Домашняя работа.

Св-ва ф-й Эрмита: $\tilde{H}_n(x) = \sqrt{\frac{\alpha}{2^n n! \sqrt{\pi}}} H_n(\alpha x) e^{-\alpha^2 x^2}$

$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$

1) $\frac{d}{dx}\tilde{H}_n(x) = \sqrt{\frac{\alpha}{2^n n! \sqrt{\pi}}} e^{-\alpha^2 x^2}\left(-2\alpha^2 x H_n(\alpha x) + \frac{dH_n(\alpha x)}{dx}\right) =$

$$= \sqrt{\frac{\alpha}{2^n n! \sqrt{\pi}}} e^{-\alpha^2 x^2}\left(-2\alpha^2 x H_n(\alpha x) + \alpha \frac{dH_n(\alpha x)}{d(\alpha x)}\right)$$

$$\frac{dH_n(x)}{dx} = (-1)^n e^{x^2}\left(2x \frac{d^n}{dx^n}(e^{-x^2}) + \frac{d^{n+1}}{dx^{n+1}}(e^{-x^2})\right) =$$

$$= 2xH_n(x) - H_{n+1}(x) \Rightarrow$$

$$\Rightarrow \frac{d}{dx}\tilde{H}_n(x) = \sqrt{\frac{\alpha}{2^n n! \sqrt{\pi}}} e^{-\alpha^2 x^2}\Big(\cancel{-2\alpha^2 x H_n(\alpha x)} +$$

$$+ \alpha\big(\cancel{2\alpha x H_n(\alpha x)} - H_{n+1}(\alpha x)\big)\Big) =$$

$$= -\sqrt{\frac{\alpha}{2^n n! \sqrt{\pi}}} e^{-\alpha^2 x^2} \alpha H_{n+1}(\alpha x) e^{-\alpha^2 x^2} =$$

$$= -\alpha\sqrt{2(n+1)}\,\tilde{H}_{n+1}(x) \quad \blacksquare$$

2) $\alpha x \tilde{H}_n(x) = \sqrt{\frac{n+1}{2}}\,\tilde{H}_{n+1}(x) + \sqrt{\frac{n}{2}}\,\tilde{H}_{n-1}(x)$

С помощью производящей функции:

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

$$\frac{\partial}{\partial t}\left(e^{2xt-t^2}\right) = (2x-2t)e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{n t^{n-1}}{n!}$$

Сравним коэф-ты при t^n:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Перейдём к $\tilde{H}_n(x)$: $\quad x \to \alpha x$

$$2(\alpha x)H_n(\alpha x) = H_{n+1}(\alpha x) + 2nH_{n-1}(\alpha x) \Big| \cdot \sqrt{\frac{\alpha}{2^n n!\sqrt{\pi}}}\,e^{-\alpha^2x^2}$$

$$\alpha x \tilde{H}_n(x) = \underbrace{\frac{1}{2}\sqrt{\frac{\alpha}{2^{n+1}(n+1)!\sqrt{\pi}}}\sqrt{2(n+1)}\,e^{-\alpha^2x^2}H_{n+1}(\alpha x)}_{\tilde{H}_{n+1}(x)} +$$

$$+\, n\cdot\underbrace{\sqrt{\frac{\alpha}{2^{n-1}(n-1)!\sqrt{\pi}}}\cdot\frac{1}{\sqrt{2n}}\,e^{-\alpha^2x^2}H_{n-1}(\alpha x)}_{\tilde{H}_{n-1}(x)} =$$

$$= \sqrt{\frac{n+1}{2}}\,\tilde{H}_{n+1}(x) + \sqrt{\frac{n}{2}}\,\tilde{H}_{n-1}(x) \quad \blacksquare$$