1. Wann heißt eine Funktion $f\colon \mathbb{R} \to \mathbb{R}$ in einem Punkt x_0 differenzierbar? Wie lässt sich die Ableitung geometisch interpretieren?

   Answer: If the $\lim_{x\to x_0} \frac{f(x_0+h)-f(x_0)}{h}$ exists, then we call the function f differentiable in the x_0 point. We call the value of $\lim_{x\to x_0} \frac{f(x_0+h)-f(x_0)}{h}$ the derivative of the function f in point x_0 and we denote it with $f'(x_0)$. Whenever the derivative of a function exists, it's unique.

   The value of the $f'(x_0)$ is the coefficient of x in the best linear approximation of f at point x_0, and it's the slope of the tangent line drawn to the function at the point $(x_0, f(x_0))$.

2. Gib Beispiele für Funtionen $f\colon \mathbb{R} \to \mathbb{R}$ an, die

   (a) stetig, aber in $x_0 = 0$ nicht differenzierbar;
   (b) differenzierbar, aber nicht gleichmäßig stetig;
   (c) differenzierbar, aber in $x_0 = 0$ nicht stetig differenzierbar sind

   Answer:

   (a) $f(x) = |x|$
   (b) $f(x) = x^2$
   (c) $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

3. Was bedeuten die Landau-Symbole $o(h)$, $\mathcal{O}(h^2)$ und $o(1)$? Wie lassen sich Stetigkeit und Differenzierbarkeit mit ihrer Hilfe ausdrücken?

   Answer:

   (a) $f(h) = o(h) \Leftrightarrow \lim_{h\to 0} \frac{f(h)}{h} = 0$
   (b) $f(h) = \mathcal{O}(h^2) \Leftrightarrow \limsup_{h\to 0} \left|\frac{f(h)}{h^2}\right| < \infty$
   (c) $f(h) = o(1) \Leftrightarrow \lim_{h\to 0} f(h) = 0$

   If there is a number $\alpha \in \mathbb{R}$ such that $f(x_0+h) = f(x_0) + \alpha h + o(h)$, then we say that the function f is differentiable in the x_0 point.

   We say that f is continuous in x_0 if $f(x_0 + h) = f(x_0) + o(1)$

4. Für welche reellen α ist $|x|^\alpha$ in $x = 0$ reell differenzierbar?

   Answer:

   (a) $\alpha = 0$: Besides the intuitive definition, it's reasonable to consider $|0|^0 = \lim_{x\to 0} x^0 = 1$ if we want the general exponentiation to be continuous as a function of its base. Since $\forall x \neq 0\colon |x|^0 = 1$, f is the constant 1 function and thus it's differentiable everywhere, also in particular in 0.
   (b) $\alpha < 0$: $\lim_{x\to 0} = \infty$ and thus f is not continuous in 0 and thus its derivative does not exists
   (c) $0 < \alpha < 1$: $f(0) = 0$ and thus $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|^\alpha}{h} = -\infty \neq +\infty = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$, thus it's not differentiable
   (d) $\alpha = 1$: $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \operatorname{sgn}(h)\,|h|^{\alpha-1} = 1 \neq -1 = \lim_{h\to 0^-} \operatorname{sgn}(h)\,|h|^{\alpha-1}$, thus it's not differentiable in 0

(e) $\alpha > 0$: $\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \operatorname{sgn}(h)\,|h|^{\alpha-1} = 0$ thus it's differentiable

5. Wie lautet die Produktregel für Ableitungen? Warum gilt sie (Beweis)?

   Answer:

   Consider two functions f and g that are both differentiable in some x_0 point of their domain. Then the fg function is also differentiable in x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

   Proof:

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x)g(x_0+h) + f(x)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{(f(x_0+h) - f(x))g(x_0+h) + f(x)(g(x_0+h) - g(x_0))}{h}
\end{aligned}$$

   Since g is continuous in x_0 and the $\lim_{h\to 0} \frac{f(x_0+h)-f(x)}{h}$ and $\lim_{h\to 0} \frac{g(x_0+h)-g(x)}{h}$ exist, thus the above limit also exists and

$$\begin{aligned}
&= \lim_{h\to 0} \frac{f(x_0+h) - f(x)}{h} g(x_0+h) + f(x)\frac{g(x_0+h) - g(x)}{h} \\
&= f'(x_0)g(x_0) + f(x_0)g'(x_0)
\end{aligned}$$

6. Wie lauten Quotienten- und Kettenregel für Ableitungen?

   Answer:

   Division: Suppose that both f and g functions are differentiable in x_0 and furthermore suppose that $g(x_0) \neq 0$. Then the function $\frac{f}{g}$ is also differentiable in x_0 and $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}$

   Chain rule: Suppose that g is differentiable in some x_0 point and f is differentiable in $y_0 = f(x_0)$. Then $f \circ g$ is also differentiable in x_0 and $(f \circ g)'(x_0) = f'(g(x_0))g'(x_0)$

7. Was sind die Ableitungen folgender Funktionen nach x?

$$e^x \sin x \qquad \frac{\sin x}{\cos x} \qquad \exp(-x^2) \qquad \log\frac{1+x}{1-x} \qquad x^x$$

   Answer:

   (a) $(e^x \sin x)' = e^x \sin x + e^x \cos x$ from the product rule because $\exp' = \exp$ and $\sin' = \cos$

   (b) Suppose that $x \neq \frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$. Then $\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$ from the rule of division

   (c) $(\exp(-x^2))' = -2x\exp(-x^2)$ from the chain rule

   (d) For $x > 1$: $(\log\frac{1+x}{1-x})' = \frac{1}{\frac{1+x}{1-x}}(\frac{1+x}{1-x})' = \frac{1-x}{1+x}\frac{(1-x)+(1+x)}{(1-x)^2} = \frac{2}{1-x^2}$

   (e) For $x > 0$: $(x^x)' = (\exp(x\log x))' = x^x(\log x + x\frac{1}{x}) = x^x(\log x + 1)$

8. Wann besitzt eine Funktion $f\colon \mathbb{R} \to \mathbb{R}$ eine differenzierbare Umkehrfunktion f^{-1}?

   Answer: Suppose that f is continuous, injective and differentiable in some $x_0 \in \mathbb{R}$ point with $f'(x_0) \neq 0$. Then the inverse function $f^{-1}\colon J \to \mathbb{R}$ $(J = f(\mathbb{R}))$ exists, also injective and continuous, and furthermore differentiable in $y_0 = f(x_0)$ with $(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}$

9. Wie lautet der Mittelwertsatz (der Differentialrechnung)? Wie lautet der Satz von Rolle?

   Answer:

Mean Value Theorem: Consider some continuous function $f\colon [a,b] \to \mathbb{R}$ which is differentiable on (a,b). Then $\exists c \in (a,b)\colon \frac{f(b)-f(a)}{b-a} = f'(c)$

Rolle: Consider some continuous function $f\colon [a,b] \to \mathbb{R}$ which is differentiable on (a,b), and suppose that $f(a) = f(b)$. Then $\exists c \in (a,b)\colon f'(c) = 0$

10. Warum gilt der Satz von Rolle (Beweisskizze)?

    Answer: Since f is continuous on $[a,b]$, it'll take on its extrema $m = \min_{x\in[a,b]} f(x)$ and $M = \max_{x\in[a,b]} f(x)$. If $m = M$, then the function is constant, and thus $f'(x) = 0$ $(\forall x \in (a,b))$. If $m \neq M$, then at least one of the place of extrema is an inner point of $[a,b]$. Without loss of generality, suppose that $m = f(c)$ for some $c \in (a,b)$ (otherwise consider $-f$). If c is a minimum point, then it's also a local minima, and we know that for inner local minimum and maximum, if the function is differentiable at that point, then the derivative is 0. Since we assumed f to be differentiable on (a,b), thus $f'(c) = 0$.

11. Wie lauten die Regeln von de l'Hôpital?

    Answer: Consider two functions f, g that are differentiable on an open interval containing some x_0 point (except maybe at this point) and suppose that $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ and that $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ also exists and $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$

    Variants

    - If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \pm\infty$ and $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists (and finite!), then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ also exists and $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$
    - Instead of a fix x_0 point we can consider the limits in $\pm\infty$
    - We can use the theorem repeatedly as long as we have $0/0$ or $\pm\frac{\infty}{\infty}$ and the functions are sufficiently many times differentiable

12. Welche Werte haben die stetigen Fortsetzungen folgender Funktionen in $x = 0$?

    $$f(x) = \frac{\sin x}{x} \qquad g(x) = \frac{\cos x - 1}{x^2} \qquad h(x) = \frac{\log(1+x)}{x} \qquad r(x) = \frac{x}{e^x - 1}$$

    Answer:

    (a) yes, with $f(x) = 0$ because $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$ (from l'Hôpital)
    (b) yes, with $g(x) = -\frac{1}{2}$ because $\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x} = -\frac{1}{2}$ (from l'Hôpital and 12a)
    (c) yes, with $h(x) = 1$ because $\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$ (from l'Hôpital)
    (d) yes, with $r(x) = 1$ because $\lim_{x\to 0} \frac{x}{e^x - 1} = \lim_{x\to 0} \frac{1}{e^x} = 1$ (from l'Hôpital)

13. Berechne

    $$\lim_{x\to+\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

    Answer: $\lim_{x\to+\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to+\infty} \frac{e^x(1 - e^{-2x})}{e^x(1 + e^{-2x})} = \lim_{x\to+\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$

14. Wie lauten die Ungleichungen von Young und Hölder?

    Answer:

    Young's inequality: Consider $x, y \geq 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. It holds that $x^{1/p} + y^{1/q} \leq \frac{x}{p} + \frac{y}{q}$.

    Hölder's inequality: For $x, y \in \mathbb{C}^n$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ it holds that $\sum_{k=1}^n |x_k|\,|y_k| \leq \|x\|_p \|y\|_q$ where $\|x\|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$ (that is: the p-norm $(1 \leq p \leq \infty)$)

15. Skizziere die Funktionen $\sin x$ und $\cos x$, beschreibe ihre Nullstellen, Ableitungen, Monotonie, Konvexität und Konkavität, und erläutere unsere Definition von π.

    Answer: *(not relevant for the first exam)*

    - $\sin(x) = 0 \Leftrightarrow x = k\pi \ (k \in \mathbb{Z})$
    - $\cos(x) = 0 \Leftrightarrow x = \frac{\pi}{2} + k\pi \ (k \in \mathbb{Z})$
    - $\sin' = \cos, \cos' = -\sin$

16. Sei $f : \mathbb{R} \to \mathbb{R}$ eine zweimal differenzierbare Funktion. Welche (notwendige) Bedingung ist erfüllt, wenn f an der Stelle x_0 ein lokales Maximum besitzt? Unter welcher (hinreichenden) Bedingung besitzt f an der Stelle x_0 ein lokales Maximum?

    Answer: (*these conditions are related to strict local maxima*)

    Necessary condition: $f'(x_0) = 0$

    Sufficient condition: $f'(x_0) = 0$ and $f''(x_0) < 0$

17. Wann heißt eine Funktion $f : (a, b) \to \mathbb{R}$ konvex? Wann heißt sie strikt konvex?

    Answer: f is convex if $\forall x, y \in (a, b)$ such that $x < y, \forall t \in (0, 1)$: $f(ty + (1 - t)x) \leq tf(y) + (t - 1)f(x)$. For strictly convex this holds with $<$ instead of $\leq$.

18. Die Funktion $f : (a, b) \to \mathbb{R}$ sei zweimal diferenzierbar. Wie lassen sich Konvexität und strikte Konvexität durch Bedingungen an die zweite Ableitung ausdrücken?

    Answer:

    (a) f is convex if and only if $f'' \geq 0$

    (b) if $f'' > 0$ then f is strictly convex (it doesn't hold in the other direction: for example x^4 is strictly convex, but $(x^4)'' = 12x^2 = 0$ for $x = 0$)

19. Wieviele Minima bzw. Maxima kann eine strikt konvexe Funktion $f : [a, b] \to \mathbb{R}$ haben? (Gib alle möglichen Zahlen an.)

    Answer: A convex function is also continuous (see sheet 9. problem 34.). A continuous function on a bounded and closed interval takes on its extremal values. Suppose that $m = \min_{x\in[a,b]} f(x) = f(p)$ and $M = \max_{x\in[a,b]} f(x)$ (for some corresponding $p \in [a, b]$ value). $m \neq M$, otherwise the function would be constant, and thus it would not be strictly convex.

    Now suppose that the function has two local minima with $m_1 = f(p_1)$ and $m_2 = f(p_2)$. If a function is strictly convex, then it's nowhere constant, and thus if m_1 and m_2 are local minima, then they are also strict local minima and thus $\exists q$ between p_1 and p_2 such that $f(q) > \max(m_1, m_2)$. But then $f(q) > f((1 - t)p_1 + tp_2)$ $(\forall t \in (0, 1))$ and this would contradict the strict convexity criteria. Consequently the function cannot have more than one local minimum, and since a global minimum is also a local minimum, they are necessarily the same.

    Since the function is nowhere constant, and cannot have other local minimum than the global minimum, it must be strictly monotone descreasing on $[a, p]$ and strictly monotone increasing on $[p, b]$. If $a \neq p$ then f has a local maximum at a, and if $p \neq b$ then f has a local maximum at b. If $f(a) = f(b)$ then they are both global maxima, otherwise the largest of them is the global maximum of f on $[a, b]$.

    Thus there is exactly one local minimum and it corresponds to the global minimum. If the global minimum is not at one of the endpoints, then the function will have two local maxima at the endpoints of the interval. If they are the same, then they are both global maxima, otherwise the largest of them will be the global maximum. If the global minimum is at one of the endpoints, then the global maxima will be at the other endpoint.

20. Wo sind (reelle) Potenzreihen differenzierbar? Wie lautet die Ableitung?

    Answer: A real power series $p(x) = \sum_{n=0}^{\infty} a_n x^n$ with convergence radius $\rho \in [0, \infty)$ is differentiable infinitely many times for any $|x| < \rho$, and $p'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$. We can define the n^{th} $(n > 1)$ derivative recursively as $p^{(n)}(x) = (p^{(n-1)}(x))'$ and with induction we get $p^{(n)}(x) = \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} a_{k+n} x^k$

21. Wie ist der Raum $\mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ definiert? Was bedeutet seine Vollständigkeit für die Vertauschbarkeit von Differentiation und Grenzwertbildung einer Funktionenfolge $f_n \colon \mathbb{R} \to \mathbb{R}$?

    Answer $\mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ is the vectorspace of continuously differentiable, bounded and continuous functions defined on $\mathbb{R}$ with bounded derivative. $\mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ is a Banach-space (that is: normed and complete) with the $\|f\|_{C^1} = \|f\|_{C^0} + \|f'\|_{C^0}$ $(f \in \mathcal{BC}^1(\mathbb{R}, \mathbb{R}))$ norm (with $\|.\|_{C^0}$ being the supremum-norm with which $\mathcal{BC}(\mathbb{R}, \mathbb{R})$ is complete).

    Since $\mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ with the norm $\|.\|_{C^1}$ is complete, any Cauchy sequence $(f_n) \in \mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ is also convergent (under the norm), thus $\exists f \in \mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ such that $\lim_{n\to\infty} \|f_n - f\|_{C^1} = 0$, or equivalently: $\|f_n - f\|_{C^0} \to 0$ and $\|f_n' - f'\|_{C^0} \to 0$ $(n \to \infty)$. If we consider the convergence under the norm this also means that $(\lim_{n\to\infty} f_n)' = f' = \lim_{n\to\infty} f_n'$, or: the derivative of the limit is the limit of the derivatives, that is: for any convergent $(f_n) \in \mathcal{BC}^1(\mathbb{R}, \mathbb{R})$ the order of differentiation and taking the limit can be exchanged with regards to the $\|.\|_{C^1}$.

22. Wie lautet das n-te Taylor-Polynom? Wie kann das Restglied ausgedrückt werden?

    Answer: *(not relevant for the first exam)*

23. Wann (und wo) wird eine reelle Funktion durch ihre Taylor-Reihe dargestellt? Gib ein Beispiel und ein Gegenbeispiel.

    Answer: *(not relevant for the first exam)*

24. Wie lauten die Taylor-Reihen folgender Funktionen in $x_0 = 0$?

    $$e^x,\ \sin x,\ \arctan x,\ (1+x)^\alpha,\ \log(1+x)$$

    Answer: *(not relevant for the first exam)*