1. Wann heißt eine Reihe konvergent, wann absolut konvergent?

   Answer: The series $\sum_{n=0}^{\infty} a_n$ converges when the sequence of partial sums $s_n = \sum_{k=0}^{n} a_k$ converges. The series $\sum_{n=0}^{\infty} a_n$ converges absolutely, when the series $\sum_{n=0}^{\infty} |a_n|$ converges.

2. Für welche komplexen q existiert $\sum_{n=0}^{\infty} q^n$? Welchen Wert hat die Summe?

   Answer: It exists for $|q| < 1$, and $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$.

$$s_n = s_{n-1} + 1, s_{n-1} = s_n - q^n \Rightarrow s_n = q(s_n - q^n) + 1$$

   thus $s_n = \frac{1-q^{n+1}}{1-q}$. s_n converges exactly when $|q| < 1$.

3. Warum divergiert die harmonische Reihe?

   Answer: For a similar argument that is used in the proof of the Verdichtungs-Kriterium: $\sum_{k=1}^{2^n} \frac{1}{k} > \sum_{k=0}^{n} 2^n \frac{1}{2^n} = n$ (note: indexes might be off-by-one, but this is the main idea).

4. Wann konvergiert eine Reihe positiver Summanden?

   Answer: When the sequence of its partial sums is bounded.

5. Wie lauten Cauchy-, Majoranten-, Verdichtungs- und Leibniz-Kriterium für die Konvergenz unendlicher Reihen?

   Answer:

   - Cauchy-criterium: $\sum_{n=0}^{\infty} a_n$ converges exactly if $\forall \epsilon > 0 \colon \exists N \in \mathbb{N} \colon \forall m, n > N \colon \ |\sum_{k=0}^{n} a_k - \sum_{k=m}^{m} a_k| = \left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$
   - Majorant: consider two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. If $|a_n| \leq b_n$ for almost all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} b_n$ converges, then so is $\sum_{n=0}^{\infty} a_n$, and furthermore it converges absolutely.
   - Verdichtungs: Consider $(a_n) \geq 0$ monoton decreasing sequence that converges to 0. Then $\sum_{n=0}^{\infty} a_n$ converges exactly when $\sum_{n=0}^{\infty} 2^n a_{2^n}$
   - Leibniz: Consider $(a_n) \geq 0$ monoton decreasing sequence that converges to 0. Then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

6. Wie lauten Wurzel- und Quotientenkriterium für die Konvergenz unendlicher Reihen?

   Answer:

- Root-test: if $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$ then $\sum_{n=0}^{\infty} a_n$ converges absolutely.
- Ratio-test: if $a_n = 0$ for at most finitely many $n \in \mathbb{N}$ and $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

7. Bei welchen der folgenden Reihen gibt das Quotientenkriterium Aufschluss über Konvergenz oder Divergenz?

$$\sum_{n=0}^{\infty} \frac{n!}{n^n},\ \sum_{n=0}^{\infty} \frac{1}{n^2},\ \sum_{n=0}^{\infty} \frac{1}{(3+(-1)^n)^n}$$

Answer:

- $\limsup_{n\to\infty} \left|\frac{(n+1)!}{(n+1)^{n+1}} / \frac{n!}{n^n}\right| = \limsup_{n\to\infty} \left|\frac{n}{n+1}\right|^n = 1/e < 1 \Rightarrow$ converges
- $\limsup_{n\to\infty} \left|\frac{1/(n+1)^2}{1/n^2}\right| = 1 \Rightarrow$ inconclusive
- $\limsup_{n\to\infty} \left|\frac{(3+(-1)^n)^n}{(3+(-1)^{n+1})^{n+1}}\right| = \limsup_{n\to\infty} \left|\frac{4^n}{2^{n+1}}\right| = 1 \Rightarrow$ inconclusive

8. Wie lautet der kleine Umordnungssatz absolut konvergenter Reihen?

Answer: Consider any $\sum_{n=0}^{\infty} a_n$ absolut convergent series, $\tau\colon \mathbb{N} \to \mathbb{N}$ permutation and define $b_n = a_{\tau^{-1}(n)}$. Then $\sum_{n=0}^{\infty} b_n$ is also absolutely convergent and $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n$

9. Wie lautet der große Umordnungssatz absolut konvergenter Reihen?

Answer: Consider $a_{ij} : \mathbb{N}^2 \to \mathbb{K}$, $\tau : \mathbb{N}^2 \to \mathbb{N}$ bijection and let $b_n = a_{ij}$ for corresponding i, j such that $n = \tau(i, j)$. Suppose furthermore that $\sum_{n=0}^{\infty} b_n$ converges absolutely. Then the series $\sigma_i = \sum_{j=0}^{\infty} a_{ij}$ $(\forall i \in \mathbb{N})$ and $s = \sum_{i=0}^{\infty} \sigma_i = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}$ converge, moreover they converge absolutely, and furthermore $\sum_{i=0}^{\infty} \sigma_i = s = \sum_{n=0}^{\infty} b_n$

10. Welche der folgenden Reihen konvergieren, welche konvergieren absolut?

$$\sum_{n=0}^{\infty} \frac{1}{n},\ \sum_{n=0}^{\infty} \frac{(-1)^n}{n},\ \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2},\ \sum_{n=0}^{\infty} \frac{x^n}{n!} \in \mathbb{C}$$

Answer:

- $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Verdichtungs-Kriterium)
- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (Leibniz), but not absolutely, see previous point
- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely, since $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges (Verdichtungs-Kriterium)

- if $x \neq 0$ then $\limsup_{n\to\infty} \left|\frac{x^{n+1}/(n+1)!}{x^n/(n)!}\right| = \limsup_{n\to\infty} \left|\frac{x}{n+1}\right| = 0 \Rightarrow$ converges (from Quotientenkriterium). If $x = 0$ then it's converges trivially

11. Für welche reellen/komplexen s konvergiert die Reihe $\sum_{n=0}^{\infty} n^{-s}$ der Riemannschen ζ-Funktion?

    Answer: Consider first $q \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{1}{n^q} \Leftrightarrow \sum_{n=0}^{\infty} \frac{2^n}{(2^n)^q} = \sum_{n=0}^{\infty} (2^n)^{1-q} = \sum_{n=0}^{\infty} (2^{1-q})^n$ which converges for $1 < q \in \mathbb{R}$ (from geometric series) and diverges for $1 \leq q \in \mathbb{R}$.

    Now for $q \in \mathbb{C}, q = a + ib$ $(a, b \in \mathbb{R})$: $|n^{-q}| = |n^{-a}| \left|(e^{-ib})^{\log n}\right| = |n^{-a}|$, thus $\zeta(q)$ converges absolutely for $\Re(q) > 1$, and consequently it'll also converge conditionally.

12. Was ist eine Potenzreihe? Was ist ihr Konvergenzradius? Wie berechnet er sich?

    Answer: The formal powerseries centered in $c \in \mathbb{C}$ with coefficients $(a_n) \in \mathbb{C}$ is defined as the $p(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ series. The radius of convergence $\rho \in [0, +\infty)$ is defined as

    $$\rho = 1/\limsup_{n\to\infty} |a_n|^{1/n}$$

    If $\rho > 0$ then we talk about power series. A $p(x)$ power series converges absolutely $\forall x \in \mathbb{C}$: $|x - c| < \rho$

13. Wann ist das Produkt zweier Potenzreihen wieder eine Potenzreihe? Wie lautet sie? Wie hängen die Konvergenzradien der Potenzreihen und ihres Produktes zusammen?

    Answer:

    From the product theorem for absolutely convergent series we know that whenever two series are absolutely convergent, then so is their product. Their product is given by such a $c_n = a_i b_j$ sequence, that contains every $(i, j) \in \mathbb{N}^2$ exactly once, and since absolutely convergent series can be rearranged arbitrarily, the value of $\sum_{n=0}^{\infty} c_n$ does not depend on the order of $a_j b_j$. One such sequence is defined by the Cauchy product with $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, which in case of power series will look like $c_k = \sum_{k=0}^{n} a_k x^k b_{n-k} x^{n-k} = x^n \sum_{k=0}^{n} a_k b_{n-k}$, thus the product of two power series is again a power series. In order the product of two power series to be well defined, both of them should only be considered inside their convergence radiuses, thus

$|x| < \min(\rho_1, \rho_2)$ must hold, where ρ_1, ρ_2 are the convergence radiuses of the first and second power series respectively, and for such x the product will again be abslutely convergent.

14. Wie lauten die Dastellungen von $\exp(x), \sin(x), \cos(x), \sinh(x), \cosh(x)$ als Potenzreihen?

    Answer:

    - $\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
    - $\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$
    - $\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$
    - $\sinh z = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$
    - $\cosh z = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$

15. Wie hängen $e^z, \sin(z), \cos(z), \sinh(z), \cosh(z)$ im Komplexen zusammen?

    - $e^{iz} = \cos z + i \sin z$
    - $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$
    - $\cos z = \frac{e^{iz} + e^{-iz}}{2}$
    - $\sinh z = \frac{e^{z} - e^{-z}}{2}$
    - $\cosh z = \frac{e^{z} + e^{-z}}{2}$
    - $\cosh z = \cos iz$
    - $\sinh z = -i \sin iz$