WALTER RUDIN'S STONE-WEIERSTRAUSS THEORY EXTENDED TO MULTICOMPLEX-VALUED FUNCTION SPACES

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Abstract. In this project we give an introduction to Multicomplex spaces. Furthermore, we extend Stone-Weierstrauss Theorem to Multicomplex spaces.

1. Terminology

Definition 1.1. Let $n \in \mathbb{N}$, $\mathbb{C}_0 \equiv \mathbb{R}$, $\mathbb{C}_1 \equiv \mathbb{C}$ and for $n \geq 2$,

$$\mathbb{C}_n \equiv \{z_n : z_n = z_{n-1,1} + j_n z_{n-1,2} : z_{n-1,1}, z_{n-1,2} \in \mathbb{C}_{n-1}\}$$

where

$$j_n^2 = 1$$

for all $n \geq 2$ and

$$j_n \neq \; j_{n-1}, j_{n-2}, j_{n-2}, ..., j_2, j_1$$

where

$$j_1 = i$$

the i denotes the usual imaginary unit.

Lemma 1.2. Let $z \in \mathbb{C}_n$. Let j_n be the hyperbolic imaginary unit introduced first in $\mathbb{C}_n$, and let $i_n = ij_n$. Then there are unique z_1, z_2, w_1, and w_2 in $\mathbb{C}_{n-1}$ such that $z = z_1 + j_n z_2 = w_1 + w_2 i_n$.

Definition 1.3. The spaces $\mathbb{C}_n$ for $n \in \mathbb{N}$ are linear spaces. In $\mathbb{C}_{n-1}$, the norm in $\mathbb{C}_{n-1}$ and the operations of addition and multiplication have been defined as:

Norm:

$$\|z_n\|_n^2 = \|z_{n-1,1}\|_{n-1}^2 + \|z_{n-1,2}\|_{n-1}^2$$

Addition:

$$\begin{aligned} z_n^1 + z_n^2 &= (z_{n-1,1}^1 + j_n z_{n-1,2}^1) + (z_{n-1,1}^2 + j_n z_{n-1,2}^2) \\ &= (z_{n-1,1}^1 + z_{n-1,1}^2) + j_n(z_{n-1,2}^1 + z_{n-1,2}^2) \end{aligned}$$

Key words and phrases. multicomplex, Stone-Weierstrauss, algebras.

Multiplication:

$$\begin{aligned} z_n^1 * z_n^2 &= (z_{n-1,1}^1 + j_n z_{n-1,2}^1) * (z_{n-1,1}^2 + j_n z_{n-1,2}^2) \\ &= z_{n-1,1}^1 * z_{n-1,1}^2 + z_{n-1,1}^1 * j_n z_{n-1,2}^2 + j_n z_{n-1,2}^1 * z_{n-1,1}^2 + j_n^2 z_{n-1,2}^1 * z_{n-1,2}^2 \\ &= z_{n-1,1}^1 * z_{n-1,1}^2 + z_{n-1,1}^1 * j_n z_{n-1,2}^2 + j_n z_{n-1,2}^1 * z_{n-1,1}^2 + z_{n-1,2}^1 * z_{n-1,2}^2 \\ &= (z_{n-1,1}^1 * z_{n-1,1}^2 + z_{n-1,2}^1 * z_{n-1,2}^2) + j_n(z_{n-1,1}^1 * z_{n-1,2}^2 + z_{n-1,2}^1 * z_{n-1,1}^2) \end{aligned}$$

2. Principal Results

Theorem 2.1. If $f : [a,b] \to \mathbb{C}_k$ is continuous, then there is a sequence of $\mathbb{C}_k$-valued polynomial functions, $(P_{m,k})_{m\in\mathbb{N}}$, on $[a,b]$ such that $P_{m,k}$ converges uniformly to f as $m \to \infty$, i.e. $\lim_{m\to\infty} P_{m,k} = f$. We will denote uniform convergence by the symbol $\to^{uc}$. In the case that $k = 0$, there is a real polynomials functions $P_{m,0} : [a,b] \to \mathbb{C}_0$ such that $P_{n,0} \to^{uc} f$ as $m \to \infty$.

Proof. The case when $k = 0$ is well known and for $k = 1$ it can be found in Walter Rudin's Principles of Mathematical Analysis 3rd Edition on page 159. Consider the case when $k = 2$. We will do induction.

If $f : [a,b] \to \mathbb{C}_2$ is continuous, then there exist functions $f_1, f_2 : [a,b] \to \mathbb{C}$ such that f_1, f_2 are continuous and $f = f_1 + jf_2$. By the result of $k = 1$ on Pg. 159 of W. Rudin, there is a sequence of polynomial functions (complex), Q_m and R_m, for $m \in \mathbb{N}$ on $[a,b]$ where $Q_m \to^{uc} f_1$ and $R_m \to^{uc} f_2$ as $m \to \infty$.

Let $\varepsilon > 0$ be given, then there exist $N_1 \in \mathbb{N}$ such that for $m \geq N_1$ and for all $t \in [a,b]$, $|Q_m(t) - f_1(t)| < \frac{\varepsilon}{2}$. Similarly, there is $N_2 \in \mathbb{N}$ such that for $m \geq N_2$ and for all $t \in [a,b]$, $|R_m(t) - f_2(t)| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$ for all $m \in \mathbb{N}$ such that for $m \geq N$ and for all $f \in [a,b]$,

$$\begin{aligned} |(Q_m(t) + jR_m(t)) - f(t)| &= |Q_m(t) + jR_m(t) - [f_1(t) + jf_2(t)]| \\ &= |Q_m(t) + jR_m(t) - f_1(t) - jf_2(t)| \\ &= |Q_m(t) - f_1(t) + jR_m(t) - jf_2(t)| \\ &= |(Q_m - f_1)(t) + j(R_m - f_2)(t)| \\ &\leq |(Q_m - f_1)(t)| + |j(R_m - f_2)(t)| \\ &= \sqrt{[(Q_m - f_1)(t)]^2} + \sqrt{[(R_m - f_2)(t)]^2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, $Q_m + jR_m$ which is a $\mathbb{C}_2$-valued polynomial function on $[a,b]$, converges uniformly to f as $n \to \infty$. Therefore, theorem 1.2 holds in the case $k = 2$. Now assume it holds for $k = \xi$ for $\xi \geq 2$

Let $f : [a,b] \to \mathbb{C}_{\xi+1}$ be continuous and define $f = f_1 + j_{\xi+1} f_2$ where $f_1, f_2 : [a,b] \to \mathbb{C}_\xi$. By the induction hypothesis there exist two sequences, $(Q_m)_{m\in\mathbb{N}}$ and $(R_m)_{m\in\mathbb{N}}$ of $\mathbb{C}_k$-valued polynomial functions on $[a,b]$, such that $Q_m \to^{uc} f_1$ and $R_m \to^{uc} f_2$ as $m \to \infty$. For all $m \geq$

$M := \{M_1, M_2\}$, and define $P_m := Q_m + j_{\xi+1}R_m$. Let $\varepsilon > 0$ be given, then there is a $M_1, M_2 \in \mathbb{N}$ such that for all $m \geq M$ and $t \in [a,b]$,

$$|Q_m(t) - f_1(t)| < \frac{\varepsilon}{2},$$

and

$$|R_m(t) - f_2(t)| < \frac{\varepsilon}{2}.$$

Hence, for all $m \geq M$ and for all $t \in [a,b]$, we have that

$$\begin{aligned}|P_m(t) - f(t)| &= |Q_m(t) + j_{\xi+1}R_m(t) - [f_1(t) + j_{\xi+1}f_2(t)]| \\ &= |Q_m(t) + j_{\xi+1}R_m(t) - f_1(t) - j_{\xi+1}f_2(t)| \\ &= |Q_m(t) - f_1(t) + j_{\xi+1}R_m(t) - j_{\xi+1}f_2(t)| \\ &= |(Q_m - f_1)(t) + j_{\xi+1}(R_m - f_2)(t)]| \\ &\leq |(Q_m - f_1)(t)| + |j_{\xi+1}(R_m - f_2)(t)]| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

Therefore, theorem 2.1 is true for $k = \xi + 1$ thus it holds by induction for all k. □

Definition 2.2. Let $\mathcal{P}$ be any algebra and let $\|.\|$ be a norm on $\mathcal{P}$, a normed space. A family Ω of $\mathcal{P}$-valued function on a set E is called an $(\mathcal{P}, E) - algebra$ if the following are true for all functions $\phi, \sigma \in \Omega$ and scalars $\alpha \in \mathcal{P}$

- $\phi + \sigma \in \Omega$
- $\phi\sigma \in \Omega$
- $\alpha\phi \in \Omega$

We say $(\mathcal{P}, E) - algebra$ is uniformly closed when a sequence $(\phi_n)_{n\in\mathbb{N}}$ in Ω with $\phi_n \rightarrow^{uc} \phi : E \rightarrow \mathcal{P}$, where $\phi \in \Omega$. Define $\overline{\Omega}^u := \{\phi : E \rightarrow \mathcal{P}$ such that there is a sequence $(\phi)_{n\in\mathbb{N}} \in \Omega$ so that $\phi_n \rightarrow^{uc} \phi\}$ to be the uniform closure of Ω.

Theorem 2.3. Let Ω be a $(\mathcal{P}, E) - algebra$, where $\mathcal{P}$ is a normed algebra. Suppose that Ω is uniformed bounded, there is a $\Gamma > 0$ such that for all $x \in \mathcal{P}$ and $\phi \in \Omega$

$$\|\phi(x)\| \leq \Gamma.$$

Then $\overline{\Omega}^u$ is also $(\mathcal{P}, E) - algebra$.

Proof. Let $\phi, \sigma \in \overline{\Omega}^u$. By definition of $\overline{\Omega}^u$, there is two sequences, (ϕ_k) and (σ_k) for $k \in \mathbb{N}$, in $\overline{\Omega}^u$ such that $\phi_k \rightarrow^{uc} \phi$ and $\sigma_k \rightarrow^{uc} \sigma$ as $k \rightarrow \infty$. It can easily be shown that it follows the properties from definition 2.2. Therefore $\overline{\Omega}^u$ is also an algebra, $(\mathcal{P}, \psi) - algebra$. □

Corollary 2.4. *(Multicomplex Theorem 2.3)* Let Ω be a $(\mathbb{C}_n, E) - algebra$, where $\mathcal{P}$ is a normed algebra. Suppose that Ω is uniformed bounded. Then $\overline{\Omega}^u$ is also $(\mathbb{C}_n, \psi) - algebra$.

Proof. By Theorem 2.3, it is the simple case when $\mathcal{P}$ equals $\mathbb{C}_n$. Thus by Lemmas 3.1 to 3.6, it holds for $\mathcal{P} = \mathbb{C}_n$. □

3. Proof of Theorem 2.3

The following are result that we will need to prove Theorem 2.3:

Lemma 3.1. $\phi_k + \sigma_k \to^{uc} \phi + \sigma$.

Proof. If $\overline{\Omega}^u$ is a $(\mathcal{P}, E) - algebra$, then $\phi_k + \sigma_k \in \overline{\Omega}^u$ for all k. Let $\varepsilon > 0$ be given. There exist a $K_1 \in \mathbb{N}$ such that for all $k \geq K_1$, and for all $x \in \mathcal{P}$, then

$$\|\phi_k(x) - \phi(x)\| < \frac{\varepsilon}{2}.$$

Similarly, there exist $K_2 \in \mathbb{N}$ such that for all $k \geq K_2$, and for all $x \in \mathcal{P}$, we have

$$\|\sigma_k(x) - \sigma(x)\| < \frac{\varepsilon}{2}.$$

Let $K = \max\{K_1, K_2\}$. Hence, for all $k \in \mathbb{N}$ such that $k \geq K$ and for all $x \in \mathcal{P}$,

$$\begin{aligned} \|(\phi_k + \sigma_k)(x) - (\sigma + \sigma)(x)\| &\leq \|(\phi_k - \sigma)(x)\| + \|(\sigma_K - \sigma)(x)\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Therefore, $\phi_k + \sigma_k \to^{uc} \phi + \sigma$, thus $\phi + \sigma \in \overline{\Omega}^u$ ∎

Lemma 3.2. $\phi_n \cdot \sigma_n \to^{uc} \phi \cdot \sigma$.

Proof. By definition of $\overline{\Omega}^u$ being a $(\mathcal{P}, \psi) - algebra$, we have that $\phi_k \cdot \sigma_k \in \overline{\Omega}^u$ for all k. Observe that boundedness is only property required to prove this result. Recall that in a normed algebra, $\|x \cdot y\| \leq \|x\| \cdot \|y\|$ holds for all $x, y \in \mathcal{P}$.

The case when either σ or ϕ are zero, is trivial true. Hence, assume that σ and ϕ are both non-zero. Let $A \in \mathbb{R}$ such that $\|\phi_n\| \leq A$ for all n and $x \in \mathcal{P}$. Since $\overline{\Omega}^u$ is uniformly bounded, it implies that ϕ_n also uniformly bounded. If $A = 0$, then $\|\phi_n\| = 0$ for all natural number n and $x \in \mathcal{P}$. Furthermore, we have that $\phi_n = 0$ for all n, by properties of the norm. Since $\phi_n \to^{uc} \phi$, then $\phi = 0$. Hence,

$$\begin{aligned} \|(\phi_n \cdot \sigma_n)(x) - (\phi \cdot \sigma)(x)\| &= ||(0 \cdot \sigma_n)(x) - (0 \cdot \sigma)(x)\| \\ &= 0 \\ &< \varepsilon. \end{aligned}$$

Assume that $A > 0$ and let $\varepsilon > 0$. Let $\Gamma \geq A$ such that $\|\sigma(x)\| \leq \Gamma$. There exist a natural number N_1 such that for all $n \geq N_1$ and for all $x \in \mathcal{P}$,

$$\|\phi_n(x) - \phi(x)\| < \frac{\varepsilon}{2\Gamma}.$$

Similarly, there is a $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ and for all $x \in \mathcal{P}$,

$$\|\sigma_n(x) - \sigma(x)\| < \frac{\varepsilon}{2\Gamma}.$$

Let $N = \max\{N_1, N_2\}$. Thus, for all $n \geq N$ and $x \in \mathcal{P}$ we have

$$\begin{aligned}\|\phi_n(x)\cdot\sigma_n(x) - \phi(x)\cdot\sigma(x)\| &= \|\phi_n(x)\cdot\sigma_n(x) - \phi_n(x)\cdot\sigma(x) + \phi_n(x)\cdot\sigma(x) - \phi(x)\cdot\sigma(x)\| \\ &\leq \|\phi_n(x)\cdot\sigma_n(x) - \phi_n(x)\cdot\sigma(x)\| + \|\phi_n(x)\cdot\sigma(x) - \phi(x)\cdot\sigma(x)\| \\ &= \|\phi_n(x)\cdot[\sigma_n(x) - \sigma(x)]\| + \|[\phi_n(x) - \phi(x)]\cdot\sigma(x)\| \\ &\leq \|\phi_n(x)\|\cdot\|[\sigma_n(x) - \sigma(x)]\| + \|[\phi_n(x) - \phi(x)]\|\cdot\|\sigma(x)\| \\ &< A\cdot\frac{\varepsilon}{2\Gamma} + \frac{\varepsilon}{2\Gamma}\cdot\Gamma \\ &\leq \Gamma\cdot\frac{\varepsilon}{2\Gamma} + \frac{\varepsilon}{2\Gamma}\cdot\Gamma \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

Hence, $\phi\cdot\sigma \in \overline{\Omega}^u$. □

Lemma 3.3. *$\alpha\phi_n \to^{uc} \alpha\phi$.*

Proof. If $\alpha = 0$, there is nothing to show. Hence, assume that $\alpha \neq 0$, imply that $|\alpha| > 0$. Let $\varepsilon > 0$ be given. Since $\phi_n \to^{uc} \phi$, by assumption, there is a natural number N such that for all $n \geq N$ and for all $x \in \mathcal{P}$,

$$\|\phi_n(x) - \phi(x)\| < \frac{\varepsilon}{|\alpha|}.$$

For $|\alpha| > 0$. For all such n we have,

$$\begin{aligned}\|(\alpha\phi_n)(x) - (\alpha\phi)(x)\| &= \|\alpha\phi_n(x) - \alpha\phi(x)\| \\ &= \|\alpha[\phi_n(x) - \phi(x)]\| \\ &\leq |\alpha|\|[\phi_n(x) - \phi(x)]\| \\ &= |\alpha|\frac{\varepsilon}{|\alpha|} \\ &= \varepsilon\end{aligned}$$

□

Our three Lemmas have showed that $\overline{\Omega}^u$ is an algebra. Now all that remains to show is that $\overline{\Omega}^u$ is uniformly closed. That is, we need to show that if $(\phi_n)_{n\in\mathbb{N}}$ is a sequence in $\overline{\Omega}^u$ and $\phi_n \to^{uc} \phi$ where $\phi : E \to \mathcal{P}$ then $\phi \in \overline{\Omega}^u$.

Lemma 3.4. . *$\overline{\Omega}^u$ is uniformly bounded by M.*

Proof. We want to show that for all $\phi \in \overline{\Omega}^u$ and $x \in E$, there exist an M such that

$$\|\phi_n(x)\| \leq M.$$

Let $\phi \in \overline{\Omega}^u$. It will suffices to show that for all $\varepsilon > 0$ and $x \in E$,

$$\|\phi(x)\| \leq M + \varepsilon.$$

By definition of $\overline{\Omega}^u$, there is a sequence $(\phi_n)_{n\in\mathbb{N}} \in \Omega$ such that the $\phi_n \to^{uc} \phi$ and there is a natural number such that for all $n \geq N$ and $x \in \psi$, and so

$$\|\phi_n(x) - \phi(x)\| < \varepsilon.$$

If $x \in E$ then

$$\begin{aligned}
\|\phi(x)\| &= \|\phi(x) + \phi_n(x) - \phi_n(x)\| \\
&= \|\phi(x) - \phi_n(x) + \phi_n(x)\| \\
&\leq \|(\phi - \phi_n)(x)\| + \|\phi_n(x)\| \\
&\leq \|(\phi - \phi_n)(x)\| + M \\
&= \| - 1 \cdot [(\phi_n - \phi)(x)]\| + M \\
&\leq | - 1| \cdot \|(\phi_n - \phi)(x)\| + M \\
&= \|(\phi_n - \phi)(x)\| + M \\
&< \varepsilon + M
\end{aligned}$$

□

Lemma 3.5. *The closure of* Ω *with respect to* d*,* $\overline{\Omega}^d$*, is contained in* $\overline{\Omega}^u$*.*

Proof. Let $d : \overline{\Omega}^u \times \overline{\Omega}^u \to [0, \infty)$ be defined as

$$d(\phi, \sigma) := \sup_{x\in\mathcal{P}} \|\phi(x) - \sigma(x)\|.$$

Note that d is well defined by Lemma 3.4. Observe that $\Omega \subset \overline{\Omega}^u$, since if $\phi \in \Omega$ and if we let $\phi_n = \phi$ then for all $x \in E$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\|\phi_n(x) - \phi(x)\| &= \|\phi(x) - \phi(x)\| \\
&= 0 \\
&< \varepsilon
\end{aligned}$$

for all $\varepsilon > 0$. This implies that (ϕ_n) is a set in Ω that converges uniformly ϕ.

If $\phi \in \overline{\Omega}^d$, then $(\phi_n)_{n\in\mathbb{N}}$ is a sequence in Ω such that $d(\phi_n, \phi) = 0$. Hence,

$$\lim_{n\to\infty} d(\phi_n, \phi) = 0$$

and so,

$$\limsup_{n\to\infty, x\in\mathcal{P}} \|\phi_n(x) - \phi(x)\| = 0.$$

For all $\varepsilon > 0$, there is a natural number N such that for all $n \geq N$,

$$\sup_{x\in\mathcal{P}} \|\phi_n(x) - \phi(x)\| < \varepsilon.$$

In particular, for all $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that for all $x \in \mathcal{P}$,

$$\|\phi_n(x) - \phi(x)\| < \varepsilon.$$

Therefor, we have that for $\phi \in \overline{\Omega}^u$ there is some $\phi_n \to^{uc} \phi$.

Since $(\overline{\Omega}^u, d)$ is a metric space and $\Omega \subset \overline{\Omega}^d$ then $\overline{\Omega}^d$ is closed in $(\overline{\Omega}^u, d)$ by Theorem 2.27(a) in Walter Rudin's Principles of Mathematical Analysis page 35. □

Lemma 3.6. $\phi \in \overline{\Omega}^u$

Proof. Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{\Omega}^u$ and let $\phi : E \to \mathcal{P}$. Consider that case when $\mathcal{P} = \overline{\Omega}^u$ and $E = \Omega$. Suppose that $\phi_n \to^{uc} \phi$. Observe that if Ω is a $(\mathcal{P}, E) - algebra$, then it is uniformly closed iff whenever ϕ_n is a sequence in Ω and $\phi : E \to \mathcal{P}$ such that $\phi_n \to^{uc} \phi$, then $\phi \in \Omega$. Let $\overline{\Omega}^u$ be the collection of all $\phi : E \to \mathcal{P}$, there is a sequence, h_n, in Ω such that $\phi_n \to^{uc} \phi$. Then for all $\varepsilon > 0$, there exist a natural number N such that for all $n \geq N$ and for all $x \in E$,

$$\|\phi_n(x) - \phi(x)\| < \frac{\varepsilon}{2}.$$

By Lemma 3.5 it is enough to show that $\phi \in \overline{\Omega}^d$. To show $\phi \in \overline{\Omega}^d$, we only need to find a sequence $(g_k)_{k\in\mathbb{N}} \in \Omega$ such that $d(g_k, \phi) \to 0$ as $k \to \infty$. Hence, it is sufficient to show that the sequence $(g_k)_{k\in\mathbb{N}}$ that satisfies

$$\lim_{k\to\infty} \sup \|g_k(x) - \phi(x)\| = 0$$

for $x \in \mathcal{P}$. That is we need to show that for all $\varepsilon > 0$ there is a $K \in \mathbb{N}$ such that for all $k \geq K$,

$$\sup_{x\in\mathcal{P}} \|g_k(x) - \phi(x)\| < \varepsilon.$$

For each $\phi_n \in \overline{\Omega}^u$ there is a sequence $(\phi_{n,k})_{n\in\mathbb{N}} \in \Omega$, such that $\phi_{n,k} \to^{uc} \phi_n$. In particular, there exist a $K \in \mathbb{N}$ such that for all $k \geq K$ and for all $x \in E$,

$$\|\phi_{n,k}(x) - \phi_n(x)\| < \frac{\varepsilon}{2}$$

Let $g_k := \phi_{k,k}$ and let $\Theta = \max\{K, N\}$. Then for all $k \in \mathbb{N}$ such that $k \geq \Theta$ and for all $x \in \mathcal{P}$,

$$\begin{aligned}\|g_k(x) - \phi(x)\| &= \|\phi_{k,k}(x) - \phi(x)\| \\ &= ||\phi_{k,k}(x) - \phi_k(x) + \phi_k(x) - \phi(x)\| \\ &\leq \|\phi_{k,k}(x) - \phi_k(x)\| + \|\phi_k(x) - \phi(x)\| \\ &\leq \frac{2}{\varepsilon} + \frac{2}{\varepsilon} \\ &= \varepsilon\end{aligned}$$

□

4. Background Results

Lemma 4.1.

(1) *In Lemma 3.5 is d a metric?*
(2) *If all of the functions in an algebra whose values are in a $\mathbb{C}_n$-algebra, where the functions are defined on any set E are assumed to be bounded. Then the function $||\cdot||_\infty : \mathcal{A} \to \mathbb{C}_0$ given by*

$$\lim_{\mathcal{X}\to\infty} \|f\|_{\mathcal{X}} = \|f\|_\infty := \sup_{x\in E} ||f(x)||$$

is a norm on $\mathcal{A}$.
(3) *In any normed vector space, a convergent sequence is always bounded.*

Proof.
(1) From definition $d(\phi, \sigma)$ is the supremum of a set of a set of non-negative real numbers, thus $d(\phi, \sigma) \geq 0$ for all $\phi, \sigma \in \overline{\Omega}^u$. Hence,

$$d(\phi, \sigma) = 0 \text{ iff } \{|\phi(x) - \sigma(x)|\}_{x\in\mathcal{P}} = \{0\},$$

thus $\phi(x) = \sigma(x)$ for all $x \in \mathcal{P}$. Next, $d(\phi, \sigma) = d(\sigma, \phi)$ follows directly by definition for all $\phi, \sigma \in \overline{\Omega}^u$. Lastly, let ϕ, σ, h be any three elements of $\overline{\Omega}^u$, for each element x_1 of $\mathcal{P}$ we have

$$\begin{aligned} |\phi(x_1) - \sigma(x_1)| &\leq |\phi(x_1) - h(x_1)| + |h(x_1) - \sigma(x_1)| \\ &\leq \sup_{x\in\mathcal{P}} |\phi(x_1) - h(x_1)| + \sup_{x\in\mathcal{P}} |h(x_1) - \sigma(x_1)|. \end{aligned}$$

Ergo, $d(\phi, \sigma) \leq d(\phi, h) + d(h, \sigma)$ and so d is a metric on $\overline{\Omega}^u$.
(2) We need to show that $\|\cdot\|_\infty$, the sup-norm, is indeed a norm in $\mathcal{A}$. For this to be true we need to verify the following:
   (a) $\|f\|_\infty \geq 0$ for all $f \in E$
   (b) $\|f\|_\infty = 0$ iff $f = 0$
   (c) $\|\lambda f\|_\infty = |\lambda|\|f\|_\infty$ for some scalar λ
   (d) $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

   Property (a): By definition, the supremum of a set is the least upper bound of a set, a number, say M, such that no element of the set exceeds can exceed it. For any positive ε, there is a element of the set which exceeds $M - \varepsilon$. The sup-norm is the largest value of a set of absolute values, thus it is obvious that it must be greater than or equal to zero.
   Property (b): If $\|f\|_\infty = 0$, then

$$\sup_{x\in E} \|f(x)\| = 0.$$

   Hence, $\|f(x)\| = 0$ for all $x \in E$. If $f(x) = 0$, then

$$\sup_{x\in E} \|f(x)\| = 0.$$

   Property (c): Observe, $\|\lambda f\|_\infty = \sup_{x\in E} \|\lambda f(x)\|$. Let α be the supremum of $\|f(x)\|$, then $\alpha \geq x$ for all $x \in E$. Therefor, for any $\varepsilon > 0$ there is $x^* \in E$ such that $x^* > \alpha - \varepsilon$.

Now multiplying $|\lambda|$ to $\|f(x)\|$ we get, $x \leq |\lambda|\alpha$ for all $x \in |\lambda|E$ and for any $\varepsilon > 0$ there is $x^* > |\lambda|\alpha - \varepsilon$. Hence, $|\lambda|\alpha = \sup_{x\in E} (|\lambda|\|f(x)\|)$; thus we can conclude that

$$\begin{aligned}\|\lambda f\|_\infty &= |\lambda| \sup_{x\in E} \|f(x)\| \\ &= |\lambda|\|f\|_\infty.\end{aligned}$$

Property (d): Note,

$$\|f+g\|_\infty = \sup_{x\in E} \|f(x)+g(x)\|.$$

By triangle inequality,

$$\sup_{x\in E} \|f(x)+g(x)\| \leq \sup_{x\in E} \|f(x)\| + \sup_{x\in E} \|g(x)\|.$$

Thus we conclude that $\|\cdot\|_\infty$ is a norm on $\mathcal{A}$.

(3) If a sequence a_n converges to a in a norm, then observe that the sequence is bounded in norm by $\|a\|+1$. This can be proven as follows: there exist $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|a_n - a\| < 1.$$

Therefore,

$$\begin{aligned}\|a_n\| &= \|a_n - a + a\| \\ &\leq \|a_n - a\| + \|a\| < \|a\| + 1\end{aligned}$$

whenever $n \in \mathbb{N}$ satisfies $n \geq N$. Let

$$M = \max\{\|a_1 - a\|, \|a_2 - a\|, ..., \|a_{n-1} - a\|, 1\}.$$

Hence, it follows that $\|a_n\| \leq M$ for all $n \in \mathbb{N}$. Therefore, $(a_n)_{n\in\mathbb{N}}$ is bounded.

□

Lemma 4.2. *Let $f : [a,b] \to \mathbb{C}_2$ be continuous then if $f(t) = f_1(t) + jf_2(t)$ $\forall$ $t \in [a,b]$ $\exists$ $f_1, f_2 : [a,b] \to \mathbb{C}$ are uniformly continuous.*

Proof. For uniform continuity the following criteria must be met: given $\varepsilon > 0$ there exist $\delta > 0$ such that for all $x, y \in \mathbb{C}_2$ with $0 < |x-y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. We have that f_1 and f_2 are uniformly continuous iff there exist a δ_1, δ_2 such that for all $x, y \in \mathbb{C}$ with $|x-y| < \delta_1$ we have $|f_1(x) - f_1(y)| < \frac{\varepsilon}{2}$ and for $|x-y| < \delta_2$, $|f_2(x) - f_2(y)| < \frac{\varepsilon}{2}$. Let $\delta = \frac{\min\{\delta_1,\delta_2\}}{2}$. Thus,

$$\begin{aligned}|f(x) - f(y)| &= |(f_1(x) + j\cdot f_2(x)) - (f_1(y) + j\cdot f_2(y))| \\ &= |(f_1(x) - f_1(y)) + j\cdot(f_2(x) - f_2(y))| \\ &\leq |(f_1(x) - f_1(y))| + |j|\cdot|(f_2(x) - f_2(y))| \\ &= |(f_1(x) - f_1(y))| + |(f_2(x) - f_2(y))| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

□

Theorem 4.3. If $f : [a, b] \to \mathbb{C}_n$ be continuous, then there exists a sequence of polynomials functions, P_n, that converges uniformly on $[a, b]$. In other words $\lim_{n\to\infty} P_n = f(x)$ such that $P_n : [a, b] \to \mathbb{C}_n$. Furthermore if $Rng(f) \subset \mathbb{R}$ then there exist a sequence of $P_n : [a, b] \to \mathbb{R}$ such that $\lim_{n\to\infty} P_n = f(x)$ holds.

Proof. Without loss of generality consider the interval $[0, 1]$. Let g be the linear function such that $g(0) = f(0)$ and $g(1) = f(1)$. Since g is a polynomial, it suffices to prove the theorem for the function $f - g$ instead of f, but $f - g$ vanishes at the endpoints of $[0, 1]$, thus we are left proving the theorem when $f(0) = f(1) = 0$. Define $g : \mathbb{R}_n \to \mathbb{C}_n$ such that

$$g(x) = \begin{cases} f(x), & x \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that f is uniformly continuous, since $[a, b]$ is compact, then suppose that g is uniformly continuous. Consider the Landau sequence

$$Q_n(x) = \begin{cases} \frac{1}{b_n}(1 - x^2)^n, & x \in [-1, 1] \text{ and } b_n \neq 0 \\ 0, & \text{abs}(x) \geq 1. \end{cases}$$

Let $R_n(x) = (1 - x^2)^n$ for all $n \in \mathbb{N}$ where we claim that there exist b_n, a sequence in $\mathbb{C}$, such that $\int_{-1}^{1} R_n(x)dx = 1$ for all $n \in \mathbb{N}$. Note well, R_n is Riemann integrable since they are polynomial functions and are continuous. Choose b_n such that $\int_{-1}^{1} Q_n(x)dx = 1$ is satisfied. In other words $\frac{1}{b_n} \int_{-1}^{1} R_n(x)dx = 1$ iff $b_n = \int_{-1}^{1} R_n(x)dx$. Hence,

$$\frac{1}{\int_{-1}^{1} R_n(x)dx} \int_{-1}^{1} R_n(x)dx = 1.$$

Let $c_n = \frac{1}{b_n}$ for all $n \in \mathbb{N}$. Then we can rewrite the Landau sequence as

$$Q_n(x) = \begin{cases} c_n(1 - x^2)^n, & x \in [-1, 1] \\ 0, & \text{abs}(x) \geq 1. \end{cases}$$

Observe that R_n is even. Hence, $\int_{-1}^{1} R_n(x)dx = 2\int_0^1 R_n(x)dx$. Recall that $(1-x^2)^n \geq 1-nx^2$. Therefore,

$$\begin{aligned}
2\int_0^1 R_n(x)dx &\geq 2\int_0^{\frac{1}{\sqrt{n}}} R_n(x)dx \\
&\geq 2\int_0^{\frac{1}{\sqrt{n}}} (1-nx^2)dx \\
&= 2(x-\frac{nx^3}{3})|_0^{\frac{1}{\sqrt{n}}} \\
&= 2(\frac{1}{\sqrt{n}} - \frac{n(\frac{1}{n\sqrt{n}})}{3}) \\
&= \frac{4}{3\sqrt{n}} \\
&> \frac{1}{\sqrt{n}}.
\end{aligned}$$

Observe, $\int_{-1}^{1} c_n(1-x^2)^n dx = 1$ for $n \in \mathbb{N}$. Hence, if $\int_{-1}^{1}(1-x^2)^n dx > \frac{1}{\sqrt{n}}$, then $c_n(\int_{-1}^{1}(1-x^2)^n dx) > (\frac{1}{\sqrt{n}})c_n$, and thus $\int_{-1}^{1} c_n(1-x^2)^n dx > \frac{c_n}{\sqrt{n}}$. Therefore, $1 > \frac{c_n}{\sqrt{n}}$ and thus $\sqrt{n} > c_n$. If $0 < \delta \leq |x| \leq 1$, then $1-x^2 \leq 1-\delta^2 \Rightarrow Q_n(x) = c_n R_n < \sqrt{n}(1-\delta^2)^n \Rightarrow 0 \leq Q_n(x)$.

Let $P_n(x) := \int_{-1}^{1} f(x+t)Q_n(t)dt$ for $x \in [0,1]$. We can partition $P_n(x)$ as follows,

$$\int_{-1}^{-x} f(x+t)Q_n(t)dt + \int_{-x}^{1-x} f(x+t)Q_n(t)dt + \int_{1-x}^{1} f(x+t)Q_n(t)dt$$

Notice that $f(x+t) = 0$ iff $t \in [-1,-x] \cup [1-x,1]$. We have that both $\int_{-1}^{-x} f(x+t)Q_n(t)dt$ and $\int_{1-x}^{1} f(x+t)Q_n(t)dt$ are zero. Therefore, we have that $P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)dt$. Utilizing the technique of change of variables, with $u = t+x$ and thus $du = dt$. Hence,

$$\begin{aligned}
\int_{t\in[-x,1-x]} f(x+t)Q_n(t)dt &= \int_{t+x=0}^{t+x=1} f(x+t)Q_n(t)dt \\
&= \int_{u=0}^{u=1} f(u)Q_n(u-x)du \\
&= \int_{u\in[0,1]} f(u)Q_n(u-x)du.
\end{aligned}$$

Given $\varepsilon > 0$ and for all $\delta > 0$ such that $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Note that f is uniformly continuous. Let $M = \sup f(x)$, we since $Q_n(x) \geq 0$ for $x \in [0, 1]$,

$$\begin{aligned} P_n(x) - f(x) &= \int_{-1}^{1} [f(x+t)]Q_n(t)dt - f(x) \\ &= \int_{-1}^{1} [f(x+t)]Q_n(t)dt - f(x) \cdot 1 \\ &= \int_{-1}^{1} [f(x+t)]Q_n(t)dt - f(x) \int_{-1}^{1} Q_n(t)dt \\ &= \int_{-1}^{1} [f(x+t) - f(x)]Q_n(t)dt. \end{aligned}$$

Therefore,

$$\begin{aligned} |P_n(x) - f(x)| &= |\int_{-1}^{1} (f(x+t) - f(x))Q_n(t)dt| \\ &\leq \int_{-1}^{1} |(f(x+t) - f(x))Q_n(t)|dt \\ &\leq \int_{-1}^{1} |f(x+t) - f(x)| \cdot |Q_n(t)|dt \\ &= \int_{-1}^{1} |f(x+t) - f(x)| \cdot Q_n(t)dt, \\ &= \int_{-1}^{-\alpha} [f(x+t) - f(x)]Q_n(t)dt + \int_{-\alpha}^{\alpha} [f(x+t) - f(x)]Q_n(t)dt \\ &+ \int_{\alpha}^{1} [f(x+t) - f(x)]Q_n(t)dt. \end{aligned}$$

since $Q_n(t) \geq 0$ for all t.

Consider the first region of the integral domain $[-1, -\alpha]$. We have that,

$$\begin{aligned} \int_{-1}^{-\alpha} [f(x+t) - f(x)]Q_n(t)dt &\leq \int_{-1}^{-\alpha} (|f(x+t)| + |-f(x)|)Q_n(t)dt \\ &= \int_{-1}^{-\alpha} (|f(x+t)| + |f(x)|)Q_n(t)dt. \end{aligned}$$

Observe, if M is the least upper bound of $f(x)$, then M is the least upper bound of $f(x+t)$ for all t. Therefore,

$$\begin{aligned}\int_{-1}^{-\alpha}(|f(x+t)|+|f(x)|)Q_n(t)dt &\leq \int_{-1}^{-\alpha}(M+M)Q_n(t)dt\\ &= \int_{-1}^{-\alpha}(2M)Q_n(t)dt\\ &= 2M\int_{-1}^{-\alpha}Q_n(t)dt.\end{aligned}$$

If $t\in[-1,-\alpha]$, then it is equivalent to saying that $\alpha\leq|t|\leq 1$.Hence, $Q_n(t)\leq\sqrt{n}(1-\alpha^2)^n$. Therefore,

$$\begin{aligned}2M\int_{-1}^{-\alpha}Q_n(t)dt &\leq 2M\int_{-1}^{-\alpha}\sqrt{n}(1-\alpha^2)^n dt\\ &= 2M\sqrt{n}(1-\alpha^2)^n\int_{-1}^{-\alpha}dt\\ &= 2M\sqrt{n}(1-\alpha^2)^n(1-\alpha)\\ &\leq 2M\sqrt{n}(1-\alpha^2)^n,\end{aligned}$$

for sufficiently small α. If $\alpha>1$, we can use $\alpha^*=\min\{\alpha,1\}$ and for all large enough n.

Consider the second region of the integral domain $[-\alpha,\alpha]$. For $t\in[-\alpha,\alpha]$, we have $|t|=|t+x-x|=|(x+t)-x|\leq\alpha$. Then for $|f(x+t)-f(x)|<\frac{\varepsilon}{2}$, we have

$$\int_{-\delta}^{\delta}|f(x+t)-f(x)|Q_n(t)dt\leq\int_{-\delta}^{\delta}\frac{\varepsilon}{2}Q_n(t)dt.$$

Recall that if an interval E is a subset of the interval F, then for a positively defined function $g(x)$ for all $x\in F$ the following holds: $\int_E g(x)dx\leq\int_F g(x)dx$. Thus,

$$\int_{-\delta}^{\delta}\frac{\varepsilon}{2}Q_n(t)dt\leq\frac{\varepsilon}{2}\int_{-1}^{1}Q_n(t)dt\leq\frac{\varepsilon}{2}(1)=\frac{\varepsilon}{2}.$$

Consider the third and final region of the integral domain, $[\delta,1]$. For $t\in[\delta,1]$, we have

$$\begin{aligned}\int_{\delta}^{1}[f(x+t)-f(x)]Q_n(t)dt &\leq \int_{\delta}^{1}(|f(x+t)|+|f(x)|)Q_n(t)dt\\ &\leq \int_{\delta}^{1}(M+M)Q_n(t)dt\\ &= 2M\int_{\delta}^{1}Q_n(t)dt.\end{aligned}$$

However, for $t \in [\delta, 1]$ we have $Q_n(t) \leq \sqrt{n}(1-\delta^2)^n$, which implies that

$$\begin{aligned} 2M \int_\delta^1 Q_n(t)dt &\leq 2M \int_\delta^1 \sqrt{n}(1-\delta^2)^n dt \\ &= 2M\sqrt{n}(1-\delta^2)^n \int_\delta^1 dt \\ &= 2M\sqrt{n}(1-\delta^2)^n(1-\delta) \\ &\leq 2M\sqrt{n}(1-\delta^2)^n \end{aligned}$$

for sufficiently small δ.

When considering the entire integral domain, [-1,1], we have that

$$\begin{aligned} |P_n(x) - f(x)| &\leq 2M\sqrt{n}(1-\delta^2)^n + 2M\sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2} \\ &= 4M\sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2}. \end{aligned}$$

Recall that $\lim_{n\to\infty} \sqrt{n}r^n = 0$ if $r \in [0,1)$. Now if we let $r = (1-\delta^2)$ we have that $0 = \lim_{n\to\infty} \sqrt{n}(1-\delta^2)^n$. Therefore, we can conclude that $\lim_{n\to\infty} 4M\sqrt{n}(1-\delta^2)^n = 4M \lim_{n\to\infty} \sqrt{n}(1-\delta^2)^n = 4M(0) = 0$ which implies that there exist an $\varepsilon > 0$ there exist an $N \in \mathbb{N}$ such that for every $n \geq N$, we have that

$$\begin{aligned} |4M\sqrt{n}(1-\delta^2)^n - 0| &= |4M\sqrt{n}(1-\delta^2)^n| \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

Let $\varepsilon > 0$ be given for all n and x, hence,

$$\begin{aligned} |P_n(x) - f(x)| &\leq 4M\sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

We have shown that if $f : [0,1] \to \mathbb{R}$ ($resp.\mathbb{C}$) and f is continuous, then there exist a sequence of real (complex) polynomial functions P_n so that $f(0) = f(1)$ for f continuous.

Let $P_n \to f$ uniformly as $n \to \infty$, and $g : [a,b] \to \mathbb{R}$ where g(a)=g(b) is continuous. Let $f : [0,1] \to \mathbb{R}$ be given, where $f(t) = g[(1-t)a + tb]$. Observe that $f(0) = g(a)$ and $f(1) = g(b)$, and since $g(a) = g(b)$ then $f(0) = f(1)$. So by the reduced case we proved, there is a P_n such that $P_n(x) \to f(x)$ uniformly as $n \to \infty$. We want to show that there is a sequence of polynomials Q_n such that $Q_n(x) \to g(x)$ uniformly as $n \to \infty$. Observe, if

$y = (1-x)a + xb$, then $x = \frac{y-a}{b-a}$. Hence, $g(x) = g(\frac{y-a}{b-a})$. Therefore,

$$\begin{aligned}\lim_{n\to\infty} Q_n(x) &= \lim_{n\to\infty} P_n(\frac{x-a}{b-a}) \\ &= f(\frac{x-a}{b-a}) \\ &= g((1-(\frac{x-a}{b-a}))a + (\frac{x-a}{b-a})b) \\ &= g(\frac{x(b-a)}{b-a}) \\ &= g(x).\end{aligned}$$

$Q_n(x)$ is a polynomial since $P_n(\frac{x-1}{b-a})$ is a first degree polynomial and because the composition of two polynomials is a polynomial.

If $f : [a,b] \to \mathbb{R}$ is continuous with $f(a) = f(b) = 0$ there is a sequence of real (complex) polynomials, P_n, if f is real (complex) such that $P_n \to^{uc} f$. Let $g : [a,b] \to \mathbb{R}$ be continuous and let $f : [a,b] \to \mathbb{R}$ given by

$$f(x) = g(x) - g(a) - [g(b) - g(a)]\frac{x-a}{b-a}.$$

Notice that if $f(a) = 0$, then

$$\begin{aligned}f(b) &= g(b) - g(a) - \frac{b-a}{b-a}(g(b) - g(a)) \\ &= 0.\end{aligned}$$

So by the reduce case there is a P_n such that $P_n \to^{uc} f$. Observe,

$$\begin{aligned}\lim_{n\to\infty} P_n(x) &= f(x) \\ &= g(x) - g(a) - [g(b) - g(a)]\frac{x-a}{b-a}.\end{aligned}$$

Let

$$Q_n(x) = P_n(x) + g(x) - g(a) - [g(b) - g(a)]\frac{x-a}{b-a}$$

and

$$T(x) = g(x) - g(a) - [g(b) - g(a)]\frac{x-a}{b-a}.$$

So now $Q_n(x)$ is reduced to $P_n(x) + T(x)$. Additionally, $Q_n(x)$ converges to $g(x)$ and thus, $P_n \to^{uc} f$ implies that $P_n + T \to^{uc} f + T = g$.

□

Corollary 4.4. *For every interval* $[-a, a]$ *there is a sequence of real polynomials* P_n *such that* $P_n(x) = 0$ *and such that* $P_n(x) \to |x|$ *as* $n \to \infty$.

Proof. By theorem 4.3 there exist a sequence of real polynomials P_n which converges to $|x|$ uniformly on$[-a, a]$. In particular, $P_n^*(0) \to 0$ as $n \to \infty$. Then the polynomials $P_n(x) = P_n^*(x) - P_n^*(0)$ for $x \in \mathbb{N}\backslash\{0\}$ have the desired properties. □

Lemma 4.5. *If f is uniformly continuous, since $[a, b]$ is compact, then g is uniformly continuous.*

Proof. Let $\varepsilon > 0$ be given, then there exist $\delta > 0$ such that for $x, y \in \mathbb{R}$ and $|x - y| \in (0, \delta)$ we have $|g(x) - g(y)| < \varepsilon$. Since f is uniformly continuous, there exist $b_1 \in [0, 1]$ and $0 < |x - y| < \delta$ such that $|f(x) - f(y)| < \varepsilon$ and $|g(x) - g(y)| < \varepsilon$ for all x, y not in $[0, 1]$. So now, $|g(x) - g(y)| = |0 - 0| = 0 < \varepsilon$.

Suppose that x is not in $[0, 1]$ and $y \in [0, 1]$. Assume that $0 < |x - y| < \delta$ and without loss of generality, suppose that $x < 0$. Since f is uniformly continuous there exist a $\delta_n > 0$ such that $0 < |0 - y| < \delta_n$, then $|f(0) - f(y)| = |g(0) - g(y)| < \frac{\varepsilon}{2}$. The difference between 0 and y is less than $\frac{\varepsilon}{2}$. Therefore if $|g(x) - g(y)| < \varepsilon$, then

$$\begin{aligned}|g(x) - g(0) + g(0) - g(y)| &\leq |g(x) - g(0)| + |g(0) - g(y)| \\ &= 0 + |g(0) - g(y)| \\ &< \varepsilon.\end{aligned}$$

If $x \in [0, 1]$ and $y > 1$, then $|x - y| \in (0, \delta)$. Hence, $|x - 1| < \delta$ and $|1 - x| = 1 - x < y - x < \delta$.

The case when $x = 1$ is trivial, $|g(x) - g(y)| = |f(x) - f(y)| = |0 - 0| = 0 < \varepsilon$. So consider the case when $x < 1$. We have, $0 < |x - 1| < \delta$ and thus, $|f(x) - f(1)| < \varepsilon$. Hence if $|x - y| \in (0, \delta)$, then

$$\begin{aligned}|g(x) - g(y)| &= |g(x) - g(1) + g(1) - g(y)| \\ &= |g(x) - g(1)| + |g(1) - g(y)| \\ &= 0 + |g(0) - g(y)| \\ &< \varepsilon\end{aligned}$$

and,

$$\begin{aligned}|g(x) - g(1)| + |g(1) - g(y)| &= |f(x) + f(1)| + 0 \\ &= |f(x) + f(1)| \\ &< \varepsilon.\end{aligned}$$

Thus our claim that the continuous function g is uniformly continuous, holds. □

Lemma 4.6. $(1 - x^2)^n \geq 1 - nx^2$

Proof. If $a \in [0, 1]$ then,

$$\begin{aligned}1 - a^n = v^n|_a^1 &= \int_a^1 nv^{n-1}dv \\ &\leq \int_a^1 ndv \\ &= n(1 - a)\end{aligned}$$

Hence, $a^n \leq n(1-a)$ and so,

$$v^n = 1 + \int_1^v nt^{n-1}dt \geq 1 + \int_1^v ndt$$
$$\geq 1 + n(v-1).$$

Therefore, $(1-x^2)^n \geq 1 + ((1-x^2)-1)n = 1 - nx^2$. □

Lemma 4.7. *$Q_n \to^{uc} 0$ in $[\delta, 1]$.*

Proof. Q_n is uniformly convergent to 0 s.t. $[-1,1] \to \mathbb{R}$ iff for all $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq \mathbb{N}$, $x \in [-1,1]$, and $|x| \in [\delta, 1]$, then $|Q_n(x) - 0| < \varepsilon$. For all $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $b \geq N$ and for all $x \in [\delta, 1]$, then $|Q_n(x)| < \varepsilon$ and so, $Q_n(x) < \varepsilon$.

□

Lemma 4.8. *Suppose that $Q_n \to^{uc} 0$ in $[\delta, 1]$ then $|Q_n(x)| < \varepsilon$ we know $Q_n(x) \leq \sqrt{n}(1-\delta^2)^n$ for $|x| \in [\delta, 1]$. Is $Q_n \geq 0$?*

Proof. Let $c_n = \frac{1}{b_n}$ where $b_n = \int_{-1}^{1} R_n(x)dx = 1 = \int_{-1}^{1}(1-x^2)^n dx \geq 0$. Hence, $Q_n(x) = c_n(1-x^2)^n \geq 0$. Therefore, $0 \leq Q_n(x) \leq \sqrt{n}(1-\delta^2)^n$ for $|x| \in [\delta, 1]$. □

Lemma 4.9. *$\lim_{n\to\infty} \sqrt{n}r^n = 0$ if $r \in [0, 1)$.*

Proof. Note, $\sqrt{n} < n + 1$. Consider now,

$$0 \leq \sum_{n=0}^{\infty} \sqrt{n}r^n$$
$$< \sum_{n=0}^{\infty}(n+1)r^n$$
$$= \frac{d}{dr}\sum_{n=1}^{\infty} r^n$$

which by comparison test, $\sum_{n=0}^{\infty} \sqrt{n}r^n$ converges and so $\lim_{n\to\infty} \sqrt{n}r^n = 0$ for $r \in [0, 1)$. □

Lemma 4.10. *$f(x+t) = 0$ iff $t \in [-1, -x] \cup [1-x, 1]$*

Proof. Assume that $f(0) = f(1) = 0$ for $x \in [0,1]$, for $1 - x \leq t$ or $1 \leq x + t$, we have $f(x+t) = 0$ for $t \in [-1, -x]$ if $t \leq -x \Rightarrow x + t \leq 0 \Rightarrow f(x+t) = 0$. □

Lemma 4.11. *M is the least upper bound of say $g(x)$ then M is the upper bound for $g(x+t)$.*

Proof. By definition we say that M is the least upper bound, or supremum, of $g(x)$ iff for every x in a given field, say $\mathbf{F}$, $x \leq M$, i.e. let $g : \mathbf{F} \to \mathbb{R}$, then M is the supremum of g iff for every element in $\mathbf{F}$ is bounded by M. Every element implies that say $x + 1$ is in $\mathbf{F}$, thus we can call $x + 1 = y$ where $y \in \mathbf{F}$ due to the field axioms. Therefore since M is the supremum of $g(x)$ and $g(y)$ then by construction, $x < y \leq M$ as expected. Therefore for any $t \in \mathbf{F}$ we can conclude that $x + t \in \mathbf{F}$ thus if we let $z = x + t$ then we gave that $g(x+t) = g(z)$, but $z \in \mathbf{F}$ thus $z \leq M$.

□

Lemma 4.12. *Let g be a regulated real valued function on $[a, b]$ and assume $a < b$. Let $a \leq c \leq b$ and assuming that g is continuous at c and that $g(c) > 0$ and also that $g(t) \geq 0$ for all $t \in [a, b]$ then $\int_a^b f > 0$.*

Proof. Given $g(c)$, there exist some δ such that $g(t) > \frac{f(c)}{2}$ if $|t - c| < \delta$ and $t \in [a, b]$. If c is not equal to a, then with out loss of generality, let $0 < \kappa < \delta$ such that $[c - \kappa, c]$ is contained in $[a, b]$. Then,

$$\begin{aligned}\int_a^b g &= \int_a^{c-\kappa} g + \int_{c-\kappa}^{c} g + \int_c^b g \\ &\geq \int_{c-\kappa}^{c} g \\ &\geq \frac{\kappa * g(c)}{2} \\ &> 0.\end{aligned}$$

In the case that $c = a$, we take a small interval $[a, a + \kappa]$ with $0 < \kappa < \delta$ such that $[a, a + \kappa] \subset [a, b]$. Then,

$$\begin{aligned}\int_a^b g &= \int_a^{a+\kappa} g + \int_{a+\kappa}^{b} g \\ &\geq \int_a^{a+\kappa} g \\ &\geq \frac{\kappa * g(c)}{2} \\ &> 0.\end{aligned}$$

One could argue similarly in the case when $c = b$ just consider the case where we take the small interval $[b - \kappa, b]$. Note this was proven when c was not equal to a. □

Lemma 4.13. *If the interval say E is a subset of the interval F, then for a positively defined function g i.e. $g(x) \geq 0 \;\forall\; x \in F$ the following holds $\int_E g(x)dx \leq \int_F g(x)dx$.*

Proof. Let F be the interval $[a, b]$ where $a < b$ and E be the interval $[c, d]$ with $c < d$ such that $[c, d] \subset [a, b]$ where $a \leq c < d \leq b$. In the case that $a = c$ and $b = d$, E is equivalent to F, then

$$\int_F g(x)dx = \int_E g(x)dx.$$

. Thus assume that either a is not equal to c or b is not equal to d or a, b is not equal to c, d respectively, in other words at least one of the end point if F and E are not equal. If that is the case, then

$$\begin{aligned}\int_F g(x)dx &= \int_a^b g(x)dx \\ &= \int_a^c g(x)dx + \int_c^d g(x)dx + \int_d^b g(x)dx.\end{aligned}$$

From Lemma 4.15 we found that if $g(x) \geq 0$ for $x \in F$ we have $\int_b^a g(x)dx > 0$ and similarly for $x \in E$ we can conclude that $\int_E g(x)dx > 0$. Thus,

$$\begin{aligned}\int_F g(x)dx &= \int_a^b g(x)dx \\ &= \int_a^c g(x)dx + \int_c^d g(x)dx + \int_d^b g(x)dx \\ &\geq 0 + \int_c^d g(x)dx + 0 \\ &= \int_c^d g(x)dx = \int_E g(x)dx\end{aligned}$$

since $g(x)$ is positively defined on the each integral domain. Thus we can conclude that that

$$\int_F g(x)dx \geq \int_E g(x)dx.$$

□

Lemma 4.14. *The sequence of functions f_n, defined on E, converges uniformly on E iff for every $\varepsilon > 0$ there is an integer N such that $m, n \geq N$, $x \in E$ implies that $|f_n(x) - f_m(x)| \leq \varepsilon$.*

Proof. Suppose f_n converges uniformly on E, and let f be the limit function. Then there is an integer N such that $n \geq N$, $x \in E$ implies that $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2}$, so that

$$\begin{aligned}|f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &\leq \varepsilon,\end{aligned}$$

if $n, m \geq N$, $x \in E$.

Conversely, suppose that the Cauchy condition hold. By Theorem 3.11, page 62 in [Ru1], there is a sequence $f_n(x)$ converges, for every x, to a limit we may call $f(x)$. Thus the sequence f_n converges on E, to f. We have to prove the convergence is uniform. Let $\varepsilon > 0$ be given, and choose N s.t. $|f_n(x) - f_m(x)| \leq \varepsilon$ holds. Fix n, and let $m \rightarrow \infty$, this gives $f_m(x) \rightarrow f(x)$ thus $|f_n(x) - f(x)| \leq \varepsilon$ for every $n \geq N$ and every $x \in E$, which completes the proof. □

Lemma 4.15. *Let ϕ, $\phi : [a, b] \rightarrow \mathbb{R}$, be uniform continuous and if $f_n \rightarrow^{uc} f$ then the composition $f_n \circ \phi \rightarrow^{uc} f \circ \phi$.*

Proof. Let $\varepsilon > 0$ then there exist a $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in [a, b]$ we have

$$|(f_n \circ \phi)(x) - (f_m \circ \phi)(x)| \leq \varepsilon.$$

Suppose $(f_n \circ \phi)$ converges uniformly on $[a, b]$, and let $(f \circ \phi)$ be the limit function. Then there is an integer N s.t. $n \geq N$, $x \in [a, b]$ implies that

$$|(f_n \circ \phi)(x) - (f \circ \phi)(x)| \leq \frac{\varepsilon}{2},$$

so that

$$\begin{aligned}|(f_n \circ \phi)(x) - (f_m \circ \phi)(x)| &\leq |(f_n \circ \phi)(x) - (f \circ \phi)(x)| + |(f \circ \phi)(x) - (f_m \circ \phi)(x)| \\ &\leq \varepsilon,\end{aligned}$$

if $n, m \geq N$, $x \in [a, b]$.

Conversely, suppose that the Cauchy condition hold. By Theorem 3.11, page 62 in [Ru1], there is a sequence $(f_n \circ \phi)$ converges, for every $x \in [a, b]$, to a limit we may call $(f \circ \phi)$. Thus the sequence $(f_n \circ \phi)$ converges on $[a, b]$, to $(f \circ \phi)$. We have to prove the convergence is uniform. Let $\varepsilon > 0$ be given, and choose N s.t. $|(f_n \circ \phi)(x) - (f_m \circ \phi)(x)| \leq \varepsilon$ holds. Fix n, and let $m \rightarrow \infty$, this gives $(f_m \circ \phi) \rightarrow (f \circ \phi)$ thus $|(f_n \circ \phi)(x) - (f \circ \phi)(x)| \leq \varepsilon$ for every $n \geq N$ and every $x \in [a, b]$.

□

Proof. If $f_n \rightarrow^{uc} f$ on $\mathbb{R}$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ then $f_n \circ \phi \rightarrow^{uc} f \circ \phi$ if given $\varepsilon > 0$ there is an integer N s.t. for all $n \geq N$ and all $y \in \mathbb{R}$ we have $|f_n(y) - f(y)| < \varepsilon$. In particular if $y = \phi(x)$ then for all such n and all x, $|f_n(\phi(x)) - f(\phi(x))| < \varepsilon \Rightarrow f_n \circ \phi \rightarrow^{uc} f \circ \phi$. □

Lemma 4.16. *If $f_n \rightarrow^{uc} f$ on E then $f_n + T \rightarrow^{uc} f + T$ on E for the sequences of functions $\{f_n\}$ and $\{T\}$.*

Proof. We need to show that for all ε there is an integer N such that for $n \geq N$ implies that

$$|(f_n + T)(x) - (f + T)(x)| \leq \varepsilon.$$

Since $f_n \rightarrow^{uc} f$ we have that given $\varepsilon > 0$ there is an integer N such that for $n \geq N$ for all $x \in E$ we have that $|f_n(x) - f(x)| < \varepsilon$. Now,

$$\begin{aligned}|(f_n + T)(x) - (f + T)(x)| &= |f_n(x) + T(x) - (f(x) + T(x))| \\ &= |f_n(x) + T(x) - f(x) - T(x)| \\ &= |f_n(x) - f(x)| \\ &< \varepsilon,\end{aligned}$$

as needed. □

5. More Open Questions

Can Theorem 2.1 be extended to Quaternions, Octonions, Sedenions? In other words using *the Cayley − Dickson construction*, can we construct a generalized form?

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