1. Wann nennt man eine Folge x_n reeller Zahlen konvergent und wann uneigentlich konvergent?

   Answer: The $(x_n) \in \mathbb{R}$ sequence converges to $l \in \mathbb{R}$ if $\forall \epsilon > 0\ \exists N \in \mathbb{N}\colon \forall n > N\colon\ |x_n - l| < \epsilon$. If such l exists, then it's unique.

   If there is no such $l \in \mathbb{R}$ but $\forall X \in \mathbb{R}\ \exists N \in \mathbb{N}\colon \forall n > N\colon x_n > X$ then we say that (x_n) tends to infinity, and we denote it with $\lim_{x\to\infty} x_n = +\infty$. If $-x_n$ tends to $+\infty$, then we say that x_n tends to negative infinity, and we denote it with $\lim_{n\to\infty} x_n = -\infty$

2. Wie ist der Grenzwert einer Folge definiert?

   Answer: If a sequence converges, then its limit is the uniquely defined l in Question 1.

3. Was sind Supremum und Infimum der Folgen $(1 + \frac{1}{n})^n$ und $(1 + \frac{1}{n})^{n+1}$?

   Answer:

   x_n increasing:

$$\frac{x_{n+1}}{x_n} = \left(\frac{n+2}{n+1}\right)^{n+1} / \left(\frac{n+1}{n}\right)^{n} = \left(\frac{n+2}{n+1}\frac{n}{n+1}\right)^{n} \frac{n+2}{n+1} = \left(\frac{(n+1)^2-1}{(n+2)^2}\right)^{n} \frac{n+2}{n+1}$$
$$= \left(1 - \frac{1}{(n+1)^2}\right)^{n} \frac{n+2}{n+1} \geq \left(1 - \frac{n}{(n+1)^2}\right) \frac{n+2}{n+1} = 1 + \frac{1}{(n+3)^3} > 1$$

   Where the $\geq$ holds because of Bernoulli's inequality, and the $=$ after is just algebraic transformations.

   y_n decreasing:

$$\frac{y_n}{y_{n+1}} = \left(\frac{n+1}{n}\right)^{n+1} / \left(\frac{n+2}{n+1}\right)^{n+2} = \left(\frac{n+1}{n}\frac{n+1}{n+2}\right)^{n+1} \frac{n+1}{n+2} = \left(\frac{(n+1)^2}{(n+1)^2-1}\right)^{n+1} \frac{n+1}{n+2}$$
$$= \left(1 + \frac{1}{(n+1)^2-1}\right)^{n+1} \frac{n+1}{n+2} \geq \left(1 + \frac{n+1}{(n+1)^2-1}\right) \frac{n+1}{n+2} > \left(1 + \frac{n+1}{(n+1)^2}\right) \frac{n+1}{n+2} = 1$$

   Where the $\geq$ holds because of Bernoulli's inequality and the $>$ holds because of incereasing the denominator, thus decreasing the value of the whole fraction.

   It holds furthermore that $x_n = \left(1 + \frac{1}{n}\right) y_n$, thus x_n converges exactly when y_n converges. Since they are both monoton sequences and $x_n < y_n$, they are both bounded, and consequently convergent, and they have the same limit, which we denote by e. Thus $\sup_{n\in\mathbb{N}} x_n = e = \inf_{n\in\mathbb{N}} y_n$ and $\inf_{n\in\mathbb{N}} x_n = x_1 = 2$ and $\sup_{n\in\mathbb{N}} y_n = y_1 = 4$

4. Wie sind Häufungswerte einer Folge x_n komplexer Zahlen definiert?

   Answer: Let $(x_n) \in \mathbb{R}$ (or $(x_n) \in \mathbb{C}$). We call $\zeta \in \mathbb{R}$ (or $\zeta \in \mathbb{C}$) a limit point of the (x_n) sequence if exists $k_n\colon \mathbb{N} \to \mathbb{N}$ index sequence with $k_n \geq n$ such that $\lim_{n\to\infty} x_{k_n} = \zeta$

5. Was sind die Häufungswerte der Folge $(-1)^n + \frac{1}{n}$?

   Answer: Let $x_n = (-1)^n + \frac{1}{n}$, now $\limsup_{n\to\infty} = 1$ and $\liminf_{n\to\infty} = -1$, thus the sequence has at least two limit points. Since $\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\colon$ either $|x_n - 1| < \epsilon$ or $|x_n - (-1)| < \epsilon$, the sequence cannot have any other limit points.

6. Was sind Limes superior und Limes inferior einer reellwertigen Folge? Wann existieren sie? Wann stimmen Limes superior und Limes inferior überein?

   Answer: Consider an arbitrary $(x_n) \in \mathbb{R}$ sequence. The

   - Limit superior: $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\geq n}\{x_k\}$
   - Limit inferior: $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{k\geq n}\{x_k\}$

   For any bounded sequence its lim sup and lim inf exist. The lim sup and lim inf of a sequence are equal if and only if the sequence converges, and in this case

   $$\lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$$

7. Konstruiere eine reelle Folge, die jede reelle Zahl aus Häufungswert hat.

   Answer: Consider the following enumeration of the rationals: $0, \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{1}{5}, \frac{2}{4}, \ldots$ and the $\phi\colon \mathbb{N}^2 \to \mathbb{N}$ bijection given by $\phi(p,q) = \frac{(p+q-1)(p+q-2)}{2} + p$

   Define the following sequence: $x_0 = 0, x_1 = \frac{1}{1}$ and $\forall n > 1\colon x_n = \frac{p}{q}$ such that $\phi(p,q) = n$. The so defined sequence will correspond to the above enumeration and thus every non-negative rational will occur in x_n eventually, and since $\mathbb{Q}$ is dense in $\mathbb{R}$, every non-negative real will be a limit point of x_n. From x_n we can construct a new sequence with $x_0, x_1, -x_1, x_2, -x_2, x_3, -x_3, \ldots$ that will have every real as a limit point.

8. Wann heißt eine Menge $A \subseteq B$ dicht in B? Gilt Dichtheit für $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Q} \subseteq \mathbb{R}$, $\mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$, $\mathbb{Q} + i\mathbb{Q} \subseteq \mathbb{C}$, $\mathbb{Q} + i(\mathbb{R} \setminus \mathbb{Q}) \subseteq \mathbb{C}$? Warum?

   Answer: A is dense in B if $\forall b \in B\ \exists (a_n)\colon \mathbb{N} \to A\colon\ \lim_{n\to\infty} a_n = b$

   Or equivalently: $\forall b \in B\ \forall \epsilon > 0\ \exists a \in A\colon\ |a - b| < \epsilon$

   Or equivalently: every point of B is a limit point of A.

   - $\mathbb{Z} \subseteq \mathbb{Q}$: no, there is no sequence in $\mathbb{Z}$ that converges to $-1/12$
   - $\mathbb{Q} \subseteq \mathbb{R}$: yes, $\forall r \in \mathbb{R}\ \exists (x_n) \in \mathbb{Q}\colon\ \lim_{n\to\infty} x_n = r$, for example with $x_n = \frac{\lfloor rn \rfloor}{n} \in \mathbb{Q}$
   - $\mathbb{R}\backslash\mathbb{Q} \subseteq \mathbb{R}$: yes, $\forall r \in \mathbb{R}\ \exists (x_n) \in (\mathbb{R}\backslash\mathbb{Q})\colon\ \lim_{n\to\infty} x_n = r$, for example with $x_n = \frac{\lfloor n\sqrt{2}r \rfloor}{n\sqrt{2}} \in \mathbb{R}\backslash\mathbb{Q}$ (because a rational divided by an irrational is always irrational)
   - $\mathbb{Q} + i\mathbb{Q} \subseteq \mathbb{C}$ yes, because $\mathbb{Q}$ is dense an $\mathbb{R}$, thus $\forall a + ib \in \mathbb{C}(a, b \in RR)\ \exists (x_n), (y_n) \in \mathbb{Q}\colon\ \lim_{n\to\infty} x_n = a, \lim_{n\to\infty} y_n = b$ and thus $\lim_{n\to\infty} x_n + iy_n = a + bi = z$ (explanation: a sequence $(z_n) = (a_n + ib_n) \in \mathbb{C}$ $(a_n, b_n \in \mathbb{R})$ convergent if and only if a_n and b_n both converges, and in this case $\lim_{n\to\infty} z_n = \lim_{n\to\infty} a_n + i \lim_{n\to\infty} b_n$).
   - $\mathbb{Q} + i(\mathbb{R} \setminus \mathbb{Q}) \subseteq \mathbb{C}$ yes, because $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$, so the previous point hold, just with $(b_n) \in \mathbb{R} \setminus \mathbb{Q}$

9. Was besagt das Konvergenzkriterium von Cauchy? Warum gilt es?

   Answer: Consider an $(x_n) \in \mathbb{R}$ sequence. x_n is convergent if and only if $\forall \epsilon > 0\ \exists N \in \mathbb{N}\colon \forall n, m > N\colon\ |x_n - x_m| < \epsilon$. Because $\mathbb{R}$ is complete (the Cauchy criterum doesn't hold in $\mathbb{Q}$ for example).

10. Welche monotonen Folgen besitzen einen Grenzwert?

    Answer: Bounded monoton sequences.

11. Welche dieser Folgen konvergieren für $n \to \infty$? Was sind ggf. ihre Grenzwerte, einschließlich $\pm\infty$?

$$\frac{n^2}{3n-2}, \qquad \frac{3n^2-2}{2n^2+3}, \qquad \frac{2^n}{n!} \qquad \sqrt[n]{n} \qquad q^{1/n} \text{ (für reelle } q>0\text{)}, \qquad \sqrt{n+1}-\sqrt{n}$$

Answer:

(a) $\lim_{n\to\infty} \frac{n^2}{3n-2} = \lim_{n\to\infty} \frac{n}{3-2/n} = +\infty$

(b) $\lim_{n\to\infty} \frac{3n^2-2}{2n^2+3} = \lim_{n\to\infty} \frac{3-2/n^2}{2+3/n^2} = 3/2$

(c) $\lim_{n\to\infty} \frac{2^n}{n!} = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{2}{i} = \frac{4}{3} \lim_{n\to\infty} \prod_{k=4}^{n} \frac{2}{k} < \frac{4}{3} \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n-3} = 0$

(d) Consider first $q > 1$. Let $h_n = q^{1/n} - 1 > 0$, thus $(1+h_n)^n = q$ $(\forall n \in \mathbb{N})$. From Bernoulli's inequality $q \geq 1 + nh_n > 1$ $(\forall n \in \mathbb{N})$, from which it follows that $h_n \to 0$ $(n \to \infty)$. Consequently $q^{1/n} \to 1$. Now consider $0 < q < 1$: from the previous $(1/q)^{1/n} \to 1$ $(n \to \infty)$, thus from the algebraic properties of sequences we get $q^{1/n} \to 1$ as well.

(e) $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$
$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

12. Wie lautet der Satz von Bolzano-Weierstraß?

Answer: Every bounded sequence in $\mathbb{R}$ or $\mathbb{C}$ has a limit point. Conversely: every convergent sequence in $\mathbb{R}$ or $\mathbb{C}$ is bounded.

13. Wieviele Häufungswerte kann eine beschränkte, reellwertige Folge mindestens/höchstens besitzen?

Answer: A convergent sequence has exactly one limit point (independently of being bounded or not). Any non convergent bounded sequence has at least 2 limit points: its limit superior and limit inferior. Consider now a sequence that enumerates all the rationals between 0 and 1. The rationals are countable, thus such sequence exists, and since $\mathbb{Q}$ is dense in $\mathbb{R}$, every real number in the $[0,1]$ interval is a limit point of such a sequence

14. Wie kann man den Kreisumfang durch Approximation mit regulären 2^n-Ecken bestimmen?

Answer: Consider the unit circle and let the sidelenght of the inscribed and escribed regular 2^n-sided polygons respectively be s_n and t_n, and let their corresponding perimeters respectively be $\underline{u}_n = 2^n s_n$ and $\overline{u}_n = 2^n t_n$. Then the $\underline{u} = \lim_{n\to\infty} \underline{u}_n$ and $\overline{u} = \lim_{n\to\infty} \overline{u}_n$ limits exist and $2\pi = \underline{u} = \overline{u}$.

15. Gib zwei Definitionen der Eulerschen Zahl e. Warum stimmen beide Definitionen überein?

Answer: Consider sequences $\hat{e}_n = \sum_{k=0}^{n} \frac{1}{k!}$ and $e_n = \left(1 + \frac{1}{n}\right)^n$. From Question 3 we already know that e_n converges, let $e = \lim_{n\to\infty} e_n$. Furhermore from the Binomial theorem it's easy to see that $e_n \leq \hat{e}_n$ $(\forall n \in \mathbb{N})$:

$$e_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \frac{n}{n} \frac{n-1}{n} \frac{n-2}{n} \cdots \frac{n-k+1}{n} < \sum_{k=0}^{n} \frac{1}{k!} = \hat{e}_n$$

Since $\hat{e}_n$ is a sum of positive terms, it's clear that it increasing, and it's bounded:

$$\hat{e}_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!} < 2 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots < 3$$

thus $\hat{e}_n$ also converges, and let $\hat{e} = \lim_{n\to\infty} \hat{e}_n$. Since $e_n < \hat{e}_n$, it follows that $e \leq \hat{e}$. Furthermore $\forall m \leq n$ holds that

$$
\begin{aligned}
e_m = \left(1+\frac{1}{m}\right)^m &= \sum_{k=0}^{m}\binom{m}{k}\frac{1}{m^k} = \sum_{k=0}^{m}\frac{1}{k!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdots\left(1-\frac{k-1}{m}\right) \\
&\leq \sum_{k=0}^{m}\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \\
&< \sum_{k=0}^{n}\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) = e_n
\end{aligned}
$$

The important part here is

$$
\sum_{k=0}^{m}\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) < e_n
$$

By taking first the $n \to \infty$ limit we get $\hat{e}_m \leq e$, and the taking the $m \to \infty$ limit we get $\hat{e} \leq e$. Together with the previous $e = \lim_{n\to\infty} e_n = \lim_{n\to\infty} \hat{e}_n = \hat{e}$ holds.

16. Wie lauten die Abschätzungen für $n!$ von Stirling?

    Answer: $n^n/e^{n-1} \leq n! \leq n^{n+1}/e^{n-1}$ (proof: Lecture 13)