A Numerical Investigation on the Period of the Simple Pendulum Problem

K. Oskar
Department of Mathematics
*California State University Fullerton**
(Dated: May 12, 2014)

We study the one-dimensional system of a classical physics problem: the simple pendulum. The simple pendulum consist of a particle of mass m fixed to the end, suspended of a massless string of length l in a gravitational field, whose other end is pivoted from the ceiling to let is swing freely in a vertical plane. We can specify the pendulum's position by the angle from equilibrium position. We will use a numerical approximation to solve for the period. We will compare this result with a perturbed expression of the period.

INTRODUCTION

There are many methods of approaching this problem, perturbation of the Hamiltonian $\mathcal{H} = \frac{p^2}{2ml^2} + mgl(1 - \cos\theta)$, where p is the momentum of the mass, say. Applying perturbation theory reduces our Hamiltonian to $\mathcal{H} = \frac{p^2}{2ml^2} + \frac{mgl\theta^2}{2}(1 - \frac{\theta^2}{12} + \frac{\theta^4}{360} - \cdots)$. For more on this approach refer to [Gold, Chapter 12]. The small amplitude limit consist of dropping all but the first term in the parentheses. This is what is usually done in introductory mechanics and even a second course in mechanics; this equation of motion is linear for small oscillations, smaller than 15^o [MT, Chapter 3]. For this approach refer to [Tay], [MT], or any other textbook on mechanics.

This paper focuses on the determination of the period of the simple pendulum via the derivation from Lagrangian mechanics. Our Lagrangian is given by

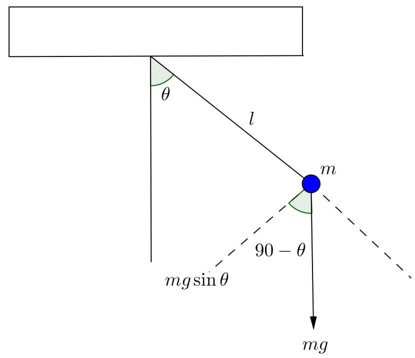
$$\mathcal{L} = \frac{1}{2}ml^2(\dot{\theta}(t))^2 + mgl(\cos\theta(t) - 1), \tag{1}$$

where $\dot{\theta}$ denotes the first time derivative of θ. From this our differential equation is given by

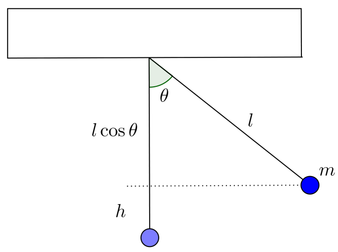
$$\frac{d^2}{dt^2}\theta(t) = -\frac{g}{l}\sin\theta(t) \tag{2}$$

where θ is to represent the angle displacement from where the pendulum pivots, and t is supposed to represent the time. Hence, $\theta(t)$ represent the motion of the pendulum. As usual g is taken to be the gravitation constant of where we decide to do this experiment and l is the length of the massless string. Of course, this equation should be referred to an ideal pendulum, had we taken into account the mass of the string our equation would be a bit more complicated since the additional mass will allows us to take into account possible non-rigid swings; for more on this approach refer to [Cio].

BACKKGROUND RESULTS



Our problem is as follows: given a object of mass m at the end of a massless string of length l find the period of oscillation. We can see that the path by the pendulum will be through an arc, hence let s denote the position; so $s = l\theta$ and thus $v = \frac{d}{dt}s$ will denote the path and velocity, respectively. However, we do impose a restriction, swings larger than $180°$ are not allowed.



This diagramme represents when we have zero potential, otherwise known as at equilibrium with the system. When the mass is not at zero potential the position, relative to zero potential, is given by $l\cos\theta$. Let h be the height from zero potential, we have $h = l - l\cos\theta$, since if the mass was at the zero potential the position would just be the length of the string, namely l. The next theorem, from [Da], allows us to find the differential equation associated to our problem.

Theorem .1. *Let* $f \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$, $f = f(x, u, \xi)$

and

$$\inf_{u\in X}\{I(u)=\int_a^b f(x,u(x),u'(x))dx\}=m \qquad (3)$$

where $X=\{u\in C^1([a,b]):u(a),\ u(b)=\beta\}$. *If* (3) *admits a minimizer* $u^*\in X\cap C^2([a,b])$, *then*

$$\frac{d}{dx}[f_\xi(x,u^*(x),(u^*)'(x))=f_u(x,u^*(x),(u^*)'(x)) \qquad (4)$$

for $x\in(a,b)$.

From our diagramms we have the potential is given by $V=mgh=mgl(1-cos\theta)$. The kinetic energy is found via the spatial coordinates, $T=\frac{1}{2}mv^2=\frac{1}{2}m(l\frac{d}{dt}\theta)^2$. Our Lagrangian is given by $\mathcal{L}=T-V$ and it reduces to (1). Using (4), our equation of motion are given by $\frac{\partial\mathcal{L}}{\partial\theta}=\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot\theta})$. Thus

$$\frac{d}{dt}(ml^2\dot\theta)+mgl\sin\theta=0$$

and we get the expression in (2).

METHODS

For small oscillations, we can use perturbation on $\sin\theta\approx\theta$. Thus (2) reduces to

$$\ddot\theta=-\omega^2\theta \qquad (5)$$

where $\omega=\sqrt{\frac{g}{l}}$ is the angular frequency, i.e. the measure of rotation rate. The period T, with $\omega=2\pi\nu$ where ν is the ordinary frequency, is given by

$$T=\frac{1}{\nu}=2\pi\sqrt{\frac{l}{g}} \qquad (6)$$

Without the small angle approximation we would need to multiply both sides (2) by $\dot\theta$ and integrate. So,

$$\dot\theta\ \ddot\theta=-\frac{g}{l}\sin\theta\ \dot\theta$$

which reduces, using $\frac{d}{dt}(\dot\theta^2)=2\dot\theta\ddot\theta$, to

$$\dot\theta\ d\dot\theta=\frac{-g}{l}\sin\theta\ d\theta \qquad (7)$$

Integrating (7) yields

$$\frac{\dot\theta^2}{2}=\frac{g}{l}\cos\theta\ +\text{constant}. \qquad (8)$$

We have three cases to consider: two special cases and the general.

180° swings

First the special case of 180°. In this case the pendulum swings back and forward from $-90°$ to $90°$. For this case $\dot\theta=0$ when $\theta=\frac{\pi}{2}$, and so the constant from (8) is zero. From our modified (8) we have

$$\dot\theta=\sqrt{\frac{2g}{l}}\sqrt{cos\theta}\ \text{ or }\ \frac{d\theta}{\sqrt{\cos\theta}}=\sqrt{\frac{2g}{l}}dt$$

From $\theta=0$ to $\theta=90°$ we covered $\frac{1}{4}$ of the period. To clarify from $\frac{\pi}{2}$ to 0, $\frac{1}{4}$ of the period is covered, from 0 to $-\frac{\pi}{2}$, another $\frac{1}{4}$ of the period. Then we return, from $-\frac{\pi}{2}$ to 0 is another $\frac{1}{4}$ of the period and lastly from 0 to $\frac{\pi}{2}$ we get the last $\frac{1}{4}$ of the period. So the period for 180° swings is given by T in

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\cos\theta}}=\sqrt{\frac{2g}{l}}\int_0^{\frac{T}{4}}dt \qquad (9)$$

So $T=4\sqrt{\frac{l}{2g}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\cos\theta}}$. Recall the beta function of the form: $B(p,q)=2\int^{\frac{\pi}{2}}(\sin\theta)^{2p-1}(\cos\theta)^{2q-1}d\theta$. This implies $T=2\sqrt{\frac{l}{2g}}B(\frac{1}{2},\frac{1}{4})$. However $B(\frac{1}{4},\frac{1}{2})=B(\frac{1}{2},\frac{1}{4})=2\sqrt{2}F(\frac{1}{\sqrt{2}},\frac{\pi}{2})=2\sqrt{2}K(\frac{1}{\sqrt{2}})$, where F and K denote the elliptical integrals of the first and second kind, respectively.

Arbitrary large swings

In the previous case we solve the special case of 180° swings. Now we solve (8) generally using elliptical integrals. Take a swing of any amplitude $\theta=\alpha$. That is we hold mass m at angle α from zerp potential and then let go; hence we would have initial velocity equal to zero, $\dot\theta=0$. Therefore (8) reduces to

$$\dot\theta^2=\frac{2g}{l}(\cos\theta-\cos\alpha) \qquad (10)$$

Taking the square root and integrating (10) yields

$$\int_0^\alpha\frac{d\theta}{\sqrt{\cos\theta-\cos\alpha}}=\sqrt{\frac{2g}{l}}\frac{T_\alpha}{4} \qquad (11)$$

where T_α denotes the period for swings from $-\alpha$ to α and back. As before during the angle $[0,\alpha]$ we would only cover a fourth of the total period. Observe, if $\alpha=\frac{\pi}{2}$, (11) reduces to (9). Our expression on the left of (11) reduces with the following transformations: $\cos\theta=1-2\sin^2\frac{\theta}{2}$;

$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$; $x = \frac{\sin\frac{\theta}{2}}{\sin\frac{\alpha}{2}}$,

$$\begin{aligned}\int_0^\alpha \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} &= \frac{2}{\sqrt{2}}\frac{1}{2}\int_0^\alpha \frac{d\theta}{\sin\frac{\alpha}{2}}\frac{1}{\sqrt{1-x^2}} \\ &= \sqrt{2}\int_0^1 \frac{1}{\sqrt{1-x^2\sin^2\frac{\alpha}{2}}}\frac{dx}{\sqrt{1-x^2}} \\ &= \sqrt{2}K(\sin\frac{\alpha}{2})\end{aligned}$$

Moreover, since $F(k, \frac{\pi}{2}) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$ if we let $k = \sin\frac{\alpha}{2}$ we have

$$T_\alpha = 4\sqrt{\frac{l}{g}}K(\sin\frac{\alpha}{2}) = \sqrt{2}F(\sin\frac{\alpha}{2}, \frac{\pi}{2}). \qquad (12)$$

our goal is to approximate this equation.

α <90°

Our calculations can be simplified to a certain extent if we restrict $\alpha < 90°$, so $\frac{\alpha}{2} < 45°$, and using the binomial expansion. Therefore, $\sin^2\frac{\alpha}{2} < \frac{1}{2}$, and so

$$\begin{aligned}K(k) &= \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \\ &= \int_0^{\frac{\pi}{2}} [1 - \frac{1}{2}(-k^2\sin^2\phi) \\ &\quad + (\frac{-1}{2})(\frac{-3}{2})\frac{1}{2!}(-k^2\sin^2\phi)^2 + \cdots]d\phi \\ &= \frac{\pi}{2}(1 + \frac{k^2}{4} + \frac{3}{8}k^4\frac{3}{8} + \cdots)\end{aligned}$$

If $k^2 = \sin^2\frac{\alpha}{2}$, then

$$K(\sin\frac{\theta}{2}) = \frac{\pi}{2}(1 + \frac{1}{4}\sin^2\frac{\alpha}{2} + (\frac{3}{8})^2\sin^4\frac{\alpha}{2} + \cdots) \qquad (13)$$

Using this in (12) we get

$$T = 4\sqrt{\frac{l}{g}}\frac{\pi}{2}(1 + \frac{1}{4}\sin^2\frac{\alpha}{2} + \cdots) \qquad (14)$$

Observe for small α, then we have $\sin\frac{\alpha}{2} \approx \frac{\alpha}{2}$ we have

$$T = 2\pi\sqrt{\frac{l}{g}}(1 + \frac{\alpha^2}{16} + \cdots)$$

which reduces to (6) for very small α.

PRINCIPAL RESULTS

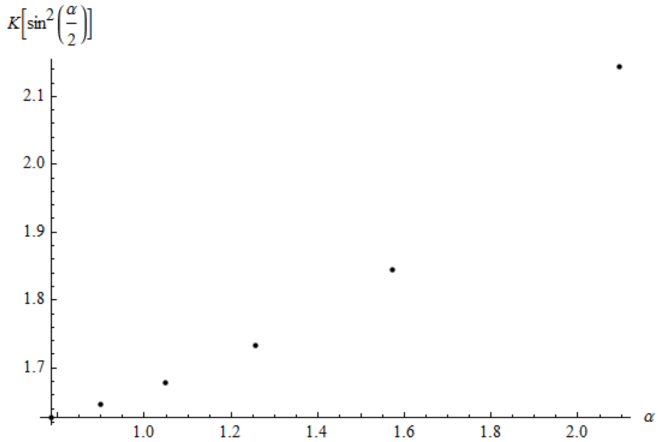
As stated before our goal is to approximate (12). We will evaluate (12) for various angles α. For the sake of argument we will need to restrict (12) even more; assume that the length of the string is 1 meter and we did this experiment on the surface of the Earth, $g = 9.8$ meters per second per second. In order to continue we need to regress a bit in our approach. We approximate $K(\sin\frac{\alpha}{2})$ using Gaussian quadriture with Legendre polynomials and 1 as the weight function. Additionally, we will use 100 nodes.

Earth

We have (12) reducing to $T_\alpha = 4\sqrt{\frac{50}{490}}K(\sin\frac{\alpha}{2})$ for arbitrary α. The programme we found gave us the following data for the elliptical integral.

α	$K(\sin^2\frac{\alpha}{2})$	T_α
$\frac{\pi}{4}$	1.627	2.0789
0.897598	1.647	2.10446
$\frac{\pi}{3}$	1.6787	2.14496
1.25664	1.7339	2.2155
$\frac{\pi}{2}$	1.8454	2.35797
2.0944	2.1443	2.73989
$\frac{-\pi}{4}$	1.6270	2.0789

We now plot $K(\sin^2\frac{\alpha}{2})$ associated to α.



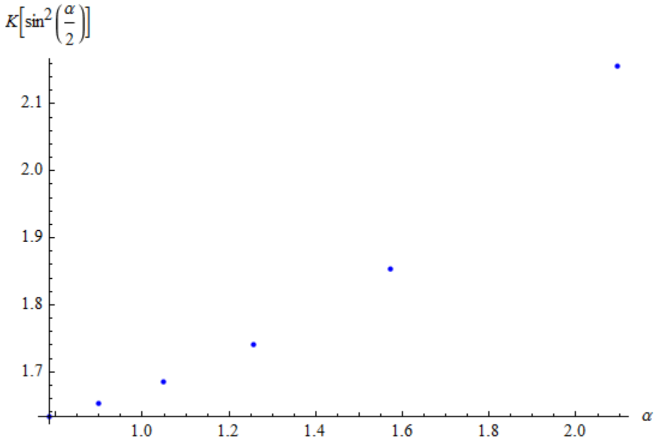
Matlab has a built-in function to evaluate elliptical integrals usign the following:

```
ellipke(m)
```

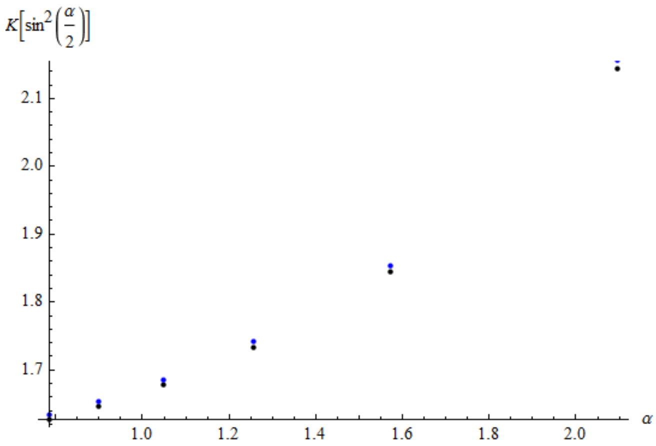
with $m = \sin^2\frac{\alpha}{2}$. The following table gives the results:

α	$K(\sin^2\frac{\alpha}{2})$	T_α
$\frac{\pi}{4}$	1.63359	2.08732
0.897598	1.65375	2.11308
$\frac{\pi}{3}$	1.68575	2.15397
1.25664	1.7415	2.22521
$\frac{\pi}{2}$	1.85407	2.36905
2.0944	2.15652	2.75549

We now plot $K(\sin^2\frac{\alpha}{2})$ associated to α.



A sided by side comparison shows:



where the bottom data points represent the data obtain from our Gaussian quadrature approximation. The results were fairly accurate. We initially were going to compare our answer to (14), however after finding MATLAB had a built-in this analysis was no longer needed. The results were within 1% accuracy. Since MATLAB has a built in interpolating function we are able to find the an expression for the period of a pendulum from $-\alpha$ to α with $\alpha \in (0, \pi)$

$$T_\alpha = 4\sqrt{\frac{50}{490}}(0.0520944\alpha^6 - 0.389324\alpha^5 + 1.20449\alpha^4 \\ -1.92873\alpha^3 + 1.80593\alpha^2 - 0.790487\alpha + 1.71409)$$

or in general,

$$T_\alpha = 4\sqrt{\frac{l}{g}}(0.0520944\alpha^6 - 0.389324\alpha^5 + 1.20449\alpha^4 \\ -1.92873\alpha^3 + 1.80593\alpha^2 - 0.790487\alpha + 1.71409).$$

ACKNOWLEDGMENTS

The author would like to thank Dr. Fearn for having this infernal problem on the homework and final for Graduate Mechanics and Mathematical Methods for Physicist. The initial diagrammes were made using Geogebra. A vast majority of the symbolic calculation and all the plots were made using Mathematica. All numeric data was obtained using MATLAB. For more information on the data please refer to the appendix.

---

* Electronic address: `koskarnegron@gmail.com`

REFERENCES

[Da] B. Dacorogna. *Introduction to the Calculus of Variation*, Imperial College Press, Shelton Street, 2004.

[Cio] V. Ciocanel/T. Witelski. *Modeling and numerical simulation of the nonlinear dynamics of the parametrically forced string pendulum* , SIAM Undergraduate Research Online, (2012), vol 5., 95-115.

[Gold] H. Goldstein/C. Poole/ J. Safko. *Classical Mechanics, Third Edition*, Addison Wesley, 2001.

[KC] D. Kincaid/W. Cheney. *Numerical Analysis*, Brooks/-Cole Publishing Company, Pacific Grove, 1991.

[Landau] L. D. Landau/E. M. Lifshitz. *Mechanics, Third Edition*, Mediterranean Press, Rende, 1989.

[MT] S. T. Thornton/J. B. Marion. *Classical Dynamics of Particles and Systems, Fifth Edition*, Brooks/Cole, Belmont, 2004.

[Tay] J. R. Taylor. *Classical Mechanics, Third Edition*, University Science Books, 2005.

APPENDIX

The following code was used

```
function [x,w]=lgwt(N,a,b)
% This script is for computing definite
% integrals using Legendre-Gauss
% Quadrature. Computes the Legendre-Gauss
% nodes and weights on an interval
% [a,b] with truncation order N

N=N-1;
N1=N+1;
N2=N+2;
xu=linspace(-1,1,N1)';

% Initial guess
y=cos((2*(0:N)'+1)*pi/(2*N+2))
       +(0.27/N1)*sin(pi*xu*N/N2);

% Legendre-Gauss Vandermonde Matrix
L=zeros(N1,N2);

% Derivative of LGVM
Lp=zeros(N1,N2);

% Compute the zeros of the N+1
% Legendre Polynomial
```

```
% using the recursion relation and
% the Newton-Raphson method

y0=2;

% Iterate until new points are uniformly
%within epsilon of old points
while max(abs(y-y0))>eps


    L(:,1)=1;
    Lp(:,1)=0;

    L(:,2)=y;
    Lp(:,2)=1;

    for k=2:N1
        L(:,k+1)=( (2*k-1)*y.*L(:,k)
                -(k-1)*L(:,k-1) )/k;
    end

    Lp=(N2)*( L(:,N1)
        -y.*L(:,N2) )./(1-y.^2);

    y0=y;
    y=y0-L(:,N2)./Lp;

end

% Linear map from[-1,1] to [a,b]
x=(a*(1-y)+b*(1+y))/2;

% Compute the weights
w=(b-a)./((1-y.^2).*Lp.^2)*(N2/N1)^2;
```

To evaluate, say at $\alpha = \frac{\pi}{2}$ we used the following command

```
[x,c]=lgwt(100, 0, 1);
 f= 1./sqrt(1-power(x,2)
       *power(sin((pi/2)/2),2))
       *1./sqrt(1-power(x,2));
value=sum(c.*f)
```

from this we recorded the data and we repeated for other α.