Calculus I

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Chapter 1

Introduction

Definition 1.1. Mathematics (according to Oxford Eng. Dictionary)

The abstract science which investigates deductively the conclusions implicit in the elementary conception of spacial and numerical relations. This science can be divided in 6 main topics:

1. **Foundations:** logic, set theory, proof theorems, etc.
2. **Algebra:** numbers, arithmetical operations, order theorems.
3. **Analysis:** differentiation, integration, measure, etc.
4. **Geometry and topology:** proporties of space, shape, position of figures.
5. **Combinatories:** graph theory, partition theory, etc.
6. **Applied Mathematics:** computational sciences, probability, the range of applications of Mathematics is wide, such as:
   (a) **Banking and Finance:** Black-Scholes equation.
   (b) **Aeronautical engineering:** Fluid mechanics, shape design.
   (c) **Chemistry:** Models for proteing folding, thermodynamics.
   (d) **Informatic:** Cryptography, computational algebra, parallel programming, etc.

Summary of the program: 1st Semester

- Real numbers
- Sequence and series of numbers
- Continuity and limit
- Differentiation
- Integration

Chapter 2

Real numbers and some basic concepts

2.1 Set of points

We recall here some basical concepts:

Definition 2.1. A **set** is a **collection of distinct objects.**

Example 2.2. 2, 5, 7 are different objects (numbers). They can compose the set $\{2,\ 5,\ 7\}$, where $\{\ldots\}$ denotes the set composed by the objects $\ldots$.

Note. If an object x is a member of a set θ, we denote:

$$x \in \theta, \text{ else we denote } x \notin \theta$$

Example 2.3.

$$\theta = \{0,\ 5,\ 7\}, \quad \text{if } x = 5 \text{ and } y = 9:$$
$$x \in \theta \text{ and } y \notin \theta$$

Definition 2.4. Considering two sets A and B. If every element of A is a member of B, A is said to be a **subset** of B, and we denote:

$$\boldsymbol{A \subseteq B}$$

, else we denote

$$\boldsymbol{A \nsubseteq B}$$

Furthermore, if it exists at least one element of B which is not a member of A, A is said to be a **proper subset** of B, and we denote

$$\boldsymbol{A \subset B}$$

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Example 2.5.

$$A = \{1,\ 2,\ 3\}$$
$$B = \{0,\ 1,\ 2,\ 3,\ 4\}$$
$$C = \{0,\ 1,\ 2\}$$
$$\therefore A \subseteq A, \quad A \subset B, \quad A \nsubseteq C$$

Definition 2.6. Set operators Let A and B be two sets.

Union: $\cup$

The **union** of $\boldsymbol{A}$ and $\boldsymbol{B}$ is the set

$$\boldsymbol{A \cup B = \{x / x \in A \vee x \in B\}}$$

Intersection: $\cap$

The **intersection** of $\boldsymbol{A}$ and $\boldsymbol{B}$ is the set

$$\boldsymbol{A \cap B = \{x / x \in A \wedge x \in B\}}$$

Complement: $\setminus$

$$\boldsymbol{A \setminus B = \{x / x \in A \wedge x \notin B\}}$$

Example 2.7. $A = \{3, 5, 7\}, B = \{5, 7, 10\}$

- $A \cup B = \{3, 5, 7, 10\}$
- $A \cap B = \{5, 7\}$
- $A \backslash B = \{3\}$
- $B \backslash A = \{10\}$

Geometrical representation

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Properties:

- $A \backslash (B \cup C) = (A \backslash B) \cup (A \backslash C)$
- $A \backslash (B \cap C) = (A \backslash B) \cap (A \backslash C)$

Graphically (Venn diagrams):

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Note. In the case of various n sets denoted by $A_1,\ A_2,\ A_n$, instead of writing:

$$A_1 \cup A_2 \cup \ldots \cup A_n \text{ we write } \cup_{k=1}^{n} A_k$$

or

$$A_1 \cap A_2 \cap \ldots \cap A_n \text{ we write } \cap_{k=1}^{n} A_k$$

Example 2.8.

$$A_1 = \{1,\ 2,\ 3\}, \quad A_2 = \{5,\ 6,\ 7\}, \quad A_3 = \{1,\ 5,\ 9\}$$
$$\cup_{k=1}^{3} A_k = A_1 \cup A_2 \cup A_3 = \{1,\ 2,\ 3,\ 5,\ 7,\ 9\}$$

Remark. We can apply the same notation in case of infinite (∞) numbers of a set $\{A_1, \ldots, A_{100}, \ldots\}$.

$$\cup_{k=1}^{\infty} A_k \quad \text{and} \quad \cap_{k=1}^{\infty} A_k$$

some examples and the concept of infinity will be defined in the next sections.

Definition 2.9. The cartesian product of two sets A and B is denoted by $\boldsymbol{A \times B}$ and defined as:

$$\boldsymbol{A \times B = \{(a,\ b) | a \in A \text{ and } b \in B\}}$$

where (a, b) is called **ordened pair.**

Example 2.10.

$$A = \{1,\ 2,\ 3\}, \quad B = \{7,\ 9\}$$

$$A \times B = \{(1,7),\ (1,9),\ (2,7),\ (2,9),\ (3,7),\ (3,9)\}$$

More properties of sets will be introduced later in this chapter.

Some common sets of real points

Here we only introduce the set of points used in next chapters.

Definition 2.11.

- $\mathbb{R} = \boldsymbol{\{\ldots, \ldots,\ -10, \ldots,\ -7, \ldots,\ 0, \ldots,\ 4, \ldots, 1000, \ldots\}}$ is called the set of **real numbers** which contains **all positive and negative numbers**.
- $\mathbb{N} = \boldsymbol{\{1,\ 2,\ 3, \ldots\}}$ is called the set of **natural numbers** which contains **all the strictly positive integer numbers.**
- $\mathbb{Z} = \boldsymbol{\{\ldots, -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3, \ldots\}}$ is called the set of **integer numbers** and contains the **positive and negative integers.**
- $\varnothing = \boldsymbol{\{\}}$ the **empty set** represents the sets without any elements.

  Example 2.12.

  $$\boldsymbol{A = \{1,\ 4\}, \quad B = \{3,\ 4\}} \qquad \boldsymbol{A \cap B} = \varnothing, \text{ ie no coincidences between A and B}$$

- $\mathbb{Q} = \boldsymbol{\{x \in \mathbb{R} \mid x = \frac{m}{n},\ m \in \mathbb{Z},\ n \in \mathbb{Z} \text{ and } n \neq 0\}}$ is called the set of **rational numbers** and contains the **real numbers that can be written as a quotient of integer numbers**
- **Odd**$=$ $\boldsymbol{\{x \in \mathbb{R} \mid \exists\ k \in \mathbb{N} \text{ st } x = 2k + 1\}}$ is the set of the **odd integer numbers**.
- **Even**$=$ $\boldsymbol{\{x \in \mathbb{R} \mid \exists\ k\ \in \mathbb{N} \text{ st } x = 2k\}}$ is the set of the **even integer numbers**.
- $\mathbb{C} = \boldsymbol{\{x + iy \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}}$ is the set of **complex numbers**. Note: i denotes the imaginary number that verifies $\boldsymbol{i^2 = -1}$.

Remark.

$$\varnothing \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

Definition 2.13. A **relation**, noted by $\leq$, is a **total order** on a set S if it verifies:

1. **Reflexivity:** $\forall a \in S,\ a \leq a$
2. **Antisymmetry:** $a \leq b$ and $b \leq a \Rightarrow\ a = b$
3. **Transitivity:** $a \leq b$ and $b \leq c \Rightarrow a \leq c$
4. **Comparability:** $\forall\ a, \_ \in S$, either $a \leq b$ or $b \leq a$

Example 2.14.

- The relation $\leq$ applied to $\mathbb{R}$ is a total order.

- The relation $\subset$ applied to a subset of $\mathbb{R}$ is **not** a total order. For example, $\{1,\ 2\}$ and $\{2,\ 4\}$ cannot be compared.

Definition 2.15. A set plus a total order relation is called a **total ordered set**.