1. Was ist ein normierter Raum? Wann sagt man, dass ein normierter Raum Banach ist?

   Answer: Let V be a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\|.\| : V \to \mathbb{R}$. The pair $(V, \|.\|)$ called a normed vectorspace if $\|.\|$ satisfies the following properties:

   (a) $\forall v \in V\colon \|v\| \geq 0$ and $\|v\| = 0 \Leftrightarrow v = 0_V$
   (b) $\forall v \in V, \lambda \in \mathbb{K}\colon \|\lambda v\| = |\lambda| \, \|v\|$
   (c) $\forall u, v \in V\colon \|u + v\| \leq \|u\| + \|v\|$

   We call furthermore $(V, \|.\|)$ normed space a Banach space, if it's complete, that is: every Cauchy sequence has a limit in V.

2. Wann sagt man, dass eine Funktion zwischen zwei normierten Räumen stetig ist?

   Answer: Condider $(U, \|.\|_U)$ and $(V, \|.\|_V)$ normed spaces, $A \subset U$ and a function $f\colon A \to V$. f is continuous in $a \in A$, if $\forall \varepsilon > 0\colon \exists \delta > 0\colon \forall x \in A\colon \|x - a\|_U < \delta \Rightarrow \|f(x) - f(a)\|_V < \varepsilon$. f is continuous on A, if it's continuous in every point of A.

3. Seien X und Z normierte Räume. Was ist die Operatornorm $\|L\|$ einer linearen Abbildung $L\colon X \to Z$? Was kann man über $\|L\|$ sagen, wenn L stetig ist?

   Answer: $\|L\| = \sup_{\|x\|_X = 1} \|Lx\|_Z = \sup_{0 \neq x \in X} \frac{\|Lx\|_Z}{\|x\|_X}$. Linear operators are continuous if and only if they are bounded.

4. Was ist eine Regelfunktion? Welche äquivalenten Charakteriesierungen gibt es (wenigstens 2)?

   Answer: Let V be a Banach space (including $\mathbb{C}$ or $\mathbb{R}$). The set $\mathcal{R}([a, b], V)$ of regulated functions is the closure of the set $\mathcal{T}([a, b], V)$ of step functions with regards to the set $B([a, b], V)$ of bounded functions under the $\|.\|_{\text{sup}}$ norm. Equivalently:

   (a) $f \in B([a, b], V)$ is a regulated function $\Leftrightarrow \forall c \in [a, b]\colon \exists \lim_{x \to c^+} f(x), \lim_{x \to c^-} f(x)$
   (b) $f \in B([a, b], V)$ is a regulated function $\Leftrightarrow \exists (f_n) \in \mathcal{T}([a, b], V)$ uniformly convergent sequence of stepfunctions such that $\lim_{n \to \infty} f_n = f$ (in $B([a, b], V)$)

5. Gib je zwei Beispiele an für

   (a) Regelfunktionen und
   (b) Funktionen, die keine Regelfunktionen sind.

   Answer:

   (a) Regelfunktionen:
      i. $f : [0, 1] \to \mathbb{R}, x \mapsto 1$
      ii. $f : [-1, 1] \to \mathbb{R}, x \mapsto \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$

   (b) Keine Regelfunktionen:
      i. $\mathbb{1}_{\mathbb{Q}}\colon [0, 1] \to \mathbb{R}, x \mapsto \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$
      ii. $f\colon [0, 1] \to \mathbb{R}, x \mapsto \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

6. Wie ist das Integral einer Regelfunktion definiert?

   Answer: Let V be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Consider any $f \in \mathcal{R}([a,b],V)$ regulated function, and $f_n \in \mathcal{T}([a,b],V)$ sequence of step functions, that converge uniformly to f. Let furthermore $P_n = \{p_0, \ldots, p_{k_n}\}$ be such a partition of $[a,b]$ for which $f_n \big|_{[p_i,p_{i+1}]} = c_i \in V$ constant (with the potential exception of the endpoints). Let the $\int_a^b : \mathcal{T}([a,b],V) \to V$ linear operator be defined as $\int_a^b f_n = \sum_{i=0}^{k_n} c_i(p_{i+1} - p_i)$. Since $\mathcal{T}([a,b],V)$ is a subspace of $\mathcal{R}([a,b],V)$, there exists a unique continuous continuation $\overline{\int_a^b} \colon \mathcal{R}([a,b],V) \to V$ of the linear operator $\int_a^b$ such that their values stays the same on $\mathcal{T}([a,b],V)$. Since $\overline{\int_a^b}$ is continuous, it "commutes" with the limit. Let thus $\int_a^b f := \overline{\int_a^b} f = \overline{\int_a^b} \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_a^b f_n$

7. Wie hängen Integration und Differentiation zusammen?

   Answer: Let V be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $f \colon [a,b] \to V$ function. Let furthermore $F(x) = \int_a^x f$. If f is continuous in $c \in [a,b]$, then F is differentiable in c, and $F'(c) = f(c)$. Furthermore if f is continuous, and F is such a function, that $F' = f$, then $\int_a^b f = F(b) - F(a)$, and we call F the primitive function of f.

8. Welchen elementaren Funktionen entsprechen folgende unbestimmte Integrale?

   $$\int \sin t \, dt \qquad \int \frac{dt}{t} \qquad \int \sqrt[n]{t+1} \qquad \int \frac{1}{1+t^2}\, dt \qquad \int t^\alpha \, dt \ (\alpha \neq -1)$$

   Answer:

   - $\int \sin t \, dt = \cos t$
   - $\int \frac{dt}{t} = \log t$
   - $\int \sqrt[n]{t+1} = \frac{n}{n+1}(t+1)^{1+\frac{1}{n}}$
   - $\int \frac{1}{1+t^2}\, dt = \arctan t$
   - $\int t^\alpha \, dt = \frac{1}{\alpha+1} t^{\alpha+1}$ $(\alpha \neq 1)$

9. Wie lauten die Regeln für partielle Integration und Substitution? Gib außerdem jeweils ein nicht-triviales Beispiel an.

   Answer:

   Partial Integration: Consider $f, g \in C^1([a,b],\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $\int_a^b f'(t)g(t)\, dt = f(t)g(t) \big|_a^b - \int_a^b f(t)g'(t)\, dt$

   Example: $\int \log x \, dx = x \log x - \int x \frac{1}{x} \, dx = x \log x - x$

   Substitution: Consider $[a,b] \subset I_1, I_2$ intervals, Z Banach space, and $f : I_2 \to Z$ continuous, and $g \colon I_1 \to I_2$ continuously differentiable. Then $\int_{g(a)}^{g(b)} f(t)\, dt = \int_a^b f(g(t))g'(t)\, dt$

   Example: Consider $\int_a^b \tan x \, dx$ and let $f(x) = \frac{1}{x}, g(x) = \cos x$. Then $\int_a^b \tan x \, dx = \int_a^b \tan x \, dx = \int_a^b \frac{\sin x}{\cos x}\, dx = \int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{f(a)} f(x)\, dx = \int_{\cos(a)}^{\cos(b)} \frac{1}{x}\, dx = \log x \big|_{\cos(a)}^{\cos(b)} = \log(\cos(b)) - \log(\cos(a))$

10. Wie integriert man rationale Funktionen? Welche elementaren Integrale muss man dazu kennen?

    Answer: **TODO** Consider $p, q \in \mathbb{R}[x]$ polynomials and the rational function $\frac{p(x)}{q(x)}$

    (a) Simplification: $\exists! r, h \in \mathbb{R}[x] : p = hq + r$ with $\deg r < \deg p$. Thus $\int_a^b \frac{p}{q} = \int_a^b h + \int_a^b \frac{r}{q}$ where we already know how to integrate $\int_a^b h$

(b) Factorization: q can be written as a unique product of linear and quadratic polynomials: $q(t) = a\prod_j (t-a_j)^{m_j} \prod_k (t^2+b_kt+c_k)^{n_k}$ $(a \neq 0, a_j, b_k, c_k \in \mathbb{R}, m_j, n_k \in \mathbb{N})$

(c) Partial fractional decomposition: $\frac{r(t)}{q(t)}$ can always be written in the form of the following sum:

$$\sum_j \sum_{m'=1}^{m_j} \frac{\alpha_{j,m'}}{(t-a_j)^{m'}} + \sum_k \sum_{n'=1}^{n_k} \frac{t\beta_{k,n'} + \gamma_{k,n'}}{(t^2+b_kt+c_k)^{n'}}$$

which we can integrate per term

(d) Integral of the previous terms:

- $\int \frac{\alpha}{t-a}\, dt = \alpha \log(t-a)$
- $m \geq 2 : \int \frac{\alpha}{(t-a)^m}\, dt = -\frac{\alpha}{m-1}\frac{1}{(t-a)^{m-1}}$
- Complete the square in the denominator:

$$\int \frac{t\beta+\gamma}{(t^2+bt+c)^n}\, dt = \int \frac{t\beta+\gamma}{((t+\frac{1}{2}b)^2 + c - \frac{b^2}{4})^n}\, dt \overset{k=c-\frac{b^2}{4}}{=} \int \frac{1}{k^n}\frac{t\beta+\gamma}{((\frac{1}{\sqrt{k}}(t+\frac{1}{2}b))^2+1)^n}\, dt$$

By substituting $u = \frac{1}{\sqrt{k}}(t+\frac{1}{2}b)$ in the integral we'll get the linear combination of the following integrals:

  - $\int \frac{1}{u^2+1}\, du = \arctan u$
  - $\int \frac{1}{(u^2+1)^n}\, du = \int \frac{1}{(u^2+1)^{n-1}}\, du - \int \frac{u^2}{(u^2+1)^n}\, du = \int \frac{1}{(u^2+1)^{n-1}}\, du - (-\frac{1}{2n-2}\frac{u}{(u^2+1)^{n-1}} + \frac{1}{2n-2}\int \frac{1}{(u^2+1)^{n-1}}\, du) = \frac{1}{2n-2}\frac{u}{(u^2+1)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{1}{(u^2+1)^{n-1}}\, du$ (recursion for $n > 1$)
  - $\int \frac{u}{(u^2+1)^n}\, du \overset{w=u^2}{=} \int \frac{dw}{2(w+1)^n}$

Consider now a rational function of $\sin t$ and $\cos t$. With the substitution $\tau = \tan\frac{t}{2}$ we get $\sin t = \frac{2\tau}{1+\tau^2}, \cos t = \frac{1-\tau^2}{1+\tau^2}$ and $d\tau = \frac{1}{2}(1+\tau^2)\, dt$, that is: a rational function in τ, integrable as seen above.

11. Was sagt das Riemann-Lebesgue Lemma? Skizziere einen Beweis für das Lemma.

Answer: Let $f\colon [a,b] \to \mathbb{R}$ continuously differentiable. Then $\lim_{|\omega|\to\infty} \int_a^b f(t)\sin(\omega t)\, dt = 0$. *Proof:*

$$\begin{aligned}\left|\int_a^b f(t)\sin(\omega t)\, dt\right| &= \left|-\frac{\cos(\omega t)}{\omega} f(t)\,\Big|_a^b + \frac{1}{\omega}\int_a^b \cos(\omega t) f'(t)\, dt\right| \\ &< \left|\frac{1}{\omega}\right| \left(|\cos(\omega b) f(b)| + |\cos(\omega a) f(a)| + \int_a^b \left|\cos(\omega t) f'(t)\right|\, dt\right) \\ &\overset{\cos\text{ bounded}}{<} \left|\frac{1}{\omega}\right| \left(|\cos(\omega b) f(b)| + |\cos(\omega a) f(a)| + \int_a^b \left|f'(t)\right|\, dt\right) \overset{|\omega|\to\infty}{\to} 0\end{aligned}$$

12. Wann dürfen Regelintegral und Grenzwert einer Funktionenfolge vertauscht werden?

Answer: Consider $f_n \in \mathcal{R}([a,b],V)$ with V Banach space. If f_n converge uniformly to some $f \in \mathcal{R}([a,b],V)$, then $\int_a^b f = \lim_{n\to\infty}\int_a^b f_n$

13. Wie ist das Riemann-Integral definiert?

Answer: Consider the interval $[a,b]$, a function $f\colon [a,b] \to \mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and for any partition $P = \{a = p_0 < p_1 < \cdots < p_n = b\}$ of $[a,b]$ let $L(f,P) = \sum_{i=1}^n \inf_{x\in(p_{i-1},p_i)} f(x)(p_i - p_{i-1})$

and $U(f,P) = \sum_{i=1}^{n} \sup_{x\in(p_{i-1},p_i)} f(x)(p_i - p_{i-1})$. Let furthermore $U^*(f) = \inf_P U(f,P)$ and $L^*(f) = \sup_P L(f,P)$. If $U^*(f) = L^*(f)$, then we say that f is Riemann integrable, and $\int_a^b f = U^*(f) = L^*(f)$.

14. Was ist eine Lebesgue-Nullmenge?

    Answer: $A \subset \mathbb{R}$ is a null set, if $\forall\epsilon > 0$ there is an at most countable collection of open intervals $I = \{I_i\}_{i\in\mathbb{N}}$ such that $A \in \cup_{I_i\in I} I_i$ and $\sum_{I_i\in I} |I_i| < \epsilon$

15. Wie ist das Lebesgue-Integral definiert?

    Answer: Consider

    $$H^u = \left\{\varphi \;\middle|\; \exists (f_n) \in \mathcal{T}([a,b], \mathbb{R}\cup\{\pm\infty\})\colon f_n(x) \le f_{n+1}(x)\ (\forall n \in \mathbb{N}) \wedge \lim_{n\to\infty} f_n(x) = \varphi(x)\ \text{(a.e)}\right\}$$

    and

    $$H^o = \left\{\varphi \;\middle|\; \exists (f_n) \in \mathcal{T}([a,b], \mathbb{R}\cup\{\pm\infty\})\colon f_n(x) \ge f_{n+1}(x)\ (\forall n \in \mathbb{N}) \wedge \lim_{n\to\infty} f_n(x) = \varphi(x)\ \text{(a.e)}\right\}$$

    $\forall\varphi \in H^u$ consider the $(f_n) \in \mathcal{T}([a,b], \mathbb{R}\cup\{\pm\infty\})$ sequence that converges from below to φ almost everywhere. Let $\int_a^b \varphi := \lim_{n\to\infty} \int_a^b f_n \in \mathbb{R}\cup\{\pm\infty\}$ the so called H^u integral of φ. The definition of H^o integral of φ is analogue.

    Define furthermore the upper Lebesgue-integral as $L^*\!\!\int_a^b f := \inf\left\{\int_a^b \varphi \;\middle|\; \varphi \in H^u\colon \varphi \ge f\right\}$ and the lower Lebesgue-integral as $L_*\!\!\int_a^b f := \inf\left\{\int_a^b \varphi \;\middle|\; \varphi \in H^u\colon \varphi \ge f\right\}$

    When $L^*\!\!\int_a^b f = L_*\!\!\int_a^b f$ and they are finite, then f is Lebesgue-integrable and $\int_a^b f = L^*\!\!\int_a^b f = L_*\!\!\int_a^b f \in \mathbb{R}$. Let $\mathrm{Leb}_{[a,b]} := \left\{f\colon [a,b] \to \mathbb{R} \;\middle|\; f \text{ Lebesgue-integrable}\right\}$

16. Gib je ein Beispiel für eine Funktion an, die

    (a) Riemann integrierbar ist, aber ist keine Regelfunktion.

    (b) Lebesgue integrierbar ist, aber nicht Riemann integrierbar.

    Answer:

    (a) $f\colon [0,1] \to \mathbb{R}, x \mapsto \begin{cases} \sin\frac{1}{x} & x \in (0,1) \\ 0 & x = 0 \end{cases}$

    (b) The Dirichlet function

17. Was sagt der Satz von Beppo Levi über monotone Folgen Lebesgue-integrierbarer Funktionen?

    Answer: Suppose that $(f_n) \in \mathrm{Leb}_{[a,b]}$, $f_n(x)$ non-decreasing and converges to $f(x)$ a.e. and the $\left\{\int_a^b f_n\ (n \in \mathbb{N})\right\}$ set is bounded. Then $f \in \mathrm{Leb}_{[a,b]}$ and $\int_a^b \lim_{n\to\infty} f_n = \int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

    The analogue holds for a sequence $(f_n) \in \mathrm{Leb}_{[a,b]}$ that converges pointwise non-increasingly to f a.e.

18. Was sagt der Satz zur majorisierten Konvergenz über die Vertauschbarkeit von Lebesgue-Integral und Grenzwert einer Funktionenfolge?

    Answer: Let $g \in \mathrm{Leb}$ and $(f_n) : \mathbb{N} \to \mathrm{Leb}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all x, and $|f_n| \le g$ $(\forall n \in \mathbb{N})$ almost everywhere. Then $\lim_{n\to\infty} \int_a^b f_n$ exists, $f \in \mathrm{Leb}$ and

    $$\int_a^b \lim_{n\to\infty} f_n = \int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

19. Wie lautet die Hölder-Ungleichung für integrierbare Funktionen?

    Answer: Let $1 < p, q < \infty$ with $\frac{1}{q} + \frac{1}{p} = 1$ and $f \in \mathcal{L}^p, g \in \mathcal{L}^q$. Then $\int_a^b fg \leq \|f\|_p \|g\|_q$

20. Wie lautet der Schrankensatz? Warum gilt der Mittelwertsatz (der Differentialrechnung) nicht in höheren Dimensionen?

    Answer:

    Let Z Banach space and $f \in C^1([a,b], Z)$. Then $\exists \zeta \in (a,b)\colon \|f(b) - f(a)\| \leq \|f'(\zeta)\| (b-a)$. The mean value theorem of $f(b) - f(a) = f'(\zeta)(b-a)$ does not hold in general. Consider $Z = \mathbb{R}^3$ and $f(t) := \begin{pmatrix} \cos t \\ \sin t \\ \varepsilon t \end{pmatrix}$ with some $\varepsilon > 0$. Then $f'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ \varepsilon \end{pmatrix}$. Now if we consider the $[0, 2\pi]$ interval, then $f(2\pi k) - f(0) = 2\pi \begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix} \neq 2\pi f'(\zeta)$ $(\forall \zeta \in [0, 2\pi])$

21. Wie lautet die Integraldarstellung von Lagrange für das Restglied der Taylorentwicklung?

    Answer: Let $I = (a,b) \subset \mathbb{R}$ open interval, $n \in \mathbb{N}_0, Z$ Banach space and $f \in C^{n+1}(I, Z)$. If $x_0 \in (a,b)$, then $\forall x \in (a,b)\colon f(x) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k + \frac{1}{n!} \int_{x_0}^x (x-t)^n f^{(n+1)}(t)\, dt$

22. Wie lautet die Trapezregel?

    Answer: Consider $f \in C^2([0,1], \mathbb{R})$. Then $\exists \zeta \in (0,1)\colon \int_0^1 f = \frac{1}{2}(f(0) + f(1)) - \frac{1}{12} f''(\zeta)$

23. Wie lassen sich Integrale dank der Trapezregel approximieren?

    Answer: Let $f \in C^2([a,b], \mathbb{R}), C := \max_{x \in [a,b]} |f''|, n \in \mathbb{N}, h := \frac{b-a}{n}$. Then it holds that

$$\left| \int_a^b f - \left(\frac{1}{2} f(a_0) + \sum_{k=1}^{n-1} f(a_k) + \frac{1}{2} f(a_n) \right) h \right| \leq \frac{1}{12} C(b-a)h^2$$

24. Was sind uneigentliche Integrale?

    Answer: Let $-\infty \leq a < b \leq +\infty$ and $f\colon (a,b) \to \mathbb{R}$ integrable over every finite $[\alpha, \beta] \subset (a,b)$ interval. Consider furthermore $(a_n), (b_n) \in \mathbb{R}$ respectively monotone decreasing and increasing sequences such that $a < a_n$ and $b_n < b$ $(\forall n \in \mathbb{N})$ and suppose furthermore that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. If for every such sequence the limit $\lim_{n\to\infty} \int_{a_n}^{b_n} f$ exists and has the same value, then let $\int_a^b f = \lim_{n\to\infty} \int_{a_n}^{b_n} f$ and call it the improper integral of f over (a,b).

25. Für welche reellen Exponenten α konvergiert das uneigentliche Integral $\int_0^1 t^\alpha\, dt$, für welche das uneigentliche Integral $\int_1^\infty t^\alpha\, dt$?

    Answer:

    - $\int_0^1 t^\alpha\, dt = \frac{1}{\alpha+1}$ $(-1 < \alpha \in \mathbb{R})$
    - $\int_1^\infty t^\alpha\, dt = -\frac{1}{\alpha+1}$ $(-1 > \alpha \in \mathbb{R})$

26. Was bedeutet absolute Konvergenz uneigentlicher Integrale? Gib Beispiele

    (a) absolut konvergenter;

    (b) konvergenter, aber nicht absolut konvergenter;

    (c) nicht konvergenter

uneigentlicher Integrale.

Answer: Consider the improper integral $\int_a^b f$. If $\int_a^b |f|$ converges, then so is $\int_a^b f$, and we call the improper integral absolulte convergent.

(a) absolute convergent: $\int_0^1 \sin \frac{1}{t}\, dt$

(b) convergent, but not absolutely: $\int_0^{+\infty} \frac{1}{t} \sin t\, dt$

(c) not convergent: $\int_1^{+\infty} \frac{1}{t}\, dt$

27. Sei $f : [0, \infty) \to (0, \infty)$ monoton fallend. Wie hängen $\sum_{k=1}^{\infty} f(k)$ und $\int_1^{\infty} f(t)\, dt$ zusammen?

Answer: $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(t)\, dt$ converges. Furthermore

$$\sum_{k=2}^{\infty} f(k) \leq \int_1^{\infty} f(t)\, dt \leq \sum_{k=1}^{\infty} f(k)$$

28. Welche der folgenden uneigentlichen Riemann-Integrale existieren? Welche konvergieren absolut?

(a) $\int_1^{\infty} \cos t\, dt$

(b) $\int_1^{\infty} \cos(t^2)\, dt$

(c) $\int_1^{\infty} \frac{\cos t}{t}\, dt$

(d) $\int_1^{\infty} \frac{\cos(t^2)}{t}\, dt$

(e) $\int_1^{\infty} \frac{\cos t}{t^2}\, dt$

(f) $\int_1^{\infty} \frac{\cos(t^2)}{t^2}\, dt$

Answer: *Proposition*: $\int_1^{\infty} \frac{\cos t}{t^\alpha}\, dt \begin{cases} \text{diverges} & \alpha = 0 \\ \text{converges, but not absolutely} & \alpha > 0 \\ \text{converges absolutely} & \alpha > 1 \end{cases}$

Proof:

- Consider first $\alpha = 0$. On one hand $\limsup_{x\to\infty} \int_1^x \frac{\cos t}{t^0}\, dt = \limsup_{x\to\infty} \int_1^x \cos t\, dt = \int_1^{\pi/2} \cos t\, dt + \limsup_{x\to\infty} \int_{\pi/2}^x \cos t\, dt = \int_1^{\pi/2} \cos t\, dt + 1$. On the other hand $\liminf_{x\to\infty} \int_1^x \frac{\cos t}{t^0}\, dt = \int_1^{\pi/2} \cos t\, dt + \liminf_{x\to\infty} \int_{\pi/2}^x \cos t\, dt = \int_1^{\pi/2} \cos t\, dt - 1$, thus $\int_1^{\infty} \cos t\, dt$ cannot exist.
- Let now $\alpha > 1$: $\int_1^{\infty} \left|\frac{\cos t}{t^\alpha}\right| dt \leq \int_1^{\infty} \left|\frac{1}{t^\alpha}\right| dt = \int_1^{\infty} \frac{1}{t^\alpha}\, dt = \frac{1}{\alpha - 1}$, thus absolute convergent.
- Let now $\alpha \in (0, 1)$: $\int_1^{n\pi} \left|\frac{\cos t}{t^\alpha}\right| dt = \int_1^{\pi} \left|\frac{\cos t}{t^\alpha}\right| dt + \int_\pi^{n\pi} \left|\frac{\cos t}{t^\alpha}\right| dt \geq \sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi} \left|\frac{\cos t}{t^\alpha}\right| dt \geq \sum_{k=1}^{n-1} \frac{1}{((k+1)\pi)^\alpha} \int_{k\pi}^{(k+1)\pi} |\cos t|\, dt = \sum_{k=1}^{n-1} \frac{2}{((k+1)\pi)^\alpha} > \frac{2}{\pi^\alpha} \sum_{k=1}^{n-1} \frac{1}{(k+1)\alpha} > \frac{2}{\pi^\alpha} \sum_{k=1}^{n-1} \frac{1}{(k+1)} \overset{n\to\infty}{\to} \infty$, thus it's **not** absolutely convergent. On the other hand $\int_1^x \frac{\cos t}{t^\alpha}\, dt = -\frac{\sin t}{t^\alpha}\Big|_1^x + \int_1^x \frac{\sin t}{t^{\alpha+1}}\, dt$ converges, since $\lim_{x\to\infty} \frac{\sin t}{t^\alpha}\Big|_1^x = \sin 1$ and $\lim_{x\to\infty} \int_1^x \frac{\sin t}{t^{\alpha+1}}\, dt$ converges, since $\alpha + 1 > 1$.

Note: The same result holds for $\int_1^{\infty} \frac{\sin t}{t^\alpha}\, dt$ regarding the parameter α and the proof is identical.$\blacksquare$

Using the previous we get:

(a) Diverges

(b) $\lim_{x\to\infty} \int_1^x \cos(t^2)\, dt \overset{u=t^2}{=} \lim_{x\to\infty} \int_1^{x^2} \frac{1}{2} \frac{\cos u}{u^{1/2}}\, du$ converges, but not absolutely

(c) Converges, but not absolutely

(d) $\lim_{x\to\infty} \int_1^x \frac{\cos(t^2)}{t} \overset{u=t^2}{=} \lim_{x\to\infty} \frac{1}{2}\int_1^{x^2} \frac{\cos u}{u}\, du$ converges, but not absolutely

(e) Converges absolutely

(f) $\lim_{x\to\infty} \int_1^x \frac{\cos(t^2)}{t^2} \overset{u=t^2}{=} \lim_{x\to\infty} \frac{1}{2}\int_1^{x^2} \frac{\cos u}{u^{3/2}}\, du$ converges absolutely

29. Warum konvergiert die Reihe der Riemannsche ζ-Funktion $\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}$ für $\operatorname{Re} s > 1$?

    Answer: Let $q = \operatorname{Re} s$. Then $\sum_{k=1}^{\infty} \left|\frac{1}{k^s}\right| = \sum_{k=1}^{\infty} \frac{1}{k^q}$ converges iif $\int_1^{\infty} \frac{1}{t^q}\, dt$ converges. But this improper integral exists exactly when $1 < q \in \mathbb{R}$. Thus if $\operatorname{Re} s > 1$ then $\sum_{k=1}^{\infty} \frac{1}{k^s}$ is absolutely convergent, and thus also converges conditionally.

30. Wie ist die Gamma-Funktion definiert, und welche Funktionalgleichung erfüllt sie?

    Answer: Let $\Gamma(\alpha) := \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt$ $(\forall \alpha \in \mathbb{C}, \operatorname{Re} \alpha > 0)$. Then $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and since $\Gamma(1) = 1 = 0!$ it holds follows from a simple induction that $\Gamma(n+1) = n!$ $(\forall n \in \mathbb{N}_0)$.

31. Wie lautet die Stirling-Formel zur Approximation von $n!$?

    Answer: For $n \to \infty$ it holds, that $\sqrt{2\pi n}\frac{n^n}{e^n} \sim n!$

32. Wie ist die Faltung $\varphi \star f$ von zwei Funktionen $\varphi, f : \mathbb{R} \to \mathbb{R}$ definiert?

    Answer: $(\varphi \star f)(x) := \int_{-\infty}^{\infty} \varphi(x-t) f(t)\, dt$ $(\forall x \in \mathbb{R})$

33. Was ist eine Dirac-Folge? Gib eine Definition und wenigstens ein Beispiel an.

    Answer: Consider the $\varphi_n : \mathbb{R} \to [0, \infty)$ sequence of functions. It's called Dirac sequence, if it satisfies the following conditions:

    - φ_n is integrable $(\forall n \in \mathbb{N})$
    - $\int_{-\infty}^{\infty} \varphi_n = 1$
    - $\forall \varepsilon, \delta > 0\colon \exists N \in \mathbb{N}\colon \forall n > N\colon \int_{|x|\geq\delta} \varphi_n(t)\, dt \leq \epsilon$

    Examples:

    - **TODO**

34. Sei $\varphi_n : \mathbb{R} \to \mathbb{R}$ eine Dirac-Folge und sei $f : \mathbb{R} \to \mathbb{R}$ stetig mit $f(x) = 0$ für $|x| \geq 1$. Bestimme $\lim_{n\to\infty}(\varphi_n * f)$ für jedes $x \in \mathbb{R}$.

    Answer: f must be bounded, and let $B \in \mathbb{R}_+$ such that $|f| \leq B$. Consider $x \in [-1, 1]$ and $\forall \varepsilon > 0$ let $\delta > 0$ such that $\delta < B$ and $\forall t \in (x-\delta, x+\delta)\colon |f(t) - f(x)| < \varepsilon$. Let furthermore

$N \in \mathbb{N} \colon \forall n > N \colon \int_{|t| \geq \delta} \varphi_n \leq \varepsilon$. Now

$$|f(x) - (\varphi_n \star f)(x)| = \left| f(x) - \int_{-\infty}^{\infty} \varphi(x-t) f(t)\, dt \right| = \left| f(x) \underbrace{\int_{-\infty}^{\infty} \varphi(x-t)\, dt}_{=1} - \int_{-\infty}^{\infty} \varphi(x-t) f(t)\, dt \right|$$

$$= \left| \int_{-\infty}^{\infty} \varphi_n(x-t)(f(x) - f(t))\, dt \right|$$

$$\leq \left| \int_{-\infty}^{x-\delta} \varphi_n(x-t)(f(x) - f(t))\, dt \right| + \left| \int_{x-\delta}^{x+\delta} \varphi_n(x-t)(f(x) - f(t))\, dt \right| + \left| \int_{x+\delta}^{\infty} \varphi_n(x-t)(f(x) - f(t))\, dt \right|$$

$$\leq \int_{-\infty}^{x-\delta} |\varphi_n(x-t)(f(x) - f(t))|\, dt + \int_{x-\delta}^{x+\delta} |\varphi_n(x-t)(f(x) - f(t))|\, dt + \int_{x+\delta}^{\infty} |\varphi_n(x-t)(f(x) - f(t))|\, dt$$

$$\leq \int_{-\infty}^{x-\delta} \varphi_n(x-t) 2B\, dt + \int_{x-\delta}^{x+\delta} |\varphi_n(x-t)(f(x) - f(t))|\, dt + \int_{x+\delta}^{\infty} \varphi_n(x-t) 2B\, dt$$

$$\leq \frac{2B\varepsilon}{2} + \int_{x-\delta}^{x+\delta} \varphi_n(x-t)\varepsilon\, dt + \frac{2B\varepsilon}{2} < 2B\varepsilon + 2\delta\varepsilon < 4B\varepsilon$$

Thus $\lim_{n\to\infty} (\varphi_n \star f)(x) = f(x)$. Furthermore the approximation does not depend on the choice of x (since f is uniformly continuous), $\lim_{n\to\infty} \varphi_n \star f = f$ (in $\|.\|_{\sup}$).

35. Wie lautet den Approximationssatz von Weierstrass?

    Answer: Consider $(\mathcal{C}([a,b], \mathbb{R}), \|.\|)$, the normed vectorspace of continuous functions on the $[a,b]$ closed interval with the supremum norm. Then the $\mathcal{P} \subset \mathcal{C}([a,b], \mathbb{R})$ subspace of the polynomials is dense in $\mathcal{BC}([a,b], \mathbb{R})$ under the supremum norm.

36. Welche Funktionen können durch Polynome gleichmäßig approximiert werden? Mit welcher Grundidee lassen sich approximierende Polynome zu einer gegebenen Funktion f konstruieren?

    Answer: Every continuous function defined on a closed interval can be approximated uniformly with polynomials. Suppose without loss of generality, that we want to approximate the $f : [0,1] \to \mathbb{R}$ function, for which it holds that $f(0) = f(1) = 0$. Extend this function to the reals by letting $f(x) = 0$ $(\forall x \notin [0,1])$. Since the approximation in 34. does not depend on the choice of x (since a continuous function on a closed interval is also uniformly continuous), the convolution of f with a Dirac-sequence is a uniform approximation of f. If we chose the Landau-kernel as our Dirac-sequence, then in particular the resulting function $(\varphi_n \star f)(x)$ will be a polynomial in x, thus we found a uniformly approximating polynomial sequence.

    If the function f was originally defined on the $[a,b]$ interval, then let $\tilde{f} \colon [0,1] \to \mathbb{R}, \tilde{f}(x) = f(a + x(b-a))$. If furthermore $f(0) = f(1) = 0$ doesn't hold, then let $\tilde{f}(x) = f(x) - f(1) - (f(1) - f(0))x$.

37. Was ist ein Hilbertraum? Gib zwei unendlich-dimensionale Beispiele.

    Answer: The H vectorspace over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a Hilbert space, if H is an inner product space, and furthermore it's a complete metric space with regards to the metric induced by the scalar product: $d(x,y) := \|x - y\| := \sqrt{\langle x, y \rangle}$

    The $L^2 = \mathcal{L}^2/_\sim$ space with $\langle f, g \rangle = \int_a^b \overline{f} g$, where $\mathcal{L}^2 = \left\{ f \colon [a,b] \to \mathbb{K} \;\middle|\; |f|^2 \in \text{Leb} \right\}$ and $f \sim g \Leftrightarrow f = g$ almost everywhere.

    The $\ell^2 = \left\{ c \colon \mathbb{Z} \to \mathbb{K} \mid \|c\| < \infty \right\}$ space with the norm induced through the $\langle c, c' \rangle = \sum_{k=-\infty}^{\infty} \overline{c_k} c'_k$ scalar product.

38. Was sind die Fourierkoeffizienten einer 2π-periodischen Funktion $f : \mathbb{R} \to \mathbb{C}$? Wie lautet die Fourier-Reihe zu f ?

    Answer: $c_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t)\, dt$ and $\tilde{f}(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$

39. Beschreibe wie 2π-periodische $f \in L^2$, die Fourier-Reihe $\hat{f}(t) := \sum_{k\in\mathbb{Z}} c_k e^{ikt}$, und Fourier-Koeffizienten $(c_k)_{k\in\mathbb{Z}} \in \ell^2$ miteinander zusammenhängen.

    Answer: Consider $\varphi : \ell^2 \to L^2$ function with $\varphi\left((c_k)_{k\in\mathbb{Z}}\right) = \sum_{k\in\mathbb{Z}} c_k e^{ikt}$. ℓ^2 and L^2 are isometrically isomorphic through the φ bijection and furthermore $\lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ikt} = f$ in the $\|.\|_{L^2}$ norm.

40. Wie hängen Fourier-Reihen mit Obertönen in der Musik zusammen?

    Answer: TODO