1. Wann heißt eine Funktion f in einem Punkt x_0 stetig? Welche äquivalente Definitionen der Stetigkeit gibt es (wenigstens drei verschiedene)?

   f is continuous in x_0 whenever one of the following equivalent conditions holds

   - $\forall (x_n) \in D\colon \lim_{n\to\infty} x_n = x \Rightarrow f(\lim_{n\to\infty} x_n) = f(x) = \lim_{n\to\infty} f(x_n)$
   - $\forall \epsilon > 0\colon \exists \delta > 0\colon \forall x \in D, |x - x_0| < \delta\colon |f(x) - f(x_0)| < \epsilon$
   - for any neighbourhood V of $f(x_0)$ there is a neighbourhood U of x_0 in D such that $f(U) \subset V$

2. Wann heißt eine Funktion f auf einer Menge $D \subseteq \mathbb{R}$ bzw. $D \subseteq \mathbb{C}$ stetig?

   f is continuous on the set D whenever f is continuous at all points $a \in D$

3. Sei $U = \{a_1, a_2, \dots, a_N\} \subset \mathbb{R}$ eine endliche Menge reeller Zahlen. Gib eine Funktion $f\colon \mathbb{R} \to \mathbb{R}$ an, die auf $D = \mathbb{R} \setminus U$ stetig, auf U aber unstetig ist.

   $f(x) = \begin{cases} 1 & x \in U \\ 0 & \text{otherwise} \end{cases}$

   Since any convergent $(x_n) \in \mathbb{R}$ will only contain at most finitely many element from U, and consequently if $x = \lim_{n\to\infty} x_n$, then $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1 \Leftrightarrow x \notin U$

4. Gib eine Fuktion $f\colon \mathbb{R} \to \mathbb{R}$ an, die nirgends stetig ist.

   $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

5. Wie lautet der Zwischenwertsatz?

   Consider any $D = [a, b] \subset R$ interval, and let $f\colon D \to \mathbb{R}$ be continuous. Then f takes on any value in $[f(a), f(b)] \cup [f(b), f(a)]$

6. Warum hat jede durch eine stetige Funktion $g\colon [0,1] \to [0,1]$ gegebene Iteration $x_{n+1} = g(x_n)$ (mindestens) einen Fixpunkt?

   Consider $f(x) = g(x) - x$, $f(0) = g(0) - 0 \geq 0, f(1) = f(1) - 1 \leq 0$. Thus from the intermediate value theorem $\exists c \in [0,1]\colon f(c) = g(c) - c = 0 \Leftrightarrow g(c) = c$. If a so defined $(x_n) \in [0,1]$ converges, then it'll converge to the fixpoint:

   $$f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1}$$

   But the convergence is not guaranteed, consider $g(x) = 1 - x$ and $x_0 = 0$, which is divergent.

7. Wie lässt sich unter Benutzung des Zwischenwertsatzes zeigen, dass die Gleichung $\exp(x) = -x$ eine reelle Lösung besitzt?

   Answer: Consider the $g(x) = e^x + x$ function on the $[-1, 0]$ interval. $g(-1) = e^{-1} - 1 = \frac{1}{e} - 1 < 1/2 - 1 < -1/2$ and $g(0) = e^0 - 0 = 1 > 0$, thus from the intermediate value theorem there must be some $c \in [-1, 0]\colon g(c) = 0$.

8. Wann heißt eine Funktion $f\colon D \to \mathbb{R}, D \subseteq \mathbb{R}$, gleichmäßig stetig? Unter welcher (hinreichenden) Bedingung sind stetige Funktionen gleichmäßig stetig?

   Answer: f is uniformly continuous if $\forall \epsilon > 0\colon \exists \delta > 0\colon \forall x, y \in D\colon |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

   If D is closed and bounded, and f is continuous on D, then f is also uniformly continuous on D.

9. Welche dieser Funktionen $f\colon \mathbb{R} \to \mathbb{R}$ sind stetig, welche gleichmäßig stetig?

$$|x|,\ \exp(x),\ x^2,\ \sin(x),\ \tfrac{x^3+1}{x^4-1},\ \lceil x \rceil - x$$

Hierbei bezeichnet die Gauß-Klammer, $\lceil x \rceil$, die größte ganze Zahl, die kleiner oder gleich x ist.

Answer:

(a) $|x|$ is continuous and furthermore is absolute continuous: consider $\epsilon > 0$ and some $x, y \in \mathbb{R}\colon |x-y| < \epsilon$ then $\epsilon > |x-y| > |x| - |y|$ and $\epsilon > |y-x| > |y| - |x|$ and thus $||x| - |y|| < \epsilon$. So $|f(x) - f(y)| = ||x| - |y|| < \epsilon$ and consequently it's absolute continuous.

(b) $\exp(x)$ is continuous, since it's defined by a powerseries with convergence radius $\rho = \infty$. A powerseries is continuous at every point inside it's convergence radius. It's not absolutely continuous: suppose indirectly that it was: $\forall \epsilon > 0\colon \exists \delta_\epsilon > 0\colon \forall x, y \in \mathbb{R} : |x-y| < \delta_\epsilon\colon |f(x) - f(y)| < \epsilon$, or $|e^x - e^y| < \epsilon$. This must hold in particular for $y = x + \delta_\epsilon/2$ $(\forall x \in \mathbb{R})$, and due to the strict monotonity of e^x it also holds that $0 < e^{x+\delta/2} - e^x < \epsilon$ or equivalently $0 < e^x(e^{\delta/2} - 1) < \epsilon$. But $\lim_{x\to\infty} e^x(e^{\delta/2} - 1) = +\infty$, contradicion.

(c) x^2 is continuous, since x is continuous, and since the multiple of two continuous function is continuous, thus so is x^2. On the other hand it's not uniformly continuous: suppose it was, that is $\forall \epsilon > 0\colon \exists \delta_\epsilon > 0\colon x, y \in \mathbb{R}\colon |x-y| < \delta_\epsilon \Rightarrow \left|x^2 - y^2\right| < \epsilon$. Consider now some $x > \delta_\epsilon > 0$ and $y = x + \delta_\epsilon/2$, the condition of uniform continuity must hold for these x, y as well:

$$0 < (x + \delta_\epsilon/2)^2 - x^2 < \epsilon$$

or

$$0 < x\delta_\epsilon + \delta_\epsilon^2/4 < \epsilon \tag{1}$$

But since $\lim_{x\to\infty} x\delta_\epsilon + \delta_\epsilon^2/4 = \infty$, (1) cannot hold for every $x \in \mathbb{R}$.

(d) sin is defined by a powerseries with a radius of convergence $\rho = \infty$ thus it's continuous on $\mathbb{R}$. Consider now sin on the closed and bounded interval $[0, 4\pi]$ (4π was chosen here intentionally, so $\forall x, y \in \mathbb{R}$ at least one of $x - 2k\pi, y - 2k\pi \in [0, 4\pi]$ or $x - 4k\pi, y - 4k\pi \in [0, 4\pi]$ will hold for some corresponding $k \in \mathbb{Z}$), where it's now uniformly continuous, thus $\forall \epsilon > 0\colon \exists \delta > 0\colon \forall x, y \in [0, 4\pi]\colon\ |x-y| < \delta \Rightarrow |\sin x - \sin y| < \epsilon$. sin is furthermore periodic with a period of 2π, thus $\forall x, y \in \mathbb{R}\colon\ |x-y| < \delta \Rightarrow |\sin x - \sin y| < \epsilon$ will also hold.

(e) $x^3 + 1$ and $x^4 - 1$ are continuous on $\mathbb{R}$, so f is continuous exactly when $x^4 - 1 \neq 0$, thus f is continuous on $\mathbb{R} \setminus \{1\}$. On the other hand $\lim_{h\to 1^-} f(1+h) = -\infty$ and $\lim_{h\to 1^+} f(1+h) = +\infty$, so there will be $x < 1 < y$ arbitrarily close to each other such that $|f(x) - f(y)|$ is arbitrarily large.

(f) $f(x) = x$ $(\forall x \in [0,1))$ and f has a periodicity of 1, thus $\lim_{x\to 1-} = 1$ but $\lim_{x\to 1+} = 0$ thus f is continuous on $\mathbb{R} \setminus \mathbb{Z}$. On the other hand f is not uniformly continuous, because we can make arbitrarily close $3/4 < x < 1 < y < 5/4$ but $|f(x) - f(y)| > 1/2$.

10. Gib stetige Funktionen $f\colon D \to \mathbb{R}, D \subseteq \mathbb{R}$ an, die ihr Supremum annehmen, und solche, die ihr Supremum nicht annehmen. Unter welcher (hinreichenden) Bedingung nimmt eine stetige Funktion ihr Supremum an?

Answer: Let $D = [0,1)$ and $f\colon D \to \mathbb{R}, f(x) = x^2$. $sup f = 1$, but f does not take on it's supremum (because $f(x) = 1 \Leftrightarrow x = \pm 1 \notin D$).

Let $D = [0,1]$ and $f\colon D \to \mathbb{R}, f(x) = x^2$. $\sup f = \max f = 1$, thus f takes on it's supremum.

If D is bounded and closed, then $f\colon D \to \mathbb{R}$ takes on its supremum.

11. Sind die Bilder von Intervallen unter stetigen Abbildungen $f\colon \mathbb{R} \to \mathbb{R}$ wieder Intervalle? Sind stetige Bilder offener Intervalle wieder offene Intervalle?

    Answer: Yes (**Why?**). The image of $f\colon (0,1) \to \mathbb{R}$ with $f(x) = 1$ is a closed inteval $[1,1]$.

12. Wo sind Potenzreihen stetig? Wo sind sie gleichmäßig stetig?

    Answer: Consider $p(x)$ powerseries centered at 0 with convergence radius of $\rho \in [0, \infty)$, and circle of convergence $C = \{x\colon |x| < \rho\}$. p is continuous at each point of C. Consider any $D \subset C$ that is closed and bounded. Then f is uniformly continuous on D. Contrary to the normal continuity, uniform continuity cannot be extended to the whole C by considering a closed circle of radius $0 \leq r < \rho$ inside C, and taking the limit $r \to \rho$.

13. Wann existiert die Inverse f^{-1} einer stetigen Funktion $f\colon [a,b] \to \mathbb{R}$? Wann ist die Inverse stetig?

    Answer: f^{-1} exists exactly when f is strictly monotonous. Whenever the inverse of a continuous function exists, it's always continuous.

14. Was ist ein normierter Vektorraum über $\mathbb{R}$ bzw. $\mathbb{C}$? Was ist ein Banachraum?

    Answer: Consider V vectorspace over $\mathbb{K}$. Then the $\|.\| : V \to \mathbb{R}$ function is a norm, if it satisfies the following conditions ($\forall x, y \in V, \lambda \in \mathbb{K}$):

    - $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$
    - $\|\lambda x\| = |\lambda| \, \|x\|$
    - $\|x + y\| \leq \|x\| + \|y\|$

    A normed vectorspace is complete, if every Cauchy-sequence is convergent (both property considered under the norm). A Banach-space is a complete normed vectorspace.

15. Was bedeutet Konvergenz in einem normierten Vektorraum?

    Answer: Let V is a vectorspace with norm $\|.\| : V \to \mathbb{R}$. We say that $(x_n) \in V$ converges if $\exists v \in V\colon \lim_{n\to\infty} \|v_n - v\| = 0$

16. Wie ist die Supremums-Norm für beschränkte, stetige Funktionen $f\colon D \to \mathbb{R}, D \subseteq \mathbb{R}$, definiert? Warum ist sie tatsächlich eine Norm?

    Answer: $\|f\|_{sup} = \sup\{|f(x)|\colon x \in D\}$.

    The above defined $\|.\|_{sup}$ function satisfies the norm properties:

    $\forall f, g : D \to \mathbb{R}, \forall \lambda \in \mathbb{R}$:

    - $\|f\| \geq 0$, $\|f\| = 0 \Leftrightarrow f = 0$
    - $\|\lambda f\| = \sup\{|\lambda f(x)|\colon x \in D\} = \sup\{|\lambda||f(x)|\colon x \in D\}$
      $= |\lambda| \sup\{|f(x)|\colon x \in D\} = |\lambda| \, \|f\|$
    - $\|f + g\| = \sup\{|f(x) + g(x)|\colon x \in D\} \leq \sup\{|f(x)| + |g(x)|\colon x \in D\} \leq \sup\{|f(x)| + \sup\{|g(y)|\colon y \in D$
      $\sup\{|f(x)| + \|g\| : x \in D\} = \sup\{|f(x)|\colon x \in D\} + \|g\| = \|f\| + \|g\|$

17. Warum ist der Raum der beschränkten, stetigen Funktionen, $\mathcal{BC}(D, \mathbb{R})$, mit der Supremums-Norm ein Banachraum?

    Answer: The $\|.\|_{\text{sup}}$ is indeed a norm, as proved in the previous point. Now we only have to prove, that $\mathcal{BC}(D, \mathbb{R})$ is complete, that is: any $(f_n) \in \mathcal{BC}(D, \mathbb{R})$ Cauchy-sequence converges.

    For $x \in D$ fixed:

    $$|f_n(x) - f_m(x)| = |(f_n - f_m)(x)| \leq \sup\{|(f_n - f_m)(x)| : x \in D\} = \|f_n - f_m\|$$

Since f_n is a Cauchy-sequence in norm,

$$\forall \epsilon > 0 \colon \exists N \in \mathbb{N} \colon n, m > N \colon \; \|f_n - f_m\| < \epsilon$$

thus for such $n, m \colon \; |f_n(x) - f_m(x)| < \epsilon$, thus f_n converges pointwise.

Let $f(x) = \lim_{n\to\infty} f_n(x)$ be defined as the pointwise limit of f_n. It remains to prove that $\lim_{n\to\infty} \|f_n - f\| = 0$ and that $f \in \mathcal{BC}(D, \mathbb{R})$.

Since f_n is a Cauchy sequence, $\|f_n - f_m\| < \epsilon$ holds for large enough n, m. It'll also hold for any $x \in D$ in particular that $|f_n(x) - f_m(x)| < \epsilon$, and by taking m to the limit we get $|f_n(x) - f(x)| \leq \epsilon$, so $\|f_n - f\| \leq \epsilon$ will hold as well, and thus $\lim_{n\to\infty} \|f_n - f\| = 0$.

As for the continuity, we have to prove that $\forall a \in D \colon \forall \epsilon > 0 \colon \exists \delta > 0 \colon \; |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$.

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| < 3\epsilon$$

Because of the continuity of each f_n $(n \in \mathbb{N})$ and because of the uniform convergence of (f_n) sequence.

18. Gib ein Beispiel einer Funktionenfolge $f_n \colon [0,1] \to [0,1]$ an, die punktweise aber nicht gleichmäßig konvergiert.

    Answer: $f_n(x) = x^n$

19. Auf welchen (möglichst großen) Intervallen konvergieren folgende Funktionenfolgen gleichmäßig?

    $$f_n(x) = \frac{1}{1+n^2x^2},\; g_n(x) = \exp(-nx^2),\; h_n(x) = \sum_{k=0}^{n} (-1)^k x^k$$

    - $f_n \in \mathcal{BC}(\mathbb{R}, \mathbb{R})$, but $\lim_{n\to\infty} f_n(0) = 1$ and $x \neq 0 \colon \; \lim_{n\to\infty} f_n(x) = 0$ thus $f(x) = \lim_{n\to\infty} f_n(x)$ is not continuous, and consequently f_n cannot converge uniformly on the whole $\mathbb{R}$, otherwise f would be continuous as well. Thus if ther is any interval I on which f_n converge uniformly, it cannot contain 0. For the same reason I cannot have 0 as its limit point, because $\lim_{x\to 0} f_n(x) = 1$ thus $\|f_n - f\| = 1$. On the other hand $\forall \delta > 0$ it converges uniformly on $[\delta, \infty)$ and on $(-\infty, -\delta]$, because $\|f_n - f\| = \|f_n\| = \frac{1}{1+n^2\delta^2} \to 0 \; (n \to \infty)$
    - $g_n \in \mathcal{BC}(\mathbb{R}, \mathbb{R})$ and $g_n(0) = 1$ but $\forall x \neq 0 \colon \; \lim_{n\to\infty} g_n(x) = 0$, thus $g(x) = \lim_{n\to\infty} g_n(x)$ is not continuous, thus g_n cannot converge on the whole $\mathbb{R}$. For similar reasons as in the previous point, g_n will converge uniformly on $[\delta, \infty)$ and on $(-\infty, -\delta]$ $(\delta > 0)$.
    - It's a powerseries with a convergence radius of

      $$\rho = 1 / \limsup_{n\to\infty} |(-1)^n|^{1/n} = \; 1$$

      thus it'll converge on $(-1, 1)$. It'll furthermore converge unformly on $[-r, r]$ $(\forall \rho > r > 0)$ (from theorem in 3.9 of Continuity).

20. Was ist die Umkehrfunktion von $\exp(x)$. Welche Funktionalgleichung erfüllt sie? Wo ist sie definiert? Wo ist sie stetig?

    Answer: Since exp is strictly monotonous and continuous, it's inverse exists and it's also continuous at its respective domain of definition:

    $$\log \colon \mathbb{R}^+ \to \mathbb{R}, \log := \exp^{-1}$$

    Since it's the inverse of a continuous function, it's continuous everywhere.

21. Wie ist die allgemeine Potenz x^α für $\alpha \in \mathbb{C}$ und $x \in \mathbb{R}^+$ definiert?

    Answer: $x^\alpha = e^{\log x \alpha}$