Of Stability and Other Minimal Surfaces

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Abstract. This project/thesis is the result of research conducted with Dr. Thomas Murphy from Fall of 2015 to Spring 2016. Our initial goal was to find critical functions that destabilize the expression for the stability for the Scherk surface. While it is well known among geometers that the Scherk is not a stable surface, it has not been explicitly done. Moreover, we derive said stability operator using perturbations along the normal of a general surface in 3 space. As of now we are looking to find the index of the Scherk surface i.e. the number of independent directions in which the surface decreases. We start with some basic definitions and Theorems needed to study minimal surfaces.

Key words: Minimal surfaces, harmonic functions, differential geometry, Scherk surface, stability operator, index of minimal surface

1 Differential Geometry

Definition 1. ***Conformal map.*** *Let S_1 and S_2 be surfaces. A conformal map $f : S_1 \to S_2$ is a local diffeomorphsim such that if γ_1 and $\tilde{\gamma}_1$ are any two curves on S_1 that intersects, say at point $\boldsymbol{p} \in S_1$, and if γ_2 and $\tilde{\gamma}_2$ are their images under f, the angles of intersection of γ_1 and $\tilde{\gamma}_1$ at $\boldsymbol{p}$ is equal to the angles of intersection of γ_2 and $\tilde{\gamma}_2$ at $f(\boldsymbol{p})$.*

Theorem 1. *A local diffeomorphism $f : S_1 \to S_2$ is conformal iff there is a function $\lambda : S_1 \to \mathbb{R}$ such that*

$$f^*\langle \mathbf{v}|\mathbf{w}\rangle_{\mathbf{p}} = \lambda(\mathbf{p})\langle \mathbf{v}|\mathbf{w}\rangle_{\mathbf{p}}$$

for all $\mathbf{p} \in \mathbf{S_1}$ and $\boldsymbol{v},\boldsymbol{w} \in T_{\mathbf{p}}S_1$.

Proof. See [7, p. 134]

Definition 2. ***Area.*** *The Area $A(R)$ of the part $\boldsymbol{\sigma}(R)$ of the surface patch $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ corresponding to a region $R \subset U$ is*

$$A(R) = \int_R \|\boldsymbol{\sigma}_u \wedge \boldsymbol{\sigma}_v\| du dv.$$

Proposition 1. $\|\boldsymbol{\sigma}_u \wedge \boldsymbol{\sigma}_v\| = (EG - F^2)^{\frac{1}{2}}$

Proof. Noting that $\langle \mathbf{a} \wedge \mathbf{b}|\mathbf{c} \wedge \mathbf{d}\rangle = \langle \mathbf{a}|\mathbf{c}\rangle\langle \mathbf{b}|\mathbf{d}\rangle - \langle \mathbf{a}|\mathbf{d}\rangle\langle \mathbf{b}|\mathbf{c}\rangle$ for vectors $\mathbf{a},\ \mathbf{b},\ \mathbf{c},\ \mathbf{d}$ in $\mathbb{R}^3$ then

$$\begin{aligned}\|\boldsymbol{\sigma}_u \wedge \boldsymbol{\sigma}_v\|^2 &= \langle \boldsymbol{\sigma}_u|\boldsymbol{\sigma}_u\rangle\langle \boldsymbol{\sigma}_v|\boldsymbol{\sigma}_v\rangle - \langle \boldsymbol{\sigma}_u|\boldsymbol{\sigma}_v\rangle^2 \\ &= EG - F^2.\end{aligned}$$

Moreover, for a regular surface $\boldsymbol{\sigma}_u \wedge \boldsymbol{\sigma}_v$ is non-zero. ■

Exercise 1. *Suppose that the first fundamental form of a surface patch* $\boldsymbol{\sigma}(u,v)$ *is of the form* $E(du^2+dv^2)$*. Prove that* $\boldsymbol{\sigma}_{uu}+\boldsymbol{\sigma}_{vv}$ *is perpendicular to* $\boldsymbol{\sigma}_u$ *and* $\boldsymbol{\sigma}_v$*. Deduce that the mean curvature* $H=0$ *everywhere if and only if the Laplacian* $\boldsymbol{\sigma}_{uu}+\boldsymbol{\sigma}_{vv}=0$*.*

Proof. If $\boldsymbol{\sigma}$ is conformal, the mean curvature of $\boldsymbol{\sigma}$ is $H=\frac{L+N}{2E}$. Ergo, if $\boldsymbol{\sigma}$ is minimal, then $L+N=0$, i.e.,

$$0=\langle\boldsymbol{\sigma}_{uu}|\mathbf{N}\rangle+\langle\boldsymbol{\sigma}_{\mathbf{vv}}|\mathbf{N}\rangle=\langle\boldsymbol{\sigma}_{\mathbf{uu}}+\boldsymbol{\sigma}_{\mathbf{vv}}|\mathbf{N}\rangle=\langle\boldsymbol{\Delta}\boldsymbol{\sigma}|\mathbf{N}\rangle$$

All that remains to be shown is $\Delta\boldsymbol{\sigma}=0$. Since $\{\boldsymbol{\sigma}_u,\ \boldsymbol{\sigma}_v,\ \mathbf{N}\}$ is a basis for $\mathbb{R}^3$,

$$\begin{aligned}\langle\Delta\boldsymbol{\sigma}|\boldsymbol{\sigma}_u\rangle&=\langle\boldsymbol{\sigma}_{uu}|\boldsymbol{\sigma}_u\rangle+\langle\boldsymbol{\sigma}_{vv}|\boldsymbol{\sigma}_u\rangle\\&=\frac{1}{2}(2\langle\boldsymbol{\sigma}_{uu}|\boldsymbol{\sigma}_u\rangle)+(\langle\boldsymbol{\sigma}_{vv}|\boldsymbol{\sigma}_u\rangle+\langle\boldsymbol{\sigma}_{uv}|\boldsymbol{\sigma}_v\rangle)-\langle\boldsymbol{\sigma}_{uv}|\boldsymbol{\sigma}_v\rangle\\&=\frac{1}{2}\frac{\partial\langle\boldsymbol{\sigma}_u|\boldsymbol{\sigma}_u\rangle}{\partial u}+\frac{\partial\langle\boldsymbol{\sigma}_u|\boldsymbol{\sigma}_v\rangle}{\partial v}-\langle\boldsymbol{\sigma}_{uv}|\boldsymbol{\sigma}_v\rangle\\&=\frac{1}{2}(\langle\boldsymbol{\sigma}_u|\boldsymbol{\sigma}_u\rangle-\langle\boldsymbol{\sigma}_v|\boldsymbol{\sigma}_v\rangle)_u+(\langle\boldsymbol{\sigma}_u|\boldsymbol{\sigma}_v\rangle)_v.\end{aligned}$$

For conformal maps, $\|\boldsymbol{\sigma}_u\|^2=\|\boldsymbol{\sigma}_v\|^2$ and $\langle\boldsymbol{\sigma}_u|\boldsymbol{\sigma}_v\rangle=0$ and so $\langle\Delta\boldsymbol{\sigma}|\boldsymbol{\sigma}_u\rangle=0$. By a similar fashion, $\langle\Delta\boldsymbol{\sigma}|\boldsymbol{\sigma}_v\rangle=0$. Thus $\Delta\boldsymbol{\sigma}=0$.

Conversely, if $\Delta\boldsymbol{\sigma}=0$, it should be clear that $H=0$. ■

Proposition 2. *If* Σ *is a compact surface, there is a point of* Σ *at which its Gaussian curvature* $K>0$*.*

Proof. See [7, p. 212].

1.1 Weingarten map

Definition 3. Weingarten map. *Let* $\boldsymbol{\sigma}$ *be a regular curve, i.e.* $\boldsymbol{\sigma}_u\wedge\boldsymbol{\sigma}_v$ *is non-vanishing. The Weingarten map* $S=S(u,v)$ *is a linear map of the tangent space* T_p *into itself defined as follows: if* $a=a_1\boldsymbol{\sigma}_u+a_2\boldsymbol{\sigma}_v$ *then,*

$$Sa=-a_1N_u-a_2N_v.$$

S *is also know as the shape operator.*

Proposition 3. *The Weingarten map is self-adjoint, i.e.* $\langle Sv|w\rangle=\langle v|Sw\rangle$*.*

Definition 4. Fundamental Forms using the shape operator *We can define the following three symmetric bilinear forms:*

$$\begin{aligned}I(v,w)&=\langle v|w\rangle\\II(v,w)&=\langle Sv|w\rangle\\III(v,w)&=\langle Sv|Sw\rangle.\end{aligned}$$

We will use the following notation, when appropriate, for the coeffcients of the fundamental forms:

$$\begin{aligned}g_{ij}&=\langle\boldsymbol{\sigma}_i|\boldsymbol{\sigma}_j\rangle\\b_{ij}&=\langle -N_i|\boldsymbol{\sigma}_j\rangle\\c_{ij}&=\langle N_i|N_j\rangle.\end{aligned}$$

Note that b_{ij} *are the entries in the matrix of the shape operator* S*; that is why they are defined with a minus sign. Since* $\langle N|\boldsymbol{\sigma}_j\rangle=0$ *and when differentiating,* $\langle N_i|\boldsymbol{\sigma}_j\rangle+\langle N|\boldsymbol{\sigma}_{ij}\rangle=0$*, yields,*

$$b_{ij}=\langle\boldsymbol{\sigma}_{ij}|N\rangle.$$

2 Complex Analysis

2.1 Holomorphic Functions

Definition 5. Holomorphic function. *If a complex function $f(z) = f(x+\mathrm{i}y) = u(x,y)+\mathrm{i}v(x,y)$ is holomorphic (or analytical), then u and v have first partial derivatives with respect to x and y, and satisfy the Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Equivalently, if the Writinger derivative of f with respect to the complex conjugate of z is zero: $\frac{\partial f}{\partial \bar{z}}$=0.

Proposition 4. *Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a conformal surface patch. Consider the complex coordinates in the plane of which U is an open subset by setting $z = u+\mathrm{i}v$ for $(u,v) \in U$. We define $\phi(z) = \boldsymbol{\sigma}_u - \mathrm{i}\boldsymbol{\sigma}_v$ and we say $\boldsymbol{\sigma}$ is minimal if and if the function ϕ is holomorphic on U.*

Proof. Let $\phi(u,v)$ be a complex-valued smooth function. Let $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ and let ϕ_k be the k-th component of $\phi(z)$ so that

$$\phi_k = (\sigma^k)_u - \mathrm{i}(\sigma^k)_v.$$

Let α and β be the real and imaginary parts of ϕ, $\phi = \alpha + \mathrm{i}\beta$, with components

$$\phi_k = \alpha_k + \mathrm{i}\beta_k.$$

If ϕ is holomorphic, then ϕ, and its components ϕ_k, must satisfy the Cauchy-Riemann equations, i.e.,

$$\frac{\partial \alpha_k}{\partial u} = \frac{\partial \beta_k}{\partial v} \text{ and } \frac{\partial \alpha_k}{\partial v} = -\frac{\partial \beta_k}{\partial u}.$$

However, this means that ϕ must also satisfy,

$$\frac{\partial (\sigma^k)_u}{\partial u} = \frac{\partial(-(\sigma^k)_v)}{\partial v} \text{ and } \frac{\partial (\sigma^k)_u}{\partial v} = -\frac{\partial(-(\sigma^k)_{v)}}{\partial u}.$$

The first equation is equivalent to $\boldsymbol{\sigma}_{uu} + \boldsymbol{\sigma}_{vv} = 0$ and the second to $\boldsymbol{\sigma}_{uv} = \boldsymbol{\sigma}_{vu}$. The result follows by Exercise 1. ■

Definition 6. Simply-connected. *An open subset $U \subset \mathbb{R}^2$ is said to be simply-connected if every simple closed curve in U can be shrunk to a point staying inside of U.*

Proposition 5. *If f is a holomorphic function of $z = x + \mathrm{i}y$, then*

$$f_x = \frac{df}{dz},\ f_y = \mathrm{i}\frac{df}{dz},\ (\bar{f})_x = \frac{d\bar{f}}{dz},\ (\bar{f})_y = -\mathrm{i}\frac{d\bar{f}}{dz}.$$

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and write $f(z) = u(x,y) + \mathrm{i}v(x,y)$ with harmonic u, v. Define the two Wirtinger derivatives as,

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}) \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y})$$

we can deduce:

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}} \text{ and } \frac{\partial}{\partial y}=\mathrm{i}(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}).$$

Now, by a direct computation,

$$\begin{aligned}\frac{\partial f}{\partial z}&=\frac{1}{2}(\frac{\partial f}{\partial x}-\mathrm{i}\frac{\partial f}{\partial y})\\&=\frac{1}{2}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y})+\frac{\mathrm{i}}{2}(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y})\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}}&=\frac{1}{2}(\frac{\partial f}{\partial x}+\mathrm{i}\frac{\partial f}{\partial y})\\&=\frac{1}{2}(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y})+\frac{\mathrm{i}}{2}(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}).\end{aligned}$$

Moreover, f must satisfy the Riemann-Cauchy equations:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

As an immediate consequence,

$$\frac{\partial f}{\partial \bar{z}}=0.$$

So, if f is holomorphic then $\frac{df}{dz}=\frac{\partial f}{\partial z}+\frac{\partial f}{\partial \bar{z}}$, and thus

$$\begin{aligned}\frac{\partial f}{\partial x}&=\frac{\partial f}{\partial z}+\frac{\partial f}{\partial \bar{z}}=\frac{\partial f}{\partial z}\\\frac{\partial f}{\partial y}&=\mathrm{i}(\frac{\partial f}{\partial z}-\frac{\partial f}{\partial \bar{z}})=\mathrm{i}\frac{\partial f}{\partial z}\\\frac{\partial \bar{f}}{\partial x}&=\frac{\partial \bar{f}}{\partial z}\\\frac{\partial \bar{f}}{\partial y}&=-\mathrm{i}\frac{\partial \bar{f}}{\partial z}.\end{aligned}$$

■

Theorem 2. *If $\boldsymbol{\sigma}: U \to \mathbb{R}^3$ is a conformal minimal surface, the vector-valued holomorphic function $\phi=(\phi_1,\phi_2,\phi_3)$ defined in Proposition 4 satisfies the following:*

(a.) ϕ is nowhere zero
(b.) $\Sigma\phi_k^2=0$ for $k=1,2,3$.

Conversely, if U is simple-connected and if ϕ_k are holomorphic functions on U satisfying (a.) and (b.), there is a conformally parametrized minimal surface $\boldsymbol{\sigma}: U \to \mathbb{R}^3$ such that ϕ satisfies $\phi(z)=\boldsymbol{\sigma}_u-\mathrm{i}\boldsymbol{\sigma}_v$. Moreover, $\boldsymbol{\sigma}$ is uniquely determined by ϕ_k up to a translation.

Proof. Suppose that $\boldsymbol{\sigma} = (\sigma^k)_{k=1}^3$ is minimal. We need to show that $\phi = (\phi_k)_{k=1}^3$ satisfies $(a.)$ and $(b.)$. Since $\phi_k = (\sigma^k)_u - \mathfrak{i}(\sigma^k)_v$,

$$\sum_{k=1}^{3} \phi_k^2 = \sum(((\sigma^k)_u)^2 - ((\sigma^k)_v)^2 - 2\mathfrak{i}(\sigma^k)_u(\sigma^k)_v) = g_{uu} - g_{vv} - 2\mathfrak{i}g_{uv} \tag{1}$$

However, since $\boldsymbol{\sigma}$ is conformal, $g_{uu} = g_{vv}$ and $g_{uv} = 0$, the result follows immediately. Lastly, $\phi = 0$ if and only if $\boldsymbol{\sigma}_u = \boldsymbol{\sigma}_v = 0$, however this is impossible since $\boldsymbol{\sigma}$ is regular.

We now solve the converse. Suppose ϕ satisfies $(a.)$ and $(b.)$. We need to show that ϕ gives a minimal surface as described. Fix $(u_0, v_0) \in U$ and define $\boldsymbol{\sigma}$ as follows (Weierstrass−Enneper parameterization):

$$\sigma^k(u, v) = \mathfrak{Re} \int_C \phi_k(\epsilon)d\epsilon \tag{2}$$

where C is any curve in U from (u_0, v_0) to $(u, v) \in U$. Since U is simply-connected, then by Cauchy's Theorem, $\int_C \phi(\epsilon)d\epsilon$ is path independent. Define $\Lambda_k(z) = \int_C \phi_k(\epsilon)d\epsilon$ to be a holomorphic function of $z = u + \mathfrak{i}v$ and $\frac{d\Lambda_k(z)}{dz} = \phi_k(z)$. Hence,

$$\sigma_u^k = \mathfrak{Re}((\Lambda_k)_u) = \mathfrak{Re}(\frac{d\Lambda_k}{dz}) = \mathfrak{Re}(\phi_k) \tag{3}$$

$$\sigma_v^k = \mathfrak{Re}((\Lambda_k)_v) = \mathfrak{Re}(\mathfrak{i}\frac{d\Lambda_k}{dz}) = -\mathfrak{Im}(\phi_k) \tag{4}$$

and so, $\phi = \boldsymbol{\sigma}_u - \mathfrak{i}\boldsymbol{\sigma}_v$. It remains to be shown that $\boldsymbol{\sigma}$ is conformal surface. Now, by $(b.)$ and the equations (3) and (4), $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are not both zero. By $(a.)$ and (1), $g_{uu} = g_{vv}$ and $g_{uv} = 0$. Since, $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are both not zero, this proves that $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are both non-zero and perpendicular, hence linearly independent so that $\boldsymbol{\sigma}$ is a regular surface patch and thus conformal. ■

Remark 1. The previous theorem allows the construction of complex functions ϕ from the parametrization of our surface. Effectively, this allows us to study the minimal surface over the complex plane.

2.2 The Maximum Principle

Theorem 3. The Maximum Principle. *If $f(z)$ is analytic and non-constant in a region Ω, the its absolute value $|f(z)|$ has no maximum in Ω.*

Proof. If $w_0 = f(z_0)$ is any value in Ω, then there exist $w \in B(w_o, \epsilon)$ in the image of Ω, where B denotes a metric ball. In $B(w_o, \epsilon)$ there are points of modulus greater than $|w_0|$ and hence $|f(z_0)|$ is not the maximum of $|f(z)|$. ■

Proof. Suppose $|f(z_0)| = \max_{x \in \Omega} |f(z)|$. Then the ball $B(z_0, \epsilon) \subset \Omega$ yields

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for all $z \in B(z_0, \epsilon)$. Now consider the following,

$$\begin{aligned}\sum_{n=0}^{\infty}|a_n|^2r^{2n} &= \sum_{n=0}^{\infty}\overline{a}_n a_n r^n r^n \\ &= \sum_{n,m\in\mathbb{N}\cup\{0\}} \overline{a}_n a_m r^n r^m \delta_{n,m} \\ &= \sum_{n,m\in\mathbb{N}\cup\{0\}} \overline{a}_n a_m r^n r^m \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}(m-n)\theta}d\theta \\ &= \frac{1}{2\pi}\int_o^{2\pi} \sum_{n,m\in\mathbb{N}\cup\{0\}} \overline{a}_n r^n e^{-\mathrm{i}n\theta} a_m r^m e^{\mathrm{i}m\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(re^{\mathrm{i}\theta})\overline{f(re^{\mathrm{i}\theta})}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} |f(re^{\mathrm{i}\theta})|^2.\end{aligned}$$

where $\delta_{n,m}$ denotes the kronecker delta. So,

$$\begin{aligned}|f(z)|^2 = |a_0|^2 + \sum_{n=1}^{\infty}|a_n|^2\epsilon^{2n} &\leq \frac{1}{2\pi}\int_0^{2\pi}\max_{x\in\Omega}|f(z)|^2d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}|f(z_0)|^2d\theta \\ &= |a_0|^2\end{aligned}$$

Therefore, $a_n = 0$ and $f(z) = a_0$. ■

Theorem 4. *If $f(z)$ is defined and continuous on a closed bounded set E and analytic on the interior of E, then the maximum of $|f(z)|$ on E is assumed on the boundary of E.*

Proof. See [1, p. 134].

Theorem 5. Maximum Principle for Harmonic Functions. *Let u be a harmonic function defined in domain Ω with non-constant u. Then u does not attain the maximum in the interior of Ω.*

Proof. Suppose u attains the maximum at a point $a \in \Omega$. We define a holomorphic function f defined in the ball $B(a,\epsilon) \subset \Omega$ such that $\mathfrak{Re}(f) = u$. So,

$$|e^f| = e^{\mathfrak{Re}(f)} = e^u$$

attains a maximum at a. This implies e^f is constant so $e^u = \textit{constant}$ and so u is constant. ■

3 First Variation

From this point onward, unless stated otherwise, we will be using Einstein's summation convention, i.e,

$$c_ix^i = c_1x^1 + c_2x^2 + \ldots + c_nx^n$$

for all values of the index i. Moreover, we define

$$g_{ij} = (g_{ij}) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{5}$$

and

$$b_{ij} = (b_{ij}) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}. \tag{6}$$

Do not confuse this N with the unit normal. Also, note from Proposition 1,

$$W = \sqrt{g} = \sqrt{EG - F^2} = \sqrt{|g_{ij}|} \tag{7}$$

where $|g_{ij}|$ denotes the determinant of (g_{ij}). We also define (g^{ij}) to be the coefficient of the inverse of (g_{ij}), so,

$$g^{ij} = (g^{ij}) = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \frac{1}{W^2} \begin{bmatrix} g_{22} & -g_{21} \\ -g_{12} & g_{11} \end{bmatrix}. \tag{8}$$

It follows from (8),

$$|g^{ij}| = \frac{1}{W^2}|g_{ij}|. \tag{9}$$

Proposition 6. *The Gaussian curvature K is given by*

$$K = k_1 k_2 = |b_i^j| = |g^{jk} b_{ki}|$$

and the mean curvature H is given by

$$2H = k_1 + k_2 = b_{ij} g^{ij}.$$

Proof. See [2, p. 10]

It follows from Proposition 6 that we can define the Gaussian curvature as the determinant of the shape operator S i.e. $K = |S| = \frac{|II|}{|I|}$ the ratio of the determinant of the second fundamental form to the determinant of first fundamental form. Similarly, we can define the mean curvature in terms of the trace of the shape operator, $H = \frac{1}{2} tr(S)$.

Theorem 6. *The surface $\boldsymbol{\sigma} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a critical point of the area function A if and only if its mean curvature H is identically zero.*

Proof. Choose a bounded domain $D \subset U$ and a differentiable function $\phi : \bar{D} \to \mathbb{R}$ where $\bar{D} = D \cup \partial D$ and $(u, v) \in D$. Define

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + t\phi\mathbf{N} \tag{10}$$

for some fixed $t \in (-\epsilon, \epsilon)$. Let $\tilde{g}$ and g be the metric tensor associated with $\tilde{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}$, respectively. From this the area function is given by,

$$A(\tilde{\boldsymbol{\sigma}}) = \int_{\bar{D}} \sqrt{\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}^2}\, du\, dv. \tag{11}$$

From a direct computation we have

$$\begin{aligned}
\tilde{g}_{11} &= \langle \tilde{\boldsymbol{\sigma}}_u | \tilde{\boldsymbol{\sigma}}_u \rangle \\
&= \langle \boldsymbol{\sigma}_u + t\phi_u \boldsymbol{N} + t\phi \boldsymbol{N}_u | \tilde{\boldsymbol{\sigma}}_u \rangle \\
&= \boldsymbol{\sigma}_u^2 + 2t\phi \langle \boldsymbol{N}_u | \boldsymbol{\sigma}_u \rangle + O(t^2) \\
&= g_{11} + 2t\phi b_{11} + O(t^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{g}_{22} &= \langle \tilde{\boldsymbol{\sigma}}_v | \tilde{\boldsymbol{\sigma}}_v \rangle \\
&= \langle \boldsymbol{\sigma}_v + t\phi_v \boldsymbol{N} + t\phi \boldsymbol{N}_v | \tilde{\boldsymbol{\sigma}})v \rangle \\
&= \boldsymbol{\sigma}_v^2 + 2t\phi \langle \boldsymbol{N}_v | \boldsymbol{\sigma}_v \rangle + O(t^2) \\
&= g_{22} + 2t\phi b_{22} + O(t^2),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{g}_{12} &= \langle \tilde{\boldsymbol{\sigma}}_u | \tilde{\boldsymbol{\sigma}}_u \rangle \\
&= \langle \boldsymbol{\sigma}_u + t\phi_u \boldsymbol{N} + t\phi \boldsymbol{N}_u | \boldsymbol{\sigma}_v + t\phi_v \boldsymbol{N} + t\phi \boldsymbol{N}_v \rangle \\
&= g_{12} + t\phi(\langle \boldsymbol{N}_u | \boldsymbol{\sigma}_v \rangle + \langle \boldsymbol{N}_v | \boldsymbol{\sigma}_u \rangle) + O(t^2).
\end{aligned}$$

Now,

$$\tilde{g}_{11}\tilde{g}_{22} = g_{11}g_{22} + t\phi(g_{11}b_{22} + g_{22}b_{11}) + O(t^2),$$

$$\tilde{g}_{12}\tilde{g}_{21} = g_{12}g_{21} + 2t\phi(g_{12}b_{12} + g_{21}b_{21}) + O(t^2),$$

and so,

$$\begin{aligned}
\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}^2 &= g_{11}g_{22} - g_{12}g_{21} + 2t\phi(g_{11}b_{22} + g_{22}b_{11} - g_{12}b_{12} + g_{21}b_{21}) + O(t^2) \\
&= g_{11}g_{22} - g_{12}^2 + 2t\phi W^2 g^{ij} b_{ij} + O(t^2) \\
&= g_{11}g_{22} - g_{12}^2 + 4t\phi W^2 H + O(t^2)
\end{aligned}$$

So the area function is given by, using $W^2 = g_{11}g_{22} - g_{12}^2$,

$$A(\tilde{\boldsymbol{\sigma}}) = \int_{\bar{D}} \sqrt{W^2(1 + 4t\phi H)}\ du\ dv + O(t^2)$$

and using the expansion $\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2)$, then,

$$\begin{aligned}
A(\tilde{\boldsymbol{\sigma}}) &= \int_{\bar{D}} W(1 + 2t\phi H)\ du\ dv + O(t^2) \\
&= \int_{\bar{D}} W\ du\ dv + 2t \int_{\bar{D}} HW\phi\ du\ dv + O(t^2)
\end{aligned}$$

Ergo, the derivative with respect to t,

$$D[A(\boldsymbol{\sigma})](\phi) = \frac{d}{dt} A(\tilde{\boldsymbol{\sigma}}) = \int_{\bar{D}} 2\phi HW\ du\ dv. \tag{12}$$

It should be clear that if we want $D[A(\boldsymbol{\sigma})](\phi) = 0$ then $\int_{\bar{D}} 2\phi HW\ du\ dv = 0$. Since this is true for every ϕ, we can choose $\phi = H$ and so

$$\int_{\bar{D}} H^2 W\ du\ dv = 0.$$

However, since $H^2 \geq 0$ everywhere we must have $H = 0$. ■

Definition 7. ***Minimal surface.*** *We say $\boldsymbol{\sigma}$ is a minimal surface if it has zero mean curvature.*

4 Second Variation

Proposition 7. $c_{ij} - 2Hb_{ij} + Kg_{ij} = 0$. *This equation is sometimes written as an equation between bilinear forms on the tangent space: $III = 2HII + KI = 0$.*

Proof. See [2, p. 57].

We now derive and expression for the stability of a surface.

Theorem 7. *Suppose $\boldsymbol{\sigma} = \boldsymbol{\sigma}(u, v)$ is regular in the closure of Ω, then*

$$D^2[A(\boldsymbol{\sigma})](\phi) = \int_{\Omega} (|\nabla\phi|^2 + 2KW\phi^2) dudv.$$

Proof. We begin as before,

$$\tilde{\boldsymbol{\sigma}}_i = \boldsymbol{\sigma}_i + t\phi_i \boldsymbol{N} + t\phi \boldsymbol{N}_i$$

and so,

$$\begin{aligned}
\tilde{g}_{ij} &= \langle \boldsymbol{\sigma}_i | \boldsymbol{\sigma}_j \rangle \\
&= \langle \boldsymbol{\sigma}_i + t\phi_i \boldsymbol{N} + t\phi \boldsymbol{N}_i | \boldsymbol{\sigma}_j + t\phi_j \boldsymbol{N} + t\phi \boldsymbol{N}_j \rangle \\
&= g_{ij} - 2t\phi \langle \boldsymbol{N}_i | \boldsymbol{\sigma}_j \rangle + t^2(\phi_i\phi_j + \phi^2 \langle \boldsymbol{N}_i | \boldsymbol{N}_j \rangle) \\
&= g_{ij} - 2t\phi b_{ij} + t^2(\phi_i\phi_j + \phi^2 c_{ij}).
\end{aligned}$$

We write $|g_{ij}| = g_{11}g_{22} - g_{12}^2 = W^2$ and by Proposition 6

$$2HW^2 = b_{ij} g^{ij} |g_{ij}| = b_{ij} \frac{g_{ij}}{|g_{ij}|} |g_{ij}| = b_{ij} g_{ij}, \tag{13}$$

which expands to $b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}$. Moreover,

$$K|g_{jm}| = |b_i^j||g_{jm}| = |g^{jk}b_{ki}||g_{jm}| = |b_{im}| \tag{14}$$

which expands to $b_{11}b_{22} - b_{12}^2$. We used lowering and rising index $A^i = g^{ij}A_j$ and $A_i = g_{ij}A^j$. Now,

$$\begin{aligned}
|\tilde{g}_{ij}| =& \tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}^2 \\
=& |g_{ij}| - 2t\phi(b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}) + t^2(\phi_u^2 g_{22}\phi_v^2 g_{11} - 2\phi_u\phi_v g_{12}) \\
&+ t^2\phi^2(4b_{11}b_{22} - 4b_{12}^2 + g_{11}c_{22} + g_{22}c_{11} - 2g_{12}c_{12}) + O(t^3) \\
=& |g_{ij}| - 4t\phi H|g_{ij}| + t^2\phi_i\phi_j g^{ij} + t^2\phi^2(4K|g_{ij}| + g^{ij}c_{ij}) + O(t^3).
\end{aligned}$$

Hence, by Proposition 7 and using $2|g_{ij}| = g^{ij}g_{ij}$,

$$g^{ij}c_{ij} = 2Hg^{ij}b_{ij} - Kg^{ij}g_{ij} = 4H^2|g_{ij}| - 2K|g_{ij}|. \tag{15}$$

Ergo,

$$\begin{aligned} |\tilde{g}_{ij}| =& |g_{ij}| - 4t\phi H|g_{ij}| + t^2\phi_i\phi_j g^{ij} + t^2\phi^2(4K|g_{ij}| + 4H^2|g_{ij}| - 2K|g_{ij}|) + O(t^3) \\ =& |g_{ij}|[1 - 4t\phi H + t^2(\phi_i\phi_j + \phi^2(4H^2 + 2K))] + O(t^3). \end{aligned}$$

Consider the first Beltrami operator $\langle \nabla_{e_i}\phi | \nabla_{e_j}\phi \rangle = \phi_i\phi_j g^{ij} = |g_{ij}||\nabla_{\boldsymbol{\sigma}}\phi|^2$ where $\nabla_{\boldsymbol{\sigma}}$ denotes the covariant derivative in the ambient space of $\boldsymbol{\sigma}$. Together with the expansion $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + O(x^3) = 1 + \frac{\alpha}{2}t + (\frac{\beta}{2} - \frac{\alpha^2}{8})t^2 + O(t^3)$ for small $x = t\alpha + t^2\beta$, we have,

$$\sqrt{|\tilde{g}_{ij}|} = \sqrt{|g_{ij}|}[1 - 2t\phi H + t^2(\frac{1}{2}|\nabla_g\phi|^2 + K\phi^2 + 2H^2\phi^2 - 2H^2\phi^2)] + O(t^3)$$

Now suppose $\boldsymbol{\sigma}$ is regular in the closure of Ω. The area is given by

$$A(\tilde{\boldsymbol{\sigma}}) = \int_\Omega \sqrt{|\tilde{g}_{ij}|}\, du\, dv$$

and so when we differentiate $\sqrt{|\tilde{g}_{ij}|}$

$$\begin{aligned} D^2[A(\boldsymbol{\sigma})](\phi) &= \frac{d^2}{dt^2}A(\tilde{\boldsymbol{\sigma}}) \\ &= \int_\Omega (|\nabla_{\boldsymbol{\sigma}}\phi|^2 + 2K\phi^2)\sqrt{|g_{ij}|}\, du\, dv \\ &= \int_\Omega (|\nabla_{\boldsymbol{\sigma}}\phi|^2 + 2K\phi^2)\, dA \\ &= \int_\Omega (|\nabla\phi|^2 + 2KW\phi^2)\, du\, dv. \end{aligned}$$

■

5 Exercises

Exercise 2. Find the associated holomorphic function for the parametrization of the catenoid $\boldsymbol{\sigma}(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$.

Proof.

$$\begin{aligned} \phi(z) &= \boldsymbol{\sigma}_u - \mathfrak{i}\boldsymbol{\sigma}_v \\ &= (\sinh u \cos v + \mathfrak{i}\cosh u \sin v, \sinh u \sin v - \mathfrak{i}\cosh u \cos v, 1) \\ &= (\sinh(u + \mathfrak{i}v), -\mathfrak{i}\cosh(u + \mathfrak{i}v), 1) \\ &= (\sinh z, -\mathfrak{i}\cosh z, 1). \end{aligned}$$

■

Proposition 8. *Show that there are no compact minimal surfaces.*

Proof. Let $\boldsymbol{\sigma} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. If M is compact then (refer to Exercise 1),

$$\int_M \langle \nabla x_i | \nabla x_i \rangle = \int_{\partial M} x_i \Delta x_i = 0$$

thus, $\|\nabla x_i\| = 0$ imply that x_i is constant. ■

Proof. If Σ is compact, then by Proposition 2 there exist at point $p \in \Sigma$ such that the Gaussian curvature $K > 0$. Now if Σ is connected, for umbilic points, then Σ is an open subset of a plane or sphere. However, if Σ is minimal, then $H = 0$ implies that $k_1 = -k_2$. Ergo, $K = -k_1^2 \leq 0$ and for umbilic points $K = 0$. ■

Proof. Let $\boldsymbol{\sigma}$ be a minimal surface with a holomorphic function ϕ, as defined in Proposition 4. Then each coordinate, ϕ_k, is holomorphic. If $\boldsymbol{\sigma}$ is compact, then ϕ_k will have a maximum and thus ϕ_k is constant. However, this is impossible as ϕ is non-constant. ■

Exercise 3. Let Σ be a regular surface without umbilical points. Prove that Σ is a minimal surface if and only if the Gauss map $N : S \to S^2$ satisfies, for all $\in S$ and all $w_1,\ w_2 \in T_p(s)$,

$$\langle dN_p(w_1) | dN_p(w_2) \rangle_{N(p)} = \lambda(p) \langle w_1 | w_2 \rangle_p$$

Proof. Since Σ is minimal then $H = 0$ and thus the principal curvature k_1, k_2 are such that $k_1 = -k_2$. Since Σ has no umbilical points then k_1 and k_2 are distinct (since they are not both zero) and are the eigenvalues to dN_p with eigenvectors e_1 and e_2 that span $T_p(\Sigma)$, respectively. Hence, for $w_1, w_2 \in T_p(\Sigma)$, $w_1 = (a_1, a_2)$ and $w_2 = (b_1.b_2)$. By a direct computation,

$$\langle dN_p(w_1) | dN_p(w_2) \rangle_{N(p)} = a_1 b_1 k_1^2 + a_2 b_2 k_1^2$$

and

$$\langle w_1 | w_2 \rangle_p = a_1 b_1 + a_2 b_2$$

since $\langle e_i | e_j \rangle = \delta_{ij}$. Therefore, $\lambda(p) = k_1^2$.

Conversely, let k_1 and k_2 be the principal curvature (and eigenvalues) of dN_p. Since there are no umbilical points then $k_1, k_2 \neq 0$. So,

$$\langle dN_p(e_1) | dN_p(e_1) \rangle_{N(p)} = \lambda(p) \langle e_1 | e_1 \rangle_p$$

and so, $k_1^2 = \lambda(p)$ and similarly $k_2^2 = \lambda(p)$ (for e_2). It follows that, $\lambda(p) = k_1^2 = k_2^2$ and thus $k_1 = -k_2$ since there are no umbilical points. Ergo, $H = 0$. ■

Exercise 4. Let $\boldsymbol{\sigma} : \Sigma \to S^2$ be a parametrization of the unit sphere S^2 by $(\theta, \phi) \in \Sigma$, where θ is the colatitude and ϕ is the arc length of the parallel determined by θ. Consider a neighborhood V of a point p of the minimal surface S in part such that $N : S \to S^2$ restricted to V is diffeomorphism (since $K(p) = det(dN_p) \neq 0$, such a V exist by the inverse function theorem). Prove that the parametrization $y = N^{-1} \circ \boldsymbol{\sigma} : \Sigma \to S$ is isothermal (conformal). This gives a way of introducing isothermal (conformal) parametrization on minimal surfaces without planar points.

Proof. For $p \in S^2$, let $\pi : S^2/\{p\} \to \mathbb{R}^2$ be a stereographic projection. The inverse will give an conformal parametrization of the sphere ($g_{11} = g_{22}$ and $g_{12} = 0$). If S is minimal without umbilic points then $y = N^{-1} \circ \boldsymbol{\sigma}$ gives a local parametrization of S and $\boldsymbol{\sigma}_u = dN(y_u)$ and $\boldsymbol{\sigma}_v = dN(y_v)$. Now, by Exercise 3,

$$\langle y_i | y_j \rangle = \lambda(p)^{-1} \langle \boldsymbol{\sigma}_i | \boldsymbol{\sigma}_j \rangle$$

Ergo, $\langle y_u | y_v \rangle = \langle y_v | y_u \rangle$ and $\langle y_u | y_v \rangle = 0$. ■

6 Weierstrass-Enneper Representation, Scherk's Surface, and Stability

6.1 Weierstrass-Enneper Representation

Theorem 2 allows us to study the behaviour of minimal surfaces in the complex plane.

Theorem 8. ***Weierstrass-Enneper Representation.*** *Let $f(z)$ be a holomorphic function on an open set $U \subset \mathbb{C}$, not identically zero, and let $g(z)$ be a meromorphic function on U such that if $z_0 \in U$ is a m^{th} pole of of g, for $m \geq 1$, then z_0 is also a zero of f of order greater than $2m$. Then,*

$$\phi = (\frac{1}{2}f(1-g^2),\ \frac{\mathrm{i}}{2}f(1+g^2),\ fg)$$

and it satisfies Theorem 2. Conversely, every holomorphic function ϕ satisfying these conditions arises in this way.

Proof. See [7, p. 329]

Remark 2. Observe that if $\phi_1 - \mathrm{i}\phi_2$ is not identically zero, we can define

$$f = \phi_1 - \mathrm{i}\phi_2 \text{ and } g = \frac{\phi_3}{\phi_1 - \mathrm{i}\phi_2}.$$

Since ϕ is holomorphic, f will also be holomorphic and g meromorphic. Furthermore noting that (with condition (b.) of Theorem 2)

$$\begin{aligned}(\phi_1 + \mathrm{i}\phi_2)(\phi_1 - \mathrm{i}\phi_2) &= \phi_1^2 + \phi_2^2 \\ &= -\phi_3^2\end{aligned}$$

and so,

$$\phi_1 + \mathrm{i}\phi_2 = -fg^2.$$

It is clear from construction that fg^2 is holomorphic. If on the other hand, $\phi_1 - \mathrm{i}\phi_2 = 0$, we can choose $f = \phi_1 + \mathrm{i}\phi_2$ and $g = \frac{\phi_3}{\phi_1 + \mathrm{i}\phi_2}$. That is because both $\phi_1 - \mathrm{i}\phi_2$ and $\phi_1 + \mathrm{i}\phi_2$ cannot be zero because if that were the case we would have $\phi_1 = \phi_2 = 0$ and so $\phi_3 = 0$ which will violate condition (a.) of Theorem 2.

Exercise 5. Using the Weierstrass-Enneper representation we can calculate the Gaussian curvature in terms of the f and g,

$$K = \frac{-16|\frac{dg}{dz}|^2}{|f|^2(1+|g|^2)^4}.$$

Proof. Recall that $\phi(z) = \boldsymbol{\sigma}_u - \mathrm{i}\boldsymbol{\sigma}_v$. From this we can define the following,

$$\boldsymbol{\sigma}_u = \frac{1}{2}(\phi + \bar{\phi}) \text{ and } \boldsymbol{\sigma}_v = \frac{1}{2\mathrm{i}}(\bar{\phi} - \phi).$$

It should be obvious that $\langle \phi | \phi \rangle = \langle \bar{\phi} | \bar{\phi} \rangle = 0$. Moreover,

$$\begin{aligned}\langle\phi|\bar{\phi}\rangle &= f\bar{f}g\bar{g} + \frac{1}{4}(f\bar{f}(1-g^2)(1-\bar{g}^2)) + f\bar{f}(1+g^2)(1+\bar{g}^2)\\ &= \frac{1}{2}f\bar{f}(1+g\bar{g})^2\\ &= \frac{1}{2}|f|^2(1+|g|^2)^2.\end{aligned}$$

Thus the first fundamental form is given by

$$\begin{aligned}I &= g_{11}(du^2+dv^2)\\ &= \langle\boldsymbol{\sigma}_u|\boldsymbol{\sigma}_v\rangle(du^2+dv^2)\\ &= \frac{1}{2}\langle\phi|\bar{\phi}\rangle(du^2+dv^2)\\ &= \frac{1}{4}|f|^2(1+|g|^2)^2(du^2+dv^2)\end{aligned}$$

We now compute the second fundamental form.

$$\begin{aligned}\boldsymbol{\sigma}_u\wedge\boldsymbol{\sigma}_v &= \frac{1}{4\mathrm{i}}(\phi+\bar{\phi})\wedge(\bar{\phi}-\phi)\\ &= \frac{1}{2\mathrm{i}}(\phi\wedge\bar{\phi})\\ &= \frac{1}{2\mathrm{i}}\{\frac{\mathrm{i}}{2}|f|^2(\bar{g}+g^2\bar{g}+g(1+\bar{g}^2)),\ \frac{1}{2}|f|^2(g-\bar{g}+g^2\bar{g}-g\bar{g}^2),\ \frac{\mathrm{i}}{2}|f|^2(|g|^4-1)\}.\end{aligned}$$

and the norm is given by

$$\begin{aligned}\|\boldsymbol{\sigma}_u\wedge\boldsymbol{\sigma}_v\|^2 &= \frac{-1}{4}\langle(\phi\wedge\bar{\phi})|(\phi\wedge\bar{\phi})\rangle\\ &= \frac{-1}{4}(\langle\phi|\phi\rangle\langle\bar{\phi}|\bar{\phi}\rangle-\langle\phi|\bar{\phi}\rangle^2)\\ &= \frac{1}{4}\langle\phi|\bar{\phi}\rangle^2.\end{aligned}$$

From this we can find our normal

$$\begin{aligned}\mathbf{N} &= \frac{1}{\mathrm{i}}\frac{\phi\wedge\bar{\phi}}{\langle\phi|\bar{\phi}\rangle}\\ &= \frac{1}{1+|g|^2}\{g+\bar{g},\ \mathrm{i}(\bar{g}-g),\ |g|^2-1\}.\end{aligned}$$

The coefficients of the second fundamental form are given by (using Proposition 5),

$$\begin{aligned}b_{11} &= \langle-\boldsymbol{\sigma}_u|\mathbf{N}_u\rangle\\ &= -\frac{1}{2}(f\ g'+\bar{f}\ \bar{g}')\\ b_{21} &= \langle-\boldsymbol{\sigma}_u|\mathbf{N}_v\rangle\\ &= -\mathrm{i}(f\ g'-\bar{f}\ \bar{g}')\end{aligned}$$

and thus,

$$II = \frac{-1}{2}((f\ g' + \bar{f}\ \bar{g}')(du^2 + dv^2) + 2\mathfrak{i}(f\ g' - \bar{f}\ \bar{g}')du\ dv).$$

From this we can easily easily deduce,

$$K = \frac{|II|}{|I|} = \frac{-16|g'|^2}{|f|^2(1+|g|^2)^4}$$

as desired. ■

Exercise 6. We prove the claim made in [8, p. 114]

$$K = -\frac{1}{2}\Delta \log g_{11}$$

where $\Delta = \frac{1}{g_{11}}(\frac{\partial^2}{du^2} + \frac{\partial^2}{dv^2})$.

Proof. Recall, from above, that $g_{11} = \frac{1}{2}\langle\phi|\bar{\phi}\rangle = \frac{1}{4}|f|^2(1+|g|^2)^2$. Using chain rule we have,

$$\begin{aligned}\frac{\partial \log g_{11}}{\partial u} &= \frac{\partial \log g_{11}}{\partial g_{11}}\frac{\partial g_{11}}{\partial u}\\ &= \frac{\frac{\partial g_{11}}{\partial u}}{g_{11}}\end{aligned}$$

(similarly when we take the partial derivative with respect to v) and thus we get the following expression,

$$\begin{aligned}-\frac{1}{2}\Delta \log g_{11} &= -\frac{1}{2\ g_{11}}(\frac{\partial^2 g_{11}}{\partial u^2} + \frac{\partial^2 g_{11}}{\partial v^2})\\ &= -\frac{1}{2\ g_{11}}(\frac{\partial}{\partial u}(\frac{\frac{\partial g_{11}}{\partial u}}{g_{11}}) + \frac{\partial}{\partial v}(\frac{\frac{\partial g_{11}}{\partial v}}{g_{11}})).\end{aligned}$$

Now we calculate the partials derivatives in the direction of u and v (recall Proposition 5),

$$\begin{aligned}\frac{\partial g_{11}}{\partial u} &= \frac{1}{4}(f'\bar{f}(1+|g|^2)^2 + f\bar{f}'(1+|g|^2)^2 + 2|f|^2(1+|g|)(g'\bar{g} + g\bar{g}'))\\ &= \frac{1}{4}(1+|g|^2)(f'\bar{f}(1+|g|^2) + f\bar{f}'(1+|g|^2) + 2|f|^2(g'\bar{g} + g\bar{g}'))\end{aligned}$$

and

$$\frac{\partial g_{11}}{\partial v} = \frac{\mathfrak{i}}{4}(1+|g|^2)(f'\bar{f}(1+|g|^2) - f(\bar{f}'(1+|g|^2) + 2\bar{f}(g\bar{g}' - \bar{g}g'))).$$

Hence,

$$\frac{\frac{\partial g_{11}}{\partial u}}{g_{11}} = \frac{f'}{f} + \frac{\bar{f}'}{\bar{f}} + 2\frac{(g'\bar{g} + g\bar{g}')}{(1+|g|^2)}$$

and

$$\frac{\frac{\partial g_{11}}{\partial v}}{g_{11}} = \mathrm{i}(\frac{f'}{f} - \frac{\bar{f}'}{\bar{f}} - 2\frac{(g\bar{g}' - \bar{g}g')}{1+|g|^2}).$$

We differentiate the expression above in the direction of u and v, respectively,

$$\frac{\partial}{\partial u}(\frac{\frac{\partial g_{11}}{\partial u}}{g_{11}}) = -\frac{\left(\bar{f}'\right)^2}{\bar{f}^2} + \frac{\bar{f}''}{\bar{f}} - \frac{2\left(\bar{g}g' + g\bar{g}'\right)^2}{\left(g\bar{g}+1\right)^2} + \frac{2\left(\bar{g}g'' + 2g'\bar{g}' + g\bar{g}''\right)}{g\bar{g}+1} + \frac{f''}{f} - \frac{\left(f'\right)^2}{f^2}$$

and

$$\frac{\partial}{\partial u}(\frac{\frac{\partial g_{11}}{\partial v}}{g_{11}}) = \frac{\left(\bar{f}'\right)^2}{\bar{f}^2} - \frac{\bar{f}''}{\bar{f}} + \frac{2\left(g\bar{g}' - \bar{g}g'\right)^2}{\left(g\bar{g}+1\right)^2} - \frac{2\left(\bar{g}g'' - 2g'\bar{g}' + g\bar{g}''\right)}{g\bar{g}+1} - \frac{f''}{f} + \frac{\left(f'\right)^2}{f^2}.$$

Ergo,

$$\begin{aligned}
-\frac{1}{2}\Delta \log g_{11} &= -\frac{1}{2\ g_{11}}(\frac{\partial}{\partial u}(\frac{\frac{\partial g_{11}}{\partial u}}{g_{11}}) + \frac{\partial}{\partial v}(\frac{\frac{\partial g_{11}}{\partial v}}{g_{11}})) \\
&= -\frac{1}{2\ g_{11}}(\frac{8g'\bar{g}'}{(1+\bar{g}g)^2}) \\
&= \frac{-16|g'|^2}{|f|^2(1+|g|^2)^4} \\
&= -(\frac{4|g'|}{|f|(1+|g|^2)^2})^2 \\
&= K
\end{aligned}$$

which is the expression we had for the Gaussian Curvature using the Weierstrass representation. ■

Remark 3. The reader should be interested in the last result as it shows that the Gaussian curvature of a minimal surface depends only on the first fundamental form.

Exercise 7. Scherk's surface. Scherek's surface can be given by the parametric equation

$$\boldsymbol{\sigma}(u,v) = (arg\frac{z+\mathrm{i}}{z-\mathrm{i}},\ arg\frac{z+1}{z-1},\ log|\frac{z^2+1}{z^2-1}|)$$

with $z \neq \{\pm 1, \pm \mathrm{i}\}$ and where $\arg z$ is the angle that the real axis makes with z. Show that it is a minimal surface.

Proof. We begin with the tedious calculations. Let $z = u + \mathrm{i}v$,

$$(z-i)(\overline{z-\mathrm{i}}) = u^2 + (v-1)^2.$$

We can compute,

$$\frac{z+\mathrm{i}}{z-\mathrm{i}}\frac{\overline{z-\mathrm{i}}}{\overline{z-\mathrm{i}}} = \frac{u^2+v^2-1+\mathrm{i}2u}{u^2+(v-1)^2}$$

and thus

$$\arg\frac{z+\mathrm{i}}{z-\mathrm{i}} = \arctan(\frac{2u}{u^2+v^2-1}).$$

Similarly,

$$\begin{aligned}\arg\frac{z+1}{z-1} &= \arg\frac{z+1}{z-1}\frac{\overline{z-1}}{\overline{z-1}}\\ &= \arg\frac{u^2+(v-\mathfrak{i})^2}{1-2u+u^2+v^2}\\ &= \arctan(\frac{-2v}{u^2+v^2-1}).\end{aligned}$$

Now,

$$\begin{aligned}(z^2-1)(\overline{z^2-1}) &= ((u^2-v^2)+\mathfrak{i}(2uv)-1)((u^2-v^2)-\mathfrak{i}(2uv)-1)\\ &= u^4+2u^2(v^2-1)+(1+v^2)^2,\\ \frac{z^2+1}{z^2-1}\frac{\overline{z^2-1}}{\overline{z^2-1}} &= \frac{v^4+2(uv)^2+u^4-1-\mathfrak{i}(4uv)}{u^4+2u^2(v^2-1)+(1+v^2)^2}\\ &= \frac{v^4+2(uv)^2+u^4-1-\mathfrak{i}(4uv)}{(u^2-v^2-1)^2+4u^2v^2}\end{aligned}$$

and so

$$|\frac{z^2+1}{z^2-1}| = \sqrt{(\frac{v^4+2(uv)^2+u^4-1}{(u^2-v^2-1)^2+4u^2v^2})^2+(\frac{-4uv}{(u^2-v^2-1)^2+4u^2v^2})^2},$$

which implies,

$$\begin{aligned}\log|\frac{z^2+1}{z^2-1}| &= \frac{1}{2}\log(\frac{(v^4+2(uv)^2+u^4-1)^2+(4uv)^2}{((u^2-v^2-1)^2+4u^2v^2)^2})\\ &= \frac{1}{2}\log(\frac{u^2+(v^2-1)^2+2u^2(1+v^2)}{u^4+2u^2(v^2-1)+(1+v^2)^2})\\ &= \frac{1}{2}\log(\frac{(u^2-v^2+1)^2+4u^2v^2}{(u^2-v^2-1)^2+4u^2v^2}).\end{aligned}$$

Thus we can write our surface as follows,

$$\boldsymbol{\sigma}(u,v) = (\arctan(\frac{2u}{u^2+v^2-1}),\ \arctan(\frac{-2v}{u^2+v^2-1}),\ \frac{1}{2}\log(\frac{(u^2-v^2+1)^2+4u^2v^2}{(u^2-v^2-1)^2+4u^2v^2}))$$

Proposition 4 allows us to construct a complex function $\phi_k = (\sigma^k)_u - \mathfrak{i}(\sigma^k)_v$

$$\phi_1 = -\frac{2}{1+z^2},\ \phi_2 = -\frac{2\mathfrak{i}}{1-z^2},\ \text{and } \phi_3 = \frac{4z}{1-z^4}.$$

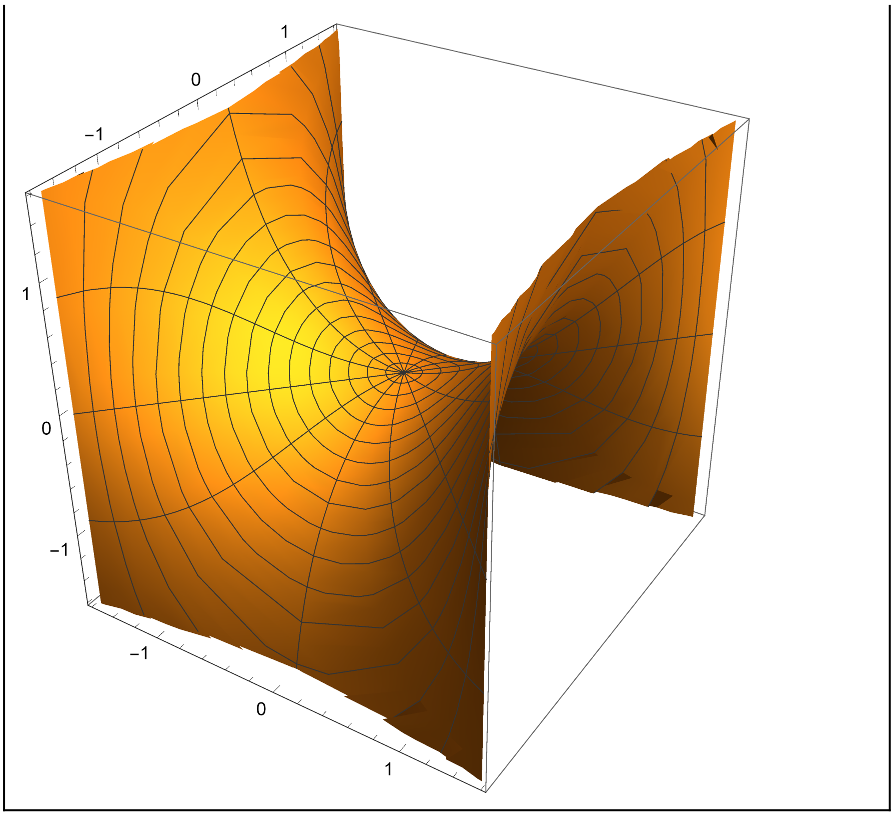
It clear that $\sum \phi_k^2 = 0$ and thus, by Theorem 2, $\sigma(u,v)$ is a minimal surface. ■

Example 1. Weierstrass representation of Scherks surface. Using our construction of f and g from Remark 2 and the ϕ_k from Exercise 7,

$$f = \frac{4}{z^4-1} \text{ and } g = -z.$$

Example 2. Plotting with Mathematica. Using construction from Theorem 2 and the f and g found above, we can plot one period using the following Mathematica code:

```
Clear[f, g, x1, y1, z1];
f[w_] = 4/(w^4 - 1);
g[w_] = -w;
x1[w_] = Integrate[f[w]*((1 - g[w]^2)/2), w];
y1[w_] = Integrate[I*f[w]*((1 + g[w]^2)/2), w];
z1[w_] = Integrate[f[w]*g[w], w];
ParametricPlot3D[
 Re[{x1[r*Exp[I*a]], y1[r*Exp[I*a]], z1[r*Exp[I*a]]}], {r, 0, 1},
   {a, -Pi, Pi}]
```

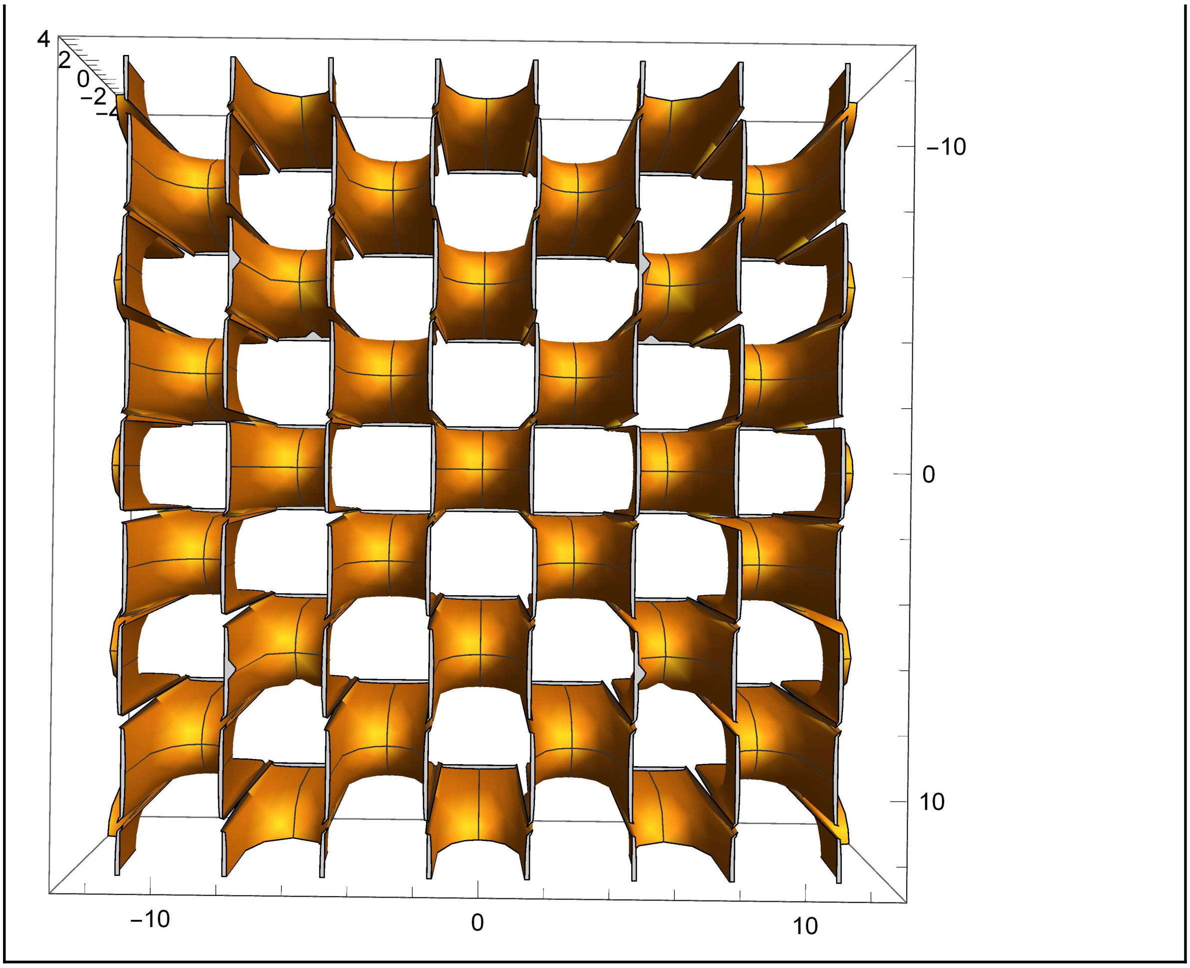


Example 3. Plotting with Mathematica 2. Another parametrization is given by $w : (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$

$$w(x, y) = \log(\frac{\cos y}{\cos x}).$$

It easily be seen that this function is periodic, namely $\frac{\cos y + n\pi}{\cos x + n\pi} = \frac{\cos y}{\cos x}$. We plot multiple periods,

```
Plot3D[Log[Cos[x]/Cos[y]], {x, -4 Pi, 4 Pi}, {y, -4 Pi, 4 Pi}]
```



6.2 the Laplacian

Let M be an n-dimensional connected, C^∞, Riemannian manifold.

Definition 8. ***Directional Derivative*** *Let $p \in M$ and let $f \in C^1(\mathbb{R})$ defined in the neighborhood of p then for each $\eta \in TM_p$, the tangent space to M at point p, is associated a directional derivative of f at p in the direction η, which we will denote by ηf and define by,*

$$\eta f = (f \circ w)'(0)$$

where $w(t)$ is any path in M that satisfies $w(0) = p$ and $w'(0) = \eta$.

Definition 9. ***Gradient*** *Given $f \in C^k(\mathbb{R})$ on M the gradient of f, denoted by* grad$f$ *or ∇f, is defined to be the vector field on M in which*

$$\langle \mathrm{grad} f | \eta \rangle = \eta f$$

for all $\eta \in TM$.

Definition 10. ***Connection and Covariant Derivative*** *In order to take a derivative a vector field we include a connection, a rule which associates each $p \in M$, $\eta \in TM_p$, and C^1 vector field X defined on a neighborhood of p, the vector $\nabla_\eta X \in TM_p$ satisfying,*

$$\nabla_\eta (X + Y) = \nabla_\eta X + \nabla_\eta Y$$

and

$$\nabla_\eta (fX) = (\eta f)X(p) + f(p)\nabla_\eta X$$

where X, Y are C^1 vector fields and f is a $C^1(\mathbb{R})$ function. $\nabla_\eta X$ is referred to as the covariant derivative of X with respect to η.

Definition 11. ***Divergence*** *Let X be a C^k vector field on M, the divergence of X, $\mathrm{div}X$, is defined as*

$$(\mathrm{div}X)(p) = tr(\eta \to \nabla_\eta X)$$

where tr is the trace.

Example 4. **Levi-Civita connection.** The Lie bracket of X and Y is given by $[X, Y] = \nabla_X Y - \nabla_Y X$ (for proof see [6, pp. 25-26]) and

$$\eta\langle X|Y\rangle = \langle \nabla_\eta X|Y\rangle + \langle X|\nabla_\eta Y\rangle.$$

Moreover, we can see that

$$\mathrm{div}(X + Y) = \mathrm{div}X + \mathrm{div}Y$$

and

$$\mathrm{div}(fX) = f\mathrm{div}X + \langle \mathrm{grad}f|X\rangle$$

Definition 12. ***Laplacian.*** *For a function $f \in C^k$ on M we define the Laplacian of f, Δf, by*

$$\Delta f = \mathrm{div}(\mathrm{grad}f).$$

Example 5. We can see that

$$\begin{aligned} \Delta(f + h) &= \Delta f + \Delta h \\ \mathrm{div}(h(\mathrm{grad}f)) &= h(\Delta f) + \langle \mathrm{grad}h|\mathrm{grad}f\rangle \\ \Delta(fh) &= h(\Delta f) + 2\langle \mathrm{grad}f|\mathrm{grad}h\rangle + f(\Delta h). \end{aligned}$$

Exercise 8. We wish to compute an expression of $\mathrm{div}f$ in local coordinates.

Proof. In a local coordinate the metric is given by $g = g_{ij}dx^i dx^j$. If $X = a^i\partial_i$ and $Y = b^j\partial_j$, then

$$g(X, Y) = g_{ij}a^i b^j$$

and so in coordinates

$$\begin{aligned} \mathrm{div}X &= tr(\mathrm{grad}X) \\ &= dx^i(\nabla_{\partial_i} X) \end{aligned}$$

and with respect of a orthonormal basis e_i

$$tr(\mathrm{grad}X) = \sum g(\nabla_{e_i} X,\ e_i).$$

■

Proposition 9. ***Laplacian in General Manifolds***

$$\Delta f = \frac{1}{\sqrt{|(g_{jk})|}} \sum_{j,k} \partial_j (g^{jk} \sqrt{|(g_{jk})|} \partial_k f).$$

where (g_{jk}) *the metric on* M.

Proof. See [4, pp. 4-5]

6.3 Stability

We wish to study the stability of Scherk's surface. For this task we have derived and expression we could use to analyze this problem, namely Theorem 7,

$$\begin{aligned} D^2[A(\boldsymbol{\sigma})](\phi) &= \int_{\Omega} (|\nabla\phi|^2 + 2KW\phi^2)\ du\ dv \\ &= \int_{\Omega} (\phi\ \Delta\phi + 2KW\phi^2)\ du\ dv \\ &= \int_{\Omega} (\phi\ \frac{1}{\sqrt{|(g_{jk})|}} \sum_{j,k} \partial_j (g^{jk} \sqrt{|(g_{jk})|} \partial_k \phi) + 2|g^{jk} b_{ki}| \sqrt{|(g_{ij})|} \phi^2)\ du\ dv \\ &= \int_{\Omega} (\phi\ \frac{1}{\sqrt{|(g_{jk})|}} \sum_{j,k} \partial_j (g^{jk} \sqrt{|(g_{jk})|} \partial_k \phi) + 2|g^{jk} b_{ki}| \sqrt{|(g_{ij})|} \phi^2)\ du\ dv. \end{aligned}$$

Exercise 9. We show that $\phi = K$ destabilizes the integral.

Proof. We initially tried running the following code to give us analytical solutions, however after a day we decided to just run a numerical simulation (the code reflects that change).

```
Clear[gs, ugs, sigma, NS, FI, FII, Shape, \[Eta]]
(*One of the items to modify for different minimal surfaces*)
sigma = {ArcTan[(2 u)/(u^2 + v^2 - 1)],
   ArcTan[(-2 v)/(u^2 + v^2 - 1)],
   1/2*Log[((u^2 - v^2 + 1)^2 + 4 u^2 v^2)/((u^2 - v^2 - 1)^2 +
      4 u^2 v^2)]};
(*We define the lower metric*)
gs[u_, v_] := {{D[sigma, u].D[sigma, u],
   D[sigma, u].D[sigma, v]}, {D[sigma, v].D[sigma, u],
   D[sigma, v].D[sigma, v]}}
(*We define the upper metric*)
ugs[u_, v_] := Inverse[gs[u, v]]
(*This is the Laplacian for a general surface*)
LapS[f_] :=
 Det[gs[u, v]]^(1/2)*(D[
      ugs[u, v][[1]][[1]]*Det[ugs[u, v]]^(1/2) D[f, u], u] +
    D[ugs[u, v][[1]][[2]]*Det[ugs[u, v]]^(1/2) D[f, u], u] +
    D[ugs[u, v][[2]][[1]]*Det[ugs[u, v]]^(1/2) D[f, v], v] +
    D[ugs[u, v][[2]][[2]]*Det[ugs[u, v]]^(1/2) D[f, v], v])
(*The following is the normal to the surface*)
```

```
NS = 1/Sqrt[
      Cross[D[sigma, u], D[sigma, v]].Cross[D[sigma, u],
        D[sigma, v]]] Cross[D[sigma, u], D[sigma, v]] // Simplify;
(*First fundamental form*)
FI = {{D[sigma, u].D[sigma, u],
      D[sigma, u].D[sigma, v]}, {D[sigma, v].D[sigma, u],
      D[sigma, v].D[sigma, v]}} // Simplify;
(*Second fundamental form*)
FII = {{D[sigma, {u, 2}].NS, D[sigma, u, v].NS}, {D[sigma, v, u].NS,
      D[sigma, {v, 2}].NS}} // Simplify;
(*Shape operator*)
Shape = 1/Det[FI] FII.FI;
(*This is Gauss curvature K=Det[Shape].*)
(*Alternatively, we could of defined it as Det[FII]/Det[FI]*)
S = Det[Shape] // Simplify;
(*Definition we used in the derivation of the first variation*)
W = Sqrt[Det[gs[u, v]]] // Simplify;
(*Modify to use different functions to potentially destabilize the integral*)
\[Eta] = S;
G = LapS[\[Eta]] // Simplify;
(*Calculating the intergral, finally*)
NIntegrate[\[Eta] G + 2*S*W*\[Eta]^2, {u, -1, 1}, {v, -1, 1}]
(*Be sure to modify the range if you change the surface*)
```

We find that

$$\int_{\Omega}(\phi \ \frac{1}{\sqrt{|(g_{jk})|}} \sum_{j,k} \partial_j(g^{jk}\sqrt{|(g_{jk})|}\partial_k\phi) + 2|g^{jk}b_{ki}|\sqrt{|(g_{ij})|}\phi^2) du \ dv \approx -2.4439.$$

This implies that $\phi = K$ is one direction in which the surface is unstable.

Remark 4. While it can be shown that various powers of K destabilize the integral our goal was to find one. What remains to be found is the index, the number of different directions the integral is unstable. After a quick investigation on the powers of K we find that the they lie on the same directions and thus provide us with no further information on the index of Scherks surface.

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