

1. Was ist eine Metrik? Gib drei verschiedene Beispiele!

Answer: Consider E arbitrary set. The function $d : E \times E \rightarrow \mathbb{R}$ is a metric, if satisfies the following conditions:

- $\forall x, y \in E: d(x, y) = 0 \Leftrightarrow x = y$
- $\forall x, y \in E: d(x, y) = d(y, x)$
- $\forall x, y, z \in E: d(x, z) \leq d(x, y) + d(y, z)$

Examples:

- Consider $E = \mathbb{R}$ and $d(x, y) = |x - y|$
- Consider arbitrary set E . The discrete metric is defined as $d(x, y) = \begin{cases} 0 & x = y \\ 1 & \text{otherwise} \end{cases}$
- Consider $\mathcal{BC}([0, 1], \mathbb{R})$ and $d(f, g) = \|f - g\|_{sup}$

2. Wie sind offene Teilmengen eines metrischen Raumes definiert, wie abgeschlossene Teilmengen?

Answer: Consider (E, d) metric space, and let $x \in E$. $\forall \varepsilon > 0$ let's define $B_\varepsilon(x) := \{y \in E \mid d(x, y) < \varepsilon\}$, the open ball of radius ε centered around x . The $A \subset E$ set is open, if $\forall x \in A: \exists \varepsilon > 0: B_\varepsilon(x) \subset A$. The $B \subset E$ set is closed, if its complement $E \setminus B$ is open.

3. Sind $\left\{ \begin{array}{c} \text{endliche} \\ \text{abzählbare} \\ \text{beliebige} \end{array} \right\} \left\{ \begin{array}{c} \text{Durchschnitte} \\ \text{Vereinigungen} \end{array} \right\} \left\{ \begin{array}{c} \text{offener} \\ \text{abgeschlossener} \end{array} \right\}$ Mengen wieder offen bzw. abgeschlossen?

Answer: Arbitrary union of open sets is open. Finite intersection of open sets is open. Arbitrary intersection of open sets is in general not open: consider \mathbb{R} with the standard metric, then $\bigcap_{n \in \mathbb{N}} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ closed.

Arbitrary intersection of closed sets is closed. Finite union of closed sets is closed. Arbitrary union of closed sets is in general not closed: consider \mathbb{R} with the standard metric, then $\bigcup_{n \in \mathbb{N}} [\frac{1}{n}, 1 - \frac{1}{n}] = (0, 1)$ open.

4. Wie sind Abschluss, Inneres und Rand einer Teilmenge eines metrischen Raumes definiert?

Answer: Consider (E, d) metric space, and let $A \subset E$.

The closure of A is $\overline{A} = \{x \in E \mid \forall \varepsilon > 0: B_\varepsilon(x) \cap A \neq \emptyset\}$

The inner of A is $\mathring{A} = \{x \in A \mid \exists \varepsilon > 0: B_\varepsilon(x) \subset A\}$

The boundary of A is $\partial A = \overline{A} \setminus \mathring{A}$

5. Was sind Abschluss, Inneres und Rand folgender Teilmengen der reellen Zahlen (mit deren Standardmetrik)?

$$\mathbb{Z}$$

$$\mathbb{Q}$$

$$\bigcup_{k \in \mathbb{N}} \left(\frac{1}{k+1}, \frac{1}{k} \right)$$

Answer:

- $\overline{\mathbb{Z}} = \mathbb{Z}$ because \mathbb{Z} only consists of isolated points. $\mathring{\mathbb{Z}} = \emptyset$, since $\mathring{\mathbb{Z}} = \mathbb{R} \setminus \overline{\mathbb{R} \setminus \mathbb{Z}} = \mathbb{R} \setminus \mathbb{R} = \emptyset$. $\partial \mathbb{Z} = \overline{\mathbb{Z}} \setminus \mathring{\mathbb{Z}} = \mathbb{Z}$
- \mathbb{Q} is dense in \mathbb{R} , thus $\overline{\mathbb{Q}} = \mathbb{R}$. $\mathring{\mathbb{Q}} = \mathbb{R} \setminus \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R} \setminus \mathbb{R} = \emptyset$, since the irrationals are also dense in \mathbb{R} . $\partial \mathbb{Q} = \overline{\mathbb{Q}} \setminus \mathring{\mathbb{Q}} = \mathbb{R}$

- $A := \cup_{k \in \mathbb{N}} \left(\frac{1}{k+1}, \frac{1}{k} \right) = (0, 1) \setminus \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$. $\bar{A} = [0, 1]$, $\overset{\circ}{A} = A$ since it's an union of open sets, thus it's also open and so the interior is itself. $\partial A = \bar{A} \setminus \overset{\circ}{A} = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \cup \{0\}$

6. Wann ist eine Teilmenge eines metrischen Raumes dicht?

Answer: Consider (E, d) metric space and $A \subset E$. A is dense in E , if $\bar{A} = E$.

7. Wann ist ein metrischer Raum zusammenhängend?

Answer: Consider (E, d) metric space. It's connected, if $\forall A, B \subset E$: A, B open, $A \cup B = E$, $A \cap B = \emptyset \Rightarrow$ either $A = \emptyset$ or $B = \emptyset$

8. Wie sind Zusammenhangskomponenten eines metrischen Raumes definiert? Wann heißt ein metrischer Raum total unzusammenhängend?

Answer: Consider (E, d) metric space. For some $x \in E$ we define the connected component of the point x as the union of all $C \subset E$ connected sets, that contain x . We denote the connected component of x with $\mathcal{C}(x)$. (E, d) is totally disconnected, if $\forall x \in E$: $\mathcal{C}(x) = \{x\}$

9. Sind $\left\{ \begin{array}{c} \text{endliche} \\ \text{abzählbare} \\ \text{beliebige} \end{array} \right\} \left\{ \begin{array}{c} \text{Durchschnitte} \\ \text{Vereinigungen} \end{array} \right\}$ zusammenhängender Mengen wieder zusammenhängend?

Gib gegebenenfalls ein Gegenbeispiel!

Answer: Union of connected sets is in general not connected: consider \mathbb{R} with the standard metric, and $[0, 1]$, $[2, 3]$ connected sets. $[0, 1] \cup [2, 3]$ is disconnected. Arbitrary union of connected sets U_i is connected, as long as $\cap_i U_i \neq \emptyset$.

Intersection of connected sets is not necessarily connected (think in \mathbb{R}^2 about a C and a I shaped open set, intersecting each other only at the ends of the C , creating two disconnected sets).

10. Sind $\left\{ \begin{array}{c} \text{endliche} \\ \text{abzählbare} \\ \text{beliebige} \end{array} \right\} \left\{ \begin{array}{c} \text{Durchschnitte} \\ \text{Vereinigungen} \end{array} \right\}$ kompakter Mengen wieder kompakt? Gib gegebenenfalls ein Gegenbeispiel!

Answer: Finite union of compact sets is compact. Arbitrary union of compact sets is in general not compact: $\cup_{n \in \mathbb{N}} [n, n+1] = \mathbb{N}$ is not totally bounded, thus also not compact.

Arbitrary intersection of compact sets is compact.

11. Warum ist in einem metrischen Raum jede konvergente Folge eine Cauchy-Folge? (Beweise!)

Answer: Consider (E, d) metric space, and an $(x_n) \in E$ convergent sequence. Let $\lim_{n \rightarrow \infty} x_n = x \in E$. Since x_n converges, $\forall \varepsilon > 0$: $\exists N \in \mathbb{N}$: $\forall n > N$: $d(x_n, x) < \frac{\varepsilon}{2}$. Now $\forall n, m > N$: $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) = d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$, thus (x_n) is a Cauchy sequence.

12. Wann heißt ein metrischer Raum vollständig?

Answer: Consider (E, d) metric space. E is complete, if every Cauchy sequence converges.

13. Was sind generische Mengen?

Answer: Consider (E, d) complete metric space and countable many $U_i \subset E$ ($i \in \mathbb{N}$) open and dense sets. Then $M = \cap_{i \in \mathbb{N}} U_i$ is dense in E . Any $G \supset M$ is called generic, residual or of Baire second category.

14. Wie lautet der Satz von Baire?

Answer: Consider (E, d) complete metric space and countable many $U_i \subset E$ ($i \in \mathbb{N}$) open and dense sets. Then $\bigcap_{i \in \mathbb{N}} U_i$ is dense in E .

15. Wann heißt ein metrischer Raum perfekt?

Answer: (E, d) is perfect, if it doesn't contain isolated points. Or equivalently: $\forall x \in E: x \in \overline{E \setminus \{x\}}$

16. Zeige, dass jeder nichtleere, vollständige, perfekte metrische Raum überabzählbar ist.

Answer: Consider E not empty, complete and perfect metric space. Suppose indirectly, that it's countable: let $(x_n): \mathbb{N} \rightarrow E$ bijection. Let furthermore $A_n = E \setminus \{x_n\}$. Since E is perfect, every A_n is dense in E . Since $\{x_n\}$ is closed, A_n is open. From Baire's theorem $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ should be dense in E , contradiction.

17. Was versteht man unter einer Cantor-Menge? Gib ein Beispiel an!

Answer: Let $\emptyset \neq C \subset [0, 1]$. We call C a Cantor set, if C is complete, totally disconnected and perfect.

18. Wann heißt eine Folge in einem metrischen Raum konvergent? Wann heißt sie Cauchy-Folge?

Answer: Consider (E, d) metric space, and $(x_n) \in E$ sequence. The (x_n) sequence is convergent, if $\exists x \in E: \forall \varepsilon > 0 \exists N \in \mathbb{N}: \forall n > N: d(x, x_n) < \varepsilon$. We call x the limit of the (x_n) sequence and we say that (x_n) converges against x .

The $(x_n) \in E$ sequence is a Cauchy sequence, if $\forall \varepsilon > 0: \exists N \in \mathbb{N}: \forall m, n > N: d(x_n, x_m) < \varepsilon$.

19. Wann heißt eine Abbildung zwischen metrischen Räumen stetig? Gib vier verschiedene (aber natürlich äquivalente) Definitionen!

Answer: Consider (E_1, d_1) and (E_2, d_2) metric spaces. The $f: E_1 \rightarrow E_2$ function is continuous in $a \in E_1$ if $\forall \varepsilon > 0: \exists \delta > 0: \forall x \in E_1: d_1(a, x) < \delta: d_2(f(a), f(x)) < \varepsilon$.

The $f: E_1 \rightarrow E_2$ is continuous if and only if one of the following equivalent characterisation holds (and if one holds, then all the other hold as well):

- (a) f is continuous in $\forall a \in E_1$
- (b) $\forall A \subset E_1: f(\overline{A}) \subset \overline{f(A)}$
- (c) $\forall A \subset E_2$ closed: $f^{-1}(A)$ is closed
- (d) $\forall A \subset E_2$ open: $f^{-1}(A)$ is open

20. Sind unter einer stetigen Abbildung $f: E \rightarrow E'$ zwischen metrischen Räumen die $\left\{ \begin{array}{l} \text{Bilder} \\ \text{Urbilder} \end{array} \right\}$

$\left\{ \begin{array}{l} \text{offener} \\ \text{abgeschlossener} \\ \text{vollständiger} \\ \text{zusammenhängender} \\ \text{kompakter} \end{array} \right\}$ Teilmengen wieder $\left\{ \begin{array}{l} \text{offener} \\ \text{abgeschlossener} \\ \text{vollständiger} \\ \text{zusammenhängender} \\ \text{kompakter} \end{array} \right\}$? Gib gegebenenfalls ein Gegenbeispiel!

Answer:

- Image of an open subset is not necessarily open. Let $E = \mathbb{R}$ with the standard metric, $f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto 1$ the constant one function and consider $A \subset E$ open. $f(A) = \{1\}$ is closed.

- Image of a closed subset is not necessarily closed. Consider $f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \frac{1}{1-e^{-x}}$. The subset $\mathbb{R} \subseteq \mathbb{R}$ is closed, but $f(\mathbb{R}) = (0, 1)$ is open.
 - Image of a complete subset is not necessarily complete. See previous example.
 - Image of connected or compact sets are again connected or compact, respectively.
 - Preimage of an open set is open.
 - Preimage of a closed set is closed.
 - Preimage of a complete set is not necessarily complete. Consider $(0, 1)$ metric space, let $f: (0, 1) \rightarrow \mathbb{R}$ be the constant 1 function. The preimage of the complete set $\{1\}$ is $(0, 1)$, which is not complete.
 - Preimage of connected sets is not necessarily complete. Consider the $(0, 1) \cup (2, 3)$ metric space, and the constant 1 function defined on this space. The preimage of the connected $\{1\}$ set is disconnected.
 - Preimage of compact sets is not necessarily compact. Consider the \mathbb{R} metric space, and the constant 1 function. The preimage of the compact $\{1\}$ set is not bounded, thus also not compact.
21. Wann heißen zwei Metriken äquivalent?
- Answer:** Consider (E_1, d_1) and (E_2, d_2) metric spaces. The $f: E_1 \rightarrow E_2$ function is a homeomorphism, if f is bijective, and both f and f^{-1} are continuous. We say, that E_1 and E_2 metric spaces are equivalent, if the $id: E_1 \rightarrow E_2, x \mapsto x$ function is a homeomorphism.
22. Wann heißen zwei Normen äquivalent?
- Answer:** Consider V vectorspace with two different $\|\cdot\|_1$ and $\|\cdot\|_2$. We say, that they are equivalent, if $\exists c, C \in \mathbb{R}^+$, such that $\forall v \in V: c \|v\|_2 < \|v\|_1 < C \|v\|_2$
23. Sei E ein metrischer Raum. Formuliere und beweise den Zwischenwertsatz für stetige Abbildungen $f: E \rightarrow \mathbb{R}$
- Answer:** Let E connected, $f: E \rightarrow \mathbb{R}$ continuous and $a, b \in E$. Then f takes on every value between $f(a)$ and $f(b)$. *Proof:* E is connected, thus so is $f(E)$. In \mathbb{R} connected sets are exactly the intervals, thus it'll take on any value between $f(a)$ and $f(b)$.
24. Gib drei verschiedene (aber natürlich äquivalente) Definitionen von kompakten metrischen Räumen! Gib außerdem je ein Beispiel eines kompakten und eines nicht kompakten Raumes!
- Answer:** Consider (E, d) metric space. The following statements are equivalent characterisation of the compactness of E :
- Every open cover of E has a finite subcover.
 - Every sequence in E has a convergent subsequence.
 - E is complete and totally bounded.
- The $[0, 1]$ set is compact in \mathbb{R} , but the $(0, 1)$ set is not.
25. Sei E ein metrischer Raum. Wann heißt eine Teilmenge $A \subset E$ relativ kompakt?
- Answer:** A is relative compact, if \overline{A} is compact.
26. Wie lassen sich die kompakten Teilmengen des \mathbb{R}^n charakterisieren?
- Answer:** In \mathbb{R}^n the subset $K \subset \mathbb{R}^n$ is bounded if and only if it's totally bounded. Thus closed and bounded subsets $K \subset \mathbb{R}^n$ are compact.

27. Wie lautet der Satz von Arzela-Ascoli?

Answer: Consider $(E_1, d_1), (E_2, d_2)$ metric spaces and $\mathcal{F} := \{f: E_1 \rightarrow E_2 \mid f \text{ continuous}\}$. We say, that the set \mathcal{F} is equicontinuous, if $\forall a \in E_1: \forall \varepsilon > 0: \exists \delta > 0: \forall a \in E_1: |x - a| < \delta: |f(x) - f(a)| < \varepsilon$ ($\forall f \in \mathcal{F}$). That is: δ only depends on $a \in E_1$ and $\varepsilon > 0$, but not on $f \in \mathcal{F}$.

Consider now (E, d) compact metric space, and $(F, \|\cdot\|)$ Banach space. The $\mathcal{F} \subset \mathcal{BC}^0(E, F)$ set is relative compact, if and only if both of the following statement hold:

- \mathcal{F} is equicontinuous
- $\forall x \in E: \{f(x) \mid f \in \mathcal{F}\}$ is relative compact

28. Wann sagt man, dass eine Abbildung Lipschitz-stetig ist? Wann ist eine Lipschitzstetige Abbildung eine Kontraktion?

Answer: Consider (E_1, d_1) and (E_2, d_2) metric spaces. $f: E_1 \rightarrow E_2$ is Lipschitz continuous, if $\exists L > 0: \forall x, y \in E_1: d_2(f(x), f(y)) < Ld_1(x, y)$. If $L < 1$, then we call f a contraction.

29. Wie lautet der Banachsche Fixpunktsatz?

Answer: Consider (E, d) complete metric space and $f: E \rightarrow E$ contraction with Lipschitz constant $L < 1$. Then

- f has exactly one fixed point $x_* \in E: f(x_*) = x_*$
- $\forall x_0 \in E$ converges the $x_{n+1} = f(x_n)$ ($\forall n \in \mathbb{N}$) against x_*
- the following error estimates hold:
 - $d(x_n, x_*) \leq L^n d(x_0, x_*)$
 - $d(x_n, x_*) \leq \frac{L^n}{1-L} d(x_0, x_1)$

30. Sei E ein metrischer Raum und $f_i: E \rightarrow E$ stetig. Wann sagt man, dass $\emptyset \neq A \subset E$ selbstähnlich ist? Was ist ein *iterated function system*?

Answer: Consider $f_i: E \rightarrow E$ stetig ($i = 1, \dots, m \geq 2$). A is self similar, if $A = \cup_{i=1}^m f_i(A)$. If E is a complete metric space and f_i 's are contractions ($\forall i = 1, \dots, m \geq 2$), then $\exists! A \subset E$ compact, such that $A = \cup_{i=1}^m f_i(A)$ and we call such a set of functions and iterated function system.