Quantum computing assignments.

(Dated: March 16, 2021)

Useful literature: 1

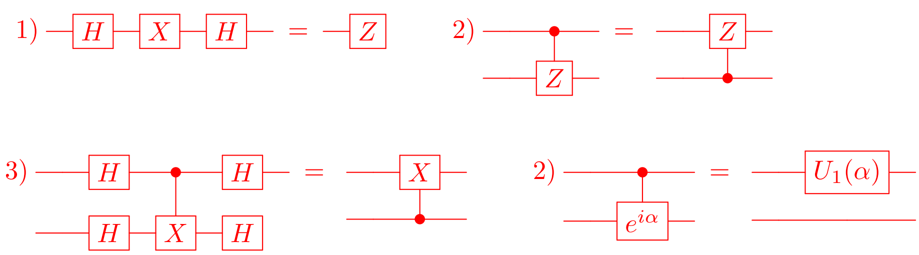
- **Problem 0 [2 points]:** Install Qiskit (see instructions in the Lecture 1 slides)
- **Problem 1 [4 points]: Extracting Qubit phase** Suppose that we have the Qubit in the state

$$|\psi\rangle = \sin\theta|0\rangle + e^{i\varphi}\cos\theta|1\rangle \tag{1}$$

  Show how using the measurement in computational basis to measure the angle φ provided θ is known. Write the code using Qiskit which prepares the state (3) and performs this measurement. Visualise the circuit. Give link to jupyter notebook.

- **Problem 2 [8 points]: Single Qubit state preparation** Suggest a circuit which uses fixed number of standard gates: X, H, S and $R_z(\gamma)$ where $R_z(\gamma)$ is rotation around z axis by an arbitrary angle γ to prepare the state (3) modulo the total phase of the wave function. Compare the accuracy of this procedure with custom U_3 gate at IMB Quantum for the angles $\varphi = \theta = \pi/3$. Write the code using Qiskit, give link to jupyter notebook.
- **Problem 3 [4 points]: Useful identities for 1-Qubit and 2-Qubit gates**

  Prove the following identities



- **Problem 4 [4 points]: Toffoli gate** Write the algorithm which realized reversible AND operation using the Toffoli gate. Give the link to the
- **Problem 5 [4 points]: Separable and entangled states**

  Determine which of the following states are entangled. If the state is not entangled, show how to write that as a tensor product; if it is entangled, prove it.

  - (a) $\frac{4}{5}|00\rangle + \frac{2}{5}|01\rangle - \frac{2}{5}|10\rangle + \frac{2\sqrt{2}}{5}|11\rangle$
  - (b) $\frac{1}{2}\left(|00\rangle - i|01\rangle + |10\rangle + |11\rangle\right)$
  - (c) $\frac{1}{2}\left(|00\rangle + |01\rangle - |10\rangle + |11\rangle\right)$

- **Problem 6 [8 points]: Preparation of the general 2-Qubit pure state**

  Using Schmidt decomposition and its relation to the Singular value decomposition of matrices, suggest the algorithm to prepare general 2-Qubit pure state

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle \tag{2}$$

  Apply this procedure to prepare the example 2-Qbit state

  - (a) $|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$
  - (b) $|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$
  - (c) $|\Phi\rangle = \frac{1}{\sqrt{12}}(3|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

The states $|\Phi^{\pm}\rangle$ and $|\Psi^{\pm}\rangle$ are called Bell states, they are used in quantum cryptography algorithms. Write code in Qiskit, give the link to the Jupyter notebook.

- **Problem 7 [8 points]: Measuring gate**

  Let us consider the two-Qubit state $|\Phi\rangle = \frac{1}{\sqrt{12}}(3|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ What is the result of (a) measuring first Qubit, (b) measuring second Qubit, (c) measuring both Qubits?

- **Problem 8: Deutsch's problem. [4 points]**

  Suppose one tried to solve Deutsch's problem not by using the trick that we considered during the lecture, but by applying the standard procedure: Start with the output and input registers in the state $|0\rangle|0\rangle$, apply the Hadamard to the input register and then apply $\mathbf{U_f}$, thereby transforming to the state

  $$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle|f(0)\rangle + \frac{1}{\sqrt{2}}|1\rangle|f(1)\rangle \tag{3}$$

  Given the two Qbits in this state, a direct measurement only reveals the value of f at either 0 or 1 (randomly), but gives no information about whether $f(0) = f(1)$. But there is a way (noticed by Deutsch) to do this with 50% probability by applying other unitary transformation before measuring.

  Show that if one applies Hadamard $\mathbf{H}$ to each of the Qbit prior to measurement, then regardless of which of the four possible states (3) one has been given (corresponding of the four possible choices of the function $f(x)$ that brings on bit into one bit), there is a 50% chance that measurement will enable one to conclude whether or not $f(0) = f(1)$. But the other 50% one will learn nothing whatever from the measurement outcome, neither about whether $f(0) = f(1)$ nor about the value of either $f(0)$ or $f(1)$.

- **Problem 9 [8 points]: Useful identities** Prove that

  - **[4 points]**

    $$H^{\otimes n}|\boldsymbol{x}\rangle = \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{z}\in\{0,1\}^n} (-1)^{\boldsymbol{x}\cdot\boldsymbol{z}}|\boldsymbol{z}\rangle \tag{4}$$

  - **[4 points]** Using Eq.4 show that

    $$H^{\otimes n}\left(\frac{1}{\sqrt{2}}|\boldsymbol{x}\rangle + \frac{1}{\sqrt{2}}|\boldsymbol{y}\rangle\right) = \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{z}\in\{\boldsymbol{s}\}^{\perp}} (-1)^{\boldsymbol{x}\cdot\boldsymbol{z}}|\boldsymbol{z}\rangle \tag{5}$$

    where $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^n$, $\boldsymbol{s} = \boldsymbol{x} \oplus \boldsymbol{y}$ and $\boldsymbol{s}^{\perp} = \{\boldsymbol{z} \in \{0,1\}^n | \boldsymbol{s}\cdot\boldsymbol{z} = 0\}$

- **Problem 10 [8 points]: Probabilities for solving Simon's problem**

  As discussed in lectures, to estimate how many times a quantum computer has to invoke the subroutine $\mathbf{U_f}$ to solve Simon's problem, on has to answer a purely mathematical question. We have an n-dimensional space of vectors whose components are either 0 or 1 whose addition and inner products are carried out with the modulo 2 arithmetic. We are interested in the $(n-1)$-dimensional subspace of vectors orthogonal to a given vector a. We have a quantum computer program which gives us a random vector y in this subspace. If we run the program $n + x$ times, what is the probability q that $n - 1$ of the vector will be linearly independent ? We have discussed in the lectures that

  $$q = \left(1 - \frac{1}{2^{2+x}}\right)\left(1 - \frac{1}{2^{3+x}}\right)\cdots\left(1 - \frac{1}{2^{n+x}}\right) \tag{6}$$

  Consider the case $n = 3$, $x = 1$ and $a = 111$ (in binary representation). Prove that the expression for probability (6) is correct by the direct computation.

- **Problem 11 [8 points]: Grover' search** Design and test the Grovers' search with 2 Qbits.

  - **[4 points]** Explain the basic elements of the quantum circuit including oracles and other operators.
  - **[2 points]** Design and test the algorithm with Qiskit.

- [**2 points**] Find out what happens if one applies Grover' iterations more times than needed.

- **Problem 12: Quantum Fourier Transform [8 points]**

  Write the code in Qiskit for QFT and check it for the case of $n = 3$ Qubits.

- **Problem 13: Quantum phase estimation [10 points]** Suppose that we have gate $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$ with unknown phase θ. Using the algorithm of quantum phase estimation, write the code in Qiskit which determines θ.

- **Problem 14: Quantum error correction [10 points]**

  Suppose the only types of errors are the bit-flip errors but we would like to correct not only single bit-flips $\mathbf{X}_i$ but also the double bit-flips $\mathbf{X}_i\mathbf{X}_j$ with $j \neq i$.

  - **(i)** [**4 points**] Find the size of the codewords n such that the dimension of the n-Qbit state space is just large enough to accommodate mutually orthogonal two-dimensional subspaces for the uncorrupted code words and all codewords produced by single- and double- bit flip corruptions.
  - **(ii)** [**6 points**] Show for the n found in **(i)** that there is indeed a perfect n-Qbit code that corrects all single and double bit-flip errors by writing down the states that encode $|\bar{0}\rangle$ and $|\bar{1}\rangle$, and writing down a set of commuting hermitian operators whose squares are unity, that preserve both codewords and have distinct patterns of commutations and anticommutations for each of the operators that produce all single and double bit-flip errors.

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[1] Quantum Algorithm Implementations for Beginners, Abhijith J., Adetokunbo Adedoyin, arXiv:1804.03719