1. Was ist eine Metrik? Gib drei verschiedene Beispiele!

   Answer: Consider E arbitrary set. The function $d : E \times E \to \mathbb{R}$ is a metric, if satisfies the following conditions:

   - $\forall x, y \in E\colon d(x,y) = 0 \Leftrightarrow x = y$
   - $\forall x, y \in E\colon d(x,y) = d(y,x)$
   - $\forall x, y, z \in E\colon d(x,z) \leq d(x,y) + d(y,z)$

   Examples:

   - Consider $E = \mathbb{R}$ and $d(x,y) = |x-y|$
   - Consider arbitrary set E. The discrete metric is defined as $d(x,y) = \begin{cases} 0 & x = y \\ 1 & \text{otherwise} \end{cases}$
   - Consider $\mathcal{BC}([0,1],\mathbb{R})$ and $d(f,g) = \|f-g\|_{sup}$

2. Wie sind offene Teilmengen eines metrischen Raumes definiert, wie abgeschlossene Teilmengen?

   Answer: Consider (E,d) metric space, and let $x \in E$. $\forall \varepsilon > 0$ let's define $B_\varepsilon(x) := \left\{ y \in E \,\middle|\, d(x,y) < \varepsilon \right\}$, the open ball of radius ε centered around x. The $A \subset E$ set is open, if $\forall x \in A\colon \exists \varepsilon > 0\colon B_\varepsilon(x) \subset A$. The $B \subset E$ set is closed, if its complement $E \setminus B$ is open.

3. Sind $\left\{ \begin{array}{c} \text{endliche} \\ \text{abzählbare} \\ \text{beliebige} \end{array} \right\} \left\{ \begin{array}{c} \text{Durchschnitte} \\ \text{Vereinigungen} \end{array} \right\} \left\{ \begin{array}{c} \text{offener} \\ \text{abgeschlossener} \end{array} \right\}$ Mengen wieder offen bzw. abgeschlossen?

   Answer: Arbitrary union of open sets is open. Finite intersection of open sets is open. Arbitrary intersection of open sets is in general not open: consider $\mathbb{R}$ with the standard metric, then $\cap_{n\in\mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$ closed.

   Arbitrary intersection of closed sets is closed. Finite union of closed sets is closed. Arbitrary union of closed sets is in general not closed: consider $\mathbb{R}$ with the standard metric, then $\cup_{n\in\mathbb{N}} \left[\frac{1}{n}, 1 - \frac{1}{n}\right] = (0,1)$ open.

4. Wie sind Abschluss, Inneres und Rand einer Teilmenge eines metrischen Raumes definiert?

   Answer: Consider (E,d) metric space, and let $A \subset E$.

   The closure of A is $\overline{A} = \left\{ x \in E \,\middle|\, \forall \varepsilon > 0\colon B_\varepsilon(x) \cap A \neq \emptyset \right\}$

   The inner of A is $\mathring{A} = \left\{ x \in A \,\middle|\, \exists \varepsilon > 0\colon B_\varepsilon(x) \subset A \right\}$

   The boundary of A is $\partial A = \overline{A} \setminus \mathring{A}$

5. Was sind Abschluss, Inneres und Rand folgender Teilmengen der reellen Zahlen (mit deren Standardmetrik)?

   $$\mathbb{Z} \qquad\qquad \mathbb{Q} \qquad\qquad \cup_{k\in\mathbb{N}} \left(\tfrac{1}{k+1}, \tfrac{1}{k} \right)$$

   Answer:

   - $\overline{\mathbb{Z}} = \mathbb{Z}$ because $\mathbb{Z}$ only consists of isolated points. $\mathring{\mathbb{Z}} = \emptyset$, since $\mathring{\mathbb{Z}} = \mathbb{R} \setminus \overline{\mathbb{R} \setminus \mathbb{Z}} = \mathbb{R} \setminus \mathbb{R} = \emptyset$. $\partial\mathbb{Z} = \overline{\mathbb{Z}} \setminus \mathring{\mathbb{Z}} = \mathbb{Z}$
   - $\mathbb{Q}$ is dense in $\mathbb{R}$, thus $\overline{\mathbb{Q}} = \mathbb{R}$. $\mathring{\mathbb{Q}} = \mathbb{R} \setminus \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R} \setminus \mathbb{R} = \emptyset$, since the irrationals are also dense in $\mathbb{R}$. $\partial\mathbb{Q} = \overline{\mathbb{Q}} \setminus \mathring{\mathbb{Q}} = \mathbb{R}$

- $A := \cup_{k\in\mathbb{N}} \left(\frac{1}{k+1}, \frac{1}{k}\right) = (0,1) \setminus \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. $\overline{A} = [0,1]$, $\mathring{A} = A$ since it's an union of open sets, thus it's also open and so the interior is itself. $\partial A = \overline{A} \setminus \mathring{A} = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$

6. Wann ist eine Teilmenge eines metrischen Raumes dicht?

   Answer: Consider (E,d) metric space and $A \subset E$. A is dense in E, if $\overline{A} = E$.

7. Wann ist ein metrischer Raum zusammenhängend?

   Answer: Consider (E,d) metric space. It's connected, if $\forall A, B \subset E\colon A, B$ open, $A \cup B = E, A \cap B = \emptyset \Rightarrow$ either $A = \emptyset$ or $B = \emptyset$

8. Wie sind Zusammenhangskomponenten eines metrischen Raumes definiert? Wann heißt ein metrischer Raum total unzusammenhängend?

   Answer: Consider (E,d) metric space. For some $x \in E$ we define the connected component of the point x as the union of all $C \subset E$ connected sets, that contain x. We denote the connected component of x with $\mathcal{C}(x)$. (E,d) is totally disconnected, if $\forall x \in E\colon \mathcal{C}(x) = \{x\}$

9. Sind $\left\{\begin{array}{c}\text{endliche}\\ \text{abzählbare}\\ \text{beliebige}\end{array}\right\} \left\{\begin{array}{c}\text{Durchschnitte}\\ \text{Vereinigungen}\end{array}\right\}$ zusammenhängender Mengen wieder zusammenhängend? Gib gegebenfalls ein Gegenbeispiel!

   Answer: Union of connected sets is in general not connected: consider $\mathbb{R}$ with the standard metric, and $[0,1]$, $[2,3]$ connected sets. $[0,1] \cup [2,3]$ is disconnected. Arbitrary union of connected sets U_i is connected, as long as $\cap_i U_i \neq \emptyset$.

   Intersection of connected sets is not necessarily connected (think in $\mathbb{R}^2$ about a C and a I shaped open set, intersecting each other only at the ends of the C, creating two disconnected sets).

10. Sind $\left\{\begin{array}{c}\text{endliche}\\ \text{abzählbare}\\ \text{beliebige}\end{array}\right\} \left\{\begin{array}{c}\text{Durchschnitte}\\ \text{Vereinigungen}\end{array}\right\}$ kompakter Mengen wieder kompakt? Gib gegebenfalls ein Gegenbeispiel!

    Answer: Finite union of compact sets is compact. Arbitrary union of compact sets is in general not compact: $\cup_{n\in\mathbb{N}}[n, n+1] = \mathbb{N}$ is not totally bounded, thus also not compact.

    Arbitrary intersection of compact sets is compact.

11. Warum ist in einem metrischen Raum jede konvergente Folge eine Cauchy-Folge? (Beweise!)

    Answer: Consider (E,d) metric space, and an $(x_n) \in E$ convergent sequence. Let $\lim_{n\to\infty} x_n = x \in E$. Since x_n converges, $\forall \varepsilon > 0\colon \exists N \in \mathbb{N}\colon \forall n > N\colon d(x_n, x) < \frac{\varepsilon}{2}$. Now $\forall n, m > N\colon d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) = d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$, thus (x_n) is a Cauchy sequence.

12. Wann heißt ein metrischer Raum vollständig?

    Answer: Consider (E,d) metric space. E is complete, if every Cauchy sequence converges.

13. Was sind generische Mengen?

    Answer: Consider (E,d) complete metric space and countable many $U_i \subset E$ $(i \in \mathbb{N})$ open and dense sets. Then $M = \cap_{i\in\mathbb{N}} U_i$ is dense in E. Any $G \supset M$ is called generic, residual or of Baire second category.

14. Wie lautet der Satz von Baire?

    Answer: Consider (E, d) complete metric space and countable many $U_i \subset E$ $(i \in \mathbb{N})$ open and dense sets. Then $\cap_{i\in\mathbb{N}} U_i$ is dense in E.

15. Wann heißt ein metrischer Raum perfekt?

    Answer: (E, d) is perfect, if it doesn't contain isolated points. Or equivalently: $\forall x \in E\colon x \in \overline{E \setminus \{x\}}$

16. Zeige, dass jeder nichtleere, vollständige, perfekte metrische Raum überabzählbar ist.

    Answer: Consider E not empty, complete and perfect metric space. Suppose indirectly, that it's countable: let $(x_n)\colon \mathbb{N} \to E$ bijection. Let furthermore $A_n = E \setminus \{x_n\}$. Since E is perfect, every A_n is dense in E. Since $\{x_n\}$ is closed, A_n is open. From Baire's theorem $\cap_{n\in\mathbb{N}} A_n = \emptyset$ should be dense in E, contradiction.

17. Was versteht man unter einer Cantor-Menge? Gib ein Beispiel an!

    Answer: Let $\emptyset \neq C \subset [0, 1]$. We call C a Cantor set, if C is complete, totally disconnected and perfect.

18. Wann heißt eine Folge in einem metrischen Raum konvergent? Wann heißt sie Cauchy-Folge?

    Answer: Consider (E, d) metric space, and $(x_n) \in E$ sequence. The (x_n) sequence is convergent, if $\exists x \in E\colon \forall \varepsilon > 0 \exists N \in \mathbb{N}\colon \forall n > N\colon d(x, d_n) < \varepsilon$. We call x the limit of the (x_n) sequence and we say that (x_n) converges agains x.

    The $(x_n) \in E$ sequence is a Cauchy sequence, if $\forall \varepsilon > 0\colon \exists N \in \mathbb{N}\colon \forall m, n > N\colon d(x_n, x_m) < \varepsilon$.

19. Wann heißt eine Abbildung zwischen metrischen Räumen stetig? Gib vier verschiedene (aber natürlich äquivalente) Definitionen!

    Answer: Consider (E_1, d_1) and (E_2, d_2) metric spaces. The $f\colon E_1 \to E_2$ function is continuous in $a \in E_1$ if $\forall \varepsilon > 0\colon \exists \delta > 0\colon \forall x \in E_1\colon d_1(a, x) < \delta\colon d_2(f(a), f(x)) < \varepsilon$.

    The $f\colon E_1 \to E_2$ is continuous if and only if one of the following equivalent characterisation holds (and if one holds, then all the other hold as well):

    (a) f is continuous in $\forall a \in E_1$
    (b) $\forall A \subset E_1\colon f(\overline{A}) \subset \overline{f(A)}$
    (c) $\forall A \subset E_2$ closed: $f^{-1}(A)$ is closed
    (d) $\forall A \subset E_2$ open: $f^{-1}(A)$ is open

20. Sind unter einer stetigen Abbildung $f : E \to E'$ zwischen metrischen Räumen die $\left\{\begin{array}{c}\text{Bilder}\\ \text{Urbilder}\end{array}\right\}$ $\left\{\begin{array}{c}\text{offener}\\ \text{abgeschlossener}\\ \text{vollstängiger}\\ \text{zusammenhängender}\\ \text{kompakter}\end{array}\right\}$ Teilmengen wieder $\left\{\begin{array}{c}\text{offener}\\ \text{abgeschlossener}\\ \text{vollständiger}\\ \text{zusammenhängender}\\ \text{kompakter}\end{array}\right\}$? Gib gegebenfalls ein Gegenbeispiel!

    Answer:

    - Image of an open subset is not necessarily open. Let $E = \mathbb{R}$ with the standard metric, $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto 1$ the constant one function and consider $A \subset E$ open. $f(A) = \{1\}$ is closed.

- Image of a closed subset is not necessarily closed. Consider $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto \frac{1}{1-e^{-x}}$. The subset $\mathbb{R} \subseteq \mathbb{R}$ is closed, but $f(\mathbb{R}) = (0,1)$ is open.
- Image of a complete subset is not necessarily complete. See previous example.
- Image of connected or compact sets are again connected or compact, respectively.
- Preimage of an open set is open.
- Preimage of a closed set is closed.
- Preimage of a complete set is not necessarily complete. Consider $(0,1)$ metric space, let $f\colon (0,1) \to \mathbb{R}$ be the constant 1 function. The preimage of the complete set $\{1\}$ is $(0,1)$, which is not complete.
- Preimage of connected sets is not necessarily
- Preimage of connected sets is not necessarily complete. Consider the $(0,1) \cup (2,3)$ metric space, and the constant 1 function defined on this space. The preimage of the connected $\{1\}$ set is disconnected. . Consider the $(0,1) \cup (2,3)$ metric space, and the constant 1 function defined on this space. The preimage of the connected $\{1\}$ set is disconnected.
- Preimage of compact sets is not necessarily compact. Consider the $\mathbb{R}$ metric space, and the constant 1 function. The preimage of the compact $\{1\}$ set is not bounded, thus also not compact.

21. Wann heißen zwei Metriken äquivalent?

    Answer: Consider (E_1, d_1) and (E_2, d_2) metric spaces. The $f\colon E_1 \to E_2$ function is a homeomorphism, if f is bijective, and both f and f^{-1} are continuous. We say, that E_1 and E_2 metric spaces are equivalent, if the $id : E_1 \to E_2, x \mapsto x$ function is a homeomorphism.

22. Wann heißen zwei Normen äquivalent?

    Answer: Consider V vectorspace with two different $\|.\|_1$ and $\|.\|_2$. We say, that they are equivalent, if $\exists c, C \in \mathbb{R}^+$, such that $\forall v \in V\colon c\,\|v\|_2 < \|v\|_1 < C\,\|v\|_2$

23. Sei E ein metrischer Raum. Formuliere und beweise den Zwischenwertsatz für stetige Abbildungen $f\colon E \to \mathbb{R}$

    Answer: Let E connected, $f\colon E \to \mathbb{R}$ continuous and $a, b \in E$. Then f takes on every value between $f(a)$ and $f(b)$. *Proof:* E is connected, thus so is $f(E)$. In $\mathbb{R}$ connected sets are exactly the intervals, thus it'll take on any value between $f(a)$ and $f(b)$.

24. Gib drei verschiedene (aber natürlich äquivalente) Definitionen von kompakten metrischen Räumen! Gib außerdem je ein Beispiel eines kompakten und eines nicht kompakten Raumes!

    Answer: Consider (E, d) metric space. The following statements are equivalent characterisation of the compactness of E:

    - Every open cover of E has a finite subcover.
    - Every sequence in E has a convergent subsequence.
    - E is complete and totally bounded.

    The $[0,1]$ space with the standard metric of $\mathbb{R}$ is compact, but the $(0,1)$ space with the same metric is not.

25. Sei E ein metrischer Raum. Wann heißt eine Teilmenge $A \subset E$ relativ kompakt?

    Answer: A is relative compact, if $\overline{A}$ is compact.

26. Wie lassen sich die kompakten Teilmengen des $\mathbb{R}^n$ charakterisieren?

    Answer: In $\mathbb{R}^n$ the subset $K \subset \mathbb{R}^n$ is bounded if and only if it's totally bounded.Thus closed and bounded subsets $K \subset \mathbb{R}^n$ are compact.

27. Wie lautet der Satz von Arzela-Ascoli?

    Answer: Consider $(E_1, d_1), (E_2, d_2)$ metric spaces and $\mathcal{F} := \{f \colon E_1 \to E_2 \mid f \text{ continuous}\}$. We say, that the set $\mathcal{F}$ is equicontinuous, if $\forall a \in E_1 \colon \forall \varepsilon > 0 \colon \exists \delta > 0 \colon \forall a \in E_1 \colon |x - a| < \delta \colon |f(x) - f(a)| < \varepsilon \ (\forall f \in \mathcal{F})$. That is: δ only depends on $a \in E_1$ and $\epsilon > 0$, but not on $f \in \mathcal{F}$.

    Consider now (E, d) compact metric space, and $(F, \|.\|)$ Banach space. The $\mathcal{F} \subset \mathcal{BC}^0(E, F)$ set is relative compact, if and only if both of the following statement hold:

    - $\mathcal{F}$ is equicontinuous
    - $\forall x \in E : \{f(x) \mid f \in \mathcal{F}\}$ is relative compact

28. Wann sagt man, dass eine Abbildung Lipschitz-stetig ist? Wann ist eine Lipschitzstetige Abbildung eine Kontraktion?

    Answer: Consider (E_1, d_1) and (E_2, d_2) metric spaces. $f \colon E_1 \to E_2$ is Lipschitz continuous, if $\exists L > 0 \colon \forall x, y \in E_1 \colon d_2(f(x), f(y)) < L d_2(x, y)$. If $L < 1$, then we call f a contraction.

29. Wie lautet der Banachsche Fixpunktsatz?

    Answer: Consider (E, d) complete metric space and $f \colon E \to E$ contraction with Lipschitz constant $L < 1$. Then

    - f has exactly one fixed point $x_\star \in E \colon f(x_\star) = x_\star$
    - $\forall x_0 \in E$ converges the $x_{n+1} = f(x_n)$ $(\forall n \in \mathbb{N})$ against $x_\star$
    - the following error estimates hold:
        - $d(x_n, x_\star) \leq L^n d(x_0, x_\star)$
        - $d(x_n, x_\star) \leq \frac{L^n}{1-L} d(x_0, x_1)$

30. Sei E ein metrischer Raum und $f_i \colon E \to E$ stetig. Wann sagt man, dass $\emptyset \neq A \subset E$ selbstähnlich ist? Was ist ein *iterated function system*?

    Answer: Consider $f_i \colon E \to E$ stetig $(i = 1, \ldots, m \geq 2)$. A is self similar, if $A = \cup_{i=1}^m f_i(A)$. If E is a complete metric space and f_i's are contractions $(\forall i = 1, \ldots, m \geq 2)$, then $\exists! A \subset E$ compact, such that $A = \cup_{i=1}^m f_i(A)$ and we call such a set of functions and iterated function system.