BLACK-SCHOLES EQUATION

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ABSTRACT. I IS MATH DO. This is a draft and not even close to being a final product. We are exploring the Black-Scholes Equations to see if we can somehow use spectral methods to solve the equation, or at least a subset.

1. Introduction

The Black-Scholes equation is a partial differential equation, that describes the price of the option over time,

$$\frac{\partial V(s,t)}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V(s,t)}{\partial s^2} + rs\frac{\partial V(s,t)}{\partial s} - rV(s,t) = 0. \tag{1}$$

s is the price of the stock, t is time, $V(s,t)$ is the price of the security, r is the annualized risk-free interest rate, and σ is the standard deviation of the stock's returns (i.e. volatility).

2. Solutions

Suppose that a solution of the form $V(s,t) = S(s)T(t)$ solves (1), then

$$S\frac{\partial T}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 S}{\partial s^2}T + rs\frac{\partial S}{\partial s}T - rST = 0.$$

Rearranging the previous equation and dividing by $S(s)T(t)$ yields,

$$\frac{T'}{T} + \frac{1}{2}\sigma^2 s^2 \frac{S''}{S} + rs\frac{S'}{S} - r = 0. \tag{2}$$

Let

$$\frac{1}{2}\sigma^2 s^2 \frac{S''}{S} + rs\frac{S'}{S} = -m^2 \text{ and } \frac{T'}{T} - r = m^2$$

so that (1) is satisfied. We have two differential equations to solve. Staritng with

$$\frac{T'}{T} - r = m^2 \iff \frac{T'}{T} = (r + m^2)$$

solutions to this form are given by,

$$\ln T = (r + m^2)t$$

or

$$T(t) = e^{(r+m^2)t}. \tag{3}$$

Table 1. Conditions of Call and Put securities

Type	$V(s,t)$	Payoff $(t=T)$	Boundary Condition	
Call	$C(s,t)$	$\max(s-K,0)$	$C(0,t)=0$	$\lim_{s\to\infty} C(s,t) = s - Ke^{-r(T-t)}$
Put	$P(s,t)$	$\max(K-s,0)$	$P(0,t)=Ke^{-r(T-t)}$	$\lim_{s\to\infty} P(s,t) = 0$

Now we solve,

$$\frac{1}{2}\sigma^2 s^2 \frac{S''}{S} + rs\frac{S'}{S} = -m^2 \iff \frac{1}{2}\sigma^2 s^2 S'' + rsS' + m^2 S = 0.$$

We guess that the solutions are of the following form $S(s) = s^\lambda$ and so

$$\frac{1}{2}\sigma^2 s^2 \lambda(\lambda-1)s^{\lambda-2} + rs\lambda s^{\lambda-1} + m^2 s^\lambda = 0$$

$$\Rightarrow s^\lambda(\frac{1}{2}\sigma^2\lambda(\lambda-1) + r\lambda + m^2) = 0$$

Hence, either $S^\lambda = 0$ or

$$\frac{1}{2}\sigma^2\lambda(\lambda-1) + r\lambda + m^2 = 0. \tag{4}$$

Solving for λ in (4),

$$\lambda_\pm = \frac{\frac{1}{2}\sigma^2 - r \pm \sqrt{(r-\frac{1}{2}\sigma^2)^2 - 4(\frac{1}{2}\sigma^2)(m^2)}}{\sigma^2}$$

and so our our characteristic solution for $S(s)$ is given by,

$$S(s) = s^{\lambda_+} + s^{\lambda_-}. \tag{5}$$

Lastly, putting (3) and (5) together gives us our solution to (1), namely,

$$V(s,t) = c_1[s^{\frac{\frac{1}{2}\sigma^2 - r + \sqrt{(r-\frac{1}{2}\sigma^2)^2 - 4(\frac{1}{2}\sigma^2)(m^2)}}{\sigma^2}} + s^{\frac{\frac{1}{2}\sigma^2 - r - \sqrt{(r-\frac{1}{2}\sigma^2)^2 - 4(\frac{1}{2}\sigma^2)(m^2)}}{\sigma^2}}]e^{(r+m^2)t} \tag{6}$$

where c_1 and m^2 will come from initial conditions. Let K be the strike price (predetermined rate) of the underlying stock and let $t = T$ be the time to maturity. Consider call and put securities. The following table gives us a summary of the conditions imposed on calls and puts.

3. Spectrum of the Black-Scholes Operator

The PDE in equation (1) can be written as

$$\frac{\partial V(s,t)}{\partial t} = -\mathcal{L}V(s,t) = -\mathcal{L}[S(s)T(t)] \tag{7}$$

where $\mathcal{L}$ is the following operator

$$\mathcal{L} = \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + rs\frac{\partial}{\partial s} - r.$$

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Consider the $h(s)$ weighted $L^2([0,\infty])$ space with norm

$$\langle f|g\rangle_h = \int_{s\in[0,\infty]} \overline{f(s)}g(s)h(s)ds$$

for $f, g \in \mathcal{L} : \{\mathcal{C}^2([0,\infty])\} \to L^2([0,\infty])$. We want to show that $\mathcal{L}$ is self-adjoint (i.e $\mathcal{L} = \overline{\mathcal{L}}$). Now,

$$\begin{aligned}
\langle f|\mathcal{L}g\rangle_h =& \int_0^\infty \overline{f}\mathcal{L}gh\ ds \\
=& \int_0^\infty \overline{f}\left(\frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + rs\frac{\partial}{\partial s} - r\right) hg\ ds \\
=& \int_0^\infty h\left(\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 g}{\partial s^2} + rs\frac{\partial g}{\partial s} - rg\right) \overline{f}\ ds \\
=& \int \left(-\frac{\partial}{\partial s}\left[\frac{1}{2}\sigma h s^2 \overline{f}\right]\frac{\partial g}{\partial s} - \frac{\partial}{\partial s}\left[rsh\overline{f}\right] g - rhg\overline{f}\right) ds + \left[\frac{1}{2}\sigma s^2 h\overline{f}\frac{\partial g}{\partial s} + rsh\overline{f}g\right]_0^\infty \\
=& \int \left(\frac{\partial^2}{\partial s^2}\left[\frac{1}{2}\sigma s^2 h\overline{f}\right] g - \frac{\partial}{\partial s}\left[rsh\overline{f}\right] g - rhg\overline{f}\right) ds + \left[\frac{1}{2}\sigma s^2 h\overline{f}\frac{\partial g}{\partial s} + rsh\overline{f}g - \frac{\partial}{\partial s}\left[\frac{1}{2}\sigma s^2 h\overline{f}\right] g\right]_0^\infty \\
=& \int \frac{1}{2}\sigma h s^2 \frac{\partial^2 \overline{f}}{\partial s^2} g + \left(\sigma^2 s^2 \frac{\partial h}{\partial s} + 2\sigma^2 hs - rsh\right)\frac{\partial \overline{f}}{\partial s} g \\
&+ \left(\frac{1}{2}\sigma^2 \frac{\partial^2 h}{\partial s^2} + \left(\sigma^2 + 2s - rs\right)\frac{\partial h}{\partial s} + \left(\sigma^2 - 2r\right) h\right)\overline{f}g\ ds \\
&+ \left[\frac{\sigma^2}{2} s^2 \overline{f} h \frac{\partial g}{\partial s} - \frac{\sigma^2}{2}\left(\frac{\partial \overline{f}}{\partial s} s^2 h + \overline{f} s^2 \frac{\partial h}{\partial s} + 2\overline{f}hs\right) g + r\overline{f}shg\right]_0^\infty \\
=& \langle \overline{\mathcal{L}} f|g\rangle_h + \text{Boundary}.
\end{aligned}$$

If we pick $h(s) = e^{-\phi(t)}$, then we have the following

$$\overline{\mathcal{L}} = \frac{\sigma^2}{2} s^2 \frac{\partial^2}{\partial s^2} + \left(2\sigma^2 s - rs\right)\frac{\partial}{\partial s} + \left(\sigma^2 - 2r\right).$$

If we assume that the boundary vanishes, then in the special case $\sigma^2 = r$ we have that $\mathcal{L}$ is self-adjoint.

References

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