1. Wann heißt eine Funktion $f\colon \mathbb{R} \to \mathbb{R}$ in einem Punkt x_0 differenzierbar? Wie lässt sich die Ableitung geometisch interpretieren?

   Answer: If the $\lim_{x\to x_0} \frac{f(x_0+h)-f(x_0)}{h}$ exists, then we call the function f differentiable in the x_0 point. We call the value of $\lim_{x\to x_0} \frac{f(x_0+h)-f(x_0)}{h}$ the derivative of the function f in point x_0 and we denote it with $f'(x_0)$. Whenever the derivative of a function exists, it's unique.

   The value of the $f'(x_0)$ is the coefficient of x in the best linear approximation of f at point x_0, and it's the slope of the tangent line drawn to the function at the point $(x_0, f(x_0))$.

2. Gib Beispiele für Funtionen $f\colon \mathbb{R} \to \mathbb{R}$ an, die

   (a) stetig, aber in $x_0 = 0$ nicht differenzierbar;
   (b) differenzierbar, aber nicht gleichmäßig stetig;
   (c) differenzierbar, aber in $x_0 = 0$ nicht stetig differenzierbar sind

   Answer:

   (a) $f(x) = |x|$
   (b) $f(x) = x^2$
   (c) $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

3. Was bedeuten die Landau-Symbole $\mathcal{o}(h)$, $\mathcal{O}(h^2)$ und $\mathcal{o}(1)$? Wie lassen sich Stetigkeit und Differenzierbarkeit mit ihrer Hilfe ausdrücken?

   Answer:

   (a) $f(h) = \mathcal{o}(h) \Leftrightarrow \lim_{h\to 0} \frac{f(h)}{h} = 0$
   (b) $f(h) = \mathcal{O}(h^2) \Leftrightarrow \limsup_{h\to 0} \left|\frac{f(h)}{h^2}\right| < \infty$
   (c) $f(h) = \mathcal{o}(1) \Leftrightarrow \lim_{h\to 0} f(h) = 0$

   If there is a number $\alpha \in \mathbb{R}$ such that $f(x_0+h) = f(x_0) + \alpha h + \mathcal{o}(h)$, then we say that the function f is differentiable in the x_0 point.

   We say that f is continuous in x_0 if $f(x_0 + h) = f(x_0) + \mathcal{o}(1)$

4. Für welche reellen α ist $|x|^\alpha$ in $x = 0$ reell differenzierbar?

   Answer: TODO

5. Wie lautet die Produktregel für Ableitungen? Warum gilt sie (Beweis)?

   Answer:

   Consider two functions f and g that are both differentiable in some x_0 point of their domain. Then the fg function is also differentiable in x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

   Proof:

$$
\begin{aligned}
&\lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x)g(x_0+h) + f(x)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{(f(x_0+h) - f(x))g(x_0+h) + f(x)(g(x_0+h) - g(x_0))}{h}
\end{aligned}
$$

Since g is continuous in x_0 and the $\lim_{h\to 0}\frac{f(x_0+h)-f(x)}{h}$ and $\lim_{h\to 0}\frac{g(x_0+h)-g(x)}{h}$ exist, thus the above limit also exists and

$$\begin{aligned}
&= \lim_{h\to 0}\frac{f(x_0+h)-f(x)}{h}g(x_0+h) + f(x)\frac{g(x_0+h)-g(x)}{h} \\
&= f'(x_0)g(x_0) + f(x_0)g'(x_0)
\end{aligned}$$

6. Wie lauten Quotienten- und Kettenregel für Ableitungen?

   Answer:

   Division: Suppose that both f and g functions are differentiable in x_0 and furthermore suppose that $g(x_0) \neq 0$. Then the function $\frac{f}{g}$ is also differentiable in x_0 and $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}$

   Chain rule: Suppose that g is differentiable in some x_0 point and f is differentiable in $y_0 = f(x_0)$. Then $f \circ g$ is also differentiable in x_0 and $(f \circ g)'(x_0) = f'(g(x_0))g'(x_0)$

7. Was sind die Ableitungen folgender Funktionen nach x?

   $$e^x \sin x \qquad \frac{\sin x}{\cos x} \qquad \exp(-x^2) \qquad \log\frac{1+x}{1-x} \qquad x^x$$

   Answer:

   (a) $(e^x \sin x)' = e^x \sin x + e^x \cos x$ from the product rule because $\exp' = \exp$ and $\sin' = \cos$
   (b) Suppose that $x \neq \frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$. Then $\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$ from the rule of division
   (c) $(\exp(-x^2))' = -2x\exp(-x^2)$ from the chain rule
   (d) For $x > 1$: $(\log\frac{1+x}{1-x})' = \frac{1}{\frac{1+x}{1-x}}(\frac{1+x}{1-x})' = \frac{1-x}{1+x}\frac{(1-x)+(1+x)}{(1-x)^2} = \frac{2}{1-x^2}$
   (e) For $x > 0$: $(x^x)' = (\exp(x\log x))' = x^x(\log x + x\frac{1}{x}) = x^x(\log x + 1)$

8. Wann besitzt eine Funktion $f\colon \mathbb{R} \to \mathbb{R}$ eine differenzierbare Umkehrfunktion f^{-1}?

   Answer: Suppose that f is continuous, injective and differentiable in some $x_0 \in \mathbb{R}$ point with $f'(x_0) \neq 0$. Then the inverse function $f^{-1}\colon J \to \mathbb{R}$ $(J = f(\mathbb{R}))$ exists, also injective and continuous, and furthermore differentiable in $y_0 = f(x_0)$ with $(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}$

9. Wie lautet der Mittelwertsatz (der Differentialrechnung)? Wie lautet der Satz von Rolle?

   Answer:

   Mean Value Theorem: Consider some continuous function $f\colon [a,b] \to \mathbb{R}$ which is differentiable on (a,b). Then $\exists c \in (a,b)\colon \frac{f(b)-f(a)}{b-a} = f'(c)$

   Rolle: Consider some continuous function $f\colon [a,b] \to \mathbb{R}$ which is differentiable on (a,b), and suppose that $f(a) = f(b)$. Then $\exists c \in (a,b)\colon f'(c) = 0$

10. Warum gilt der Satz von Rolle (Beweisskizze)?

    Answer: Since f is continuous on $[a,b]$, it'll take on its extrema $m = \min_{x\in[a,b]} f(x)$ and $M = \max_{x\in[a,b]} f(x)$. If $m = M$, then the function is constant, and thus $f'(x) = 0$ $(\forall x \in (a,b))$. If $m \neq M$, then at least one of the place of extrema is an inner point of $[a,b]$. Without loss of generality, suppose that $m = f(c)$ for some $c \in (a,b)$ (otherwise consider $-f$). If c is a minimum point, then it's also a local minima, and we know that for inner local minimum and maximum, if the function is differentiable at that point, then the derivative is 0. Since we assumed f to be differentiable on (a,b), thus $f'(c) = 0$.

11. Wie lauten die Regeln von de l'Hôpital?

    Answer: Consider two functions f, g that are differentiable on an open interval containing some x_0 point (except maybe at this point) and suppose that $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ and that $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ also exists and $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$

    Variants

    - If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \pm\infty$ and $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists (and finite!), then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ also exists and $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$
    - Instead of a fix x_0 point we can consider the limits in $\pm\infty$
    - We can use the theorem repeatedly as long as we have $0/0$ or $\pm\frac{\infty}{\infty}$ and the functions are sufficiently many times differentiable

12. Welche Werte haben die stetigen Fortsetzungen folgender Funktionen in $x = 0$?

    $$f(x) = \frac{\sin x}{x} \qquad g(x) = \frac{\cos x - 1}{x^2} \qquad h(x) = \frac{\log(1+x)}{x} \qquad r(x) = \frac{x}{e^x - 1}$$

    Answer:

    (a) yes, with $f(x) = 0$ because $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$ (from l'Hôpital)
    (b) yes, with $g(x) = -\frac{1}{2}$ because $\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x} = -\frac{1}{2}$ (from l'Hôpital and 12a)
    (c) yes, with $h(x) = 1$ because $\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$ (from l'Hôpital)
    (d) yes, with $r(x) = 1$ because $\lim_{x\to 0} \frac{x}{e^x - 1} = \lim_{x\to 0} \frac{1}{e^x} = 1$ (from l'Hôpital)

13. Berechne

    $$\lim_{x\to+\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

    Answer: $\lim_{x\to+\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to+\infty} \frac{e^x(1 - e^{-2x})}{e^x(1 + e^{-2x})} = \lim_{x\to+\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$

14. Wie lauten die Ungleichungen von Young und Hölder?

    Answer:

    Young's inequality: Consider $x, y \geq 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. It holds that $x^{1/p} + y^{1/q} \leq \frac{x}{p} + \frac{y}{q}$.

    Hölder's inequality: For $x, y \in \mathbb{C}^n$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ it holds that $\sum_{k=1}^n |x_k| |y_k| \leq \|x\|_p \|y\|_q$ where $\|x\|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$ (that is: the p-norm $(1 \leq p \leq \infty)$)

15. Skizziere die Funktionen $\sin x$ und $\cos x$, beschreibe ihre Nullstellen, Ableitungen, Monotonie, Konvexität und Konkavität, und erläutere unsere Definition von π.

    Answer: *(not relevant for the first exam)*

    - $\sin(x) = 0 \Leftrightarrow x = k\pi \ (k \in \mathbb{Z})$
    - $\cos(x) = 0 \Leftrightarrow x = \frac{\pi}{2} + k\pi \ (k \in \mathbb{Z})$
    - $\sin' = \cos, \cos' = -\sin$

16. Sei $f\colon \mathbb{R} \to \mathbb{R}$ eine zweimal differenzierbare Funktion. Welche (notwendige) Bedingung ist erfüllt, wenn f an der Stelle x_0 ein lokales Maximum besitzt? Unter welcher (hinreichenden) Bedingung besitzt f an der Stelle x_0 ein lokales Maximum?

    Answer: (*these conditions are related to strict local maxima*)

    Necessary condition: $f'(x_0) = 0$

    Sufficient condition: $f'(x_0) = 0$ and $f''(x_0) < 0$

17. Wann heißt eine Funktion $f\colon (a,b) \to \mathbb{R}$ konvex? Wann heißt sie strikt konvex?

    Answer: f is convex if $\forall x, y \in (a,b)$ such that $x < y, \forall t \in (0,1)\colon f(ty + (1-t)x) \leq tf(y) + (t-1)f(x)$. For strictly convex this holds with $<$ instead of $\leq$.

18. Die Funktion $f\colon (a,b) \to \mathbb{R}$ sei zweimal diferenzierbar. Wie lassen sich Konvexität und strikte Konvexität durch Bedingungen an die zweite Ableitung ausdrücken?

    Answer:

    (a) f is convex if and only if $f'' \geq 0$

    (b) if $f'' > 0$ then f is strictly convex (it doesn't hold in the other direction: for example x^4 is strictly convex, but $(x^4)'' = 12x^2 = 0$ for $x = 0$)

19. Wieviele Minima bzw. Maxima kann eine strikt konvexe Funktion $f\colon [a,b] \to \mathbb{R}$ haben? (Gib alle möglichen Zahlen an.)

    Answer: TODO A convex function is also continuous (see sheet 9. problem 34.). A continuous function on a bounded and closed interval takes on it's extremal values. Suppose that $m = \min_{x\in[a,b]} f(x) = f(p)$ and $M = \max_{x\in[a,b]} f(x) = f(q)$.

20. Wo sind (reelle) Potenzreihen differenzierbar? Wie lautet die Ableitung?

    Answer: A real power series $p(x) = \sum_{n=0}^{\infty} a_n x^n$ with convergence radius $\rho \in [0, \infty)$ is differentiable infinitely many times for any $|x| < \rho$, and $p'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$. We can define the n^{th} $(n > 1)$ derivative recursively as $p^{(n)}(x) = (p^{(n-1)}(x))'$ and with induction we get $p^{(n)}(x) = \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} a_{k+n} x^k$

21. Wie ist der Raum $\mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ definiert? Was bedeutet seine Vollständigkeit für die Vertauschbarkeit von Differentiation und Grenzwertbildung einer Funktionenfolge $f_n\colon \mathbb{R} \to \mathbb{R}$?

    Answer $\mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ is the vectorspace of continuously differentiable, bounded and continuous functions defined on $\mathbb{R}$ with bounded derivative. $\mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ is a Banach-space (that is: normed and complete) with the $\|f\|_{C^1} = \|f\|_{C^0} + \|f'\|_{C^0}$ $(f \in \mathcal{BC}^1(\mathbb{R}, \mathbb{R}))$ norm (with $\|.\|_{C^0}$ being the supremum-norm with which $\mathcal{BC}(\mathbb{R}, \mathbb{R})$ is complete).

    Since $\mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ with the norm $\|.\|_{C^1}$ is complete, any Cauchy sequence $(f_n) \in \mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ is also convergent (under the norm), thus $\exists f \in \mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ such that $\lim_{n\to\infty} \|f_n - f\|_{C^1} = 0$, or equivalently: $\|f_n - f\|_{C^0} \to 0$ and $\|f_n' - f'\|_{C^0} \to 0$ $(n \to \infty)$. If we consider the convergence under the norm this also means that $(\lim_{n\to\infty} f_n)' = f' = \lim_{n\to\infty} f_n'$, or: the derivative of the limit is the limit of the derivatives, that is: for any convergent $(f_n) \in \mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ the order of differentiation and taking the limit can be exchanged with regards to the $\|.\|_{C^1}$.

22. Wie lautet das n-te Taylor-Polynom? Wie kann das Restglied ausgedrückt werden?

    Answer: *(not relevant for the first exam)*

23. Wann (und wo) wird eine reelle Funktion durch ihre Taylor-Reihe dargestellt? Gib ein Beispiel und ein Gegenbeispiel.

    Answer: *(not relevant for the first exam)*

24. Wie lauten die Taylor-Reihen folgender Funktionen in $x_0 = 0$?

    $$e^x,\ \sin x,\ \arctan x,\ (1+x)^\alpha,\ \log(1+x)$$

    Answer: *(not relevant for the first exam)*