

1. Was ist ein normierter Raum? Wann sagt man, dass ein normierter Raum Banach ist?

Answer: Let V be a vector space over \mathbb{K} ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}) and $\|\cdot\| : V \rightarrow \mathbb{R}$. The pair $(V, \|\cdot\|)$ called a normed vectorspace if $\|\cdot\|$ satisfies the following properties:

- (a) $\forall v \in V: \|v\| \geq 0$ and $\|v\| = 0 \Leftrightarrow v = 0_V$
- (b) $\forall v \in V, \lambda \in \mathbb{K}: \|\lambda v\| = |\lambda| \|v\|$
- (c) $\forall u, v \in V: \|u + v\| \leq \|u\| + \|v\|$

We call furthermore $(V, \|\cdot\|)$ normed space a Banach space, if it's complete, that is: every Cauchy sequence has a limit in V .

2. Wann sagt man, dass eine Funktion zwischen zwei normierten Räumen stetig ist?

Answer: Consider $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ normed spaces, $A \subset U$ and a function $f: A \rightarrow V$. f is continuous in $a \in A$, if $\forall \varepsilon > 0: \exists \delta > 0: \forall x \in A: \|x - a\|_U < \delta \Rightarrow \|f(x) - f(a)\|_V < \varepsilon$. f is continuous on A , if it's continuous in every point of A .

3. Seien X und Z normierte Räume. Was ist die Operatornorm $\|L\|$ einer linearen Abbildung $L: X \rightarrow Z$? Was kann man über $\|L\|$ sagen, wenn L stetig ist?

Answer: $\|L\| = \sup_{\|x\|_X=1} \|Lx\|_Z = \sup_{0 \neq x \in X} \frac{\|Lx\|_Z}{\|x\|_X}$. Linear operators are continuous if and only if they are bounded.

4. Was ist eine Regelfunktion? Welche äquivalenten Charakterisierungen gibt es (wenigstens 2)?

Answer: Let V be a Banach space (including \mathbb{C} or \mathbb{R}). The set $\mathcal{R}([a, b], V)$ of regulated functions is the closure of the set $\mathcal{T}([a, b], V)$ of step functions with regards to the set $B([a, b], V)$ of bounded functions under the $\|\cdot\|_{\sup}$ norm. Equivalently:

- (a) $f \in B([a, b], V)$ is a regulated function $\Leftrightarrow \forall c \in [a, b]: \exists \lim_{x \rightarrow c^+} f(x), \lim_{x \rightarrow c^-} f(x)$
- (b) $f \in B([a, b], V)$ is a regulated function $\Leftrightarrow \exists (f_n) \in \mathcal{T}([a, b], V)$ uniformly convergent sequence of stepfunctions such that $\lim_{n \rightarrow \infty} f_n = f$ (in $B([a, b], V)$)

5. Gib je zwei Beispiele an für

- (a) Regelfunktionen und
- (b) Funktionen, die keine Regelfunktionen sind.

Answer:

- (a) Regelfunktionen:

i. $f: [0, 1] \rightarrow \mathbb{R}, x \mapsto 1$

ii. $f: [-1, 1] \rightarrow \mathbb{R}, x \mapsto \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$

- (b) Keine Regelfunktionen:

i. $\mathbb{1}_{\mathbb{Q}}: [0, 1] \rightarrow \mathbb{R}, x \mapsto \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

ii. $f: [0, 1] \rightarrow \mathbb{R}, x \mapsto \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

6. Wie ist das Integral einer Regelfunktion definiert?

Answer: Let V be a Banach space over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Consider any $f \in \mathcal{R}([a, b], V)$ regulated function, and $f_n \in \mathcal{T}([a, b], V)$ sequence of step functions, that converge uniformly to f . Let furthermore $P_n = \{p_0, \dots, p_{k_n}\}$ be such a partition of $[a, b]$ for which $f_n|_{[p_i, p_{i+1}]} = c_i \in V$ constant (with the potential exception of the endpoints). Let the $\int_a^b : \mathcal{T}([a, b], V) \rightarrow V$ linear operator be defined as $\int_a^b f_n = \sum_{i=0}^{k_n} c_i(p_{i+1} - p_i)$. Since $\mathcal{T}([a, b], V)$ is a subspace of $\mathcal{R}([a, b], V)$, there exists a unique continuous continuation $\overline{\int_a^b} : \mathcal{R}([a, b], V) \rightarrow V$ of the linear operator \int_a^b such that their values stays the same on $\mathcal{T}([a, b], V)$. Since $\overline{\int_a^b}$ is continuous, it "commutes" with the limit. Let thus $\int_a^b f := \overline{\int_a^b} f = \overline{\int_a^b} \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int_a^b f_n$

7. Wie hängen Integration und Differentiation zusammen?

Answer: Let V be a Banach space over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , and $f : [a, b] \rightarrow V$ function. Let furthermore $F(x) = \int_a^x f$. If f is continuous in $c \in [a, b]$, then F is differentiable in c , and $F'(c) = f(c)$. Furthermore if f is continuous, and F is such a function, that $F' = f$, then $\int_a^b f = F(b) - F(a)$, and we call F the primitive function of f .

8. Welchen elementaren Funktionen entsprechen folgende unbestimmte Integrale?

$$\int \sin t \, dt \quad \int \frac{dt}{t} \quad \int \sqrt[n]{t+1} \quad \int \frac{1}{1+t^2} \, dt \quad \int t^\alpha \, dt \quad (\alpha \neq -1)$$

Answer:

- $\int \sin t \, dt = -\cos t$
- $\int \frac{dt}{t} = \log t$
- $\int \sqrt[n]{t+1} = \frac{n}{n+1} (t+1)^{1+\frac{1}{n}}$
- $\int \frac{1}{1+t^2} \, dt = \arctan t$
- $\int t^\alpha \, dt = \frac{1}{\alpha+1} t^{\alpha+1} \quad (\alpha \neq -1)$

9. Wie lauten die Regeln für partielle Integration und Substitution? Gib außerdem jeweils ein nicht-triviales Beispiel an.

Answer:

Partial Integration: Consider $f, g \in C^1([a, b], \mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Then $\int_a^b f'(t)g(t) \, dt = f(t)g(t) \Big|_a^b - \int_a^b f(t)g'(t) \, dt$

Example: $\int \log x \, dx = x \log x - \int x \frac{1}{x} \, dx = x \log x - x$

Substitution: Consider $[a, b] \subset I_1, I_2$ intervals, Z Banach space, and $f : I_2 \rightarrow Z$ continuous, and $g : I_1 \rightarrow I_2$ continuously differentiable. Then $\int_{g(a)}^{g(b)} f(t) \, dt = \int_a^b f(g(t))g'(t) \, dt$

Example: Consider $\int_a^b \tan x \, dx$ and let $f(x) = \frac{1}{x}, g(x) = \cos x$. Then $\int_a^b \tan x \, dx = \int_a^b \frac{\sin x}{\cos x} \, dx = \int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(x) \, dx = \int_{\cos(a)}^{\cos(b)} \frac{1}{x} \, dx = \log x \Big|_{\cos(a)}^{\cos(b)} = \log(\cos(b)) - \log(\cos(a))$

10. Wie integriert man rationale Funktionen? Welche elementaren Integrale muss man dazu kennen?

Answer: Consider $p, q \in \mathbb{R}[x]$ polynomials and the rational function $\frac{p(x)}{q(x)}$

- (a) Simplification: $\exists! r, h \in \mathbb{R}[x] : p = hq + r$ with $\deg r < \deg q$. Thus $\int_a^b \frac{p}{q} = \int_a^b h + \int_a^b \frac{r}{q}$ where we already know how to integrate $\int_a^b h$

- (b) Factorization: q can be written as a unique product of linear and quadratic polynomials:
 $q(t) = a \prod_j (t - a_j)^{m_j} \prod_k (t^2 + b_k t + c_k)^{n_k}$ ($a \neq 0, a_j, b_k, c_k \in \mathbb{R}, m_j, n_k \in \mathbb{N}$)
- (c) Partial fractional decomposition: $\frac{r(t)}{q(t)}$ can always be written in the form of the following sum:

$$\sum_j \sum_{m'=1}^{m_j} \frac{\alpha_{j,m'}}{(t - a_j)^{m'}} + \sum_k \sum_{n'=1}^{n_k} \frac{t\beta_{k,n'} + \gamma_{k,n'}}{(t^2 + b_k t + c_k)^{n'}}$$

which we can integrate per term

- (d) Integral of the previous terms:

- $\int \frac{\alpha}{t-a} dt = \alpha \log(t-a)$
- $m \geq 2$: $\int \frac{\alpha}{(t-a)^m} dt = -\frac{\alpha}{m-1} \frac{1}{(t-a)^{m-1}}$
- Complete the square in the denominator:

$$\int \frac{t\beta + \gamma}{(t^2 + bt + c)^n} dt = \int \frac{t\beta + \gamma}{((t + \frac{1}{2}b)^2 + c - \frac{b^2}{4})^n} dt \stackrel{k=c-\frac{b^2}{4}}{=} \int \frac{1}{k^n} \frac{t\beta + \gamma}{((\frac{1}{\sqrt{k}}(t + \frac{1}{2}b))^2 + 1)^n} dt$$

By substituting $u = \frac{1}{\sqrt{k}}(t + \frac{1}{2}b)$ in the integral we'll get the linear combination of the following integrals:

$$\begin{aligned} & - \int \frac{1}{u^2+1} du = \arctan u \\ & - \int \frac{1}{(u^2+1)^n} du = \int \frac{1}{(u^2+1)^{n-1}} du - \int \frac{u^2}{(u^2+1)^n} du = \int \frac{1}{(u^2+1)^{n-1}} du - \left(-\frac{1}{2n-2} \frac{u}{(u^2+1)^{n-1}} + \frac{1}{2n-2} \int \frac{1}{(u^2+1)^{n-1}} du\right) \text{ (recursion for } n > 1) \\ & - \int \frac{u}{(u^2+1)^n} du \stackrel{w=u^2}{=} \int \frac{dw}{2(w+1)^n} \end{aligned}$$

Consider now a rational function of $\sin t$ and $\cos t$. With the substitution $\tau = \tan \frac{t}{2}$ we get $\sin t = \frac{2\tau}{1+\tau^2}$, $\cos t = \frac{1-\tau^2}{1+\tau^2}$ and $d\tau = \frac{1}{2}(1+\tau^2) dt$, that is: a rational function in τ , integrable as seen above.

11. Was sagt das Riemann-Lebesgue Lemma? Skizziere einen Beweis für das Lemma.

Answer: Let $f: [a, b] \rightarrow \mathbb{R}$ continuously differentiable. Then $\lim_{|\omega| \rightarrow \infty} \int_a^b f(t) \sin(\omega t) dt = 0$.
Proof:

$$\begin{aligned} \left| \int_a^b f(t) \sin(\omega t) dt \right| &= \left| -\frac{\cos(\omega t)}{\omega} f(t) \Big|_a^b + \frac{1}{\omega} \int_a^b \cos(\omega t) f'(t) dt \right| \\ &< \left| \frac{1}{\omega} \right| \left(|\cos(\omega b) f(b)| + |\cos(\omega a) f(a)| + \int_a^b |\cos(\omega t) f'(t)| dt \right) \\ &\stackrel{\cos \text{ bounded}}{<} \left| \frac{1}{\omega} \right| \left(|\cos(\omega b) f(b)| + |\cos(\omega a) f(a)| + \int_a^b |f'(t)| dt \right) \xrightarrow{|\omega| \rightarrow \infty} 0 \end{aligned}$$

12. Wann dürfen Regelintegral und Grenzwert einer Funktionenfolge vertauscht werden?

Answer: Consider $f_n \in \mathcal{R}([a, b], V)$ with V Banach space. If f_n converge uniformly to some $f \in \mathcal{R}([a, b], V)$, then $\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$

13. Wie ist das Riemann-Integral definiert?

Answer: Consider the interval $[a, b]$, a function $f: [a, b] \rightarrow \mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or \mathbb{C}), and for any partition $P = \{a = p_0 < p_1 < \dots < p_n = b\}$ of $[a, b]$ let $L(f, P) = \sum_{i=1}^n \inf_{x \in (p_{i-1}, p_i)} f(x) (p_i - p_{i-1})$

and $U(f, P) = \sum_{i=1}^n \sup_{x \in (p_{i-1}, p_i)} f(x)(p_i - p_{i-1})$. Let furthermore $U^*(f) = \inf_P U(f, P)$ and $L^*(f) = \sup_P L(f, P)$. If $U^*(f) = L^*(f)$, then we say that f is Riemann integrable, and $\int_a^b f = U^*(f) = L^*(f)$.

14. Was ist eine Lebesgue-Nullmenge?

Answer: $A \subset \mathbb{R}$ is a null set, if $\forall \epsilon > 0$ there is an at most countable collection of open intervals $I = \{I_i\}_{i \in \mathbb{N}}$ such that $A \subset \cup_{i \in \mathbb{N}} I_i$ and $\sum_{i \in \mathbb{N}} |I_i| < \epsilon$

15. Wie ist das Lebesgue-Integral definiert?

Answer: Consider

$$H^u = \left\{ \varphi \mid \exists (f_n) \in \mathcal{T}([a, b], \mathbb{R} \cup \{\pm\infty\}) : f_n(x) \leq f_{n+1}(x) \ (\forall n \in \mathbb{N}) \wedge \lim_{n \rightarrow \infty} f_n(x) = \varphi(x) \text{ (a.e.)} \right\}$$

and

$$H^o = \left\{ \varphi \mid \exists (f_n) \in \mathcal{T}([a, b], \mathbb{R} \cup \{\pm\infty\}) : f_n(x) \geq f_{n+1}(x) \ (\forall n \in \mathbb{N}) \wedge \lim_{n \rightarrow \infty} f_n(x) = \varphi(x) \text{ (a.e.)} \right\}$$

$\forall \varphi \in H^u$ consider the $(f_n) \in \mathcal{T}([a, b], \mathbb{R} \cup \{\pm\infty\})$ sequence that converges from below to φ almost everywhere. Let $\int_a^b \varphi := \lim_{n \rightarrow \infty} \int_a^b f_n \in \mathbb{R} \cup \{\pm\infty\}$ the so called H^u integral of φ . The definition of H^o integral of φ is analogue.

Define furthermore the upper Lebesgue-integral as $L^* \int_a^b f := \inf \left\{ \int_a^b \varphi \mid \varphi \in H^u : \varphi \geq f \right\}$ and the lower Lebesgue-integral as $L_* \int_a^b f := \sup \left\{ \int_a^b \varphi \mid \varphi \in H^o : \varphi \leq f \right\}$

When $L^* \int_a^b f = L_* \int_a^b f$ and they are finite, then f is Lebesgue-integrable and $\int_a^b f = L^* \int_a^b f = L_* \int_a^b f \in \mathbb{R}$. Let $\text{Leb}_{[a, b]} := \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ Lebesgue-integrable}\}$

16. Gib je ein Beispiel für eine Funktion an, die

- (a) Riemann integrierbar ist, aber ist keine Regelfunktion.
- (b) Lebesgue integrierbar ist, aber nicht Riemann integrierbar.

Answer:

$$(a) f : [0, 1] \rightarrow \mathbb{R}, x \mapsto \begin{cases} \sin \frac{1}{x} & x \in (0, 1) \\ 0 & x = 0 \end{cases}$$

- (b) The Dirichlet function

17. Was sagt der Satz von Beppo Levi über monotone Folgen Lebesgue-integrierbarer Funktionen?

Answer: Suppose that $(f_n) \in \text{Leb}_{[a, b]}$, $f_n(x)$ non-decreasing and converges to $f(x)$ a.e. and the $\left\{ \int_a^b f_n \ (n \in \mathbb{N}) \right\}$ set is bounded. Then $f \in \text{Leb}_{[a, b]}$ and $\int_a^b \lim_{n \rightarrow \infty} f_n = \int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$.

The analogue holds for a sequence $(f_n) \in \text{Leb}_{[a, b]}$ that converges pointwise non-increasingly to f a.e.

18. Was sagt der Satz zur majorisierten Konvergenz über die Vertauschbarkeit von Lebesgue-Integral und Grenzwert einer Funktionenfolge?

Answer: Let $g \in \text{Leb}$ and $(f_n) : \mathbb{N} \rightarrow \text{Leb}$ such that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for almost all x , and $|f_n| \leq g \ (\forall n \in \mathbb{N})$ almost everywhere. Then $\lim_{n \rightarrow \infty} \int_a^b f_n$ exists, $f \in \text{Leb}$ and

$$\int_a^b \lim_{n \rightarrow \infty} f_n = \int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$$

19. Wie lautet die Hölder-Ungleichung für integrierbare Funktionen?

Answer: Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in \mathcal{L}^p, g \in \mathcal{L}^q$. Then $\int_a^b fg \leq \|f\|_p \|g\|_q$

20. Wie lautet der Schrankensatz? Warum gilt der Mittelwertsatz (der Differentialrechnung) nicht in höheren Dimensionen?

Answer:

Let Z Banach space and $f \in C^1([a, b], Z)$. Then $\exists \zeta \in (a, b): \|f(b) - f(a)\| \leq \|f'(\zeta)\| (b - a)$. The mean value theorem of $f(b) - f(a) = f'(\zeta)(b - a)$ does not hold in general. Consider $Z = \mathbb{R}^3$ and

$f(t) := \begin{pmatrix} \cos t \\ \sin t \\ \varepsilon t \end{pmatrix}$ with some $\varepsilon > 0$. Then $f'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ \varepsilon \end{pmatrix}$. Now if we consider the $[0, 2\pi]$

interval, then $f(2\pi k) - f(0) = 2\pi \begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix} \neq 2\pi f'(\zeta) \ (\forall \zeta \in [0, 2\pi])$

21. Wie lautet die Integraldarstellung von Lagrange für das Restglied der Taylorentwicklung?

Answer: Let $I = (a, b) \subset \mathbb{R}$ open interval, $n \in \mathbb{N}_0, Z$ Banach space and $f \in C^{n+1}(I, Z)$. If $x_0 \in (a, b)$, then $\forall x \in (a, b): f(x) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k + \frac{1}{n!} \int_{x_0}^x (x - t)^n f^{(n+1)}(t) dt$

22. Wie lautet die Trapezregel?

Answer: Consider $f \in C^2([0, 1], \mathbb{R})$. Then $\exists \zeta \in (0, 1): \int_0^1 f = \frac{1}{2}(f(0) + f(1)) - \frac{1}{12}f''(\zeta)$

23. Wie lassen sich Integrale dank der Trapezregel approximieren?

Answer: Let $f \in C^2([a, b], \mathbb{R}), C := \max_{x \in [a, b]} |f''|, n \in \mathbb{N}, h := \frac{b-a}{n}$. Then it holds that

$$\left| \int_a^b f - \left(\frac{1}{2}f(a_0) + \sum_{k=1}^{n-1} f(a_k) + \frac{1}{2}f(a_n) \right) h \right| \leq \frac{1}{12} C(b-a)h^2$$

24. Was sind uneigentliche Integrale?

Answer: Let $-\infty \leq a < b \leq +\infty$ and $f: (a, b) \rightarrow \mathbb{R}$ integrable over every finite $[\alpha, \beta] \subset (a, b)$ interval. Consider furthermore $(a_n), (b_n) \in \mathbb{R}$ respectively monotone decreasing and increasing sequences such that $a < a_n$ and $b_n < b \ (\forall n \in \mathbb{N})$ and suppose furthermore that $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$. If for every such sequence the limit $\lim_{n \rightarrow \infty} \int_{a_n}^{b_n} f$ exists and has the same value, then let $\int_a^b f = \lim_{n \rightarrow \infty} \int_{a_n}^{b_n} f$ and call it the improper integral of f over (a, b) .

25. Für welche reellen Exponenten α konvergiert das uneigentliche Integral $\int_0^1 t^\alpha dt$, für welche das uneigentliche Integral $\int_1^\infty t^\alpha dt$?

Answer:

- $\int_0^1 t^\alpha dt = \frac{1}{\alpha+1} \ (-1 < \alpha \in \mathbb{R})$
- $\int_1^\infty t^\alpha dt = -\frac{1}{\alpha+1} \ (-1 > \alpha \in \mathbb{R})$

26. Was bedeutet absolute Konvergenz uneigentlicher Integrale? Gib Beispiele

- (a) absolut konvergenter;
- (b) konvergenter, aber nicht absolut konvergenter;
- (c) nicht konvergenter

uneigentlicher Integrale.

Answer: Consider the improper integral $\int_a^b f$. If $\int_a^b |f|$ converges, then so is $\int_a^b f$, and we call the improper integral absolute convergent.

- (a) absolute convergent: $\int_0^1 \sin \frac{1}{t} dt$
- (b) convergent, but not absolutely: $\int_0^{+\infty} \frac{1}{t} \sin t dt$
- (c) not convergent: $\int_1^{+\infty} \frac{1}{t} dt$

27. Sei $f : [0, \infty) \rightarrow (0, \infty)$ monoton fallend. Wie hängen $\sum_{k=1}^{\infty} f(k)$ und $\int_1^{\infty} f(t) dt$ zusammen?

Answer: $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(t) dt$ converges. Furthermore

$$\sum_{k=2}^{\infty} f(k) \leq \int_1^{\infty} f(t) dt \leq \sum_{k=1}^{\infty} f(k)$$

28. Welche der folgenden uneigentlichen Riemann-Integrale existieren? Welche konvergieren absolut?

- (a) $\int_1^{\infty} \cos t dt$
- (b) $\int_1^{\infty} \cos(t^2) dt$
- (c) $\int_1^{\infty} \frac{\cos t}{t} dt$
- (d) $\int_1^{\infty} \frac{\cos(t^2)}{t} dt$
- (e) $\int_1^{\infty} \frac{\cos t}{t^2} dt$
- (f) $\int_1^{\infty} \frac{\cos(t^2)}{t^2} dt$

Answer: Proposition: $\int_1^{\infty} \frac{\cos t}{t^{\alpha}} dt \begin{cases} \text{diverges} & \alpha = 0 \\ \text{converges, but not absolutely} & \alpha > 0 \\ \text{converges absolutely} & \alpha > 1 \end{cases}$

Proof:

- Consider first $\alpha = 0$. On one hand $\limsup_{x \rightarrow \infty} \int_1^x \frac{\cos t}{t^0} dt = \limsup_{x \rightarrow \infty} \int_1^x \cos t dt = \int_1^{\pi/2} \cos t dt + \limsup_{x \rightarrow \infty} \int_{\pi/2}^x \cos t dt = \int_1^{\pi/2} \cos t dt + 1$. On the other hand $\liminf_{x \rightarrow \infty} \int_1^x \frac{\cos t}{t^0} dt = \int_1^{\pi/2} \cos t dt + \liminf_{x \rightarrow \infty} \int_{\pi/2}^x \cos t dt = \int_1^{\pi/2} \cos t dt - 1$, thus $\int_1^{\infty} \cos t dt$ cannot exist.
- Let now $\alpha > 1$: $\int_1^{\infty} \left| \frac{\cos t}{t^{\alpha}} \right| dt \leq \int_1^{\infty} \left| \frac{1}{t^{\alpha}} \right| dt = \int_1^{\infty} \frac{1}{t^{\alpha}} dt = \frac{1}{\alpha-1}$, thus absolute convergent.
- Let now $\alpha \in (0, 1)$: $\int_1^{n\pi} \left| \frac{\cos t}{t^{\alpha}} \right| dt = \int_1^{\pi} \left| \frac{\cos t}{t^{\alpha}} \right| dt + \int_{\pi}^{n\pi} \left| \frac{\cos t}{t^{\alpha}} \right| dt \geq \sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi} \left| \frac{\cos t}{t^{\alpha}} \right| dt \geq \sum_{k=1}^{n-1} \frac{1}{((k+1)\pi)^{\alpha}} \int_{k\pi}^{(k+1)\pi} |\cos t| dt = \sum_{k=1}^{n-1} \frac{2}{((k+1)\pi)^{\alpha}} = \frac{2}{\pi^{\alpha}} \sum_{k=1}^{n-1} \frac{1}{(k+1)^{\alpha}} > \frac{2}{\pi^{\alpha}} \sum_{k=1}^{n-1} \frac{1}{(k+1)} \xrightarrow{n \rightarrow \infty} \infty$, thus it's **not** absolutely convergent. On the other hand $\int_1^x \frac{\cos t}{t^{\alpha}} dt = -\frac{\sin t}{t^{\alpha}} \Big|_1^x + \int_1^x \frac{\sin t}{t^{\alpha+1}} dt$ converges, since $\lim_{x \rightarrow \infty} \frac{\sin t}{t^{\alpha}} \Big|_1^x = \sin 1$ and $\lim_{x \rightarrow \infty} \int_1^x \frac{\sin t}{t^{\alpha+1}} dt$ converges, since $\alpha + 1 > 1$.

Note: The same result holds for $\int_1^{\infty} \frac{\sin t}{t^{\alpha}} dt$ regarding the parameter α and the proof is identical. ■

Using the previous we get:

- (a) Diverges
- (b) $\lim_{x \rightarrow \infty} \int_1^x \cos(t^2) dt \stackrel{u=t^2}{=} \lim_{x \rightarrow \infty} \int_1^{x^2} \frac{1}{2} \frac{\cos u}{u^{1/2}} du$ converges, but not absolutely
- (c) Converges, but not absolutely

(d) $\lim_{x \rightarrow \infty} \int_1^x \frac{\cos(t^2)}{t} \stackrel{u=t^2}{=} \lim_{x \rightarrow \infty} \frac{1}{2} \int_1^{x^2} \frac{\cos u}{u} du$ converges, but not absolutely

(e) Converges absolutely

(f) $\lim_{x \rightarrow \infty} \int_1^x \frac{\cos(t^2)}{t^2} \stackrel{u=t^2}{=} \lim_{x \rightarrow \infty} \frac{1}{2} \int_1^{x^2} \frac{\cos u}{u^{3/2}} du$ converges absolutely

29. Warum konvergiert die Reihe der Riemannsche ζ -Funktion $\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}$ für $\operatorname{Re} s > 1$?

Answer: Let $q = \operatorname{Re} s$. Then $\sum_{k=1}^{\infty} \left| \frac{1}{k^s} \right| = \sum_{k=1}^{\infty} \frac{1}{k^q}$ converges iff $\int_1^{\infty} \frac{1}{t^q} dt$ converges. But this improper integral exists exactly when $1 < q \in \mathbb{R}$. Thus if $\operatorname{Re} s > 1$ then $\sum_{k=1}^{\infty} \frac{1}{k^s}$ is absolutely convergent, and thus also converges conditionally.

30. Wie ist die Gamma-Funktion definiert, und welche Funktionalgleichung erfüllt sie?

Answer: Let $\Gamma(\alpha) := \int_0^{\infty} t^{\alpha-1} e^{-t} dt$ ($\forall \alpha \in \mathbb{C}, \operatorname{Re} \alpha > 0$). Then $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$ and since $\Gamma(1) = 1 = 0!$ it holds follows from a simple induction that $\Gamma(n + 1) = n!$ ($\forall n \in \mathbb{N}_0$).

31. Wie lautet die Stirling-Formel zur Approximation von $n!$?

Answer: For $n \rightarrow \infty$ it holds, that $\sqrt{2\pi n} \frac{n^n}{e^n} \sim n!$

32. Wie ist die Faltung $\varphi \star f$ von zwei Funktionen $\varphi, f : \mathbb{R} \rightarrow \mathbb{R}$ definiert?

Answer: $(\varphi \star f)(x) := \int_{-\infty}^{\infty} \varphi(x-t)f(t) dt$ ($\forall x \in \mathbb{R}$)

33. Was ist eine Dirac-Folge? Gib eine Definition und wenigstens ein Beispiel an.

Answer: Consider the $\varphi_n : \mathbb{R} \rightarrow [0, \infty)$ sequence of functions. It's called Dirac sequence, if it satisfies the following conditions:

- φ_n is integrable ($\forall n \in \mathbb{N}$)
- $\int_{-\infty}^{\infty} \varphi_n = 1$
- $\forall \varepsilon, \delta > 0: \exists N \in \mathbb{N}: \forall n > N: \int_{|x| \geq \delta} \varphi_n(t) dt \leq \varepsilon$

Examples:

- Landau kernel: Let $\varphi_n(t) = \frac{1}{c_n} (1 - t^2)_+^n = \begin{cases} \frac{1}{c_n} (1 - t^2)^n & t \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$

$$\text{with } c_n = \int_{-1}^1 (1 - t^2)^n dt$$

34. Sei $\varphi_n : \mathbb{R} \rightarrow \mathbb{R}$ eine Dirac-Folge und sei $f : \mathbb{R} \rightarrow \mathbb{R}$ stetig mit $f(x) = 0$ für $|x| \geq 1$. Bestimme $\lim_{n \rightarrow \infty} (\varphi_n * f)$ für jedes $x \in \mathbb{R}$.

Answer: f must be bounded, and let $B \in \mathbb{R}_+$ such that $|f| \leq B$. Consider $x \in [-1, 1]$ and $\forall \varepsilon > 0$ let $\delta > 0$ such that $\delta < B$ and $\forall t \in (x - \delta, x + \delta): |f(t) - f(x)| < \varepsilon$. Let furthermore

$N \in \mathbb{N}: \forall n > N: \int_{|t| \geq \delta} \varphi_n \leq \varepsilon$. Now

$$\begin{aligned}
|f(x) - (\varphi_n \star f)(x)| &= \left| f(x) - \int_{-\infty}^{\infty} \varphi(x-t)f(t) dt \right| = \left| f(x) \underbrace{\int_{-\infty}^{\infty} \varphi(x-t) dt}_{=1} - \int_{-\infty}^{\infty} \varphi(x-t)f(t) dt \right| \\
&= \left| \int_{-\infty}^{\infty} \varphi_n(x-t)(f(x) - f(t)) dt \right| \\
&\leq \left| \int_{-\infty}^{x-\delta} \varphi_n(x-t)(f(x) - f(t)) dt \right| + \left| \int_{x-\delta}^{x+\delta} \varphi_n(x-t)(f(x) - f(t)) dt \right| + \left| \int_{x+\delta}^{\infty} \varphi_n(x-t)(f(x) - f(t)) dt \right| \\
&\leq \int_{-\infty}^{x-\delta} |\varphi_n(x-t)(f(x) - f(t))| dt + \int_{x-\delta}^{x+\delta} |\varphi_n(x-t)(f(x) - f(t))| dt + \int_{x+\delta}^{\infty} |\varphi_n(x-t)(f(x) - f(t))| dt \\
&\leq \int_{-\infty}^{x-\delta} \varphi_n(x-t)2B dt + \int_{x-\delta}^{x+\delta} |\varphi_n(x-t)(f(x) - f(t))| dt + \int_{x+\delta}^{\infty} \varphi_n(x-t)2B dt \\
&\leq \frac{2B\varepsilon}{2} + \int_{x-\delta}^{x+\delta} \varphi_n(x-t)\varepsilon dt + \frac{2B\varepsilon}{2} < 2B\varepsilon + 2\delta\varepsilon < 4B\varepsilon
\end{aligned}$$

Thus $\lim_{n \rightarrow \infty} (\varphi_n \star f)(x) = f(x)$. Furthermore the approximation does not depend on the choice of x (since f is uniformly continuous), $\lim_{n \rightarrow \infty} \varphi_n \star f = f$ (in $\|\cdot\|_{\sup}$).

35. Wie lautet den Approximationssatz von Weierstrass?

Answer: Consider $(\mathcal{C}([a, b], \mathbb{R}), \|\cdot\|)$, the normed vectorspace of continuous functions on the $[a, b]$ closed interval with the supremum norm. Then the $\mathcal{P} \subset \mathcal{C}([a, b], \mathbb{R})$ subspace of the polynomials is dense in $\mathcal{BC}([a, b], \mathbb{R})$ under the supremum norm.

36. Welche Funktionen können durch Polynome gleichmäßig approximiert werden? Mit welcher Grundidee lassen sich approximierende Polynome zu einer gegebenen Funktion f konstruieren?

Answer: Every continuous function defined on a closed interval can be approximated uniformly with polynomials. Suppose without loss of generality, that we want to approximate the $f: [0, 1] \rightarrow \mathbb{R}$ function, for which it holds that $f(0) = f(1) = 0$. Extend this function to the reals by letting $f(x) = 0$ ($\forall x \notin [0, 1]$). Since the approximation in 34. does not depend on the choice of x (since a continuous function on a closed interval is also uniformly continuous), the convolution of f with a Dirac-sequence is a uniform approximation of f . If we chose the Landau-kernel as our Dirac-sequence, then in particular the resulting function $(\varphi_n \star f)(x)$ will be a polynomial in x , thus we found a uniformly approximating polynomial sequence.

If the function f was originally defined on the $[a, b]$ interval, then let $\tilde{f}: [0, 1] \rightarrow \mathbb{R}, \tilde{f}(x) = f(a + x(b-a))$. If furthermore $f(0) = f(1) = 0$ doesn't hold, then let $\tilde{f}(x) = f(x) - f(1) - (f(1) - f(0))x$.

37. Was ist ein Hilbertraum? Gib zwei unendlich-dimensionale Beispiele.

Answer: The H vectorspace over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} is a Hilbert space, if H is an inner product space, and furthermore it's a complete metric space with regards to the metric induced by the scalar product: $d(x, y) := \|x - y\| := \sqrt{\langle x, y \rangle}$

The $L^2 = \mathcal{L}^2 / \sim$ space with $\langle f, g \rangle = \int_a^b \bar{f}g$, where $\mathcal{L}^2 = \left\{ f: [a, b] \rightarrow \mathbb{K} \mid |f|^2 \in \text{Leb} \right\}$ and $f \sim g \Leftrightarrow f = g$ almost everywhere.

The $\ell^2 = \{c: \mathbb{Z} \rightarrow \mathbb{K} \mid \|c\| < \infty\}$ space with the norm induced through the $\langle c, c' \rangle = \sum_{k=-\infty}^{\infty} \bar{c}_k c'_k$ scalar product.

38. Was sind die Fourierkoeffizienten einer 2π -periodischen Funktion $f : \mathbb{R} \rightarrow \mathbb{C}$? Wie lautet die Fourier-Reihe zu f ?

Answer: $c_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt$ and $\tilde{f}(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$

39. Beschreibe wie 2π -periodische $f \in L^2$, die Fourier-Reihe $\hat{f}(t) := \sum_{k \in \mathbb{Z}} c_k e^{ikt}$, und Fourier-Koeffizienten $(c_k)_{k \in \mathbb{Z}} \in \ell^2$ miteinander zusammenhängen.

Answer: Consider $\varphi : \ell^2 \rightarrow L^2$ function with $\varphi((c_k)_{k \in \mathbb{Z}}) = \sum_{k \in \mathbb{Z}} c_k e^{ikt}$. ℓ^2 and L^2 are isometrically isomorphic through the φ bijection and furthermore $\lim_{n \rightarrow \infty} \sum_{k=-n}^n c_k e^{ikt} = f$ in the $\|\cdot\|_{L^2}$ norm.

40. Wie hängen Fourier-Reihen mit Obertönen in der Musik zusammen?

Answer: Consider a periodic waveform f , with minimal period $p \in \mathbb{R}^+$. We say, that the fundamental frequency is $\frac{1}{p}$. Consider now $f(t) = \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} c_k \sin(k(t + \varphi_k))$, the amplitude-phase form of the Fourier series. We call $f_1 = c_1 \sin(t + \varphi_1)$ the fundamental, and f_k ($k > 1$) the overtones. The amplitude of the overtones determine the timbre.