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Homotopy perturbation method for solving boundary value problems

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Abstract

Homotopy perturbation method is applied to nonlinear boundary value problems. Comparison of the result obtained by the present method with that obtained by Adomian method [A.M. Wazwaz, Found. Phys. Lett. 13 (2000) 493] reveals that the present method is very effective and convenient.

Recently Adomian method [1–5] has been widely applied to nonlinear problems. Liu [6] and He [7] found that Adomian method could not always satisfy all its boundary conditions, leading to an error at its boundaries. Wazwaz [8] overcame the difficulty arising in boundary value problems, but it is still a very intricate problem to calculate the so-called Adomian polynomials, if it is not a difficult problem. Abdou, Soliman [9] and Momani, Abuasad [10] found that the variational iteration method [11] can completely overcome the difficulty arising in calculating Adomian polynomials. In this Letter, we will use the homotopy perturbation method [12–15], which is proved to be very effective, simple, and convenient to solve nonlinear boundary value problems.

Consider a special nonlinear PDE [8]

$$\nabla^2 u + \left(\frac{\partial u}{\partial y}\right)^2 = 2y + x^4, \tag{1}$$

subject to boundary conditions

$$\begin{aligned} &u(0,y) = 0, \qquad u(1,y) = y + a, \\ &u(x,0) = ax, \qquad u(x,1) = x(x+a), \end{aligned} \tag{2}$$

where a is a constant.

Liu [6] obtained the following approximate solution by Adomian method

$$u(x,y) = x(xy+a) + \frac{1}{2}\left[y(y-1)\left(\frac{x^4}{2} + \frac{y+1}{3}\right) + \frac{x}{30}\left(x^5 - 1\right)\right]. \tag{3}$$

It is obvious that the obtained solution does not satisfy the given boundary condition. Liu [6] suggested a weighted residual method and obtained the following approximation

$$u(x,y) = x(xy+a) + \frac{1}{4}xy(y-1)\left(1 - x^3\right) \tag{4}$$

which satisfies all boundary conditions. The accuracy of Liu's method, however, depends upon the choice of weighted factors.

Hereby we suggest a simple but powerful homotopy perturbation method [12–15] to the discussed problem. According to the homotopy perturbation, we construct the following simple homotopy:

$$\nabla^2 u - 2y = p\left[x^4 - \left(\frac{\partial u}{\partial y}\right)^2\right]. \tag{5}$$

The homotopy parameter p always changes from zero to unity. In case $p = 0$, Eq. (5) becomes a linear equation, $\nabla^2 u - 2y = 0$, which is easy to be solved; and when it is one, Eq. (5) turns out to be the original one, Eq. (1).

In view of homotopy perturbation method, we use the homotopy parameter p to expand the solution

$$u = u_0 + pu_1 + p^2 u_2 + \cdots. \tag{6}$$

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The approximate solution can be obtained by setting $p = 1$:

$$u = u_0 + u_1 + u_2 + \cdots. \tag{7}$$

The convergence of the method has been proved in [1,14]. Substituting Eq. (6) into Eq. (5), and equating the terms with the identical powers of p, we can obtain a series of linear equations, and we write only the first two linear equations:

$$\nabla^2 u_0 - 2y = 0, \tag{8}$$

$$\nabla^2 u_1 = x^4 - \left(\frac{\partial u_0}{\partial y}\right)^2. \tag{9}$$

The special solution of Eq. (7) can be easily obtained, which reads

$$u_0 = x^2 y. \tag{10}$$

Substituting u_0 into Eq. (9), we obtain a differential equation for u_1,

$$\nabla^2 u_1 = 0. \tag{11}$$

If the first-order approximate solution is sought, $u = u_0 + u_1$, then the boundary conditions for u_1 are

$$\begin{aligned}
&u_1(0, y) = u(0, y) - u_0(0, y) = 0,\\
&u_1(1, y) = u(1, y) - u_0(1, y) = y + a - y = a,\\
&u_1(x, 0) = u(x, 0) - u_0(x, 0) = ax,\\
&u_1(x, 1) = u(x, 1) - u_0(x, 1) = x(x + a) - x^2 = ax.
\end{aligned} \tag{12}$$

Considering the boundary conditions (12), we can solve Eq. (11) easily. The solution reads

$$u_1 = ax. \tag{13}$$

So the first-order approximate solution is

$$u = u_0 + u_1 = x^2 y + ax, \tag{14}$$

which happens to be the exact solution.

Wazwaz [8] applied Adomian method and obtained an infinite series which converges to the exact solution.

Compared with Adomian method [8], the present method has some obvious merits: (1) the method needs not to calculate Adomian polynomials; (2) the method is very straightforward, and the solution procedure can be done by pencil-and-paper only.

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