]$ and $f(x_2) = x_1$. Thus f is biyective.

Definition 2.31. Let $\boldsymbol{f : A \longrightarrow B}$ and $\boldsymbol{f : B \longrightarrow C}$, the **composition of** $\boldsymbol{f}$ with $\boldsymbol{g}$ is a function denoted as $\boldsymbol{g \circ f}$ and defined by:

$$\begin{aligned} \boldsymbol{g \circ f : A} &\boldsymbol{\longrightarrow C} \\ \boldsymbol{a} &\boldsymbol{\longrightarrow g(f(a))} \end{aligned}$$

Example 2.32.

$$g : \mathbb{R} \longrightarrow \mathbb{R} \qquad f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longrightarrow \cos(x) \qquad x \longrightarrow x^2$$

$gof(x) = \cos(x^2) \neq fog(x) = (\cos(x))^2$

Definition 2.33. The **identity function** $\boldsymbol{f}$ on $\boldsymbol{A}$, is the function:

$$\boldsymbol{id_A : A \longrightarrow A}$$
$$\boldsymbol{x \longrightarrow x}$$

Definition 2.34. Let $\boldsymbol{f : A \longrightarrow B}$ (biyective). The inverse function of $\boldsymbol{f}$, denoted by $\boldsymbol{f^{-1}}$, is the function $\boldsymbol{f^{-1} : B \longrightarrow A}$ such that:

$$\boldsymbol{b = f(a) \Leftrightarrow a = f^{-1}(b)}$$

Note that this is the same as saying $\boldsymbol{f \circ f^{-1}(b) = id_B}$ and $\boldsymbol{f \circ f^{-1}(a) = id_A}$.

Example 2.35. $fof^{-1}(x) = f(x-1) = (x-1)+1 = x$ and $f^{-1}of(x) = (x+1)-1 = x$.

Definition 2.36. We call **real function** with real variable a function of the type $f : \mathbb{R} \longrightarrow \mathbb{R}$

2.3 Some properties of particular real sub-sets

2.3.1 Odd and Even sets

Proposition. Let $\boldsymbol{O_1, O_2} \in$ Odd, and $\boldsymbol{e_1, e_2} \in$ Even.

a) $\boldsymbol{O_1 + O_2}$ is even.

b) $\boldsymbol{e_1 + e_2}$ is even.

c) $\boldsymbol{e_1 + O_1}$ is odd.

d) $\boldsymbol{e_1 \cdot e_2}$ is even.

e) $\boldsymbol{O_1 \cdot O_2}$ is odd.

f) $\boldsymbol{e_1 \cdot O_1}$ is even.

Proof. Let $e_1 = 2k_1, e_2 = 2k_2, O_1 = 2k_3 + 1, O_2 = 2k_4 + 1$ with $k_1, k_2, k_3, k_4 \in \mathbb{Z}$.

a) $O_1 + O_2 = 2k_3 + 1 + 2k_4 + 1 = 2(k_3 + k_4) + 2 = 2(k_3 + k_4 + 1) \in$ Even.

b) $e_1 + e_2 = 2k_1 + 2k_2 = 2(k_1 + k_2) \in$ Even.

c) $e_1 + O_1 = 2k_1 + 2k_3 + 1 = 2(k_1 + k_3) + 1 \in$ Odd.

d) $e_1 \cdot e_2 = 2k_1 \cdot 2k_2 = 2(2k_1k_2) \in$ Even.

e) $e_1 \cdot O_1 = 2k_1 \cdot (2k_3 + 1) = 2k_1k_3 + 2k_1 = 2(k_1k_3 + k_1) \in$ Even.

□

Proposition. *a*) n is even from previous proposition. $n^2 = n \cdot n$ is even.

Proof. Trivial due to section e) of previous proofs. □

b) *Proof.* $(n+p)^2$ is even $\Rightarrow (n+p)$ is even $\Rightarrow n,p$ are even $\vee$ n,p are odd $\Rightarrow$

$$\Rightarrow \begin{cases} \text{if } n,p \text{ even} & \Rightarrow n-p \text{ is even} \\ \text{if } n,p \text{ odd} & \Rightarrow n-p \text{ is even} \end{cases} \Rightarrow (n-p)^2 \text{ is even.}$$

$\Leftarrow$ would use the same idea. Justifying steps with previous proof. □

2.3.2 $\mathbb{N}$ and $\mathbb{Z}$

Proposition 2.37. *Let $n_1, n_2 \in \mathbb{N}$ and $z_1, z_2 \in \mathbb{Z}$*

a) $n_1 + n_2 \in \mathbb{N}$

b) $n_1 \cdot n_2 \in \mathbb{N}$

c) $z_1 + z_2 \in \mathbb{N}$

d) $z_1 \cdot z_2 \in \mathbb{N}$

e) $n_1 \geq n_2$ *or* $n_2 \geq n_1$

f) $z_1 \geq z_2$ *or* $z_2 \geq z_1$

Proposition 2.38. ***Well-Ordering Principle*** *Let $\boldsymbol{B} \subseteq \mathbb{N}$ and $\boldsymbol{B} \neq \emptyset$.*

It always exists $\boldsymbol{n_0} \in \boldsymbol{B}$ such that $\forall \boldsymbol{m} \in \boldsymbol{B}$, $\boldsymbol{n_0} \leq \boldsymbol{m}$. Such $\boldsymbol{n_0}$ is called minimum of $\boldsymbol{B}$ and denoted $\mathbf{min}\,\boldsymbol{B}$. **** principio pagina 13*

Definition 2.39. Mathematical Induction

We want to demonstrate a statement P_n involving $n \in \mathbb{N}$ for all values of n. We have to follow these steps:

a) We prove that the statement holds for the first value of n.

b) We prove that if the statement holds for n, then it holds for $n+1$.

Example 2.40. *Proof.* **W.O.P.** (Well-Ordering Principle). We will see the following proposition is false, proving it by absurdity.

Let $J = \mathbb{N} \backslash B$. "$P_n = \{1, \ldots, n\} \in J$". We start with

a) $P_1 =$ "$1 \in J''$. True, else we would be saying that $\min B = 1$.

b) $P_n = \{1, \ldots, n \in J\}$, then $n+1 \in J$, else $\min B = n+1$ (as $1, \ldots, n \notin B$).

$\Rightarrow \forall n \in \mathbb{N} \in J \Rightarrow B = \emptyset \Rightarrow$ ABSURD. □

Example 2.41. *a*) $P_n = \displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

- $P_1 = 1 = \dfrac{1(1+1)}{2} = 2$
- P_n true. $P_n = \displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \Rightarrow \sum_{k=1}^{n+1} k = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1)}{2} + n + 1 =$ $\dfrac{n(n+1)+2n+2}{2} = \dfrac{(n+1)((n+1)+1)}{2} \Rightarrow \displaystyle\sum_{k=1}^{n+1} k = \frac{(n+1)((n+1)+1)}{2}$ $P_{n+1} =$ true.

b) $P_n = \sum_{k=0}^{n} r^k = \dfrac{1-r^{n+1}}{1-r}$

- $P_1 = r^0 + r = 1 + r = \dfrac{1-r^2}{1-r} = \dfrac{(1+r)(1-r)}{1-r} = 1 + r$ true.
- P_n true. $P_n = \sum_{k=0}^{n+1} r^k = \sum_{k=0}^{n} r^k + r^{n+1} = \dfrac{1-r^{n+1}+r^{n+1}-r\cdot r^{n+1}}{1-r} = \dfrac{1-r^{n+2}}{1-r} \Rightarrow$ P_{n+1} true.

2.4 $\mathbb{Q}$

Proposition. $Q_1, Q_2 \in \mathbb{Q}$

a) $Q_1 + Q_2 \in \mathbb{Q}$

b) $Q_1 \cdot Q_2 \in \mathbb{Q}$

c) $\dfrac{1}{Q_1} \in \mathbb{Q}$ (if $Q_1 \neq 0$)

d) $Q_1 \leq Q_2$ or $Q_2 \leq Q_1$

Example 2.42. a) Let $p \in \mathbb{Q}, p \neq 0 \Rightarrow p = \dfrac{a}{b}, a, b \in \mathbb{Z}\backslash\{0\}$.

Let $x \notin \mathbb{Q}$. If $p + x \in \mathbb{Q}$, $\exists c, d \in \mathbb{Z} | p + x = \frac{c}{d} \Rightarrow \frac{a}{b} + x = \frac{c}{d} \Rightarrow x = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad \in \mathbb{Z}}{db \in \mathbb{Z}\backslash\{0\}} \Rightarrow x \in \mathbb{Q}$ absurd.

Example 2.43. $\sqrt{2}$ is irrational.

Proof. Lets assume that $\sqrt{2}$ is rational. Then we can write $\sqrt{2} = \dfrac{p}{q}$ where we may assume that p and q have no common factors (if there are any common factors we cancel them in the numerator and denominator) $\Rightarrow \frac{p^2}{q^2} = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p$ even $\Rightarrow p^2$ is divisible by 4 (i.e. $p^2 = 4m, m \in \mathbb{N}$) $\Rightarrow q^2 = 2m \Rightarrow q$ even (i.e. $q = 2s, s \in \mathbb{N}$). Then p and q have a common factor (2). Absurd. $\square$

Proposition 2.44. $\sqrt{m}$, *with* $n \in \mathbb{N}$ *and such that* n *is not a square number (i.e.* $\nexists\, p \in \mathbb{N} |\ p^2 = n$*), is irrational.*

Definition 2.45. An **algebraic number** is a real number that is a root of a non-zero polinomial with rational coefficients, i.e. if n is algebraic, it exists a polinomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k = \sum_{i=0}^{k} a_i x^i$$

such that $p(n) = 0$.

Example 2.46. *a)* If $\sqrt{2}-\sqrt{3}$ is rational $\Rightarrow (\sqrt{2}-\sqrt{3})^2$ is rational $\Rightarrow 5 - 2\sqrt{2}\sqrt{3}$ is rational $\Rightarrow \sqrt{6}$ is rational $\Rightarrow$ absurd.

b) $1 - \sqrt[3]{2+\sqrt{5}}$

Proof. 2 is algebraic and $\sqrt{5}$ is algebraic (since it exists the polinomial $x^2 - 5$) $\Rightarrow 2 + \sqrt{5}$ is algebraic and root of $p(x) \Rightarrow \sqrt[3]{2+\sqrt{5}}$ is root of $p(x^3)$ (i.e. all exponents of x are increased by 3). Then $p(x) = x^2 + x - 1 \Rightarrow p(x^3) = x^6 + x^3 - 1 \Rightarrow 1$ is algebraic $\Rightarrow \sqrt[3]{2+\sqrt{5}}$ is algebraic. $\square$

Definition 2.47. A decimal representation of a real number r is an expression of the form:

$$r = a_0.a_1a_2a_3a_4\dots \text{ with } a_i \in \mathbb{N}, i \in \mathbb{N}$$

$r = \sum_{i=1}^{\infty} \frac{a_i}{10^i}$

Proposition 2.48. *The decimal representation of a real number r either terminates (i.e. $\exists n_0 \in \mathbb{N} \mid a_i = 0 \ \forall i \geq n_0$) or begins to repeat a same finite sequence over and over iff r is rational.*

Example 2.49. $0.127841841841841\dots$ is rational.

$\pi = 3.14159265\dots$ is irrational.

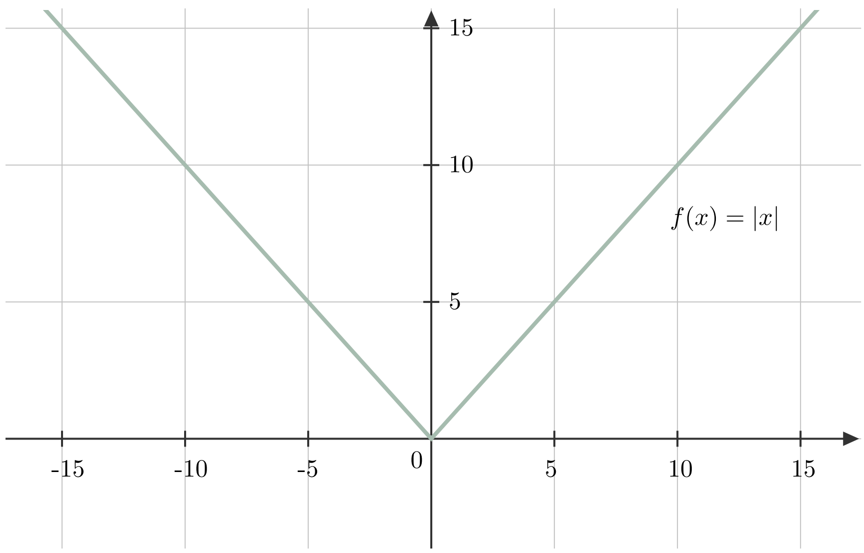
Proposition 2.50. *$\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ are dense in $\mathbb{R}$: $\forall x_1, x_2 \in \mathbb{R}, x_1 < x_2$, $\exists q \in \mathbb{Q}$ and $n \in \mathbb{R}\backslash\mathbb{Q} \mid x_1 < q < x_2$ and $x_1 < n < x_2$.*

2.5 $\mathbb{R}$

Distance in $\mathbb{R}$

Definition 2.51. The **absolute value** or a modulus of a real number x is denoted $|x|$ and defined by:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



Proposition 2.52. *Properties*

a) $|x| = \sqrt{x^2}$

b) $|xy| = |x| \cdot |y|$

c) $|x+y| \leq |x|+|y|$

Proof. $\begin{cases} x \leq |x| & \wedge\ y \leq |y| \\ -x \leq |x| & \wedge\ -y \leq |y| \end{cases} \Rightarrow \begin{cases} x+y \leq |x|+|y| \\ -(x+y) \leq |x|+|y| \end{cases} \Rightarrow |x+y| \leq |x|+|y|$ □

Definition 2.53. $\forall x, y \in \mathbb{R}$ we call $|x-y|$ the distance from x to y.

Proposition. Properties. $\forall x, y, z \in \mathbb{R}$

a) $|x-y| = 0 \Leftrightarrow x = y$

b) $|x-y| = |y-x|$

c) $|x-y| \leq |x-z| + |z-y|$ (Triangle inequality)

Example 2.54. a) $|2x+3| - 1 < |x|$ We determine all the cases of signs:

$$x < -\frac{3}{2}, 0 > x \geq -\frac{3}{2}, x \geq 0$$

- $x \geq 0 \Rightarrow |2x+3| = 2x+3 \geq 0$ and $|x| = 0 \geq 0$. The equation leads to:

  $2x+3-1 < x \Rightarrow x < -2 \Rightarrow x < -2 \Rightarrow$ absurd.
- $x < -\frac{3}{2} \Rightarrow |2x+3| = -2x-3$ and $|x| = -x$. Then $-2x-3-1 < -x \Rightarrow -2x-4 < -x \Rightarrow 2x+4 > x \Rightarrow x > -4 \Rightarrow x \in (-4, -\frac{3}{2})$
- $x \in [-\frac{3}{2}, 0) \Rightarrow |2x+3| = 2x+3,\ |x| = -x$

  $2x+3-1 < -x \Rightarrow 3x < -2 \Rightarrow x < -\frac{2}{3} \Rightarrow x \in [-\frac{3}{2}, -\frac{2}{3}]$

The solutions are $x \in (-4, -\dfrac{2}{3})$

b) Same idea: $x \geq 4;\ 4 > x \geq 0;\ x < 0$

$|2-|x|| = 2+|x|$

- $x \geq 2$: $|x| = x$, $|2-|x|| = -2+x \Rightarrow -2+x = 2+x \Rightarrow 2 = -2$ absurd.
- $2 > x \geq 0$: $|x| = x$, $|2-|x|| = 2-x \Rightarrow x-x = x+x \Rightarrow x = -x \Rightarrow x = 0$
- $0 > x \geq -2$: $|x| = -x$, $|2-|x|| = 2+x \Rightarrow 2-x = 2+x \Rightarrow x = 0$
- $x < -2$: $|x| = -x$, $|2-|x|| = -2+x \Rightarrow -2+x = 2-x \Rightarrow 2x = 4 \Rightarrow x = 2$ absurd.

The solution is $x = 0$.

Some applications of the axiomatic definition of $\mathbb{R}$

1.7

a) $ax = a$ and $a \neq 0 \Rightarrow x = \frac{a}{a} = 1$

b) $(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$

c) $(x+y)(x-y) = x^2 + xy - xy - y^2 = x^2 - y^2$

d) $x^2 = y^2 \Rightarrow x = \pm\sqrt{x^2} \Rightarrow \begin{cases} x = +|y| \\ x = -|y| \end{cases} \Rightarrow x = \pm y$

e) $(x-y)(x^2+xy+y^2) = x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3 = x^3 - y^3$

f) $(x-y)(\sum_{i=0}^{n-1} x^{n-(i+1)}y^i) = \sum_{i=0}^{n-1} x^{n-i}y^i - \sum_{i=0}^{n-1} x^{n-(i+1)}y^{i+1} = x^n + \sum_{i=1}^{n-1} x^{n-i}y^i - \sum_{i=0}^{n-2} x^{n-(i+1)}y^{i+1} - y^n = x^n + \sum_{i=1}^{n-1} x^{n-i}y^i - \sum_{i=1}^{n-1} x^{n-i}y^i - y^n = x^n - y^n$

1.8 $0 < a < b$

a) $0 < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}} \Rightarrow \frac{a^2+b^2+2ab}{4} < \frac{a^2+b^2}{2} \Rightarrow 2ab < a^2+b^2 \Rightarrow 0 < a^2+b^2-2ab \Rightarrow 0 < (a-b)^2$

b) $\sqrt{ab} < \frac{a+b}{2} \Rightarrow 2\sqrt{ab} < a+b \Rightarrow 4ab < a^2+b^2+2ab \Rightarrow 2ab < a^2+b^2$

c) $\frac{2ab}{a+b} < \sqrt{ab} \Rightarrow 2ab < \sqrt{ab}(a+b) \Rightarrow 4a^2b^2 < ab(a+b)^2 \Rightarrow 4ab < a^2+b^2+2ab$

1.8

a) $a \leq b$ and $\forall \epsilon > 0,\ a \leq b \leq a+\epsilon$

Proof. If $a < b \Rightarrow 0 < b-a \Rightarrow \exists \epsilon_b > 0 \mid \epsilon_b < b-a \Rightarrow a+\epsilon_b < b$ with $\epsilon_b > 0 \Rightarrow$ absurd $\Rightarrow a = b$. □

b) $a \leq b$ and $\forall \epsilon > 0 \quad b-\epsilon \leq a \leq b$.

Proof. If $a < b \Rightarrow a-b < 0 \Rightarrow \exists \epsilon_a > 0 \mid a-b < -\epsilon_a \Rightarrow a < b-\epsilon_a,\ \epsilon_a > 0$, absurd $\Rightarrow a = b$ □

Boundaries of real subsets

Definition 2.55. Let $\boldsymbol{A} \subseteq \mathbb{R}$. $\boldsymbol{A}$ is called **bounded from above** if $\exists \boldsymbol{k} \in \mathbb{R} \mid \forall \boldsymbol{a} \in \boldsymbol{A},\ \boldsymbol{a} \leq \boldsymbol{k}$. In this case, $\boldsymbol{k}$ is called upper bound of $\boldsymbol{A}$.

$\boldsymbol{A}$ is called **bounded from below** if $\exists \boldsymbol{k} \in \mathbb{R} \mid \forall \boldsymbol{a} \in \boldsymbol{A},\ \boldsymbol{a} \geq \boldsymbol{k}$. $\boldsymbol{k}$ is called lower bound.

If $\boldsymbol{A}$ has both upper and lower bounds, then $\boldsymbol{A}$ is called a bounded set.

Example 2.56. $(0, +\infty)$ is bounded from below by any $k \leq 0$ but not bounded from above.

Example 2.57. If we have the following set: $A = [0,\ 3)$, we have this interval of upper bounds: $[3,\ \infty]$ and this one of lower bounds: $(-\infty,\ 0]$

Then infimum: $0 \Rightarrow$ as $0 \in A \Rightarrow 0$ is minimum.

Supremum: $3 \Rightarrow$ there does not exist maximum!

Definition 2.58. $A \subseteq \mathbb{R},\ \alpha$ is called **supremum** of A (or least upper bound) if:

- α is an upper bound of A.
- $\forall k \in \mathbb{R}$ upper bound of A, $\alpha \leq k$.

We denote $\alpha = \sup A$.

Definition 2.59. $A \subseteq \mathbb{R},\ \beta$ is called **infimum** of A (or greatest lower bound) if:

- β is an upper bound of A.
- $\forall k \in \mathbb{R}$ lower bound of A, $\beta \geq k$.

We denote $\beta = \inf A$.

Example 2.60. $A = [3, 5]$, $\sup A = 5$ and $\inf A = 3$.

Proposition.

- If $A \subseteq \mathbb{R}$ is bounded from above, then A admits a supremum.
- If $A \subseteq \mathbb{R}$ is bounded from below, then A admits a infimum.

Definition 2.61. Let $A \subseteq \mathbb{R}$

- If $\sup A$ exists and $\sup A \in A$, then $\sup A$ is called **maximum** of A and is denoted $\max A$.
- If $\inf A$ exists and $\inf A \in A$, then $\inf A$ is called **minimum** of A and is denoted $\min A$.

Example 2.62. $A = [3, 4)$, $\min A = \inf A = 3$. $\sup A = 4 \notin A \Rightarrow \max A$ does not exist.

1.10: A bounded and $A_0 \subseteq A$.

- **From above**: If A_0 is not bounded from above (b.f.a.) $\forall k \in \mathbb{R},\ \exists a_0 \in A_0 \mid a_0 > k \Rightarrow \forall k \in \mathbb{R},\ \exists a_0 \in A \mid a_0 > k \Rightarrow A$ is not b.f.a. $\Rightarrow$ absurd.

  Furthermore, $\forall a \in A_0,\ a \in A$ and $a \leq \sup A \Rightarrow \sup A$ is an upper bound of $A_0 \Rightarrow \sup A \geq \sup A_0$.

- **From below**: same idea.

1.10: $A, B \subset \mathbb{R}$ bounded.

$$A + B = \{x \in \mathbb{R} \mid x = a + b,\ a \in A, b \in B\}$$

- $\forall x \in A + B,\ x = a + b \leq \sup A + \sup B$ since $a \leq \sup A \wedge b \leq \sup B \Rightarrow \sup A + \sup B$ is an upper bound of $A + B$.

  Let $k \in \mathbb{R} \mid \forall x \in A + B, \quad x \leq k \Rightarrow \forall a \in A, b \in B \quad a + b \leq k \Rightarrow \forall a \in A \quad a \leq k - b \Rightarrow \sup A \leq k - b,\ \forall b \in B \Rightarrow \forall b \in B, b \leq k - \sup A \Rightarrow \sup B \leq k - \sup A \Rightarrow \sup A + \sup B \leq k$. Thus $\sup A + B = \sup A + \sup B$.

- Same idea.
- $\alpha > 0:\ \alpha A = \{x \in \mathbb{R} \mid x = \alpha a,\ a \in A\}$
  - $\forall x \in A,\ x = \alpha a \leq \alpha \sup A$, since $a \leq \sup B$
  - Let $k \in \mathbb{R} \mid \forall x \in \alpha A, \quad x \leq k \Rightarrow \forall a \in A,\ \alpha a \leq k$.

  Thus $\alpha \sup A = \sup \alpha A$.

  $\inf \alpha A =$ same idea.

- Same idea but $\alpha < 0$ (change in inequalities)
  - $\forall x \in A,\ x = \alpha A \geq \alpha \sup A$.
  - $\forall a \in A,\ \alpha a \geq k \Rightarrow \forall a \in A,\ a \leq \frac{k}{\alpha} \Rightarrow \sup A \leq \frac{k}{\alpha} \Rightarrow \alpha \sup A \geq k$.

1.12:

a) $A = \{2; 2.2; 2.22; 2.222; \ldots\}$

- $\forall a \in A, \forall x \in (-\infty, 2], x \leq a \Rightarrow (-\infty, 2]$ is the set of lower bounds.
- $\inf A = 2$ (the greatest lower bound).
- $\inf A \in A \Rightarrow \min A = 2$.
- $\forall a \in A,\ \forall x \in [2.\widehat{2}, \infty),\ x \geq a \Rightarrow [2.\widehat{2}, \infty)$ set of upper bounds.

- $\sup A = 2.\widehat{2}$ least upper bound.
- $\sup A \notin A$, max does not exist.

b) $\forall r \in \mathbb{R}, z = \text{floor}(r) + 1 \in \mathbb{Z}$ and $z > r$.

$z = \text{floor}(r) - 1 \in \mathbb{Z}$ and $z < r \Rightarrow \mathbb{Z}$ does not admit lower bounds.

c) Study the roots.

d) $2r^3 - 1 < 15 \Rightarrow r^3 < 8 \Rightarrow r < 2$

- $[2, \infty)$ is the set of upper bounds.
- $\sup A = 2$ and $\sup A \notin A$.
- No lower bounds.

e) $x^2 - x - 2 < 0$

- $(-\infty, -1]$ set of lower bounds and $[2, \infty)$ set of upper bounds.
- $\sup A = 2 \in \mathbb{R}, \quad \inf A = -1 \in \mathbb{Q}$
- No max nor min.
- Roots are -1 and 2.