

# Optimal Stopping of BSDEs with Constrained Jumps and Related Double Obstacle PDEs

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## Abstract

We consider partial differential equations (PDEs) characterized by an upper barrier that depends on the solution itself and a fixed lower barrier, while accommodating a non-local driver. First, we show a Feynman-Kac representation for the PDE when the driver is local. Specifically, we relate the non-linear Snell envelope for an optimal stopping problem, where the underlying process is the first component in the solution to a stopped backward stochastic differential equation (BSDE) with jumps and a constraint on the jumps process, to a viscosity solution for the PDE. Leveraging this Feynman-Kac representation, we subsequently prove existence and uniqueness of viscosity solutions in the non-local setting by employing a contraction argument. In addition, the contraction argument yields existence of a new type of non-linear Snell envelope and extends the theory of probabilistic representation for PDEs.

## 1 Introduction

We consider partial differential equations (PDEs) of the type

$$\begin{cases} \min\{v(t, x) - h(t, x), \max\{v(t, x) - \mathcal{M}v(t, x), -v_t(t, x) - \mathcal{L}v(t, x) \\ -f(t, x, v(t, \cdot), \sigma^\top(t, x)\nabla_x v(t, x))\}\} = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d \\ v(T, x) = \psi(x), \end{cases} \quad (1.1)$$

where for each  $(t, x, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ , the map  $g \mapsto f(t, x, g, z) : C(\mathbb{R}^d \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$  is a functional,  $\mathcal{M}v(t, x) := \inf_{e \in E} \{v(t, x + \gamma(t, x, e)) + \chi(t, x, e)\}$  and

$$\mathcal{L} := \sum_{j=1}^d a_j(t, x) \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^\top(t, x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \quad (1.2)$$

is the infinitesimal generator related to the stochastic differential equation (SDE)

$$\check{X}_s = x + \int_0^s a(r, \check{X}_r) dr + \int_0^s \sigma(r, \check{X}_r) dW_r.$$

We establish existence and uniqueness (within the set of continuous functions of polynomial growth) of viscosity solutions to the above PDE by relating  $v(t, x)$  to  $Y_t^{t,x}$ , where

$$Y_s^{t,x} = \text{ess sup}_{\tau \in \mathcal{T}_s} Y_s^{t,x,\tau} \quad (1.3)$$

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and for each stopping time  $\tau \geq t$ , the process  $Y^{t,x,\tau}$  is the first component in the quadruple of processes  $(Y^{t,x,\tau}, Z^{t,x,\tau}, V^{t,x,\tau}, K^{t,x,\tau})$  that is the unique maximal solution to the backward stochastic differential equation (BSDE), driven by the above mentioned Brownian motion and an independent Poisson random measure  $\mu$  with intensity  $\lambda$ , that has a constraint on the jump component

$$\begin{cases} Y_s^{t,x,\tau} = \Psi(\tau, X_\tau^{t,x}) + \int_s^\tau f(r, X_r^{t,x}, \bar{Y}(r, \cdot), Z_r^{t,x,\tau}) dr - \int_s^\tau Z_r^{t,x,\tau} dW_r - \int_s^\tau \int_E V_r^{t,x,\tau}(e) \mu(dr, de) \\ \quad - (K_\tau^{t,x,\tau} - K_s^{t,x,\tau}), \quad \forall s \in [t, \tau], \\ V_s^{t,x,\tau}(e) \geq -\chi(s, X_{s-}^{t,x}), \quad d\mathbb{P} \otimes ds \otimes \lambda(de) - a.e. \end{cases} \quad (1.4)$$

Here,  $\Psi(t, x) := \mathbb{1}_{[t < T]} h(t, x) + \mathbb{1}_{[t=T]} \psi(t, x)$ , the process  $X^{t,x}$  is the unique solution to the forward SDE

$$X_s^{t,x} = x + \int_t^s a(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dW_r + \int_t^s \int_E \gamma(r, X_{r-}^{t,x}, e) \mu(dr, de) \quad (1.5)$$

and the driver at time  $t \in [0, T]$  depends on the solution to the optimal stopping problem through the map  $(t, x) \mapsto \bar{Y}(t, x) := Y_t^{t,x}$ . Our contribution is twofold in the sense that our approach implicitly provides existence of a unique solution to the optimal stopping problem (1.3)-(1.5).

**Related literature** Partial differential equations featuring non-local drivers, wherein the mapping  $g \mapsto f(t, x, g, z)$  adopts the structure of an integral, are commonly referred to as integro-partial differential equations (IPDEs). A branch of the literature that relates BSDEs with jumps to IPDEs have focused on the setting where the driver takes the form

$$f(t, x, g, z) = \bar{f}(t, x, g(x), z, \int_E \theta(t, x, e)(g(x + \zeta(t, x, e)) - g(x)) \lambda(de)),$$

for a Lipschitz function  $\bar{f}$ . In particular, [3] and [8] both assume that  $\bar{f}$  is non-decreasing in the last variable and that  $\theta \geq 0$ , the former considering the relation between regular BSDEs with jumps and IPDEs whereas the latter prove a relation between the reflected BSDEs with jumps considered in [21] and IPDEs with one-sided obstacles. These results were later extended in [13] to the setting when  $\bar{f}$  is a general Lipschitz function and  $\zeta$  may be negative. We also mention the work in [11], where a system of IPDEs with interconnected obstacles are treated.

When the lower barrier  $h$  is absent, the PDE described in (1.1) is termed a quasi-variational inequality (QVI). It is well known that, under suitable conditions on the involved parameters, value functions to impulse control problems are solutions (in viscosity sense) to standard QVIs when the driving noise process is a Brownian motion (see the seminal work in [4]) and to so called quasi-integrovariational inequalities when the driving noise is a general Lévy process [16]. Employing a contraction argument akin to the one delineated in the present study, quasi-variational inequalities (QVIs) featuring general non-local drivers were investigated in [19]. Specifically, [19] establishes a Feynman-Kac representation of such QVIs by linking their solutions to systems of reflected BSDEs (RBSDEs).

An alternative Feynman-Kac representation for solutions to standard QVIs was proposed in [14] (see also [5]), where the solution to a QVI is related to the minimal solution of a BSDE with constrained jumps. In particular, their result implies that (when  $f$  is local) the deterministic function  $v$  defined through the relation  $v(t, x) := Y_t^{t,x,T}$ , with  $Y^{t,x,T}$  as in (1.4), is the unique viscosity solution to the PDE obtained by setting  $h \equiv -\infty$  in (1.1). Following the seminal work in [14], BSDEs with positive jumps were related to fully non-linear Hamilton-Jacobi-Bellman integro-partial differential equations (HJB-IPDEs) through a Feynman-Kac representation in [15] while a correspondence between RBSDEs with positive jumps and fully non-linear variational inequalities was established in [6].

The ensemble of approaches to find probabilistic representations of PDEs or solve stochastic optimal control problems that utilize BSDEs with constrained jumps is commonly referred to as *control randomization*. A significant breakthrough in this field was achieved with the seminal work of [10], which directly

linked the value function of the randomized control problem to that of the original control problem. This eliminated the need for a Feynman-Kac representation, thereby expanding the theoretical framework to encompass stochastic systems with path-dependencies. Building upon this foundation, subsequent advancements extended their approach to the framework of partial information settings in [1] and optimal switching problems in [9]. Recently, [18] further extended the scope of control randomization to zero-sum games by (within a non-markovian framework) relating the solution to the above optimal stopping problem (1.3)-(1.5) to the upper and lower value functions in a stochastic differential game between an impulse controller and a stopper. Notable is that [18] presumes that  $f$  only depends on local values of  $Y$ , whereas it may depend on  $V$ .

An intermediate result in the present work addresses the framework of a local driver and bridges a gap left in [18]. Specifically, we establish a connection between the non-linear Snell envelope examined in [18] and viscosity solutions to (1.1). Consequently, we elucidate that, in the Markovian framework, the value function of the aforementioned zero-sum game indeed constitutes the unique viscosity solution to (1.1). Moreover, our primary findings extend those of [18] in the Markovian framework by proving the existence of the more general non-linear Snell envelope defined through equations (1.3)-(1.5).

**Outline** The remainder of the article is organised as follows. In the next section we set the notations and state the assumptions that hold throughout. In addition, we give some preliminary results that are repeatedly referred in the article. Then, in Section 3 we turn to the local setting before we, in the following section, derive the complete result. Uniqueness of solutions to (1.1) when the driver  $f$  is local appears rudimentary and resembles earlier results deduced in, for example, [12]. However, since our setting is fundamentally different and for the sake of completeness, a uniqueness proof through viscosity comparison is included as an appendix.

## 2 Preliminaries

### 2.1 Notation

We let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space on which lives a  $d$ -dimensional Brownian motion  $W$  and an independent Poisson random measure  $\mu$  defined on  $[0, T] \times E$  with intensity  $\lambda(de)$ . Here, it is assumed that  $E$  is a Borelian subset of  $\mathbb{R}^d$  endowed with its Borel  $\sigma$ -field  $\mathcal{B}(E)$ . We denote by  $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$  the augmented natural filtration generated by  $W$  and  $\mu$  and for  $t \in [0, T]$  we let  $\mathbb{F}^t := (\mathcal{F}_s^t)_{t \leq s \leq T}$  (resp.  $\mathbb{F}^{W,t} := (\mathcal{F}_s^{W,t})_{t \leq s \leq T}$ ) denote the augmented natural filtration generated by  $(W_s - W_t : t \leq s \leq T)$  and  $\mu(\cdot \cap [t, T], \cdot)$  (resp.  $(W_s - W_t, t \leq s \leq T)$ ).

Throughout, we will use the following notation, where  $d \geq 1$  is the dimension of the state-space:

- We let  $\Pi^g$  denote the set of all functions  $\varphi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  that are of polynomial growth in  $x$ , *i.e.* there are constants  $C, \rho > 0$  such that  $|\varphi(t, x)| \leq C(1 + |x|^\rho)$  for all  $(t, x) \in [0, T] \times \mathbb{R}^d$ , and let  $\Pi_c^g$  be the subset of jointly continuous functions.
- For each  $t \in [0, T]$ ,  $\mathcal{P}_t$  is the  $\sigma$ -algebra of  $\mathbb{F}^t$ -predictably measurable subsets of  $[t, T] \times \Omega$  and  $\mathcal{P} := \mathcal{P}_0$ .
- We let  $\mathcal{T}$  be the set of all  $[0, T]$ -valued  $\mathbb{F}$ -stopping times and for each  $\eta \in \mathcal{T}$ , we let  $\mathcal{T}_\eta$  be the subset of stopping times  $\tau$  such that  $\tau \geq \eta$ ,  $\mathbb{P}$ -a.s. Moreover, we let  $\mathcal{T}^t$  (resp.  $\mathcal{T}_\eta^t$ ) be the corresponding subsets of  $\mathbb{F}^t$ -stopping times, with  $\tau \geq t$  (resp.  $\tau \geq \eta$ ),  $\mathbb{P}$ -a.s.
- For  $p \geq 1$ ,  $t \in [0, T]$  and  $\tau \in \mathcal{T}^t$ , we let  $\mathcal{S}_{[t, \tau]}^p$  be the set of all  $\mathbb{R}$ -valued,  $\mathbb{F}^t$ -progressively measurable, càdlàg processes  $(Z_s : s \in [t, \tau])$  for which  $\|Z\|_{\mathcal{S}_{[t, \tau]}^p} := \mathbb{E}[\sup_{s \in [t, \tau]} |Z_s|^p] < \infty$ . Moreover, we let  $\mathcal{S}_{[t, \tau], i}^p$  be the subset of  $\mathbb{F}^t$ -predictably measurable and non-decreasing processes with  $Z_t = 0$ . Whenever  $\tau = T$  we use the notations  $\mathcal{S}_t^p$  and  $\mathcal{S}_{t, i}^p$ , respectively.

- We let  $\mathcal{H}_{[t,\tau]}^p(W)$  denote the set of all  $\mathbb{R}^d$ -valued  $\mathbb{F}^t$ -progressively measurable processes  $(Z_s : s \in [t, \tau])$  such that  $\|Z\|_{\mathcal{H}_{[t,\tau]}^p(W)} := \mathbb{E}[(\int_t^\tau |Z_s|^2 ds)^{p/2}]^{1/p} < \infty$ . Furthermore, we set  $\mathcal{H}_t^p(W) := \mathcal{H}_{[t,T]}^p(W)$ .
- We let  $\mathcal{H}_{[t,\tau]}^p(\mu)$  denote the set of all  $\mathbb{R}$ -valued,  $\mathcal{P}_t \otimes \mathcal{B}(E)$ -measurable maps  $(Z_s(e) : s \in [t, \tau], e \in E)$  such that  $\|Z\|_{\mathcal{H}_{[t,\tau]}^p(\mu)} := \mathbb{E}[(\int_t^\tau \int_E |Z_s(e)|^2 \lambda(de) ds)^{p/2}]^{1/p} < \infty$  and set  $\mathcal{H}_t^p(\mu) = \mathcal{H}_{[t,T]}^p(\mu)$ .
- For  $t \in [0, T]$ , we let  $\mathcal{A}_t$  denote the set of all  $[-1, 1]^d$ -valued,  $\mathbb{F}^t$ -progressively measurable processes  $(\alpha_s : t \leq s \leq T)$  and set  $\mathcal{A} := \mathcal{A}_0$ . Moreover, we let  $\mathcal{A}_t^W$  be the subset of  $\mathbb{F}^{W,t}$ -progressively measurable processes (resp.  $\mathcal{A}^W := \mathcal{A}_0^W$ ).
- For  $t \in [0, T]$ , we define the composition  $\oplus_t$  of  $\alpha^1 \in \mathcal{A}$  and  $\alpha^2 \in \mathcal{A}_t$  as  $(\alpha^1 \oplus_t \alpha^2)_s := \mathbb{1}_{[0,t)}(s)\alpha_s^1 + \mathbb{1}_{[t,T]}(s)\alpha_s^2$ .
- We let  $\mathcal{V}_t$  denote the set of all  $\mathcal{P}_t \otimes \mathcal{B}(E)$ -measurable, bounded maps  $\nu : [t, T] \times \Omega \times E \rightarrow [0, \infty)$  and for each  $n \in \mathbb{N}$ , we denote by  $\mathcal{V}_t^n$  the subset of maps  $\nu : [t, T] \times \Omega \times E \rightarrow [0, n]$ .

We also mention that, unless otherwise specified, all inequalities between random variables are to be interpreted in the  $\mathbb{P}$ -a.s. sense.

## 2.2 Assumptions

We make the following assumption on the intensity  $\lambda$  of the process  $\mu$ .

**Assumption 2.1.** *We assume that  $\lambda$  has full topological support on  $E$  and finite intensity, i.e.  $\lambda(E) < \infty$ .*

Throughout, we make the following assumptions on the parameters, where  $\rho > 0$  is a fixed constant:

**Assumption 2.2.** *i) We assume that  $f : [0, T] \times \mathbb{R}^d \times C(\mathbb{R}^d \rightarrow \mathbb{R}) \times \mathbb{R}^d \rightarrow \mathbb{R}$  ( $(t, x, g, z) \rightarrow f(t, x, g, z)$ ) is such that for any  $v \in \Pi_c^g$ , the map  $(t, x) \mapsto f(t, x, v(t, \cdot), z)$  is jointly continuous, uniformly in  $z$ ,  $f$  is of polynomial growth in  $x$ , i.e. there is a  $C_f > 0$  such that*

$$|f(t, x, 0, 0)| \leq C_f(1 + |x|^\rho)$$

*and that there are constants  $k_f, K_\Gamma > 0$  such that for any  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $g, \tilde{g} \in C(\mathbb{R}^d \rightarrow \mathbb{R})$  and  $z, \tilde{z} \in \mathbb{R}^d$  we have*

$$|f(t, x, \tilde{g}, \tilde{z}) - f(t, x, g, z)| \leq k_f \left( \sup_{x' \in \Lambda_f(|x|)} |\tilde{g}(x') - g(x')| + |\tilde{z} - z| \right),$$

*where for each  $\alpha \in \mathbb{R}_+$ ,  $\Lambda_f(\alpha) := \{x \in \mathbb{R}^d : \|x\| \leq \alpha \vee K_\Gamma\}$  is the closed ball of radius  $\alpha \vee K_\Gamma$  centered at the origin.*

*ii) The lower barrier  $h : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  is jointly continuous and of polynomial growth in  $x$ , i.e. there is a  $C_h > 0$  such that*

$$|h(t, x)| \leq C_h(1 + |x|^\rho).$$

*iii) The terminal reward  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$  is continuous and satisfies the growth condition*

$$|\psi(x)| \leq C_\psi(1 + |x|^\rho)$$

*for some  $C_\psi > 0$ .*

iv) The jump-barrier  $\chi : [0, T] \times \mathbb{R}^d \times E \rightarrow \mathbb{R}_+$  is jointly continuous, of polynomial growth and bounded from below, i.e.

$$\chi(t, x, e) \geq \delta > 0,$$

v) For each  $(x, e) \in \mathbb{R}^d \times E$  we have

$$h(T, x) \leq \psi(x) \leq \psi(x + \gamma(T, x, e)) + \chi(T, x, e)$$

Moreover, we make the following assumptions on the coefficients of the forward SDE:

**Assumption 2.3.** For any  $t, t' \in [0, T]$ ,  $e \in E$  and  $x, x' \in \mathbb{R}^d$  we have:

i) The function  $\gamma : [0, T] \times \mathbb{R}^d \times E \rightarrow \mathbb{R}^d$  is jointly continuous and satisfies the growth condition

$$|x + \gamma(t, x, e)| \leq K_\Gamma \vee |x| \quad (2.1)$$

Moreover, it is Lipschitz continuous in  $x$  uniformly in  $(t, e)$ , i.e.

$$|\gamma(t, x', e) - \gamma(t, x, e)| \leq k_\gamma |x' - x|$$

for all  $(t, x, x', e) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times E$ .

ii) The coefficients  $a : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^n$  and  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{n \times d}$  are jointly continuous and satisfy the growth conditions

$$|a(t, x)| + |\sigma(t, x)| \leq C_{a, \sigma}(1 + |x|),$$

for some  $C_{a, \sigma} > 0$  and the Lipschitz continuity

$$|a(t, x) - a(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq k_{a, \sigma} |x' - x|,$$

for some  $k_{a, \sigma} > 0$ .

## 2.3 Viscosity solutions

We define the upper,  $v^*$ , and lower,  $v_*$  semi-continuous envelope of a function  $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  as

$$v^*(t, x) := \limsup_{(t', x') \rightarrow (t, x), t' < T} v(t', x') \quad \text{and} \quad v_*(t, x) := \liminf_{(t', x') \rightarrow (t, x), t' < T} v(t', x').$$

Next, we introduce the limiting parabolic superjet  $\bar{J}^+ v$  and subjet  $\bar{J}^- v$ .

**Definition 2.4.** Subjets and superjets

i) For a l.s.c. (resp. u.s.c.) function  $v : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ , the parabolic subjet, denote by  $J^- v(t, x)$ , (resp. the parabolic superjet,  $J^+ v(t, x)$ ) of  $v$  at  $(t, x) \in [0, T] \times \mathbb{R}^n$ , is defined as the set of triples  $(p, q, M) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$  satisfying

$$v(t', x') \geq (\text{resp. } \leq) v(t, x) + p(t' - t) + \langle q, x' - x \rangle + \frac{1}{2} \langle x' - x, M(x' - x) \rangle + o(|t' - t| + |x' - x|^2)$$

for all  $(t', x') \in [0, T] \times \mathbb{R}^n$ , where  $\mathbb{S}^n$  is the set of symmetric real matrices of dimension  $n \times n$ .

- ii) For a l.s.c. (resp. u.s.c.) function  $v : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$  we denote by  $\bar{J}^-v(t, x)$  the parabolic limiting subjet (resp.  $\bar{J}^+v(t, x)$  the parabolic limiting superjet) of  $v$  at  $(t, x) \in [0, T] \times \mathbb{R}^n$ , defined as the set of triples  $(p, q, M) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$  such that:

$$(p, q, M) = \lim_{n \rightarrow \infty} (p_n, q_n, M_n), \quad (t, x) = \lim_{n \rightarrow \infty} (t_n, x_n)$$

for some sequence  $(t_n, x_n, p_n, q_n, M_n)_{n \geq 1}$  with  $(p_n, q_n, M_n) \in J^-v(t_n, x_n)$  (resp.  $(p_n, q_n, M_n) \in J^+v(t_n, x_n)$ ) for all  $n \geq 1$  and  $v(t, x) = \lim_{n \rightarrow \infty} v(t_n, x_n)$ .

We now give the definition of a viscosity solution for the QVI in (1.1). (see also pp. 9-10 of [7]).

**Definition 2.5.** Let  $v$  be a locally bounded function from  $[0, T] \times \mathbb{R}^d$  to  $\mathbb{R}$ . Then,

- a) It is referred to as a viscosity supersolution (resp. subsolution) to (1.1) if it is l.s.c. (resp. u.s.c.) and satisfies:

- i)  $v(T, x) \geq \psi(x)$  (resp.  $v(T, x) \leq \psi(x)$ )  
ii) For any  $(t, x) \in [0, T] \times \mathbb{R}^d$  and  $(p, q, X) \in \bar{J}^-v(t, x)$  (resp.  $\bar{J}^+v(t, x)$ ) we have

$$\begin{aligned} \min \{ & v(t, x) - h(t, x), \max \{ v(t, x) - \mathcal{M}v(t, x), -p - q^\top a(t, x) \\ & - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) X) - f(t, x, v(t, \cdot), \sigma^\top(t, x) q) \} \} \geq 0 \quad (\text{resp. } \leq 0) \end{aligned}$$

- b) It is called a viscosity solution to (1.1) if  $v_*$  is a supersolution and  $v^*$  is a subsolution.

We will sometimes use the following alternative definition of viscosity supersolutions (resp. subsolutions):

**Definition 2.6.** A l.s.c. (resp. u.s.c.) function  $v$  is a viscosity supersolution (resp. subsolution) to (1.1) if  $v(T, x) \leq \psi(x)$  (resp.  $\geq \psi(x)$ ) and whenever  $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d \rightarrow \mathbb{R})$  is such that  $\varphi(t, x) = v(t, x)$  and  $\varphi - v$  has a local maximum (resp. minimum) at  $(t, x)$ , then

$$\begin{aligned} \min \{ & v(t, x) - h(t, x), \max \{ v(t, x) - \mathcal{M}v(t, x), -\varphi_t(t, x) - \mathcal{L}\varphi(t, x) \\ & - f(t, x, v(t, \cdot), \sigma^\top(t, x) \nabla_x \varphi(t, x)) \} \} \geq 0 \quad (\text{resp. } \leq 0). \end{aligned}$$

## 2.4 Reflected BSDEs with jumps and obstacle problems

We introduce the local driver  $\tilde{f}$  that satisfies the following assumption:

**Assumption 2.7.** *i) The driver  $\tilde{f} : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is jointly continuous and of polynomial growth in  $x$ , i.e.  $|\tilde{f}(t, x, 0, 0)| \leq C(1 + |x|^\rho)$ . Moreover,  $(t, x) \mapsto \tilde{f}(t, x, y, z)$  is continuous, uniformly in  $(y, z)$ , and  $(y, z) \mapsto \tilde{f}(t, x, y, z)$  is Lipschitz continuous, uniformly in  $(t, x)$ .*

We give the following proposition, strategically formulated to streamline subsequent implementation processes. It is worth noting that comparable findings are documented in [8, 13], with the latter being the most closely aligned with our own.

**Proposition 2.8.** *For each  $(t, x) \in [0, T] \times \mathbb{R}^d$  and  $n \in \mathbb{N}$ , there is a unique quadruple  $(Y^{t,x}, Z^{t,x}, V^{t,x}, K^{+,t,x}) \in \mathcal{S}_t^2 \times \mathcal{H}_t^2(W) \times \mathcal{H}_t^2(\mu) \times \mathcal{S}_{t,i}^2$  such that*

$$\begin{cases} Y_s^{t,x} = \psi(X_T^{t,x}) + \int_s^T \tilde{f}^n(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, V_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r - \int_s^T \int_E V_r^{t,x}(e) \mu(dr, de) \\ \quad + (K_T^{+,t,x} - K_s^{+,t,x}), \quad \forall s \in [t, T], \\ Y_s^{t,x} \geq h(s, X_s^{t,x}), \quad \forall s \in [t, T] \quad \text{and} \quad \int_t^T (Y_s^{t,x} - h(s, X_s^{t,x})) dK_s^{+,t,x}, \end{cases} \quad (2.2)$$

where

$$\tilde{f}^n(t, x, y, z, v) := \tilde{f}(t, x, y, z) - n \int_E (v(e) + \chi(t, x, e))^- \lambda(de).$$

Moreover,  $Y_s^{t,x} = \text{ess sup}_{\tau \in \mathcal{T}_s^t} Y_s^{t,x;\tau}$ , where for each  $\tau \in \mathcal{T}^t$ , the triple  $(Y^{t,x;\tau}, Z^{t,x;\tau}, V^{t,x;\tau}) \in \mathcal{S}_{[t,\tau]}^2 \times \mathcal{H}_{[t,\tau]}^2(W) \times \mathcal{H}_{[t,\tau]}^2(\mu)$  satisfies

$$Y_s^{t,x;\tau} = \Psi(\tau, X_\tau^{t,x}) + \int_s^\tau \tilde{f}^n(r, X_r^{t,x}, Y_r^{t,x;\tau}, Z_r^{t,x;\tau}, V_r^{t,x;\tau}) dr - \int_s^\tau Z_r^{t,x;\tau} dW_r - \int_s^\tau \int_E V_r^{t,x;\tau}(e) \mu(dr, de) \quad (2.3)$$

and for each  $\eta \in \mathcal{T}^t$ , the stopping time

$$\tau^* := \inf\{s \geq \eta : Y_s^{t,x} = h(r, X_s^{t,x})\} \wedge T$$

is optimal in the sense that  $Y_\eta^{t,x} = Y_\eta^{t,x;\tau^*}$ . Finally, there is a function  $v_n \in \Pi_c^g$  such that  $v_n(s, X_s^{t,x}) = Y_s^{t,x}$  for all  $s \in [t, T]$  and  $v_n$  is the unique viscosity solution in  $\Pi_c^g$  to

$$\begin{cases} \min\{v_n(t, x) - h(t, x), -\frac{\partial}{\partial t}v_n(t, x) - \mathcal{L}v_n(t, x) + \mathcal{K}^n v_n(t, x) \\ - \tilde{f}(t, x, v_n(t, x), \sigma^\top(t, x) \nabla_x v_n(t, x))\} = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d \\ v_n(T, x) = \psi(x), \end{cases} \quad (2.4)$$

where  $\mathcal{K}^n \phi(t, x) := n \int_E (\phi(t, x + \gamma(t, x, e)) + \chi(t, x, e) - \phi(t, x))^- \lambda(de)$ .

*Proof.* As the parameters of the reflected BSDE (2.2) satisfy the conditions for the comparison result of [21], everything but the viscosity solution property follows by results presented therein. Existence of a unique viscosity solution in  $\Pi_c^g$  to (2.4) can be shown as a special case of the method described in [19]. We can now argue as in [13] and let  $v \in \Pi_c^g$  be defined as  $v(t, x) := Y_t^{t,x}$ . Then, it can be shown (see Proposition 3.1 in [13]) that  $V_s^{t,x}(e) = v(s, X_s^{t,x} + \gamma(s, X_s^{t,x}, e)) - v(s, X_s^{t,x})$ ,  $d\mathbb{P} \otimes ds \otimes \lambda(de)$ -a.e. Moreover, the BSDE

$$\begin{cases} \tilde{Y}_s^{t,x} = \psi(X_T^{t,x}) + \int_s^T \tilde{f}^n(r, X_r^{t,x}, \tilde{Y}_r^{t,x}, \tilde{Z}_r^{t,x}, v(r, X_r^{t,x} + \gamma(r, X_r^{t,x}, \cdot)) - v(r, X_r^{t,x})) dr \\ - \int_s^T \tilde{Z}_r^{t,x} dW_r - \int_s^T \int_E \tilde{V}_r^{t,x}(e) \mu(dr, de) + (\tilde{K}_T^{+,t,x} - \tilde{K}_s^{+,t,x}), \quad \forall s \in [t, T], \\ \tilde{Y}_s^{t,x} \geq h(s, X_s^{t,x}), \quad \forall s \in [t, T] \quad \text{and} \quad \int_t^T (\tilde{Y}_s^{t,x} - h(s, X_s^{t,x})) dK_s^{+,t,x}, \end{cases} \quad (2.5)$$

admits a unique solution and  $\tilde{v}(t, x) := \tilde{Y}_t^{t,x}$  belongs to  $\Pi_c^g$  and is the unique viscosity solution to

$$\begin{cases} \min\{\tilde{v}(t, x) - h(t, x), -\tilde{v}_t(t, x) - \mathcal{L}\tilde{v}(t, x) - \mathcal{K}^n \tilde{v}(t, x) \\ - \tilde{f}(t, x, \tilde{v}(t, x), \sigma^\top(t, x) \nabla_x \tilde{v}(t, x))\} = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d \\ \tilde{v}(T, x) = \psi(x). \end{cases}$$

On the other hand,  $(Y^{t,x}, Z^{t,x}, V^{t,x}, K^{+,t,x})$  is the unique solution to (2.5) and we conclude that  $\tilde{v} = v$  is the unique solution to (2.4).  $\square$

To emphasize the dependence of the solution to (2.2) on the parameter  $n$ , we will henceforth employ the notation  $(Y^{t,x,n}, Z^{t,x,n}, V^{t,x,n}, K^{+,t,x,n})$  for these quadruples.

## 2.5 Preliminary estimates

For  $\nu \in \mathcal{V}_t$  we let  $\mathbb{E}^\nu$  be expectation with respect to the probability measure  $\mathbb{P}^\nu$  on  $(\Omega, \mathcal{F})$  defined by  $d\mathbb{P}^\nu := \kappa_T^\nu d\mathbb{P}$  with

$$\begin{aligned}\kappa_s^\nu &:= \mathcal{E}_s \left( \int_t^\cdot \int_E (\nu_r(e) - 1)(\mu(dr, de) - \lambda(de)) dr \right) \\ &:= \exp \left( \int_t^s \int_E (1 - \nu_r(e)) \lambda(de) dr \right) \prod_{t < \eta_j \leq s} \nu_{\eta_j}(\beta_j),\end{aligned}$$

where the sequence  $(\eta_j, \beta_j)_{j \geq 1}$  is the one that appears in the Dirac decomposition  $\mu = \sum_{j \geq 1} \delta_{(\eta_j, \beta_j)}$ .

**Lemma 2.9.** *Under Assumption 2.3, the SDE (1.5) admits a unique solution for each  $(t, x) \in [0, T] \times \mathbb{R}^d$ . Furthermore, the solution has moments of all orders, in particular, for each  $p \geq 0$ , there is a constant  $C > 0$  such that*

$$\mathbb{E}^\nu \left[ \sup_{s \in [\zeta, T]} |X_s^{t,x}|^p \middle| \mathcal{F}_\zeta^t \right] \leq C(1 + |X_\zeta^{t,x}|^p), \quad (2.6)$$

$\mathbb{P}$ -a.s. for all  $(t, x) \in [0, T] \times \mathbb{R}^n$ ,  $\nu \in \mathcal{V}_t$  and  $\zeta \in [t, T]$ .

*Proof.* The proof is rather elementary and follows a similar structure to the proof of Proposition 4.2 in [17] (albeit the latter is confined to a Brownian filtration). It is provided in its entirety since some of its intermediate results are utilized later on. For each  $j \in \mathbb{N}$ , we let  $X^j$  be the unique solution to the SDE

$$X_s^j = x + \int_t^s a(r, X_r^j) dr + \int_t^s \sigma(r, X_r^j) dW_r + \int_t^s \int_E \mathbb{1}_{[\mu((0,r), E) < j]} \gamma(s, X_{s-}^j, e) \mu(dr, de), \quad \forall s \in [t, T],$$

and note that  $X^j \rightarrow X^{t,x}$ ,  $\mathbb{P}$ -a.s., since  $\mu$  has finite intensity. By Assumption 2.3.(i) we get for  $s \in [\eta_j, T]$ , using integration by parts, that

$$\begin{aligned}|X_s^j|^2 &= |X_{\zeta \vee \eta_j}^j|^2 + 2 \int_{(\zeta \vee \eta_j)+}^s X_r^j dX_r^j + \int_{(\zeta \vee \eta_j)+}^s d[X^j, X^j]_r \\ &\leq K_\Gamma^2 \vee |X_{\zeta \vee \eta_j}^{j-1}|^2 + 2 \int_{(\zeta \vee \eta_j)+}^s X_r^j dX_r^j + \int_{(\zeta \vee \eta_j)+}^s d[X^j, X^j]_r.\end{aligned}$$

Now, either  $|X_{\eta_j}^{j-1}| \leq K_\Gamma$  in which case

$$|X_s^j|^2 \leq |X_\zeta^j|^2 \vee K_\Gamma^2 + 2 \int_{(\zeta \vee \eta_j)+}^s X_r^j dX_r^j + \int_{(\zeta \vee \eta_j)+}^s d[X^j, X^j]_r.$$

or  $|X_{\eta_j}^{j-1}| > K_\Gamma$  implying that

$$\begin{aligned}|X_s^j|^2 &\leq K_\Gamma^2 \vee |X_{\zeta \vee \eta_{j-1}}^{j-2}|^2 + 2 \int_{(\zeta \vee \eta_{j-1})+}^{\eta_j} X_r^{j-1} dX_r^{j-1} + \int_{(\zeta \vee \eta_{j-1})+}^{\eta_j} d[X^{j-1}, X^{j-1}]_r \\ &\quad + 2 \int_{(\zeta \vee \eta_j)+}^s X_r^j dX_r^j + \int_{(\zeta \vee \eta_j)+}^s d[X^j, X^j]_r.\end{aligned}$$

In the latter case the same argument can be repeated and we conclude that

$$|X_s^j|^2 \leq |X_\zeta^j|^2 \vee K_\Gamma^2 + \sum_{i=j_0}^j \left\{ 2 \int_{(\zeta \vee \tilde{\eta}_i)+}^{s \wedge \tilde{\eta}_{i+1}} X_r^i dX_r^i + \int_{(\zeta \vee \tilde{\eta}_i)+}^{s \wedge \tilde{\eta}_{i+1}} d[X^i, X^i]_r \right\}, \quad (2.7)$$



where  $\tilde{\eta}_0 = -1$ ,  $\tilde{\eta}_i = \eta_i$  for  $i = 1, \dots, j$  and  $\tilde{\eta}_{j+1} = \infty$  and  $j_0 := \max\{i \in \{1, \dots, j\} : |X_{\eta_i}^{i-1}| \leq K_\Gamma\} \vee 0$ .  
Now, since  $X^i$  and  $X^j$  coincide on  $[0, \eta_{i+1} \wedge j+1)$  we have

$$\sum_{i=j_0}^j \int_{(\zeta \vee \tilde{\eta}_i)^+}^{s \wedge \tilde{\eta}_{i+1}} X_r^i dX_r^i = \int_{\zeta \vee \eta_{j_0}}^s X_r^j a(r, X_r^j) dr + \int_{\zeta \vee \eta_{j_0}}^s X_r^j \sigma(r, X_r^j) dW_r,$$

and

$$\sum_{i=j_0}^j \int_{(\zeta \vee \tilde{\eta}_i)^+}^{s \wedge \tilde{\eta}_{i+1}} d[X^i, X^i]_r = \int_{\zeta \vee \eta_{j_0}}^s \sigma^2(r, X_r^j) dr.$$

Inserted in (2.7) this gives that

$$\begin{aligned} |X_s^j|^2 &\leq |X_\zeta^j|^2 \vee K_\Gamma^2 + \int_{\eta_{j_0}}^s (2X_s^j a(r, X_r^j) + \sigma^2(r, X_r^j)) dr + 2 \int_{\eta_{j_0}}^s X_r^j \sigma(r, X_r^j) dW_r \\ &\leq |X_\zeta^j|^2 + C \left( 1 + \int_\zeta^s |X_r^j|^2 dr + \sup_{\eta \in [\zeta, s]} \left| \int_\zeta^\eta X_r^j \sigma(r, X_r^j) dW_r \right| \right) \end{aligned} \quad (2.8)$$

for all  $s \in [\zeta, T]$ . The Burkholder-Davis-Gundy inequality and the fact that the right-hand side of (2.8) does not depend on  $\mu$  now gives that for any  $\nu \in \mathcal{V}_t$  and  $p \geq 2$ ,

$$\mathbb{E}^\nu \left[ \sup_{r \in [\zeta, s]} |X_r^j|^p \middle| \mathcal{F}_\zeta^t \right] \leq |X_\zeta^j|^2 + C \left( 1 + \mathbb{E}^\nu \left[ \int_\zeta^s |X_r^j|^p dr + \left( \int_\zeta^s |X_r^j|^4 dr \right)^{p/4} \middle| \mathcal{F}_\zeta^t \right] \right).$$

We can thus apply Grönwall's lemma to conclude that for  $p \geq 4$ ,

$$\mathbb{E}^\nu \left[ \sup_{s \in [\zeta, T]} |X_s^j|^p \middle| \mathcal{F}_\zeta^t \right] \leq C(1 + |X_\zeta^j|^p),$$

$\mathbb{P}$ -a.s., where the constant  $C = C(T, p)$  does not depend on  $\nu$  and  $j$  and (2.6) follows by letting  $j \rightarrow \infty$  on both sides and using Fatou's lemma. The result for general  $p \geq 0$  is then a simple consequence of Jensen's inequality.  $\square$

**Lemma 2.10.** *There is a  $C > 0$  such that*

$$|Y_s^{t,x,n}| \leq C(1 + |X_s^{t,x}|^q) \quad (2.9)$$

for all  $(t, x) \in [0, T] \times \mathbb{R}^d$  and  $s \in [t, T]$ .

*Proof.* We have

$$h(s, X_s^{t,x}) \leq Y_s^{t,x,n} \leq Y_s^{t,x,0} \quad (2.10)$$

and the assertion follows by the polynomial growth assumptions on  $h$  and the fact that  $v_0 \in \Pi_c^g$  (see Proposition 2.8).  $\square$

We also make use of the following lemma which is given without proof as it follows immediately from the definitions:

**Lemma 2.11.** *Let  $u, v : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$  be locally bounded functions.  $\mathcal{M}$  is monotone (if  $u \leq v$  pointwise, then  $\mathcal{M}u \leq \mathcal{M}v$ ). Moreover,  $\mathcal{M}(u_*)$  (resp.  $\mathcal{M}(u^*)$ ) is l.s.c. (resp. u.s.c.).*

In particular, it follows that  $\mathcal{M}v$  is jointly continuous whenever  $v$  is.

### 3 The local setting

Our approach to obtain existence of a unique viscosity solution to (1.1) goes through a fixed point argument. Before we are ready to proceed with this argument we need to solve the problem in the case where the driver  $f$  is a local function. In this regard, we consider the PDE

$$\begin{cases} \min\{v(t, x) - h(t, x), \max\{v(t, x) - \mathcal{M}v(t, x), -v_t(t, x) - \mathcal{L}v(t, x) \\ -\tilde{f}(t, x, v(t, x), \sigma^\top(t, x)\nabla_x v(t, x))\}\} = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d \\ v(T, x) = \psi(x), \end{cases} \quad (3.1)$$

where  $\tilde{f}$  satisfies Assumption 2.7.

The arguments utilized in this section are based on penalization and use the unique viscosity solution,  $v_n$ , to (2.4). By a comparison result for solutions to RBSDEs with jumps (see *e.g.* Theorem 4.1 in [21]), we find that  $(v_n)_{n \in \mathbb{N}}$  is a non-increasing sequence in  $\Pi_c^g$  and by Lemma 2.10 this sequence is bounded from below in the sense that there is a constant  $C > 0$  such that  $v_n(t, x) \geq -C(1 + |x|^\rho)$  for all  $(t, x) \in [0, T] \times \mathbb{R}^d$  and  $n \in \mathbb{N}$ . It then immediately follows that there is an upper semi-continuous function  $v \in \Pi^g$  such that  $v_n \searrow v$ , pointwisely. We prove that  $v$  is the unique viscosity solution to (3.1) within the set of functions of polynomial growth.

We first show that  $v$  satisfies the requirements for a viscosity solution at time  $T$ .

**Lemma 3.1.** *For each  $x \in \mathbb{R}^d$  it holds that  $v^*(T, x) \leq \psi(x)$  and  $v_*(T, x) \geq \psi(x)$ .*

*Proof.* First note that

$$v^*(T, x) \leq \lim_{(t', x') \rightarrow (T, x)} v_0(t', x') = \psi(x)$$

by continuity of  $v_0$ , proving the first inequality.

We turn to the second one which requires more work. In search for a contradiction, we assume that there is an  $x'_0 \in \mathbb{R}^d$  such that  $v_*(T, x'_0) < \psi(x'_0)$ . The proof is based on the following observation:

a) If  $v_*(T, x'_0) < \psi(x'_0)$  for some  $x'_0 \in \mathbb{R}^d$ , then there is a (possible different) point  $x_0 \in \mathbb{R}^d$  such that  $v_*(T, x_0) < \psi(x_0) \wedge \mathcal{M}v_*(T, x_0)$ .

To see this, we note that if a) does not hold, then by lower semi-continuity, there is an  $e'_0 \in E$  such that

$$v_*(T, x'_0) \geq \mathcal{M}v_*(T, x'_0) = v_*(T, x'_1) + \chi(T, x'_0, e'_0),$$

with  $x'_1 := x'_0 + \gamma(T, x'_0, e'_0)$ . Now, by assumption  $\psi(x'_0) \leq \psi(x'_1) + \chi(T, x'_0, e'_0)$  and we find that

$$\begin{aligned} v_*(T, x'_1) - \psi(x'_1) &\leq v_*(T, x'_0) - \chi(T, x'_0, e'_0) - \psi(x'_1) \\ &\leq v_*(T, x'_0) - \chi(T, x'_0, e'_0) - (\psi(x'_0) - \chi(T, x'_0, e'_0)) \\ &< 0 \end{aligned}$$

Hence, if a) does not hold, then we must have  $v_*(T, x'_1) \geq \mathcal{M}v_*(T, x'_1)$ . We can repeat this argument indefinitely to find that, if a) does not hold, then there is a sequence  $(x'_j, e'_j)_{j=1}^\infty$  in  $\mathbb{R}^d \times E$  such that  $x'_{j+1} = x'_j + \gamma(T, x'_j, e'_j)$  and  $v_*(T, x'_j) \geq v_*(T, x'_{j+1}) + \chi(T, x'_j, e'_j)$  for all  $j \in \mathbb{N}$ . We conclude that for each  $k$  it holds that

$$\begin{aligned} v_*(T, x'_0) &\geq v_*(T, x'_k) + \sum_{j=0}^{k-1} \chi(T, x'_j, e'_j) \\ &\geq h(T, x'_k) + k\delta, \end{aligned}$$

a contradiction since  $h$  is of uniformly bounded from below on  $[0, T] \times \Lambda_f(|x'_0|)$  and  $v_*(T, x'_0) \leq v_0(T, x'_0)$ .

We thus assume that there is a point  $x_0$  and an  $\varepsilon > 0$  such that

$$v_*(T, x_0) \leq \psi(x_0) \wedge \mathcal{M}v_*(T, x_0) - 3\varepsilon. \quad (3.2)$$

There is a sequence  $(t_j, x_j, n_j)$  in  $[0, T] \times \mathbb{R}^d \times \mathbb{N}$  such that  $(t_j, x_j) \rightarrow (T, x_0)$  and  $v_{n_j}(t_j, x_j) \rightarrow v_*(T, x_0)$ . In particular, if (3.2) holds, then there is a  $j_0$  such that

$$v_{n_j}(t_j, x_j) \leq \psi(x_0) \wedge \mathcal{M}v_*(T, x_0) - 2\varepsilon$$

whenever  $j \geq j_0$ . On the other hand, continuity of  $\psi$  and lower semi-continuity of  $\mathcal{M}v_*$  then implies that there is a  $\delta' > 0$  such that

$$\begin{aligned} v_{n_j}(t_j, x_j) &\leq \psi(x) \wedge \mathcal{M}v_*(t, x) - \varepsilon \\ &\leq \psi(x) \wedge \mathcal{M}v_{n_j}(t, x) - \varepsilon \end{aligned}$$

whenever  $j \geq j_0$  and  $(t, x) \in B_{\delta'}(T, x_0) \cap [0, T] \times \mathbb{R}^d$  (where  $B_{\delta'}(T, x_0)$  is the open ball in  $\mathbb{R}^{d+1}$  of radius  $\delta'$  centered at  $(T, x_0)$ ).

We introduce the stopping times

$$\theta_j := \inf\{s \geq t_j : v_{n_j}(s, X_s^{t_j, x_j}) \geq \psi(X_s^{t_j, x_j}) \wedge \mathcal{M}v_{n_j}(s, X_s^{t_j, x_j})\}$$

and

$$\vartheta_j := \inf\{s \geq t_j : (s, X_s^{t_j, x_j}) \notin B_{\delta'}(T, x_0) \text{ or } \mu((t_j, s], E) \geq 1\}.$$

A standard dynamic programming result now gives that

$$\begin{aligned} v_{n_j}(t_j, x_j) &= v_{n_j}(\theta_j \wedge \vartheta_j, X_{\theta_j \wedge \vartheta_j}^{t_j, x_j}) + \int_{t_j}^{\theta_j \wedge \vartheta_j} \tilde{f}^{n_j}(r, X_r^{t_j, x_j}, Y_r^{t_j, x_j, n_j}, Z_r^{t_j, x_j, n_j}, V_r^{t_j, x_j, n_j}) dr \\ &\quad - \int_{t_j}^{\theta_j \wedge \vartheta_j} Z_r^{t_j, x_j, n_j} dW_r - \int_{t_j}^{\theta_j \wedge \vartheta_j} \int_E V_r^{t_j, x_j, n_j}(e) \mu(dr, de) + K_{\theta_j \wedge \vartheta_j}^{-, t_j, x_j, n_j} - K_{t_j}^{-, t_j, x_j, n_j}. \end{aligned}$$

Since  $V_r^{t_j, x_j, n_j}(e) = v_n(s, X_s^{t_j, x_j} + \gamma(s, X_s^{t_j, x_j}, e)) - v_n(s, X_s^{t_j, x_j})$ ,  $d\mathbb{P} \otimes ds \otimes \lambda(de)$ -a.e. (see Proposition 3.1 in [13]) we get that,

$$\begin{aligned} &\int_{t_j}^{\theta_j} \int_E (V_r^{t_j, x_j, n_j}(e) + \chi(r, X_r^{t_j, x_j}, e))^- \lambda(de) \\ &= \int_{t_j}^{\theta_j} \int_E (v_n(s, X_s^{t_j, x_j} + \gamma(s, X_s^{t_j, x_j}, e)) + \chi(r, X_r^{t_j, x_j}, e) - v_n(s, X_s^{t_j, x_j}))^- \lambda(de) \\ &= 0. \end{aligned}$$

Consequently,

$$\int_{t_j}^{\theta_j \wedge \vartheta_j} \tilde{f}^{n_j}(r, X_r^{t_j, x_j}, Y_r^{t_j, x_j, n_j}, Z_r^{t_j, x_j, n_j}, V_r^{t_j, x_j, n_j}) dr = \int_{t_j}^{\theta_j \wedge \vartheta_j} \tilde{f}(r, X_r^{t_j, x_j}, Y_r^{t_j, x_j, n_j}, Z_r^{t_j, x_j, n_j}) dr$$

and we conclude that

$$\begin{aligned} v_{n_j}(t_j, x_j) &= v_{n_j}(\theta_j \wedge \vartheta_j, X_{\theta_j \wedge \vartheta_j}^{t_j, x_j}) + \int_{t_j}^{\theta_j \wedge \vartheta_j} \tilde{f}(r, X_r^{t_j, x_j}, Y_r^{t_j, x_j, n_j}, Z_r^{t_j, x_j, n_j}) dr \\ &\quad - \int_{t_j}^{\theta_j \wedge \vartheta_j} Z_r^{t_j, x_j, n_j} dW_r - \int_{t_j}^{\theta_j \wedge \vartheta_j} \int_E V_r^{t_j, x_j, n_j}(e) \mu(dr, de) + K_{\theta_j \wedge \vartheta_j}^{-, t_j, x_j, n_j} - K_{t_j}^{-, t_j, x_j, n_j}. \end{aligned}$$

Standard arguments now give that there is a  $C > 0$  such that

$$\|Y^{t_j, x_j, n_j}\|_{S_{t, \theta_j \wedge \vartheta_j}^2} + \|Z^{t_j, x_j, n_j}\|_{\mathcal{H}_{t, \theta_j \wedge \vartheta_j}^2(W)} + \|V^{t_j, x_j, n_j}\|_{\mathcal{H}_{t, \theta_j \wedge \vartheta_j}^2(\mu)} \leq C(1 + |x_j|^\rho)$$

for all  $j \in \mathbb{N}$ , and we find that

$$v_{n_j}(t_j, x_j) \geq -C(T - t_j)^{1/2}(1 + |x_j|^\rho) + \mathbb{E}[\mathbb{1}_{[\theta_j < \vartheta_j]} v_n(\theta_j, X_{\theta_j}^{t_j, x_0})] - C\mathbb{E}[\mathbb{1}_{[\theta_j \geq \vartheta_j]}]^{1/2}(1 + |x_j|^\rho),$$

where the constant  $C > 0$  can be chosen independent of  $j$ . Now, since  $v_{n_j}(T, x) = \psi(x)$ , we have  $\theta_j \leq T$ ,  $\mathbb{P}$ -a.s. and on the subset of  $\Omega$  where  $[\theta_j < \vartheta_j]$  we have  $v_{n_j}(\theta_j, X_{\theta_j}^{t_j, x_0}) - v_{n_j}(t_j, x_j) \geq \varepsilon$ . We thus conclude that

$$\varepsilon \mathbb{P}[\theta_j < \vartheta_j] \leq C(T - t_j)^{1/2}(1 + |x_j|^\rho) + C\mathbb{E}[\mathbb{1}_{[\theta_j \geq \vartheta_j]}]^{1/2}(1 + |x_j|^\rho),$$

whenever  $j \geq j_0$ . On the other hand,  $\mathbb{P}[\theta_j \geq \vartheta_j] \rightarrow 0$  as  $j \rightarrow \infty$  and sending  $j \rightarrow \infty$  gives the sought contradiction.  $\square$

**Theorem 3.2.** *The function  $v$  is continuous and thus belongs to  $\Pi_c^g$ . Moreover, it is the unique viscosity solution to (3.1) in  $\Pi^g$ .*

*Proof.* We prove that  $v$  is a viscosity solution to (1.1), then continuity and uniqueness will follow from the comparison principle stated in Proposition A.4 of Appendix A. By continuity of  $v_n$ , we have (see *e.g.* [2], p. 91),

$$\begin{aligned} v_*(t, x) &= \liminf_{n \rightarrow \infty} v_n(t, x) := \liminf_{(n, t', x') \rightarrow (\infty, t, x)} v_n(t', x'), \\ v(t, x) &= v^*(t, x) = \limsup_{n \rightarrow \infty} v_n(t, x) := \limsup_{(n, t', x') \rightarrow (\infty, t, x)} v_n(t', x'). \end{aligned}$$

*Subsolution-property.* We first show that  $v = v^*$  is a viscosity subsolution. For  $(t, x) \in [0, T) \times \mathbb{R}^d$  and  $(p, q, M) \in \bar{J}^+ v(t, x)$  there is (by Lemma 6.1 in [7]) a sequence  $(t_j, x_j) \in [0, T) \times \mathbb{R}^d$  and sequences  $n_j \rightarrow \infty$  and  $(p_j, q_j, M_j) \in J^+ v_{n_j}(t_j, x_j)$  such that

$$(t_j, x_j, v_{n_j}(t_j, x_j), p_j, q_j, M_j) \rightarrow (t, x, v(t, x), p, q, M).$$

As  $(p_j, q_j, M_j) \in J^+ v_{n_j}(t_j, x_j)$  and  $v_{n_j}$  is a viscosity subsolution to (2.4) with  $n = n_j$ , it holds that

$$\begin{aligned} &\min\{v_{n_j}(t_j, x_j) - g(t_j, x_j), -p_j - \langle a(t_j, x_j), q_j \rangle - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t_j, x_j) M_j) \\ &+ n_j \int_E (v_{n_j}(t_j, x_j + \gamma(t_j, x_j, e)) + \chi(t_j, x_j, e) - v_{n_j}(t_j, x_j))^- \lambda(de) \\ &- \tilde{f}(t_j, x_j, v_{n_j}(t_j, x_j), \sigma^\top(t_j, x_j) q_j)\} \leq 0. \end{aligned}$$

Now, whenever  $v(t, x) > g(t, x)$  there is a  $j_0 \in \mathbb{N}$ , such that  $v_{n_j}(t_j, x_j) > g(t_j, x_j)$  for all  $j \geq j_0$ . Hence,

$$\begin{aligned} &-p_j - \langle a(t_j, x_j), q_j \rangle - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t_j, x_j) M_j) \\ &+ n_j \int_E (v_{n_j}(t_j, x_j + \gamma(t_j, x_j, e)) + \chi(t_j, x_j, e) - v_{n_j}(t_j, x_j))^- \lambda(de) \\ &- \tilde{f}(t_j, x_j, v_{n_j}(t_j, x_j), \sigma^\top(t_j, x_j) q_j) \leq 0 \end{aligned} \tag{3.3}$$

whenever  $j \geq j_0$ . Sending  $j \rightarrow \infty$  gives that

$$-p - \langle a(t, x), q \rangle - \frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) M) - \tilde{f}(t, x, v(t, x), \sigma^\top(t, x) q) \leq 0.$$

Now, assume that  $v(t, x) > \inf_{e \in E} \{v(t, x + \gamma(t, x, e)) + \chi(t, x, e)\}$ . Then there exists an  $e_0 \in E$  such that

$$v(t, x + \gamma(t, x, e_0)) + \chi(t, x, e_0) - v(t, x) < 0.$$

This in turn implies the existence of an  $\varepsilon > 0$  and an open neighborhood  $E_0 \in \mathcal{B}(E)$  of  $e_0$  in  $E$  such that

$$v_{n_j}(t_j, x_j + \gamma(t_j, x_j, e)) + \chi(t_j, x_j, e) - v_{n_j}(t_j, x_j) \leq -\varepsilon$$

for all  $e \in E_0$  and all  $j$  sufficiently large. Consequently,

$$\int_E (v_{n_j}(t_j, x_j + \gamma(t_j, x_j, e)) + \chi(t_j, x_j, e) - v_{n_j}(t_j, x_j))^- \lambda(de) \geq \varepsilon \lambda(E_0)$$

for  $j$  sufficiently large. However, since  $\lambda$  has full topological support,  $\lambda(E_0) > 0$  and this contradicts the fact that (3.3) holds for all  $j$ . We thereby conclude that  $v(t, x) \leq \inf_{e \in E} \{v(t, x + \gamma(t, x, e)) + \chi(t, x, e)\}$ .

*Supersolution-property.* We turn to the supersolution property of  $v_*$ . For  $(t, x) \in [0, T) \times \mathbb{R}^d$  and  $(p, q, M) \in \bar{J}^- v_*(t, x)$  there is (again by Lemma 6.1 in [7]) a sequence  $(t_j, x_j) \in [0, T) \times \mathbb{R}^d$  and sequences  $n_j \rightarrow \infty$  and  $(p_j, q_j, M_j) \in J^- v_{n_j}(t_j, x_j)$  such that

$$(t_j, x_j, v_{n_j}(t_j, x_j), p_j, q_j, M_j) \rightarrow (t, x, v_*(t, x), p, q, M).$$

As  $(p_j, q_j, M_j) \in J^- v_{n_j}(t_j, x_j)$  and  $v_{n_j}$  is a viscosity supersolution to (2.4) with  $n = n_j$ , it holds that

$$\begin{aligned} & \min\{v_{n_j}(t_j, x_j) - g(t_j, x_j), -p_j - < a(t_j, x_j), q_j > -\frac{1}{2} \text{Tr}(\sigma \sigma^\top(t_j, x_j) M_j) \\ & + n_j \int_E (v_{n_j}(t_j, x_j + \gamma(t_j, x_j, e)) + \chi(t_j, x_j, e) - v_{n_j}(t_j, x_j))^- \lambda(de) \\ & - \tilde{f}(t_j, x_j, v_{n_j}(t_j, x_j), \sigma^\top(t_j, x_j) q_j)\} \geq 0 \end{aligned} \quad (3.4)$$

In particular,  $v_{n_j}(t_j, x_j) \geq g(t_j, x_j)$  and it follows by continuity of  $g$  that  $v_*(t, x) \geq g(t, x)$ . Suppose now that  $v_*(t, x) < \inf_{e \in E} \{v_*(t, x + \gamma(t, x, e)) + \chi(t, x, e)\}$  implying the existence of an  $\varepsilon > 0$  such that

$$v_*(t, x) \leq v_*(t, x + \gamma(t, x, e)) + \chi(t, x, e) - \varepsilon, \quad \forall e \in E.$$

Since  $v_{n_j}(t_j, x_j) \rightarrow v_*(t, x)$  and

$$\begin{aligned} & \liminf_{j \rightarrow \infty} \inf_{e \in \mathbb{E}} \{v_{n_j}(t_j, x_j + \gamma(t_j, x_j, e)) + \chi(t_j, x_j, e)\} \\ & \geq \liminf_{j \rightarrow \infty} \inf_{e \in \mathbb{E}} \{v_*(t_j, x_j + \gamma(t_j, x_j, e)) + \chi(t_j, x_j, e)\} \\ & \geq \inf_{e \in \mathbb{E}} \{v_*(t, x + \gamma(t, x, e)) + \chi(t, x, e)\}, \end{aligned}$$

this implies that

$$v_{n_j}(t, x) \leq v_{n_j}(t_j, x_j + \gamma(t_j, x_j, e)) + \chi(t_j, x_j, e), \quad \forall e \in E,$$

whenever  $j$  is sufficiently large. In particular, we get that

$$\int_E (v_{n_j}(t_j, x_j + \gamma(t_j, x_j, e)) + \chi(t_j, x_j, e) - v_{n_j}(t_j, x_j))^- \lambda(de) = 0$$

for  $j$  sufficiently large. Taking the limit in (3.4) we thus find that in this situation

$$-p - < a(t, x), q > -\frac{1}{2} \text{Tr}(\sigma \sigma^\top(t, x) M) - \tilde{f}(t, x, v_*(t, x), \sigma^\top(t, x) q) \geq 0,$$

proving the supersolution property of  $v_*$ .  $\square$

Much like the functions  $v_n$ , for each  $(t, x) \in [0, T] \times \mathbb{R}^d$ , the processes  $(Y^{t,x,n})_{n \in \mathbb{N}}$  form a non-increasing sequence that is bounded from below by  $h(\cdot, X_\cdot^{t,x}) \in \mathcal{S}^2$ , implying the existence of an  $\mathbb{F}^t$ -progressively measurable process  $Y^{t,x}$  such that  $Y^{t,x,n} \searrow Y^{t,x}$ , pointwisely. Taking the limit on both sides of  $v_n(\eta, X_\eta^{t,x}) = Y_\eta^{t,x,n}$  gives that  $Y_\eta^{t,x} = v(\eta, X_\eta^{t,x})$  for each  $\eta \in \mathcal{T}_t$ . In particular,  $Y^{t,x}$  is càdlàg and thus belongs to  $\mathcal{S}_t^2$ . In our pursuit of a related optimal stopping problem, we let for each  $\tau \in \mathcal{T}^t$ , the process  $Y^{t,x,\tau}$  be the first component in the quadruple of processes  $(Y^{t,x,\tau}, Z^{t,x,\tau}, V^{t,x,\tau}, K^{-,t,x,\tau}) \in \mathcal{S}_{[t,\tau]}^2 \times \mathcal{H}_{[t,\tau]}^2(W) \times \mathcal{H}_{[t,\tau]}^2(\mu) \times \mathcal{S}_{[t,\tau],i}^2$  that is the unique maximal solution to the BSDE with constrained jumps,

$$\begin{cases} Y_s^{t,x,\tau} = \Psi(\tau, X_\tau^{t,x}) + \int_s^\tau \tilde{f}(r, X_r^{t,x}, Y_r^{t,x,\tau}, Z_r^{t,x,\tau}) dr - \int_s^\tau Z_r^{t,x,\tau} dW_r - \int_s^\tau \int_E V_r^{t,x,\tau}(e) \mu(dr, de) \\ \quad - (K_\tau^{-,t,x,\tau} - K_s^{-,t,x,\tau}), \quad \forall s \in [t, \tau], \\ V_s^{t,x,\tau}(e) \geq -\chi(s, X_{s-}^{t,x}, e), \quad d\mathbb{P} \otimes ds \otimes \lambda(de) - a.e. \end{cases} \quad (3.5)$$

Theorem 3.1 of [18] seamlessly transfers to the present context, allowing us to conclude the following:

**Proposition 3.3.** *For each  $(t, x) \in [0, T] \times \mathbb{R}^d$  and  $\eta \in \mathcal{T}^t$ , the process  $Y^{t,x}$  can be represented as*

$$Y_\eta^{t,x} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_\eta^t} Y_\eta^{t,x,\tau} = \operatorname{ess\,inf}_{\nu \in \mathcal{V}_t} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_\eta^t} P_\eta^{t,x;\tau,\nu}, \quad (3.6)$$

where given  $\nu \in \mathcal{V}_t$  and  $\tau \in \mathcal{T}^t$ , the triple  $(P^{t,x;\tau,\nu}, Q^{t,x;\tau,\nu}, S^{t,x;\tau,\nu}) \in \mathcal{S}_{[t,\tau]}^2 \times \mathcal{H}_{[t,\tau]}^2(W) \times \mathcal{H}_{[t,\tau]}^2(\mu)$  is the unique solution to the standard BSDE

$$\begin{aligned} P_s^{t,x;\tau,\nu} &= \Psi(\tau, X_\tau^{t,x}) + \int_s^\tau \tilde{f}^\nu(r, X_r^{t,x}, P_r^{t,x;\tau,\nu}, Q_r^{t,x;\tau,\nu}, S_r^{t,x;\tau,\nu}) dr \\ &\quad - \int_s^\tau Q_r^{t,x;\tau,\nu} dW_r - \int_s^\tau \int_E S_r^{t,x;\tau,\nu}(e) \mu(dr, de), \end{aligned} \quad (3.7)$$

with driver

$$\tilde{f}^\nu(t, x, y, z, v) := \tilde{f}(t, x, y, z) + \int_E (v(e) + \chi(t, x, e)) \nu_t(e) \lambda(de).$$

## 4 The general setting

We now turn to the general setting of a non-local driver. Existence will again follow by an approximation routine and for  $(t, x) \in [0, T] \times \mathbb{R}^d$  and  $k \in \mathbb{N}$ , we let  $Y_s^{t,x,k} := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_s^t} Y_s^{t,x,k;\tau}$ , where the quadruple  $(Y^{t,x,k;\tau}, Z^{t,x,k;\tau}, V^{t,x,k;\tau}, K^{-,t,x,k;\tau}) \in \mathcal{S}_{[t,\tau]}^2 \times \mathcal{H}_{[t,\tau]}^2(W) \times \mathcal{H}_{[t,\tau]}^2(\mu) \times \mathcal{S}_{[t,\tau],i}^2$  is the unique maximal solution to

$$\begin{cases} Y_s^{t,x,k;\tau} = \Psi(\tau, X_\tau^{t,x}) + \int_s^\tau f(r, X_r^{t,x}, \bar{Y}^{k-1}(r, \cdot), Z_r^{t,x,k;\tau}) dr - \int_s^\tau Z_r^{t,x,k;\tau} dW_r \\ \quad - \int_s^\tau \int_E V_r^{t,x,k;\tau}(e) \mu(dr, de) - (K_\tau^{-,t,x,k;\tau} - K_s^{-,t,x,k;\tau}), \quad \forall s \in [t, \tau] \\ V_s^{t,x,k;\tau}(e) \geq -\chi(s, X_{s-}^{t,x}, e), \quad d\mathbb{P} \otimes ds \otimes \lambda(de) - a.e., \end{cases} \quad (4.1)$$

for  $k \geq 1$ , with  $\bar{Y}^{k-1}(t, x) := Y_t^{t,x,k}$  and  $\bar{Y}^0 \equiv 0$ .

**Proposition 4.1.** *There is a sequence  $(\bar{Y}^k \in \Pi_c^g)_{k \geq 0}$  that satisfies the recursion above.*

*Proof.* We need to show that for each  $k \geq 1$ , there is a  $v_{k-1} \in \Pi_c^g$  such that  $Y_t^{t,x,k-1} = v_{k-1}(t, x)$  for all  $(t, x) \in [0, T] \times \mathbb{R}^n$ . However, for  $k = 1$  this is immediate by the definition. Now, the result follows by

using Proposition 3.3, Theorem 3.2 and induction.  $\square$

For  $(t, \zeta) \in [0, T] \times \mathbb{R}_+$  and  $\alpha \in \mathcal{A}_t$ , we let  $(\Upsilon^{t, \zeta; \alpha}, \Theta^{t, \zeta; \alpha}) \in \mathcal{S}_t^2 \times \mathcal{S}_{t, t}^2$  solve the one-dimensional reflected SDE

$$\begin{cases} \Upsilon_s^{t, \zeta; \alpha} = \zeta^2 \vee K_\Gamma^2 + (4C_{a, \sigma} + 2C_{a, \sigma}^2) \int_t^s (1 + \Upsilon_r^{t, \zeta; \alpha}) dr + 4C_{a, \sigma} \int_t^s (1 + \Upsilon_r^{t, \zeta; \alpha}) \alpha_r dW_r + \Theta_s^{t, \zeta; \alpha}, \\ \forall s \in [t, T], \\ \Upsilon_s^{t, \zeta; \alpha} \geq \zeta^2 \vee K_\Gamma^2 \text{ and } \int_t^T (\Upsilon_s^{t, \zeta; \alpha} - (\zeta^2 \vee K_\Gamma^2)) d\Theta_s^{t, \zeta; \alpha} = 0. \end{cases} \quad (4.2)$$

We then set  $R_s^{t, \zeta; \alpha} := \sqrt{\Upsilon_s^{t, \zeta; \alpha}}$  and note that classically, we have

$$\mathbb{E} \left[ \sup_{s \in [t, T]} |R_s^{t, \zeta; \alpha}|^p \right] \leq C(1 + |\zeta \vee K_\Gamma|^p),$$

for all  $p \geq 2$ .

**Lemma 4.2.** *For each  $(t, x) \in [0, T] \times \mathbb{R}^d$ , there is an  $\alpha \in \mathcal{A}_t$  such that  $|X_s^{t, x}| \leq R_s^{t, |x|; \alpha}$  for all  $s \in [t, T]$ ,  $\mathbb{P}$ -a.s.*

*Proof.* Since

$$|2xa(r, x) + \sigma^2(r, x)| \leq (4C_{a, \sigma} + 2C_{a, \sigma}^2)(1 + |x|^2),$$

it follows from (2.8) that we can always choose  $\alpha \in \mathcal{A}_t$  such that

$$2C_{a, \sigma}(1 + \Upsilon_r^{t, |x|; \alpha}) \alpha_r = X_r^{t, x} \sigma(r, X_r^{t, x}), \quad \forall r \in [t, T],$$

and the statement holds by (2.8).  $\square$

For each  $\varphi \in \Pi_c^g$ , we let  $\bar{Y}^\varphi \in \Pi_c^g$  be defined as  $\bar{Y}^\varphi(t, x) = \text{ess sup}_{\tau \in \mathcal{T}_t} Y_t^{t, x, \varphi; \tau}$ , where the quadruple  $(Y^{t, x, \varphi; \tau}, Z^{t, x, \varphi; \tau}, V^{t, x, \varphi; \tau}, K^{-, t, x, \varphi; \tau})$  is the unique maximal solution to

$$\begin{cases} Y_s^{t, x, \varphi; \tau} = \Psi(\tau, X_\tau^{t, x}) + \int_s^\tau f(r, X_r^{t, x}, \varphi(r, \cdot), Z_r^{t, x, \varphi; \tau}) dr - \int_s^\tau Z_r^{t, x, \varphi; \tau} dW_r \\ - \int_s^\tau \int_E V_r^{t, x, \varphi; \tau}(e) \mu(dr, de) - (K_\tau^{-, t, x, \varphi; \tau} - K_s^{-, t, x, \varphi; \tau}), \quad \forall s \in [t, \tau] \\ V_s^{t, x, k; \tau}(e) \geq -\chi(s, X_s^{t, x}, e), \quad d\mathbb{P} \otimes ds \otimes \lambda(de) - a.e., \end{cases} \quad (4.3)$$

and note that letting  $(P^{t, x, \varphi; \nu, \tau}, Q^{t, x, \varphi; \nu, \tau}, S^{t, x, \varphi; \nu, \tau}) \in \mathcal{S}_t^2 \times \mathcal{H}_t^2(W) \times \mathcal{H}_t^2(\mu)$  solve

$$\begin{aligned} P_s^{t, x, \varphi; \nu, \tau} &= \Psi(\tau, X_\tau^{t, x}) + \int_s^\tau f^\nu(r, X_r^{t, x}, P_r^{t, x, \varphi; \nu, \tau}, \varphi(r, \cdot), Q_r^{t, x, \varphi; \nu, \tau}, S_r^{t, x, \varphi; \nu, \tau}) dr \\ &\quad - \int_s^\tau Q_r^{t, x, \varphi; \nu, \tau} dW_r - \int_s^\tau \int_E S_r^{t, x, \varphi; \nu, \tau}(e) \mu(dr, de), \end{aligned} \quad (4.4)$$

where

$$f^\nu(t, x, g, z, v) := f(t, x, g, z, v) + \int_E (v(e) + \chi(t, x, e)) \nu_t(e) \lambda(de),$$

Proposition 3.3 gives that

$$\bar{Y}^\varphi(t, x) = \inf_{\nu \in \mathcal{V}_t} \sup_{\tau \in \mathcal{T}^t} P_t^{t, x, \varphi; \nu, \tau}. \quad (4.5)$$

The above definitions allow us to present the following result, which extends prior results derived in [19] (see Proposition 4.3 therein) and forms the basis for our contraction argument:

**Proposition 4.3.** *There is a  $\kappa > 0$  such that for all  $\varphi, \tilde{\varphi} \in \Pi_c^g$  and  $\zeta > 0$ , we have*

$$\begin{aligned} & \sup_{\alpha \in \mathcal{A}^W} \mathbb{E} \left[ \int_0^T e^{\kappa t} \sup_{x \in \Lambda_f(R_t^{0, \zeta; \alpha})} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x)|^2 dt \right] \\ & \leq \frac{1}{4} \sup_{\alpha \in \mathcal{A}^W} \mathbb{E} \left[ \int_0^T e^{\kappa t} \sup_{x \in \Lambda_f(R_t^{0, \zeta; \alpha})} |\tilde{\varphi}(t, x) - \varphi(t, x)|^2 dt \right]. \end{aligned} \quad (4.6)$$

Furthermore, there is a  $C > 0$  such that

$$\sup_{t \in [0, T]} \sup_{x \in \Lambda_f(\zeta)} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x)|^2 \leq C \sup_{\alpha \in \mathcal{A}^W} \mathbb{E} \left[ \int_0^T \sup_{x \in \Lambda_f(R_t^{0, \zeta; \alpha})} |\tilde{\varphi}(t, x) - \varphi(t, x)|^2 dt \right] \quad (4.7)$$

for each  $\zeta > 0$ .

*Proof.* By (4.5) we find that

$$\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x) \leq \sup_{\nu \in \mathcal{V}_t} \sup_{\tau \in \mathcal{T}^t} (P_t^{t, x, \varphi; \nu, \tau} - P_t^{t, x, \tilde{\varphi}; \nu, \tau}).$$

Since a similar inequality holds in the opposite direction, we get that

$$|\bar{Y}^{\varphi}(t, x) - \bar{Y}^{\tilde{\varphi}}(t, x)| \leq \sup_{\nu \in \mathcal{V}_t} \sup_{\tau \in \mathcal{T}^t} |P_t^{t, x, \varphi; \nu, \tau} - P_t^{t, x, \tilde{\varphi}; \nu, \tau}|. \quad (4.8)$$

For  $(\nu, \tau) \in \mathcal{V}_t \times \mathcal{T}^t$ , let  $(P, Q, S) := (P^{t, x, \varphi; \nu, \tau}, Q^{t, x, \varphi; \nu, \tau}, S^{t, x, \varphi; \nu, \tau})$  and  $(\tilde{P}, \tilde{Q}, \tilde{S}) := (P^{t, x, \tilde{\varphi}; \nu, \tau}, Q^{t, x, \tilde{\varphi}; \nu, \tau}, S^{t, x, \tilde{\varphi}; \nu, \tau})$  and note that for  $\kappa > 0$ , Itô's formula applied to  $e^{\kappa \cdot} |\tilde{P} - P|^2$  gives

$$\begin{aligned} & e^{\kappa t} |\tilde{P}_t - P_t|^2 + \int_t^\tau e^{\kappa s} |\tilde{Q}_s - Q_s|^2 ds + \int_t^\tau \int_E e^{\kappa s} |\tilde{S}_s(e) - S_s(e)|^2 d\mu(ds, de) \\ & = -2 \int_t^\tau e^{\kappa s} (\tilde{P}_s - P_s)(\tilde{Q}_s - Q_s) dW_s - 2 \int_t^\tau \int_E e^{\kappa s} (\tilde{P}_s - P_s)(\tilde{S}_s(e) - S_s(e)) d\mu(ds, de) \\ & \quad - \kappa \int_t^\tau e^{\kappa s} |\tilde{P}_s - P_s|^2 ds + 2 \int_t^\tau e^{\kappa s} (\tilde{P}_s - P_s) (f(s, X_s^{t, x}, \tilde{\varphi}(s, \cdot), \tilde{Q}_s) - f(s, X_s^{t, x}, \varphi(s, \cdot), Q_s)) ds \\ & \quad + 2 \int_t^\tau \int_E e^{\kappa s} (\tilde{P}_s - P_s)(\tilde{S}(e) - S(e)) \nu_t(e) \lambda(de). \end{aligned}$$

By assumption

$$|f(s, X_s^{t, x}, \varphi(s, \cdot), Q_s) - f(s, X_s^{t, x}, \tilde{\varphi}(s, \cdot), \tilde{Q}_s)| \leq k_f \left( \sup_{x' \in \Lambda_f(|X_s^{t, x}|)} |\tilde{\varphi}(s, x') - \varphi(s, x')| + |\tilde{Q}_s - Q_s| \right).$$

Hence, taking the expectation w.r.t. the measure  $\mathbb{P}^\nu$  and using inequalities  $2Cxy \leq (Cx)^2 + y^2$  and  $2xy \leq x^2/\sqrt{\kappa} + \sqrt{\kappa}y^2$  gives

$$\begin{aligned} e^{\kappa t} |\tilde{P}_t - P_t|^2 & \leq (C^2 + C\sqrt{\kappa} - \kappa) \mathbb{E}^\nu \left[ \int_t^\tau e^{\kappa s} |\tilde{P}_s - P_s|^2 ds \right] \\ & \quad + \frac{C}{\sqrt{\kappa}} \mathbb{E}^\nu \left[ \int_t^\tau e^{\kappa s} \sup_{x' \in \Lambda_f(|X_s^{t, x}|)} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right]. \end{aligned}$$

Now, pick  $\kappa_0 > 0$  such that  $\kappa_0 \geq C^2 + C\sqrt{\kappa_0}$  and note that for each  $\kappa \geq \kappa_0$ , we have

$$\begin{aligned} e^{\kappa t} |P_t - \tilde{P}_t|^2 & \leq \frac{C}{\sqrt{\kappa}} \mathbb{E}^\nu \left[ \int_t^\tau e^{\kappa s} \sup_{x' \in \Lambda_f(|X_s^{t, x}|)} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right] \\ & \leq \frac{C}{\sqrt{\kappa}} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}^\nu \left[ \int_t^\tau e^{\kappa s} \sup_{x' \in \Lambda_f(|R_s^{t, |x|; \alpha}|)} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right] \\ & \leq \frac{C}{\sqrt{\kappa}} \sup_{\alpha \in \mathcal{A}_t^W} \mathbb{E} \left[ \int_t^\tau e^{\kappa s} \sup_{x' \in \Lambda_f(|R_s^{t, |x|; \alpha}|)} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right], \end{aligned}$$



where the first inequality follows from Lemma 4.2 while the second one is due to the fact that the SDE in (4.2) does not depend on  $\mu$  (see *e.g.* Section 4.1 of [1]). Since the right-hand side is non-decreasing in  $|x|$  and independent of  $\nu$  and  $\tau$ , (4.8) now gives that

$$e^{\kappa t} \sup_{x \in \Lambda_f(\zeta)} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x)|^2 \leq \frac{C}{\sqrt{\kappa}} \sup_{\alpha \in \mathcal{A}_t^W} \mathbb{E} \left[ \int_t^T e^{\kappa s} \sup_{x' \in \Lambda_f(R_s^{t, \zeta; \alpha})} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right], \quad (4.9)$$

for any  $\zeta \geq 0$ . In particular, as both sides are continuous in  $\zeta$  a standard dynamic programming argument gives that for any  $\alpha_1 \in \mathcal{A}^W$ , we have

$$\mathbb{E} \left[ e^{\kappa t} \sup_{x \in \Lambda_f(R_t^{0, \zeta; \alpha_1})} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x)|^2 \right] \leq \frac{C}{\sqrt{\kappa}} \sup_{\alpha \in \mathcal{A}_t^W} \mathbb{E} \left[ \int_t^T e^{\kappa s} \sup_{x' \in \Lambda_f(R_s^{0, \zeta; \alpha_1 \oplus t \alpha})} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right].$$

Taking the supremum with respect to  $\alpha_1$  on the right-hand side and once again relying on a standard dynamic programming argument gives that

$$\mathbb{E} \left[ e^{\kappa t} \sup_{x \in \Lambda_f(R_t^{0, \zeta; \alpha_1})} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x)|^2 \right] \leq \frac{C}{\sqrt{\kappa}} \sup_{\alpha \in \mathcal{A}^W} \mathbb{E} \left[ \int_0^T e^{\kappa s} \sup_{x' \in \Lambda_f(R_s^{0, \zeta; \alpha})} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right].$$

Integrating with respect to time and using Fubini's theorem, we find that

$$\mathbb{E} \left[ \int_0^T e^{\kappa t} \sup_{x \in \Lambda_f(R_t^{0, \zeta; \alpha_1})} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x)|^2 dt \right] \leq \frac{CT}{\sqrt{\kappa}} \sup_{\alpha \in \mathcal{A}^W} \mathbb{E} \left[ \int_0^T e^{\kappa s} \sup_{x' \in \Lambda_f(R_s^{0, \zeta; \alpha})} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right]$$

after which taking the supremum with respect to  $\alpha_1 \in \mathcal{A}^W$  and choosing  $\kappa \geq (4CT)^2 \vee \kappa_0$  gives the first inequality. To get (4.7) we note that comparison gives that  $R_s^{0, \zeta; \alpha} \geq R_s^{t, \zeta; \alpha}$  for all  $s \in [t, T]$  and  $\alpha \in \mathcal{A}^W$ . From (4.9) we thus get that

$$\sup_{x \in \Lambda_f(\zeta)} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x)|^2 \leq C \sup_{\alpha \in \mathcal{A}^W} \mathbb{E} \left[ \int_0^T \sup_{x' \in \Lambda_f(R_s^{0, \zeta; \alpha})} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right]$$

from which (4.7) is immediate since the right-hand side is independent of  $t$ .  $\square$

We now introduce the norm  $\|\cdot\|_{\zeta}$  on the space of jointly continuous functions of polynomial growth,  $\Pi_c^g$ , defined as

$$\|\varphi\|_{\zeta} := \sup_{\alpha \in \mathcal{A}^W} \mathbb{E} \left[ \int_0^T e^{\kappa t} \sup_{x \in \Lambda_f(R_t^{0, \zeta; \alpha})} |\varphi(t, x)|^2 dt \right]^{1/2},$$

with  $\kappa > 0$  as in Proposition 4.3 and note that under  $\|\cdot\|_{\zeta}$ , the map  $\Phi : \Pi_c^g \rightarrow \Pi_c^g$  that maps  $\varphi$  to  $\bar{Y}^{\varphi}$  is a contraction.

**Corollary 4.4.** *There are constants  $C > 0$  and  $p \geq 0$  such that  $|\bar{Y}^k(t, x)| \leq C(1 + |x|^p)$  for all  $(t, x) \in [0, T] \times \mathbb{R}^d$  and all  $k \geq 0$ .*

*Proof.* First, we note that (4.6) and the triangle inequality implies that

$$\|\bar{Y}^k\|_{\zeta} \leq \|\bar{Y}^k - \bar{Y}^{k-1}\|_{\zeta} + \|\bar{Y}^{k-1}\|_{\zeta} \leq \frac{1}{2} \|\bar{Y}^{k-1} - \bar{Y}^{k-2}\|_{\zeta} + \|\bar{Y}^{k-1}\|_{\zeta} \leq \frac{1}{2^{k-1}} \|\bar{Y}^1 - \bar{Y}^0\|_{\zeta} + \|\bar{Y}^{k-1}\|_{\zeta}.$$

However, as a similar scheme holds for  $\|\bar{Y}^{k-1}\|_{\zeta}$  and since  $\bar{Y}^0 \equiv 0$  we conclude that

$$\|\bar{Y}^k\|_{\zeta} \leq \sum_{j=1}^k \frac{1}{2^{j-1}} \|\bar{Y}^1\|_{\zeta} \leq 2 \|\bar{Y}^1\|_{\zeta}.$$

On the other hand, as  $\bar{Y}^1 \in \Pi_c^g$  there are constants  $C > 0$  and  $p \geq 2$  such that  $|\bar{Y}^1(t, x)| \leq C(1 + |x|^p)$  and we conclude that

$$\begin{aligned} \|\bar{Y}^1\|_\zeta^2 &= \sup_{\alpha \in \mathcal{A}^W} \mathbb{E} \left[ \int_0^T e^{\kappa t} \sup_{x \in \Lambda_f(R_t^{0, \zeta; \alpha})} |\bar{Y}^1(t, x)|^2 dt \right] \\ &\leq C \left( 1 + \sup_{\alpha \in \mathcal{A}^W} \mathbb{E} \left[ \sup_{t \in [0, T]} |R_t^{0, \zeta; \alpha}|^{2p} \right] \right) \\ &\leq C(1 + |\zeta|^{2p}) \end{aligned}$$

implying the existence of a  $C > 0$  such that  $\|\bar{Y}^k\|_\zeta \leq C(1 + |\zeta|^p)$  for all  $k \geq 0$ . Now, (4.7) gives that

$$\begin{aligned} \sup_{t \in [0, T]} \sup_{x \in \Lambda_f(\zeta)} |\bar{Y}^k(t, x) - \bar{Y}^1(t, x)|^2 &\leq C \sup_{\alpha \in \mathcal{A}^W} \mathbb{E} \left[ \int_0^T \sup_{x \in \Lambda_f(R_t^{0, \zeta; \alpha})} |\bar{Y}^{k-1}(t, x)|^2 dt \right] \\ &\leq C(1 + |\zeta|^{2p}) \end{aligned}$$

where the constants  $C > 0$  and  $p \geq 2$  do not depend on  $k$  and the desired bound follows.  $\square$

Letting  $v_k(t, x) := \bar{Y}^k(t, x)$ , Proposition 4.3 and Corollary 4.4 implies that there is a  $v \in \Pi^g$  such that for each  $\zeta > 0$  we have  $\|v_k - v\|_\zeta \rightarrow 0$  as  $k \rightarrow \infty$ .

**Theorem 4.5.**  *$v$  is the unique viscosity solution in  $\Pi_c^g$  to (1.1).*

*Proof.* First, (4.7) implies that the convergence is uniform on compact subsets of  $[0, T] \times \mathbb{R}^n$  and since  $v_k$  is jointly continuous for each  $k \geq 0$  we conclude that  $v$  is also jointly continuous. This in turn gives that  $\Phi(v)$  is well defined and we conclude that  $\Phi(v) = v$  establishing existence of a solution to the optimal stopping problem (1.3)-(1.5). Moreover, if  $\tilde{Y}$  is another solution, then  $\tilde{v}(t, x) := \tilde{Y}_t^{t, x}$  must also satisfy  $\Phi(\tilde{v}) = \tilde{v}$ . However, then repeated use of the contraction property in (4.6) gives that  $\|\tilde{v} - v\|_\zeta = 0$  and by continuity we conclude that  $\tilde{v} = v$  implying by uniqueness of the non-linear Snell envelope in (3.6) as obtained in Proposition 3.3 that  $\tilde{Y}^{t, x} = Y^{t, x}$ . The solution to the optimal stopping problem (1.3)-(1.5) is thus unique.

Utilizing, once more, the connection between the optimal stopping problem (1.3)-(1.5) and PDEs with obstacles we conclude that  $v$  is a viscosity solution to (1.1). Suppose now that there exists another function  $\tilde{v} \in \Pi_c^g$  that solves (1.1) and let  $\bar{v} = \Phi(\tilde{v})$ , then by Theorem 3.2 and Proposition 3.3 we conclude that  $\bar{v} \in \Pi_c^g$  is the unique solution to

$$\begin{cases} \min\{\bar{v}(t, x) - h(t, x), \max\{\bar{v}(t, x) - \mathcal{M}\bar{v}(t, x), -\bar{v}_t(t, x) - \mathcal{L}\bar{v}(t, x) \\ -f(t, x, \bar{v}(t, \cdot), \sigma^\top(t, x)\nabla_x \bar{v}(t, x))\}\} = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d \\ \bar{v}(T, x) = \psi(x), \end{cases}$$

But then  $\bar{v} = \tilde{v}$  implying that  $\tilde{v}$  is a fixed point of  $\Phi$  and since  $v$  is the only fixed point of  $\Phi$  in the set of jointly continuous functions of polynomial growth we must have  $\tilde{v} = v$ .  $\square$

**Corollary 4.6.**  *$Y^v$  is the unique solution to the optimal stopping problem (1.3)-(1.5).*

## A Uniqueness of viscosity solutions in the local framework

In this section we prove the critical comparison principle for the PDE with local driver,

$$\begin{cases} \min\{v(t, x) - h(t, x), \max\{v(t, x) - \mathcal{M}v(t, x), -v_t(t, x) - \mathcal{L}v(t, x) \\ -\tilde{f}(t, x, v(t, x), \sigma^\top(t, x)\nabla_x v(t, x))\}\} = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d \\ v(T, x) = \psi(x), \end{cases} \quad (\text{A.1})$$

that was treated in Section 3 (see (3.1)). We need the following lemma:

**Lemma A.1.** *Let  $v$  be a supersolution to (A.1) satisfying*

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad |v(t, x)| \leq C(1 + |x|^{2\varrho})$$

*for some  $\varrho > 0$ . Then there is a  $\varpi_0 > 0$  such that for any  $\varpi > \varpi_0$  and  $\theta > 0$ , the function  $v + \theta e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2}$  is also a supersolution to (A.1).*

*Proof.* With  $w(t, x) := v(t, x) + \theta e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2}$  we note that, since  $v$  is a supersolution and  $\theta e^{-\varpi T}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2} \geq 0$ , we have

$$w(t, x) \geq v(t, x) \geq \mathbb{1}_{[t < T]} h(t, x) + \mathbb{1}_{[t = T]} \psi(x)$$

so that  $w(t, x) \geq h(t, x)$  for all  $(t, x) \in [0, T] \times \mathbb{R}^d$  and the terminal condition holds. Assume now that

$$v(t, x) - \inf_{e \in E} \{v(t, x + \gamma(t, x, e)) + \chi(t, x, e)\} \geq 0.$$

In this case,

$$\begin{aligned} & w(t, x) - \inf_{e \in E} \{w(t, x + \gamma(t, x, e)) + \chi(t, x, e)\} \\ &= v(t, x) + \theta e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2} \\ &\quad - \inf_{e \in E} \{v(t, x + \gamma(t, x, e)) + \theta e^{-\gamma t}(1 + (|x + \gamma(t, x, e)| - K_\Gamma)^+)^{2\varrho+2} + \chi(t, x, e)\} \\ &\geq v(t, x) - \inf_{e \in E} \{v(t, x + \gamma(t, x, e)) + \chi(t, x, e)\} \\ &\quad + \theta e^{-\gamma t} \{1 + (|x| - K_\Gamma)^+)^{2\varrho+2} - \sup_{e \in E} \theta e^{-\gamma t}(1 + (|x + \gamma(t, x, e)| - K_\Gamma)^+)^{2\varrho+2}\}. \end{aligned}$$

Now, either  $|x| \leq K_\Gamma$  in which case it follows by (2.1) that  $|x + \gamma(t, x, e)| \leq K_\Gamma$  or  $|x| > K_\Gamma$  and (2.1) gives that  $|x + \gamma(t, x, e)| \leq |x|$ . We conclude that

$$w(t, x) - \inf_{e \in E} \{w(t, x + \gamma(t, x, e)) + \chi(t, x, e)\} \geq 0.$$

Consider instead the case when

$$v(t, x) - \inf_{e \in E} \{v(t, x + \gamma(t, x, e)) + \chi(t, x, e)\} < 0$$

and let  $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d \rightarrow \mathbb{R})$  be such that  $\varphi - w$  has a local maximum of 0 at  $(t, x)$  with  $t < T$ . Then  $(\tilde{t}, \tilde{x}) \mapsto \tilde{\varphi}(\tilde{t}, \tilde{x}) := \varphi(\tilde{t}, \tilde{x}) - \theta e^{-\varpi \tilde{t}}(1 + (|\tilde{x}| - K_\Gamma)^+)^{2\varrho+2} \in C^{1,2}([0, T] \times \mathbb{R}^d \rightarrow \mathbb{R})$  and  $\tilde{\varphi} - v$  has a local maximum of 0 at  $(t, x)$ . Since  $v$  is a viscosity supersolution, we have

$$\begin{aligned} & -\partial_t(\varphi(t, x) - \theta e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2}) - \mathcal{L}(\varphi(t, x) - \theta e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2}) \\ & - \tilde{f}(t, x, \varphi(t, x) - \theta e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2}), \sigma^\top(t, x) \nabla_x(\varphi(t, x) - \theta e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2})) \geq 0. \end{aligned}$$

Consequently,

$$\begin{aligned} & -\partial_t \varphi(t, x) - \mathcal{L} \varphi(t, x) - \tilde{f}(t, x, \varphi(t, x), \sigma^\top(t, x) \nabla_x \varphi(t, x)) \\ & \geq \theta \varpi e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2} - \theta \mathcal{L} e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2} \\ & \quad \tilde{f}(t, x, \varphi(t, x) - \theta e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2}), \sigma^\top(t, x) \nabla_x(\varphi(t, x) - \theta e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2})) \\ & \quad - \tilde{f}(t, x, \varphi(t, x), \sigma^\top(t, x) \nabla_x \varphi(t, x)) \\ & \geq \theta \varpi e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2} - \theta C(1 + \varrho) e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2} \\ & \quad - k_f(1 + \varrho) \theta e^{-\varpi t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2}, \end{aligned}$$

where the right hand side is non-negative for all  $\theta > 0$  and all  $\varpi > \varpi_0$  for some  $\varpi_0 > 0$ .  $\square$

We have the following result, the proof of which we omit since it is classical:

**Lemma A.2.** For any  $\kappa \in \mathbb{R}$ , a locally bounded function  $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a viscosity supersolution (resp. subsolution) to (A.1) if and only if  $\check{v}(t, x) := e^{\kappa t} v(t, x)$  is a viscosity supersolution (resp. subsolution) to

$$\begin{cases} \min\{\check{v}(t, x) - e^{\kappa t} h(t, x), \max\{\check{v}(t, x) + \inf_{e \in E} \{\check{v}(t, x + \gamma(t, x, e)) + e^{\kappa t} \chi(t, x, e)\}, -\check{v}_t(t, x) \\ + \kappa \check{v}(t, x) - \mathcal{L}\check{v}(t, x) - e^{\kappa t} \tilde{f}(t, x, e^{-\kappa t} \check{v}(t, x), e^{-\kappa t} \sigma^\top(t, x) \nabla_x \check{v}(t, x))\} = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d \\ \check{v}(T, x) = e^{\kappa T} \psi(x). \end{cases} \quad (\text{A.2})$$

**Remark A.3.** Here, it is important to note that  $\check{h}(t, x) := e^{\kappa t} h(t, x)$ ,  $\check{\chi}(t, x, e) := e^{\kappa t} \chi(t, x, e)$ ,  $\check{f}(t, x, y, z) := -\kappa y + e^{\kappa t} \tilde{f}(t, x, e^{-\kappa t} y, e^{-\kappa t} z)$  and  $\check{\psi}(x) := e^{\kappa T} \psi(x)$  satisfy Assumption 2.2. In particular, this implies that Lemma A.1 holds for supersolutions to (A.2) as well.

We have the following comparison result for viscosity solutions in  $\Pi^g$ :

**Proposition A.4.** Let  $v$  (resp.  $u$ ) be a supersolution (resp. subsolution) to (A.1). If  $u, v \in \Pi^g$ , then  $u \leq v$ .

*Proof.* First, we note that it is sufficient to show that the statement holds for solutions to (A.2) for some  $\kappa \in \mathbb{R}$ . We thus assume that  $v$  (resp.  $u$ ) is a viscosity supersolution (resp. subsolution) to (A.2) for  $\kappa \in \mathbb{R}$  specified below. Furthermore, we may without loss of generality assume that  $v$  is l.s.c. and  $u$  is u.s.c.

By assumption,  $u, v \in \Pi^g$ , which implies that there are constants  $C > 0$  and  $\varrho > 0$  such that

$$|v(t, x)| + |u(t, x)| \leq C(1 + |x|^{2\varrho}). \quad (\text{A.3})$$

Now, for any  $\varpi > 0$  we only need to show that

$$\begin{aligned} w(t, x) &= w^{\theta, \varpi}(t, x) := v(t, x) + \theta e^{-\varpi t} (1 + (|x| - K_\Gamma)^+)^{2\varrho+2} \\ &\geq u(t, x) \end{aligned}$$

for all  $(t, x) \in [0, T] \times \mathbb{R}^d$  and any  $\theta > 0$ . Then the result follows by taking the limit  $\theta \rightarrow 0$ . We know from Lemma A.1 that there is a  $\varpi_0 > 0$  such that  $w$  is a supersolution to (A.2) for each  $\varpi \geq \varpi_0$  and  $\theta > 0$ . We thus assume that  $\varpi \geq \varpi_0$ .

We search for a contradiction and assume that there is a  $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$  such that  $u(t_0, x_0) > w(t_0, x_0)$ . By (A.3), there is for each  $\theta > 0$  a  $R > K_\Gamma$  such that

$$w(t, x) > u(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d, |x| > R.$$

Our assumption thus implies that there is a point  $(\bar{t}, \bar{x}) \in [0, T] \times B_R$  (the open unit ball of radius  $R$  centered at 0) such that

$$\begin{aligned} \max_{(t, x) \in [0, T] \times \mathbb{R}^d} (u(t, x) - w(t, x)) &= \max_{(t, x) \in [0, T] \times B_R} (u(t, x) - w(t, x)) \\ &= u(\bar{t}, \bar{x}) - w(\bar{t}, \bar{x}) = \eta > 0. \end{aligned}$$

We first show that there is at least one point  $(t^*, x^*) \in [0, T] \times B_R$  such that

- a)  $u(t^*, x^*) - w(t^*, x^*) = \eta$ ,
- b)  $u(t^*, x^*) > \check{h}(t^*, x^*)$  and
- c)  $w(t^*, x^*) < \inf_{e \in E} \{v(t^*, x^* + \gamma(t^*, x^*, e)) + \check{\chi}(t^*, x^*, e)\}$ .

Assume first that  $u(t, x) \leq \check{h}(t, x)$  for some  $(t, x) \in [0, T] \times \mathbb{R}^d$ . Since  $w$  is a supersolution, we have  $w(t, x) \geq \check{h}(t, x)$  and it follows that  $u(t, x) - w(t, x) \leq 0$  contradicting that  $u(t, x) - w(t, x) = \eta$ . In particular, any point satisfying *a)* must also satisfy *b)*.

We proceed by assuming that  $w(t, x) \geq \inf_{e \in E} \{w(t, x + \gamma(t, x, e)) - \check{\chi}(t, x, e)\}$  for all  $(t, x) \in A := \{(s, y) \in [0, T] \times \mathbb{R}^d : u(s, y) - w(s, y) = \eta\}$ . Indeed, as  $w$  is l.s.c. and  $\gamma$  is continuous, there is an  $e_1 \in E$  such that

$$w(\bar{t}, \bar{x}) = \inf_{e \in E} \{w(\bar{t}, \bar{x} + \gamma(\bar{t}, \bar{x}, e)) - \check{\chi}(\bar{t}, \bar{x}, e)\} = w(\bar{t}, \bar{x} + \gamma(\bar{t}, \bar{x}, e_1)) - \check{\chi}(\bar{t}, \bar{x}, e_1). \quad (\text{A.4})$$

Now, set  $x_1 = \bar{x} + \gamma(\bar{t}, \bar{x}, e_1)$  and note that since

$$|x + \gamma(t, x, e)| < R, \quad \forall (t, x, e) \in [0, T] \times B_R \times E$$

it follows that  $x_1 \in B_R$ . Moreover, as  $u$  is a subsolution with  $u(\bar{t}, \bar{x}) > \check{h}(\bar{t}, \bar{x})$  it satisfies

$$u(\bar{t}, \bar{x}) - (u(\bar{t}, \bar{x} + \gamma(\bar{t}, \bar{x}, e_1)) - \check{\chi}(\bar{t}, \bar{x}, e_1)) \leq 0,$$

that is

$$u(\bar{t}, x_1) \geq u(\bar{t}, \bar{x}) - \check{\chi}(\bar{t}, \bar{x}, e_1)$$

and we conclude from (A.4) that

$$\begin{aligned} u(\bar{t}, x_1) - w(\bar{t}, x_1) &\geq u(\bar{t}, \bar{x}) - \check{\chi}(\bar{t}, \bar{x}, e_1) - (w(\bar{t}, \bar{x}) - \check{\chi}(\bar{t}, \bar{x}, e_1)) \\ &= u(\bar{t}, \bar{x}) - w(\bar{t}, \bar{x}) = \eta. \end{aligned}$$

Hence,  $(\bar{t}, x_1) \in A$  and by our assumption it follows that there is an  $e_2 \in E$  such that

$$u(\bar{t}, x_1) = u(\bar{t}, x_1 + \gamma(\bar{t}, x_1, e_2)) - \check{\chi}(\bar{t}, x_1, e_2)$$

and a corresponding  $x_2 := x_1 + \gamma(\bar{t}, x_1, e_2) \in B_R$ . Now, this process can be repeated indefinitely to find a sequence  $(x_j, e_j)_{j \geq 1}$  in  $B_R \times E$  such that for any  $l \geq 0$  we have

$$w(\bar{t}, \bar{x}) \geq w(\bar{t}, x_l) + \sum_{j=1}^l \check{\chi}(\bar{t}, x_{j-1}, e_j),$$

with  $x_0 := \bar{x}$ . However, as  $\check{\chi} \geq \delta > 0$  we get a contradiction by letting  $l \rightarrow \infty$  while noting that  $w(t, x) \geq \check{h}(t, x)$  where the latter is bounded on  $[0, T] \times \bar{B}_R$ . We can thus find a  $(t^*, x^*) \in [0, T] \times B_R$  such that *a)-c)* above holds.

Since  $\tilde{f}$  is Lipschitz in  $y$  and  $z$  for  $(t, x) \in [0, T] \times \bar{B}_R$ , the remainder of the proof follows along the lines of the proof of Proposition 4.1 in [12] (see also Proposition 6.4 in [20]) and is included only for the sake of completeness.

Next, we assume without loss of generality that  $\varrho \geq 2$  and define

$$\Phi_n(t, x, y) := u(t, x) - w(t, x) - \varphi_n(t, x, y),$$

where

$$\varphi_n(t, x, y) := \frac{n}{2} |x - y|^{2\varrho} + |x - x^*|^2 + |y - x^*|^2 + (t - t^*)^2.$$

Since  $u$  is u.s.c. and  $w$  is l.s.c. there is a triple  $(t_n, x_n, y_n) \in [0, T] \times \bar{B}_R \times \bar{B}_R$  (with  $\bar{B}_R$  the closure of  $B_R$ ) such that

$$\Phi_n(t_n, x_n, y_n) = \max_{(t, x, y) \in [0, T] \times \bar{B}_R \times \bar{B}_R} \Phi_n(t, x, y).$$

Now, the inequality  $2\Phi_n(t_n, x_n, y_n) \geq \Phi_n(t_n, x_n, x_n) + \Phi_n(t_n, y_n, y_n)$  gives

$$n|x_n - y_n|^{2\varrho} \leq u(t_n, x_n) - u(t_n, y_n) + w(t_n, x_n) - w(t_n, y_n).$$

Consequently,  $n|x_n - y_n|^{2\varrho}$  is bounded (since  $u$  and  $w$  are bounded on  $[0, T] \times \bar{B}_R \times \bar{B}_R$ ) and  $|x_n - y_n| \rightarrow 0$  as  $n \rightarrow \infty$ . We can, thus, extract subsequences  $n_l$  such that  $(t_{n_l}, x_{n_l}, y_{n_l}) \rightarrow (\tilde{t}, \tilde{x}, \tilde{x})$  as  $l \rightarrow \infty$ . Since

$$u(t^*, x^*) - w(t^*, x^*) \leq \Phi_n(t_n, x_n, y_n) \leq u(t_n, x_n) - w(t_n, y_n),$$

it follows that

$$\begin{aligned} u(t^*, x^*) - w(t^*, x^*) &\leq \limsup_{l \rightarrow \infty} \{u(t_{n_l}, x_{n_l}) - w(t_{n_l}, y_{n_l})\} \\ &\leq u(\tilde{t}, \tilde{x}) - w(\tilde{t}, \tilde{x}) \end{aligned}$$

and as the right-hand side is dominated by  $u(t^*, x^*) - w(t^*, x^*)$  we conclude that

$$u(\tilde{t}, \tilde{x}) - w(\tilde{t}, \tilde{x}) = u(t^*, x^*) - w(t^*, x^*).$$

In particular, this gives that  $\lim_{l \rightarrow \infty} \Phi_n(t_{n_l}, x_{n_l}, y_{n_l}) = u(\tilde{t}, \tilde{x}) - w(\tilde{t}, \tilde{x})$  which implies that

$$\limsup_{l \rightarrow \infty} n_l |x_{n_l} - y_{n_l}|^{2\varrho} = 0$$

and

$$(t_{n_l}, x_{n_l}, y_{n_l}) \rightarrow (t^*, x^*, x^*).$$

We can thus extract a subsequence  $(\tilde{n}_l)_{l \geq 0}$  of  $(n_l)_{l \geq 0}$  such that  $t_{\tilde{n}_l} < T$ ,  $|x_{\tilde{n}_l}| < R$  and

$$u(t_{\tilde{n}_l}, x_{\tilde{n}_l}) - w(t_{\tilde{n}_l}, x_{\tilde{n}_l}) \geq \frac{\eta}{2}$$

for all  $l \in \mathbb{N}$ . Moreover, since  $(t, x) \mapsto \inf_{e \in E} \{w(t, x + \gamma(t, x, e)) - \check{\chi}(t, x, e)\}$  is u.s.c. (see Lemma 2.11),  $w(t_{\tilde{n}_l}, x_{\tilde{n}_l}) \rightarrow w(t^*, x^*)$  and  $u$  is u.s.c. while  $\check{h}$  is continuous, there is an  $l_0 \geq 0$  such that

$$w(t_{\tilde{n}_l}, x_{\tilde{n}_l}) - \inf_{e \in E} \{w(t_{\tilde{n}_l}, x_{\tilde{n}_l} + \gamma(t_{\tilde{n}_l}, x_{\tilde{n}_l}, e)) - \check{\chi}(t_{\tilde{n}_l}, e)\} < 0,$$

and

$$u(t_{\tilde{n}_l}, x_{\tilde{n}_l}) - \check{h}(t_{\tilde{n}_l}, x_{\tilde{n}_l}) > 0,$$

for all  $l \geq l_0$ . To simplify notation we will, from now on, denote  $(\tilde{n}_l)_{l \geq l_0}$  simply by  $n$ .

By Theorem 8.3 of [7] there are  $(p_n^u, q_n^u, M_n^u) \in \bar{J}^{2,+}u(t_n, x_n)$  and  $(p_n^w, q_n^w, M_n^w) \in \bar{J}^{2,+}w(t_n, y_n)$ , where  $\bar{J}^{2,+}$  is the limiting superjet, such that

$$\begin{cases} p_n^u - p_n^w = \partial_t \varphi_n(t_n, x_n, y_n) = 2(t_n - t^*) \\ q_n^u = D_x \varphi_n(t_n, x_n, y_n) = n\varrho(x - y)|x - y|^{2\varrho-2} + 2(x - x^*) \\ q_n^w = -D_y \varphi_n(t_n, x_n, y_n) = n\varrho(x - y)|x - y|^{2\varrho-2} + 2(x - x^*) \end{cases}$$

and for every  $\epsilon > 0$ ,

$$\begin{bmatrix} M_x^n & 0 \\ 0 & -M_y^n \end{bmatrix} \leq B(t_n, x_n, y_n) + \epsilon B^2(t_n, x_n, y_n),$$

where  $B(t_n, x_n, y_n) := D_{(x,y)}^2 \varphi_n(t_n, x_n, y_n)$ . Now, we have

$$D_{(x,y)}^2 \varphi_n(t, x, y) = \begin{bmatrix} D_x^2 \varphi_n(t, x, y) & D_{yx}^2 \varphi_n(t, x, y) \\ D_{xy}^2 \varphi_n(t, x, y) & D_y^2 \varphi_n(t, x, y) \end{bmatrix} = \begin{bmatrix} n\xi(x, y) + 2I & -n\xi(x, y) \\ -n\xi(x, y) & n\xi(x, y) + 2I \end{bmatrix}$$

where  $I$  is the identity-matrix of suitable dimension and

$$\xi(x, y) := \varrho |x - y|^{2\varrho-4} \{|x - y|^2 I + 2(\varrho - 1)(x - y)(x - y)^\top\}.$$

In particular, since  $x_n$  and  $y_n$  are bounded, choosing  $\epsilon := \frac{1}{n}$  gives that

$$\tilde{B}_n := B(t_n, x_n, y_n) + \epsilon B^2(t_n, x_n, y_n) \leq Cn|x_n - y_n|^{2\varrho-2} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + CI. \quad (\text{A.5})$$

By the definition of viscosity supersolutions and subsolutions we have that

$$\begin{aligned} & -p_n^u + \kappa u(t_n, x_n) - a^\top(t_n, x_n)q_n^u - \frac{1}{2}\text{Tr}[\sigma^\top(t_n, x_n)M_n^u\sigma(t_n, x_n)] \\ & - e^{\kappa t_n} \tilde{f}(t_n, x_n, e^{-\kappa t_n} u(t_n, x_n), e^{-\kappa t_n} \sigma^\top(t_n, x_n)q_n^u) \leq 0 \end{aligned}$$

and

$$\begin{aligned} & -p_n^w + \kappa w(t_n, y_n) - a^\top(t_n, y_n)q_n^w - \frac{1}{2}\text{Tr}[\sigma^\top(t_n, y_n)M_n^w\sigma(t_n, y_n)] \\ & - e^{\kappa t_n} \tilde{f}(t_n, y_n, e^{-\kappa t_n} w(t_n, y_n), e^{-\kappa t_n} \sigma^\top(t_n, y_n)q_n^w) \geq 0. \end{aligned}$$

Combined, this gives that

$$\begin{aligned} \kappa(u(t_n, x_n) - w(t_n, y_n)) & \leq p_n^u + a^\top(t_n, x_n)q_n^u + \frac{1}{2}\text{Tr}[\sigma^\top(t_n, x_n)M_n^u\sigma(t_n, x_n)] \\ & \quad + e^{\kappa t_n} \tilde{f}(t_n, x_n, e^{-\kappa t_n} u(t_n, x_n), e^{-\kappa t_n} \sigma^\top(t_n, x_n)q_n^u) \\ & \quad - p_n^w - a^\top(t_n, y_n)q_n^w - \frac{1}{2}\text{Tr}[\sigma^\top(t_n, y_n)M_n^w\sigma(t_n, y_n)] \\ & \quad - e^{\kappa t_n} \tilde{f}(t_n, y_n, e^{-\kappa t_n} w(t_n, y_n), e^{-\kappa t_n} \sigma^\top(t_n, y_n)q_n^w) \end{aligned}$$

Collecting terms we have that

$$p_n^u - p_n^w = 2(t_n - t^*)$$

and since  $a$  is Lipschitz continuous in  $x$  and bounded on  $\bar{B}_R$ , we have

$$\begin{aligned} a^\top(t_n, x_n)q_n^u - a^\top(t_n, y_n)q_n^w & \leq (a^\top(t_n, x_n) - a^\top(t_n, y_n))n\varrho(x_n - y_n)|x_n - y_n|^{2\varrho-2} \\ & \quad + C(|x_n - x^*| + |y_n - x^*|) \\ & \leq C(n|x_n - y_n|^{2\varrho} + |x_n - x^*| + |y_n - x^*|), \end{aligned}$$

where the right-hand side tends to 0 as  $n \rightarrow \infty$ . Let  $s_x$  denote the  $i^{\text{th}}$  column of  $\sigma(t_n, x_n)$  and let  $s_y$  denote the  $i^{\text{th}}$  column of  $\sigma(t_n, y_n)$  then by the Lipschitz continuity of  $\sigma$  and (A.5), we have

$$\begin{aligned} s_x^\top M_n^u s_x - s_y^\top M_n^w s_y & = \begin{bmatrix} s_x^\top & s_y^\top \end{bmatrix} \begin{bmatrix} M_n^u & 0 \\ 0 & -M_n^w \end{bmatrix} \begin{bmatrix} s_x \\ s_y \end{bmatrix} \\ & \leq \begin{bmatrix} s_x^\top & s_y^\top \end{bmatrix} \tilde{B}_n \begin{bmatrix} s_x \\ s_y \end{bmatrix} \\ & \leq C(n|x_n - y_n|^{2\varrho} + |x_n - y_n|) \end{aligned}$$

and we conclude that

$$\limsup_{n \rightarrow \infty} \frac{1}{2} \text{Tr}[\sigma^\top(t_n, x_n) M_n^u \sigma(t_n, x_n) - \sigma^\top(t_n, y_n) M_n^w \sigma(t_n, y_n)] \leq 0.$$

Finally, we have that

$$\begin{aligned} & e^{\kappa t_n} \tilde{f}(t_n, x_n, e^{-\kappa t_n} u(t_n, x_n), e^{-\kappa t_n} \sigma^\top(t_n, x_n) q_n^u) - e^{\kappa t_n} \tilde{f}(t_n, y_n, e^{-\kappa t_n} w(t_n, y_n), e^{-\kappa t_n} \sigma^\top(t_n, x_n) q_n^w) \\ & \leq k_f(u(t_n, x_n) - w(t_n, y_n) + |\sigma^\top(t_n, x_n) q_n^u - \sigma^\top(t_n, x_n) q_n^w|) \\ & \quad + e^{\kappa t_n} |\tilde{f}(t_n, x_n, e^{-\kappa t_n} u(t_n, x_n), e^{-\kappa t_n} \sigma^\top(t_n, x_n) q_n^u) - \tilde{f}(t_n, y_n, e^{-\kappa t_n} u(t_n, x_n), e^{-\kappa t_n} \sigma^\top(t_n, x_n) q_n^u)| \end{aligned}$$

Repeating the above argument and using that  $\tilde{f}$  is jointly continuous in  $(t, x)$  uniformly in  $(y, z)$  we get that the upper limit of the right-hand side when  $n \rightarrow \infty$  is bounded by  $k_f(u(t_n, x_n) - w(t_n, y_n))$ . Put together, this gives that

$$(\kappa - k_f) \limsup_{n \rightarrow \infty} (u(t_n, x_n) - w(t_n, y_n)) \leq 0$$

and choosing  $\kappa > k_f$  gives a contradiction.  $\square$

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