Aalto University
School of Science

From multiple SLE/GFF coupling to dynamical random matrices

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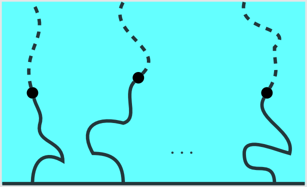
Joint work with Makoto Katori (Chuo University)

Ref: Katori-K., J Phys A **54** (2021) 325002.

Players

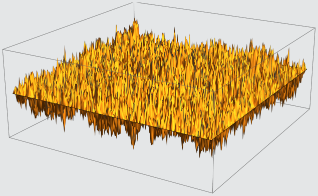
(Multiple) SLE



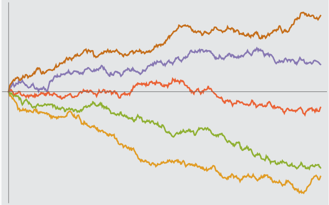
Coupling

Conformal map

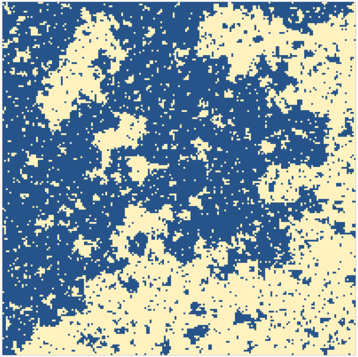
GFF



Dyson BMs



SLE background: interface in two dimensions



Critical Ising model.

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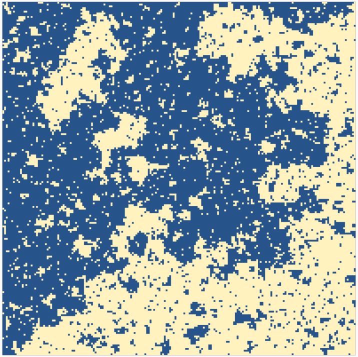

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$$\text{lattice mesh} \to 0$$

if it exists.

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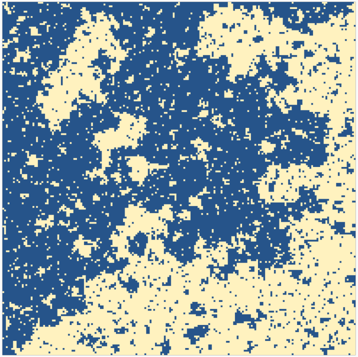

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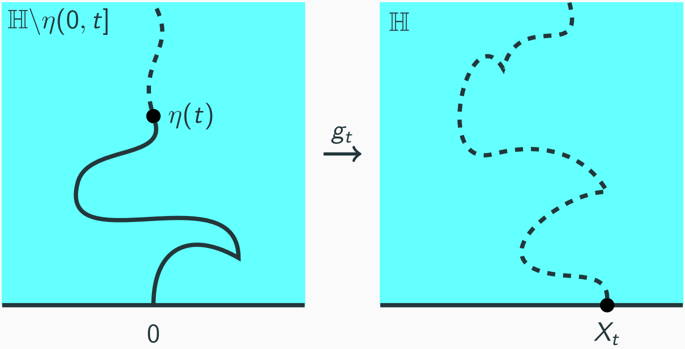
Two properties are expected (proved for the Ising model):

- Conformal invariance
- Domain Markov property

Loewner theory

- $\eta \colon [0,\infty) \to \overline{\mathbb{H}}$, $\eta(0) = 0$: simple curve in $\mathbb{H}$ starting at 0.
- Riemann's mapping theorem: at each $t \geq 0$, there exists a unique conformal map

$$g_t : \mathbb{H}\backslash\eta(0,t] \to \mathbb{H} \quad \text{such that} \quad \lim_{|z|\to\infty} |g_t(z) - z| = 0.$$



Loewner theory

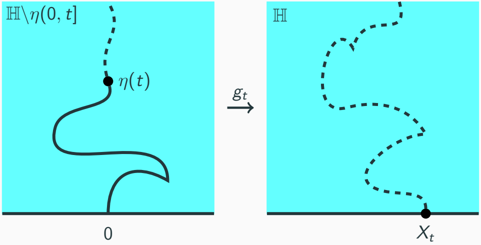
Theorem (Löwner, Kufarev–Sobolov–Sporyševa)

After possible reparametrization, the family $(g_t : t \geq 0)$ of conformal maps satisfies

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - X_t} \quad t \geq 0, \quad g_0(z) = z,$$

where

$$X_t = \lim_{z \to \eta(t)} g_t(z), \quad t \geq 0.$$



Schramm's principle

Theorem

A random curve that exhibits conformal invariance and domain Markov property is governed by the Loewner chain $(g_t : t \geq 0)$ driven by a Brownian motion:

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad t \geq 0, \quad g_0(z) = z,$$

where $\kappa \geq 0$ and $(B_t : t \geq 0)$ is a standard Brownian motion.

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Definition

The above $(g_t : t \geq 0)$ is called the Schramm–Loewner evolution (SLE) of parameter κ, or SLE(κ).

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For the Ising model, $\kappa = 3$.

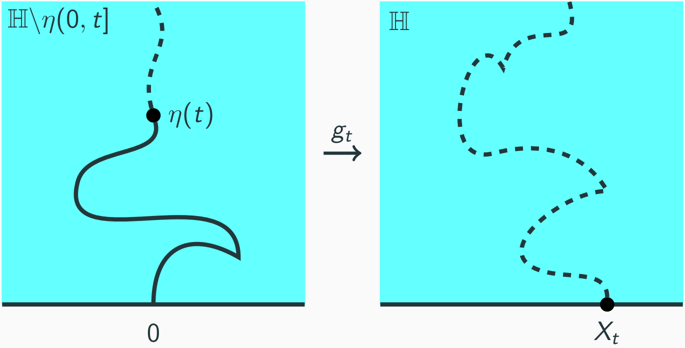
In the continuum SLE(κ) makes sense for $\kappa \geq 0$.

Upshot

(Random)
curve in 2D domain

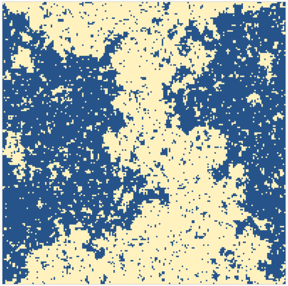
$\overset{\text{Conformal map}}{\Longleftrightarrow}$

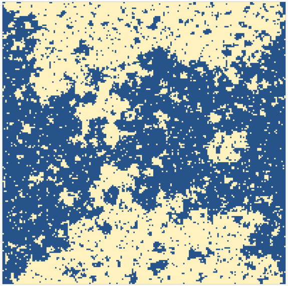
(Stochastic)
process on 1D boundary



Multiple SLE

Matters when the boundary conditions change many times.





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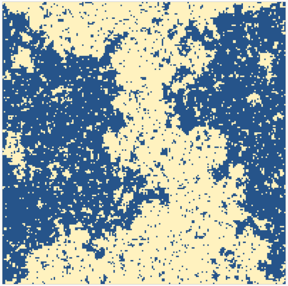

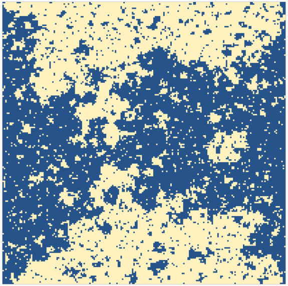

How to sample multiple random curves?

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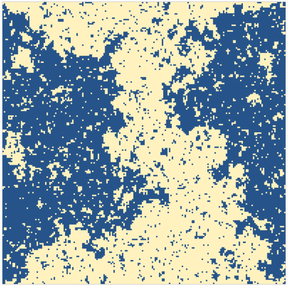

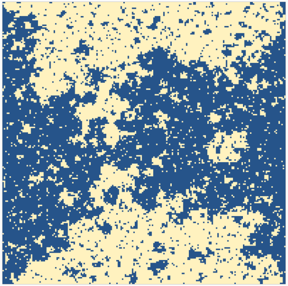

How to sample multiple random curves?

1. Commuting SLEs (Dubédat).

Multiple SLE

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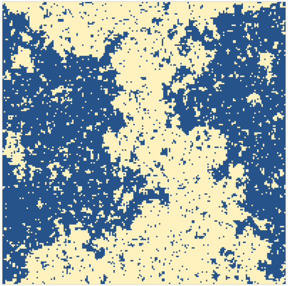
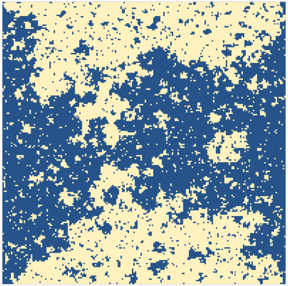
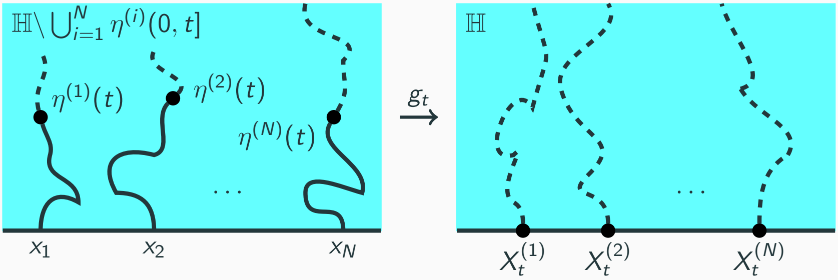




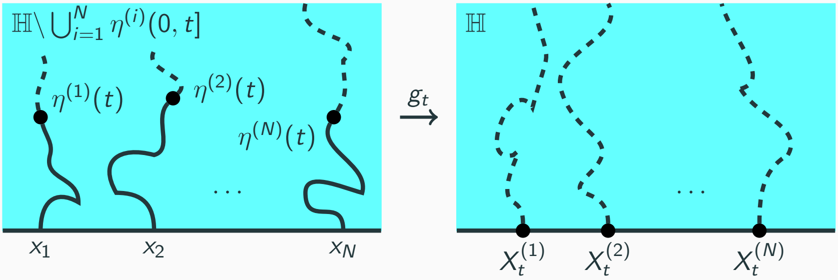
How to sample multiple random curves?

1. Commuting SLEs (Dubédat).
2. Multiple Loewner equation (Bauer–Bernard–Kytölä).

Multiple Loewner equation



Multiple Loewner equation



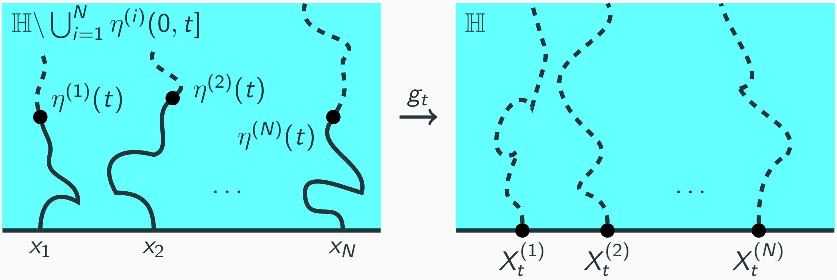
Theorem

After possible reparametrization, $(g_t : t \geq 0)$ *satisfies*

$$\frac{d}{dt}g_t(z) = \sum_{i=1}^{N} \frac{2}{g_t(z) - X_t^{(i)}}, \quad t \geq 0, \quad g_0(z) = z,$$

$$X_t^{(i)} = \lim_{z \to \eta^{(i)}(t)} g_t(z), \quad i = 1, \ldots, N.$$

Driving processes for a multiple SLE



- When $\eta^{(i)}$ are random, $(X_t^{(i)} : t \geq 0)$ are stochastic processes.

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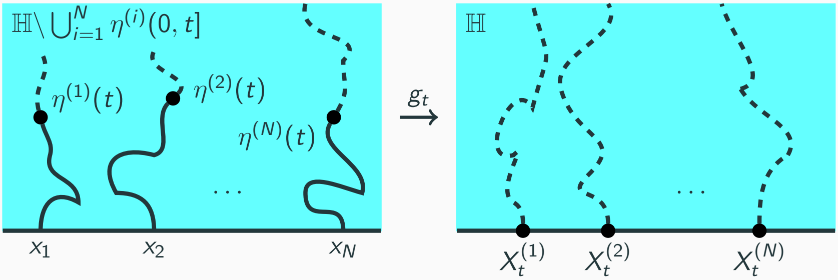

- When $\eta^{(i)}$ are random, $(X_t^{(i)} : t \geq 0)$ are stochastic processes.
- Conformal invariance and domain Markov property do not tell much about $(X_t^{(i)} : t \geq 0)$ due to nontrivial moduli.

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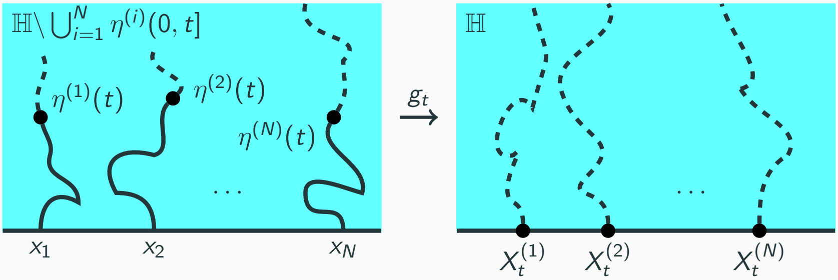

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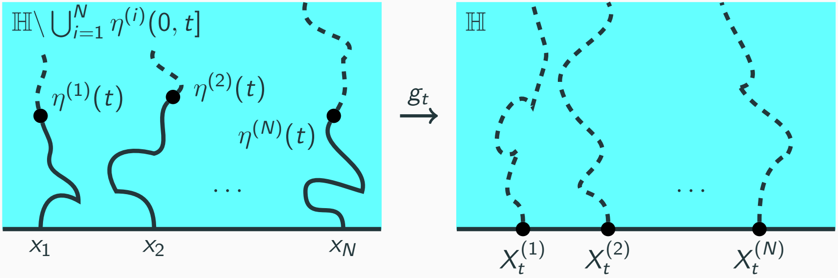
Question

What is a good choice of $(X_t^{(i)} : t \geq 0)$ for a multiple SLE?

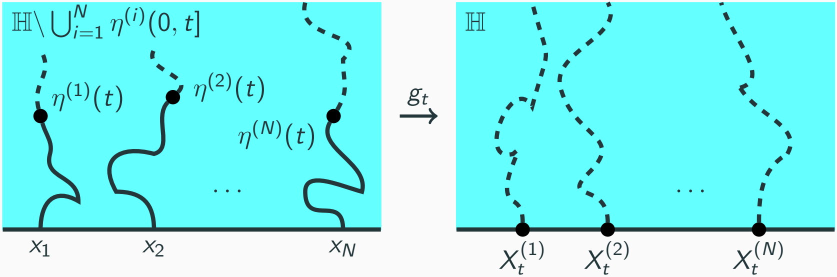
Driving processes for a multiple SLE



Idea

When we fix a statistical mechanics model in two dimensions, there should be a unique multiple SLE that describes the interfaces.

Driving processes for a multiple SLE



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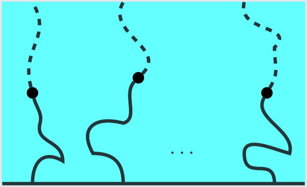
When we fix a statistical mechanics model in two dimensions, there should be a unique multiple SLE that describes the interfaces.

Finding

When we place Gaussian free field (GFF) on two dimensions, the corresponding multiple SLE must be driven by the Dyson model.

Players

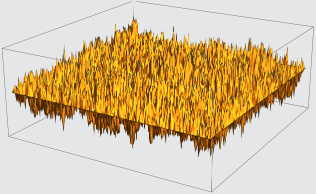
(Multiple) SLE



Coupling

Conformal map

GFF



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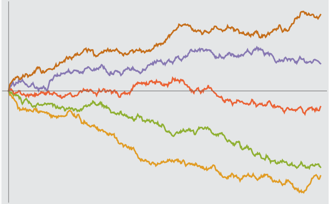

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- The Dirichlet boundary GFF

$$H = \sum_{i \in \mathbb{N}} \alpha_i \phi_i$$

converges a.s. to a distribution with test functions in $C_0^\infty(D)$.

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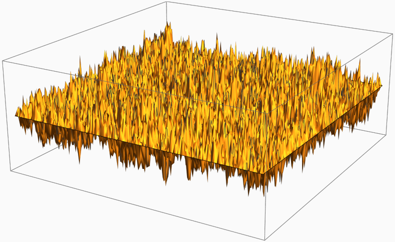

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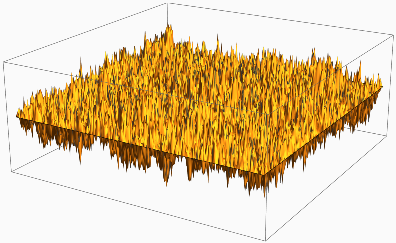


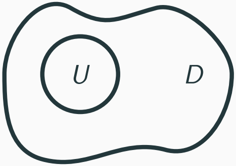
- $\{(H, f) : f \in C_0^\infty(D)\}$: Gaussian family

$$\mathbb{E}[(H, f)] = 0,$$
$$\mathrm{Cov}((H, f), (H, g)) = \int_{D \times D} f(z) G_D(z, w) g(w) dz dw,$$

where $G_D(z, w)$ is the Green's function of D.

Domain Markov property of GFF

- $U \subset D$: subdomain
  $\rightsquigarrow W(U) \subset W(D)$: closed subspace.



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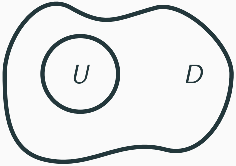

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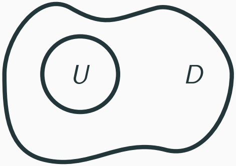

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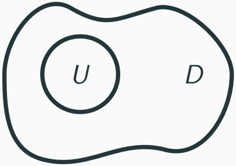

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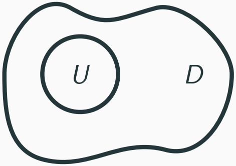
  - H_U: Ditichlet boundary GFF on U.



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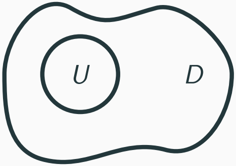
  - $H_{U^c}|_U$: "harmonic extension" of $H|_{\partial U}$.



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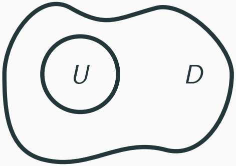


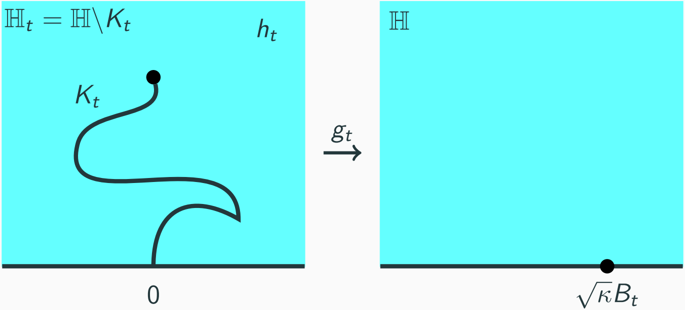
Proposition

The conditional law of $H|_U$ given $H|_{D\setminus U}$ agrees with the law of

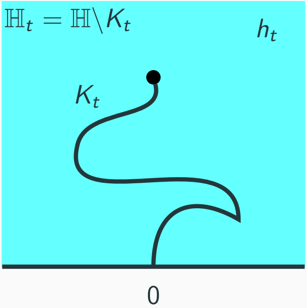
$$H_U + (\textit{harmonic extension of } H|_{\partial U}).$$

SLE/GFF-coupling

- $(g_t : t \geq 0)$: SLE(κ).
- Harmonic function
$$\mathfrak{h}_t = -\frac{2}{\sqrt{\kappa}} \arg(g_t(\cdot) - \sqrt{\kappa} B_t) - \chi \arg g_t'(\cdot) \quad \left(\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}\right)$$
on $\mathbb{H}_t$.
- $h_t := H_{\mathbb{H}_t} + \mathfrak{h}_t$, $h := h_0$.



SLE/GFF-coupling



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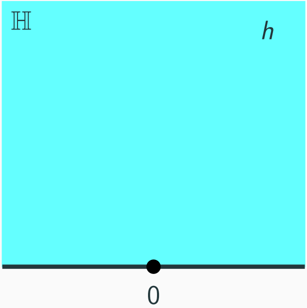

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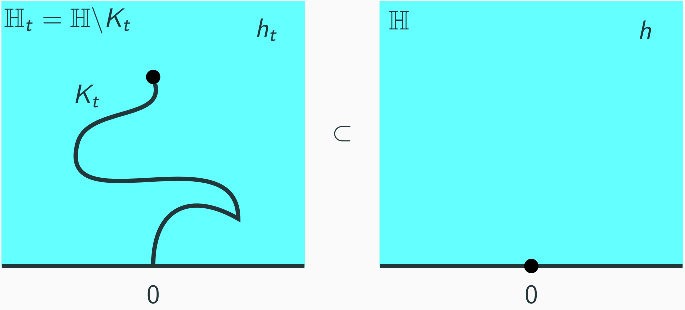


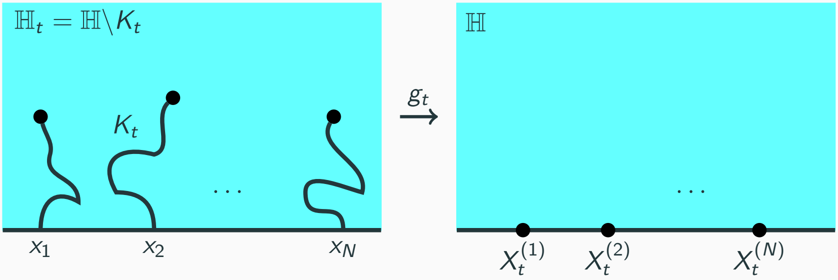
Theorem (Dubédat, Schramm–Sheffield, Miller–Sheffield)

At each $t \geq 0$, conditioned on K_t,

$$h|_{\mathbb{H}_t} \overset{(\mathrm{law})}{=} h_t.$$

$$\text{i.e., } H_{\mathbb{H}_t} + (\text{harmonic extension of } h|_{\partial \mathbb{H}_t}) \overset{(\mathrm{law})}{=} H_{\mathbb{H}_t} + \mathfrak{h}_t.$$

Multiple SLE/GFF-coupling

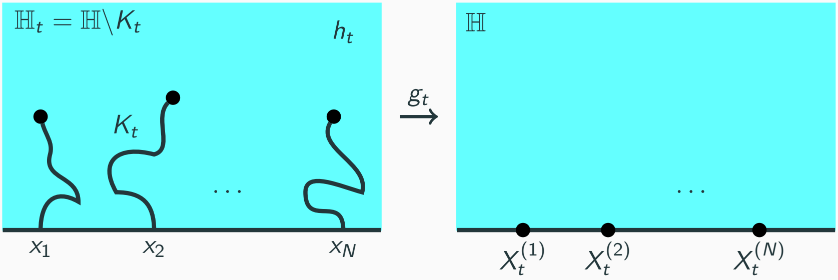


- $(g_t : t \geq 0)$: multiple SLE, i.e.,

$$\frac{d}{dt} g_t(z) = \sum_{i=1}^{N} \frac{2}{g_t(z) - X_t^{(i)}}, \quad t \geq 0, \quad g_0(z) = z,$$

where $(X_t^{(i)} : t \geq 0)$, $i = 1, \ldots, N$ are unspecified continuous stochastic processes.

Multiple SLE/GFF-coupling



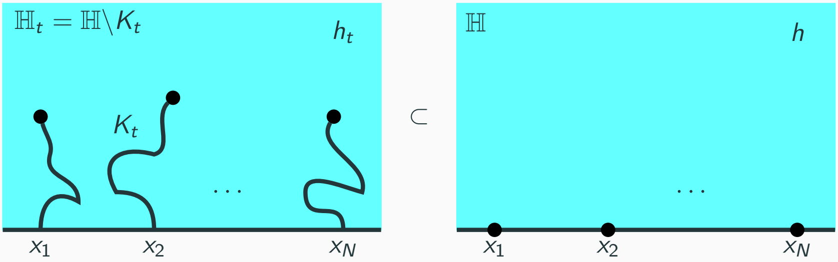
- Harmonic function

$$\mathfrak{h}_t = -\frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \arg(g_t(\cdot) - X_t^{(i)}) - \chi \arg g_t'(\cdot)$$

on $\mathbb{H}_t$ with $\kappa > 0$.

- $h_t := H_{\mathbb{H}_t} + \mathfrak{h}_t$, $h := h_0$.

Multiple SLE/GFF-coupling



Definition

We say that the multiple SLE $(g_t : t \geq 0)$ is coupled to the GFF h on $\mathbb{H}$ if at each $t \geq 0$, conditioned on K_t,

$$h|_{\mathbb{H}_t} \overset{\text{(law)}}{=} h_t.$$

Main result

Theorem (Katori–K.)

Under the above setting, the multiple SLE $(g_t : t \geq 0)$ *is coupled to the GFF h if and only if* $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$ *and the driving processes* $(X_t^{(i)} : t \geq 0)$, $i = 1, \dots, N$ *satisfy the system of stochastic differential equations*

$$dX_t^{(i)} = \sqrt{\kappa} dB_t^{(i)} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{4dt}{X_t^{(i)} - X_t^{(j)}}, \quad t \geq 0, \quad i = 1, \dots, N,$$

where $(B_t^{(i)} : t \geq 0)$ *are independent standard BMs.*

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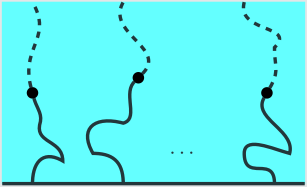
where $(B_t^{(i)} : t \geq 0)$ *are independent standard BMs.*

When we set $\lambda_t^{(i)} = X_{t/\kappa}^{(i)}$, $i = 1, \dots, N$,

$$d\lambda_t^{(i)} = dB_t^{(i)} + \frac{\beta}{2} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{dt}{\lambda_t^{(i)} - \lambda_t^{(j)}}, \quad t \geq 0, \quad i = 1, \dots, N, \; \beta = \frac{8}{\kappa}$$

Players

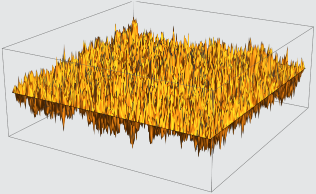
(Multiple) SLE



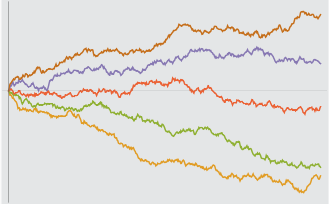
Coupling

Conformal map

GFF



Dyson BMs



Gaussian unitary ensemble (GUE)

$$A = \begin{pmatrix} \xi_{11} & \frac{1}{\sqrt{2}}(\xi_{12} + \mathbb{i}\eta_{12}) & \cdots & \frac{1}{\sqrt{2}}(\xi_{1N} + \mathbb{i}\eta_{1N}) \\ \frac{1}{\sqrt{2}}(\xi_{12} - \mathbb{i}\eta_{12}) & \xi_{22} & \cdots & \frac{1}{\sqrt{2}}(\xi_{2N} + \mathbb{i}\eta_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2}}(\xi_{1N} - \mathbb{i}\eta_{1N}) & \frac{1}{\sqrt{2}}(\xi_{2N} - \mathbb{i}\eta_{2N}) & \cdots & \xi_{NN} \end{pmatrix}$$

$$\xi_{ij}, \eta_{ij} \sim \mathrm{N}(0,1) \quad (1 \leq i \leq j \leq N) : \text{i.i.d.}$$

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- Probability distribution function for $(\lambda_1, \dots, \lambda_N)$:

$$p(\lambda_1, \dots, \lambda_N) = \prod_{1 \le i < j \le N} |\lambda_j - \lambda_i|^{\beta} e^{-\frac{\beta}{4} \sum_{i=1}^{N} \lambda_i^2}, \quad \beta = 2.$$

Dyson Brownian motions

$$A_t = \begin{pmatrix} \xi_t^{(11)} & \frac{1}{\sqrt{2}}(\xi_t^{(12)} + \mathbb{i}\eta_t^{(12)}) & \cdots & \frac{1}{\sqrt{2}}(\xi_t^{(1N)} + \mathbb{i}\eta_t^{(1N)}) \\ \frac{1}{\sqrt{2}}(\xi_t^{(12)} - \mathbb{i}\eta_t^{(12)}) & \xi_t^{(22)} & \cdots & \frac{1}{\sqrt{2}}(\xi_t^{(2N)} + \mathbb{i}\eta_t^{(2N)}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2}}(\xi_t^{(1N)} - \mathbb{i}\eta_t^{(1N)}) & \frac{1}{\sqrt{2}}(\xi_t^{(2N)} - \mathbb{i}\eta_t^{(2N)}) & \cdots & \xi_t^{(NN)} \end{pmatrix}$$

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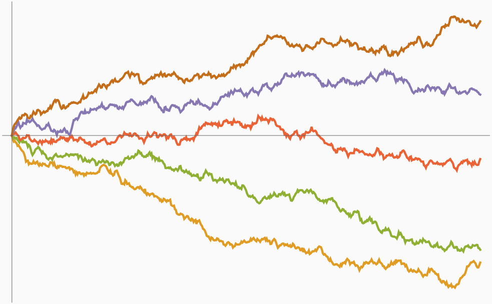

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Application: equivalence to commuting SLEs

Commuting SLEs

$=$ Reweighting of independent SLEs by a *partition function*.

Theorem

Let $0 < \kappa \leq 8$. The multiple SLE as commuting SLEs associated with the partition function $Z(x_1, \ldots, x_n) = \prod_{i<j} |x_i - x_j|^{2/\kappa}$ is equivalent to our version of multiple SLE driven by

$$dX_t^{(i)} = \sqrt{\kappa} dB_t^{(i)} + \sum_{j;j\neq i} \frac{4dt}{X_t^{(i)} - X_t^{(j)}}, \quad t \geq 0, \quad i = 1, \ldots, N.$$

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Proof.

The commuting SLEs are coupled with the GFF, and the GFF determines the multiple curves (Dubédat, Schramm–Sheffield, Miller–Sheffield). The "only if" part does the trick. □

Application: three phases of multiple SLE

Corollary

Let $0 < \kappa \leq 8$. The multiple SLE $(g_t : t \geq 0)$ driven by

$$dX_t^{(i)} = \sqrt{\kappa} dB_t^{(i)} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{4dt}{X_t^{(i)} - X_t^{(j)}}, \quad t \geq 0$$

generates multiple curves $\eta^{(i)}$ evolving towards ∞. Furthermore,

1. *when $\kappa \in (0, 4]$, $\eta^{(i)}$ are disjoint simple curves,*
2. *when $\kappa \in (4, 8)$, $\eta^{(i)}$ are intersecting,*
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Equivalence to the commuting SLEs. □

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Time change: $t \to t/N$,

$$\frac{d}{dt} g_t^N(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{2}{g_t^N(z) - X_{t/N}^{(i)}}.$$

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Assume $\frac{1}{N}\sum_{i=1}^{N} \delta_{X_0^{(i)}} \Rightarrow \delta_0$. For fixed t, and as $N \to \infty$,

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{X_{t/N}^{(i)}} \Rightarrow \frac{1}{8\pi t} \sqrt{16t - x^2}\, \mathbf{1}_{[-4\sqrt{t}, 4\sqrt{t}]} dx.$$

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Question

What happens to multiple SLE as $N \to \infty$?

Hydrodynamic limit of Dyson's Brownian motions

Assume $\frac{1}{N}\sum_{i=1}^{N}\delta_{X_0^{(i)}} \Rightarrow \exists\mu$.

Theorem

The Dyson's Brownian motions have a deterministic limit:

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{X_{t/N}^{(i)}} \Rightarrow \mu_t, \quad N \to \infty.$$

The measures $(\mu_t : t \geq 0)$ *are characterized by the complex Burgers equation*

$$\frac{\partial M_t(z)}{\partial t} = -2M_t(z)\frac{\partial M_t(z)}{\partial z}$$

of the Cauchy transform

$$M_t(z) = \int_{\mathbb{R}} \frac{2}{z-x}\mu_t(dx), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Hydrodynamic limit of multiple SLE

Theorem (del Monaco–Schleißinger)

(1) The Loewner chains $(g_t^N : t \geq 0)$ *have a deterministic limit:*

$$g_t^N \to g_t, \quad N \to \infty$$

that satisfies

$$\frac{d}{dt} g_t(z) = M_t(g_t(z)) = \int_{\mathbb{R}} \frac{2}{g_t(z) - x} \mu_t(dx).$$

(2) The growing hulls $(K_t^N : t \geq 0)$ *generated by* $(g_t^N : t \geq 0)$ *have a deterministic limit:*

$$K_t^N \to K_t, \quad N \to \infty$$

that is generated by $(g_t : t \geq 0)$*.*

Hydrodynamic limit of multiple SLE

- Hotta–Katori solved the equations in the case that $\mu = \delta_0$.

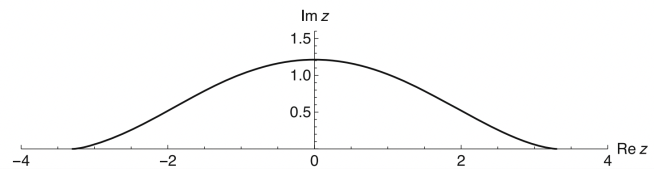

Figure 1: The boundary of K_1: $2\mathbb{i}\exp(-\mathbb{i}\varphi - \frac{e^{2\mathbb{i}\varphi}}{2})$, $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

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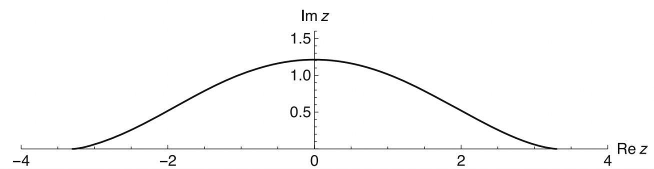


Figure 1: The boundary of K_1: $2\mathbb{i}\exp(-\mathbb{i}\varphi - \frac{e^{2\mathbb{i}\varphi}}{2})$, $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

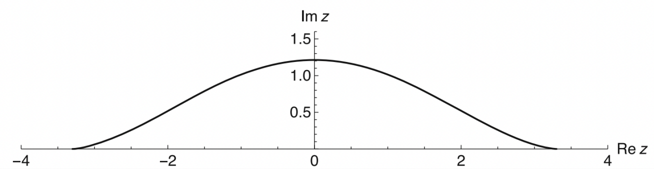
- Already for a finite N, we can view

$$\frac{d}{dt} g_t^N(z) = \int_{\mathbb{R}} \frac{2}{g_t^N(z) - x} \mu_t^N(dx), \quad \mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_{t/N}^{(i)}}$$

as a measure driven SLE. (cf: quantum Loewner evolution)

Another scaling?

Hydrodynamic limit = law of large numbers.



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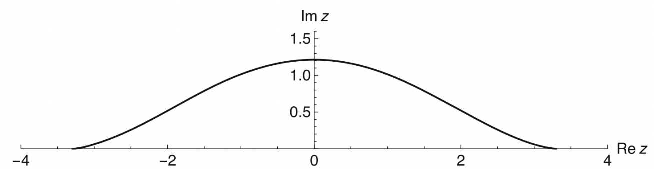

- (Gaussian) fluctuation?

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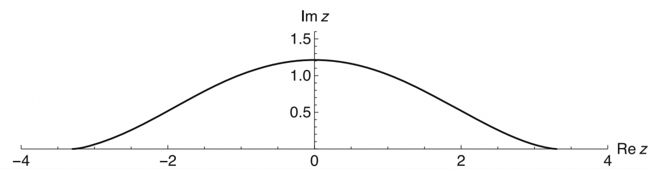


- (Gaussian) fluctuation?
- Tracy–Widom ($\beta = 8/\kappa$)?

Proof: "if" part

Goal

Assume $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$ and that $(X_t^{(i)} : t \geq 0)$ are Dyson Brownian motions, and show

$$h|_{\mathbb{H}_t} \stackrel{(\text{law})}{=} h_t.$$

Lemma

$$\mathfrak{h}_t(z) = -\frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \arg(g_t(z) - X_t^{(i)}) - \chi \arg g_t'(z), \quad z \in \mathbb{H}$$

are local martingales and

$$d\left\langle \mathfrak{h}(z), \mathfrak{h}(w) \right\rangle_t = -dG_{\mathbb{H}_t}(z, w) = -dG_{\mathbb{H}}(g_t(z), g_t(w)),$$

$$G_{\mathbb{H}}(z, w) = \log \left| \frac{z - \overline{w}}{z - w} \right|, \quad z \neq w.$$

Proof: "if" part

Proof.

Notice $\mathfrak{h}_t(z) = \mathrm{Im}\widehat{\mathfrak{h}}_t(z)$ with

$$\widehat{\mathfrak{h}}_t(z) = -\frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \log(g_t(z) - X_t^{(i)}) - \chi \log g_t'(z).$$

log is easier to differentiate than arg:

$$\begin{aligned} d\widehat{\mathfrak{h}}_t(z) &= \sum_{i=1}^{N} \frac{2}{g_t(z) - X_t^{(i)}} dB_t^{(i)}, \\ d\left\langle \mathfrak{h}(z), \mathfrak{h}(w) \right\rangle_t &= \sum_{i=1}^{N} \mathrm{Im}\frac{2}{g_t(z) - X_t^{(i)}} \mathrm{Im}\frac{2}{g_t(w) - X_t^{(i)}} dt \\ &= -dG_{\mathbb{H}}(g_t(z), g_t(w)). \end{aligned}$$

$\square$

Proof: "if" part

Let us fix $f \in C_0^\infty(\mathbb{H})$ (test function), and τ a stopping time such that

$$K_\tau \cap \mathrm{supp}(f) = \emptyset.$$

Compute the characteristic function of (h_τ, f):

$$\mathbb{E}\left[e^{\mathrm{i}\theta(h_\tau,f)}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\theta(H_{\mathbb{H}_\tau},f)}\middle|\mathcal{F}_\tau\right] e^{\mathrm{i}\theta(\mathfrak{h}_\tau,f)}\right]. \quad (\mathcal{F}_t = \sigma(K_t),\ t \geq 0.)$$

$$\mathbb{E}\left[e^{\mathrm{i}\theta(H_{\mathbb{H}_\tau},f)}\middle|\mathcal{F}_\tau\right] = e^{-\frac{\theta^2}{2}E_\tau(f)}, \quad E_\tau(f) = \int f(z)G_{\mathbb{H}_\tau}(z,w)f(w)dzdw.$$

$$\begin{aligned}\mathbb{E}\left[e^{\mathrm{i}\theta(h_\tau,f)}\right] &= \mathbb{E}\left[e^{\mathrm{i}\theta(\mathfrak{h}_\tau,f)-\frac{\theta^2}{2}E_\tau(f)}\right] \\ &= \mathbb{E}\left[e^{\mathrm{i}\theta(\mathfrak{h}_0,f)-\frac{\theta^2}{2}E_0(f)}\right] = \mathbb{E}\left[e^{\mathrm{i}\theta(h,f)}\right].\end{aligned}$$

Proof: “only if” part

Goal

Assume $h|_{\mathbb{H}_t} \overset{\text{(law)}}{=} h_t$, and show $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$ and

$$dX_t^{(i)} = \sqrt{\kappa} dB_t^{(i)} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{4dt}{X_t^{(i)} - X_t^{(j)}}, \quad t \geq 0, \quad i = 1, \ldots, N.$$

Domain Markov property of GFF: conditioned on K_t,

$$h|_{\mathbb{H}_t} \overset{\text{(law)}}{=} H_{\mathbb{H}_t} + (\text{harmonic extension of } h|_{\partial \mathbb{H}_t}).$$

From the coupling,

$$\mathfrak{h}_t = (\text{harmonic extension of } h|_{\partial \mathbb{H}_t}) = \mathbb{E}\left[h|_{\mathbb{H}_t} \middle| \mathcal{F}_t \right]$$

is a continuous local martingale.

Proof: “only if” part

Recall the definition of $\mathfrak{h}_t$:

$$\mathfrak{h}_t(z) = -\frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \arg(g_t(z) - X_t^{(i)}) - \chi \arg g_t'(z).$$

Implicit function theorem

$\rightsquigarrow$ $(X_t^{(i)} : t \geq 0)$, $i = 1, \ldots, N$ are continuous semi-martingales:

$$X_t^{(i)} = M_t^{(i)} + F_t^{(i)}, \quad t \geq 0, \quad i = 1, \ldots, N.$$

Also notice that $\mathfrak{h}_t(z) = \mathrm{Im}(\widehat{\mathfrak{h}}_t(z))$ with

$$\widehat{\mathfrak{h}}_t(z) = -\frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \log(g_t(z) - X_t^{(i)}) - \chi \log g_t'(z)$$

and that $(\widehat{\mathfrak{h}}_t : t \geq 0)$ is a continuous local martingale (Cauchy–Riemann).

Proof: "only if" part

$$
\begin{aligned}
d\widehat{\mathfrak{h}}_t(z) = &\sum_{i=1}^{N} \frac{1}{(g_t(z) - X_t^{(i)})^2} \left(\left(-\frac{4}{\sqrt{\kappa}} + 2\chi \right) dt + \frac{1}{\sqrt{\kappa}} d \left\langle M^{(i)}, M^{(i)} \right\rangle_t \right) \\
&+ \frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \frac{1}{g_t(z) - X_t^{(i)}} \left(dF_t^{(i)} - \sum_{j;j\neq i} \frac{4dt}{X_t^{(i)} - X_t^{(j)}} \right) \\
&+ \frac{2}{\sqrt{\kappa}} \sum_{i=1}^{N} \frac{1}{g_t(z) - X_t^{(i)}} dM_t^{(i)}.
\end{aligned}
$$

- $d \left\langle M^{(i)}, M^{(i)} \right\rangle_t = \kappa(1+\xi)dt$, $i = 1, \dots, N$, $\xi = \frac{2}{\sqrt{\kappa}}(\frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2} - \chi)$.
- $dF_t^{(i)} = \sum_{j;j\neq i} \frac{4dt}{X_t^{(i)} - X_t^{(j)}}$, $i = 1, \dots, N$.

$$
dX_t^{(i)} = \sqrt{\kappa(1+\xi)}dB_t^{(i)} + \sum_{j;j\neq i} \frac{4dt}{X_t^{(i)} - X_t^{(j)}}, \quad t \geq 0, \quad i = 1, \dots, N.
$$

Proof: “only if” part

Coupling also gives

$$\langle \mathfrak{h}(z), \mathfrak{h}(w) \rangle_t + G_{\mathbb{H}_t}(z, w) = G_{\mathbb{H}}(z, w).$$

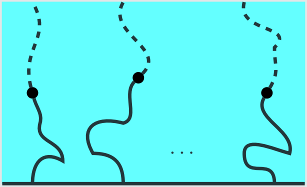
Cross variation $\langle \mathfrak{h}(z), \mathfrak{h}(w) \rangle_t$:

$$\begin{aligned} & d \langle \mathfrak{h}(z), \mathfrak{h}(w) \rangle_t + (1 + \xi) dG_{\mathbb{H}_t}(z, w) \\ = & \frac{4}{\kappa} \sum_{i \neq j} \mathrm{Im} \left(\frac{1}{g_t(z) - X_t^{(i)}} \right) \mathrm{Im} \left(\frac{1}{g_t(w) - X_t^{(j)}} \right) d \left\langle M^{(i)}, M^{(j)} \right\rangle_t \end{aligned}$$

- $\xi = 0$, i.e., $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$.
- $d \left\langle M^{(i)}, M^{(j)} \right\rangle_t = 0$, $i \neq j$.

Players

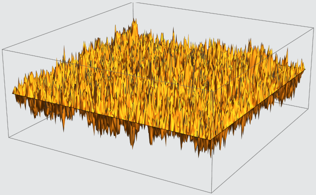
(Multiple) SLE



Coupling

Conformal map

GFF



Dyson BMs