General*General Relativity and Chameleon Action Integrals

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Abstract

We give a short introduction of chameleon theory and we derive the field equations from Khoury and Weltman. This a writeup for the completion of an independent study course on general relativity

1 Introduction

The chameleon particle is a hypothetical particle that couples to matter much weaker than gravity. Khoury and Weltman (2004) postulate this particles as a dark energy candidate. One of the properties of this particles is that it has a variable effective mass which is an increasing function of the ambient energy density. As a consequence, the chameleon particle is able to evade current constraints on equivalence principle violation derived from experiments[1]. Although this property would allow the chameleon to drive the currently observed acceleration of the universe's expansion, it also makes it very difficult to test for experimentally.

1.1 Conventions

This section is adapted from Dirac (1996). In this paper we will use the following conventions.

Definition 1.1. Metric. The metric g_{ij} is given by the diagonal matrix

$$g_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

*Not a typo

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1. Even if it couples to matter with a strength equal or greater than that of gravity

Definition 1.2. Inverse Metric Tensor. We define the inverse of the metric tensor, g^{ij}, such that

$$g^{ij} g_{ik} = \delta^j_k.$$

where δ^j_k denotes the usual indicator function. That is $\delta^j_k = 1$ when $k = j$ and 0 otherwise. From this we can see that,

$$g^{ij} = (g_{ij})^{-1} = -g_{ij}. \tag{1}$$

Lastly, we will adopt the following notation when appropriate for a coordinate system (x^0, x^1, x^2, x^3),

$$\frac{\partial Q_a}{\partial x^i} = \partial_i Q_a = Q_{a,i}$$

1.2 Christoffel Symbols

The first fundamental form is given by (i.e the distance from x^i to $x^i + dx^i$),

$$|I| = ds^2 = g_{ij} \delta x^i \delta x^j. \tag{2}$$

Consider the ambient space of x^i. For each point x^i in the surface, there is a definite point y^n in the $n-$dimensional Euclidean space. Hence, each y^n is a function of the four x^i. Moreover, for two neighboring points in the surface differing by δx^i,

$$\delta y^n = \frac{\partial y^n}{\partial x^i} \delta x^i.$$

You might recognize this as the total derivative of y^n. With this notation we have the first fundamental form be,

$$\begin{aligned} \delta s^2 &= h_{nm} \delta y^n \delta y^m \\ &= h_{nm} \frac{\partial y^n}{\partial x^i} \frac{\partial y^m}{\partial x^j} \delta x^i \delta x^j. \end{aligned}$$

From this notation, we can see that,

$$g_{ij} = h_{nm} \frac{\partial y^n}{\partial x^i} \frac{\partial y^m}{\partial x^j} \tag{3}$$

where we can take h_{nm} as a scaling factor. We can make the scaling factor h_{mn} such that

$$\begin{aligned} h_{nm} \frac{\partial y^n}{\partial x^i} \frac{\partial y^m}{\partial x^j} &= \partial_i y^n h_{nm} \partial_j y^m \\ &= \partial_i y^n \partial_j y_n. \end{aligned}$$

Now, differentiating (3),

$$\begin{aligned}\frac{\partial g_{ij}}{\partial x^s} &= \frac{\partial^2 y^n}{\partial x^i \partial x^s}\partial_j y^m + \partial_i y^n \frac{\partial^2 y^m}{\partial x^j \partial x^s} \\ &= \partial_{is} y^n \partial_j y_n + \partial_i y^n \partial_{js} y_n \\ &= \partial_j y^n \partial_{is} y_n + \partial_i y^n \partial_{js} y_n \\ &= \frac{\partial y^n}{\partial x^j}\frac{\partial^2 y_n}{\partial x^i \partial x^s} + \frac{\partial y^n}{\partial x^i}\frac{\partial^2 y_n}{\partial x^j \partial x^s}.\end{aligned} \tag{4}$$

Definition 1.3. Christoffel Symbol of the First Kind. We define Christoffel Symbol of the first kind as,

$$\begin{aligned}\Gamma_{qwe} &= \frac{1}{2}(\partial_e g_{qw} + \partial_w g_{qe} - \partial_q g_{we}) \\ &= \partial_{we} y_n \partial_q y^n.\end{aligned} \tag{5}$$

Remark. Observe the following,

$$\Gamma_{qwe} + \Gamma_{wqe} = \partial_q g_{we}.$$

Definition 1.4. Christoffel Symbol of the Second Kind. We define Christoffel Symbol of the second kind,

$$\Gamma^q_{we} = g^{q\lambda}\Gamma_{\lambda we}. \tag{6}$$

If we want to find the change in a vector A_q due to a displacement, we have

$$dA_q = A^q \Gamma_{qwe} dx^e$$

We can interpret $A^q\Gamma_{qwe}$ as an inner product. As such $A^q\Gamma_{qwe}dx^e$ looks like the directional derivative.

1.3 Ricci Tensor

Definition 1.5. Covariant Derivative Let A_u be a vector field, the covariant derivative of A_u, $A_{u:v}$, is given by

$$\begin{aligned}A_{u:v} &= \partial_v A_u - \Gamma^\alpha_{uv} A_\alpha \\ &= A_{u,v} - \Gamma^\alpha_{uv} A_\alpha\end{aligned} \tag{7}$$

If our vector field has n index, say $T_{a_1 \dots a_n}$, then

$$T_{a_1 \dots a_n : \sigma} = \partial_\sigma T_{a_1 \dots a_n} - \sum_{i=1}^n \Gamma^\lambda_{a_i \sigma} T_{a_1 \dots a_n + \{\lambda\} - \{a_i\}} \tag{8}$$

where

$$T_{a_1 \dots a_n + \{\lambda\} - \{a_i\}} = T_{a_1 \dots a_{i-1} \lambda a_{i+1} \dots a_n}.$$

Now, let us consider a scalar field S. From definition 1.5, let us apply two covariant derivatives

$$\begin{aligned}S_{:u:v} &= \partial_v S_{:u} - \Gamma^{\lambda}_{uv} S_{:\lambda}\\ &= \partial_{uv} S - \Gamma^{\lambda}_{uv}\partial_\lambda S\end{aligned}$$

where we used $S_{:i} = \partial_i S$ for scalar S. Similarly, let us apply two covariant derivatives to a vector A_q,

$$\begin{aligned}A_{q:w:e} =&\partial_e A_{q:w} - \Gamma^{\alpha}_{qe} A_{\alpha:w} - \Gamma^{\alpha}_{we} A_{q:\alpha}\\ =&\partial_e(\partial_w A_q - \Gamma^{\alpha}_{qw} A_\alpha) - \Gamma^{\alpha}_{qe}(\partial_w A_\alpha - \Gamma^{\beta}_{\alpha w} A_\beta) - \Gamma^{\alpha}_{we}(\partial_\alpha A_q - \Gamma^{\beta}_{q\alpha} A_\beta)\\ =&\partial_{we} A_q - \Gamma^{\alpha}_{qw}\partial_e A_\alpha - \partial_e \Gamma^{\alpha}_{qw} A_\alpha - \Gamma^{\alpha}_{qe}\partial_w A_\alpha + \Gamma^{\alpha}_{qe}\Gamma^{\beta} A_\beta - \Gamma^{\alpha}_{we}\partial_\alpha A_q + \Gamma^{\alpha}_{we}\Gamma^{\beta}_{q\alpha} A_\beta\\ =&\partial_{we} A_q - \Gamma^{\alpha}_{qw}\partial_e A_\alpha - \Gamma^{\alpha}_{qe}\partial_w A_\alpha - \Gamma^{\alpha}_{we}\partial_\alpha A_q + \Gamma^{\alpha}_{we}\Gamma^{\beta}_{q\alpha} A_\beta + \Gamma^{\alpha}_{qe}\Gamma^{\beta}_{\alpha w} A_\beta - \partial_e \Gamma^{\alpha}_{qw} A_\alpha\\ =&\partial_{we} A_q - \Gamma^{\alpha}_{qw}\partial_e A_\alpha - \Gamma^{\alpha}_{qe}\partial_w A_\alpha - \Gamma^{\alpha}_{we}\partial_\alpha A_q + \Gamma^{\alpha}_{we}\Gamma^{\beta}_{q\alpha} A_\beta + \Gamma^{\alpha}_{qe}\Gamma^{\beta}_{\alpha w} A_\beta - \partial_e \Gamma^{\beta}_{qw} A_\beta\\ =&\partial_{we} A_q - \Gamma^{\alpha}_{qw}\partial_e A_\alpha - \Gamma^{\alpha}_{qe}\partial_w A_\alpha - \Gamma^{\alpha}_{we}\partial_\alpha A_q + (\Gamma^{\alpha}_{we}\Gamma^{\beta}_{q\alpha} + \Gamma^{\alpha}_{qe}\Gamma^{\beta}_{\alpha w} - \partial_e \Gamma^{\beta}_{qw}) A_\beta.\end{aligned} \tag{9}$$

If we switch the indexes w and e in (9), we get

$$A_{q:e:w} = \partial_{ew} A_q - \Gamma^{\alpha}_{qe}\partial_w A_\alpha - \Gamma^{\alpha}_{qw}\partial_e A_\alpha - \Gamma^{\alpha}_{ew}\partial_\alpha A_q + (\Gamma^{\alpha}_{ew}\Gamma^{\beta}_{q\alpha} + \Gamma^{\alpha}_{qw}\Gamma^{\beta}_{\alpha e} - \partial_w \Gamma^{\beta}_{qe}) A_\beta. \tag{10}$$

Now, if we subtract (9) from (10),

$$\begin{aligned}A_{q:w:e} - A_{q:e:w} &= (\Gamma^{\alpha}_{we}\Gamma^{\beta}_{q\alpha} + \Gamma^{\alpha}_{qe}\Gamma^{\beta}_{\alpha w} - \partial_e \Gamma^{\beta}_{qw}) A_\beta - (\Gamma^{\alpha}_{ew}\Gamma^{\beta}_{q\alpha} + \Gamma^{\alpha}_{qw}\Gamma^{\beta}_{\alpha e} - \partial_w \Gamma^{\beta}_{qe}) A_\beta\\ &= (\partial_w \Gamma^{\beta}_{qe} - \partial_e \Gamma^{\beta}_{qw} + \Gamma^{\alpha}_{qe}\Gamma^{\beta}_{\alpha w} - \Gamma^{\alpha}_{qw}\Gamma^{\beta}_{\alpha e}) A_\beta\end{aligned} \tag{11}$$

Definition 1.6. Curvature Tensor. We define the curvature tensor (R^{β}_{qwe}) as follows,

$$R^{\beta}_{qwe} A_\beta = A_{q:w:e} - A_{q:e:w}. \tag{12}$$

From the expression found in (11), we get

$$R^{\beta}_{qwe} = \partial_w \Gamma^{\beta}_{qe} - \partial_e \Gamma^{\beta}_{qw} + \Gamma^{\alpha}_{qe}\Gamma^{\beta}_{\alpha w} - \Gamma^{\alpha}_{qw}\Gamma^{\beta}_{\alpha e} \tag{13}$$

Observe, if we contract R^{β}_{qwe},

$$\begin{aligned}R_{aqwe} &= g_{a\beta} R^{\beta}_{qwe}\\ &= g_{a\beta}(\partial_w \Gamma^{\beta}_{qe} - \partial_e \Gamma^{\beta}_{qw} + \Gamma^{\alpha}_{qe}\Gamma^{\beta}_{\alpha w} - \Gamma^{\alpha}_{qw}\Gamma^{\beta}_{\alpha e})\\ &= \partial_w(g_{a\beta}\Gamma^{\beta}_{qe}) - \partial_e(g_{a\beta}\Gamma^{\beta}_{qw}) + \Gamma^{\alpha}_{qe} g_{a\beta}\Gamma^{\beta}_{\alpha w} - \Gamma^{\alpha}_{qw} g_{a\beta}\Gamma^{\beta}_{\alpha e}\\ &= \partial_w(g_{a\beta}\Gamma^{\beta}_{qe}) - \partial_e(g_{a\beta}\Gamma^{\beta}_{qw}) + \Gamma^{\alpha}_{qe}\Gamma_{a\alpha w} - \Gamma^{\alpha}_{qw}\Gamma_{a\alpha e}\\ &= \partial_w \Gamma_{aqe} - \partial_w g_{a\beta}\Gamma^{\beta}_{qe} - \partial_e(g_{a\beta}\Gamma^{\beta}_{qw}) + \Gamma^{\beta}_{qe}\Gamma_{a\beta w} - \Gamma^{\beta}_{qw}\Gamma_{a\beta e}\\ &= \partial_w \Gamma_{aqe} - (\Gamma_{a\beta w} + \Gamma_{\beta aw})\Gamma^{\beta}_{qe} - \partial_e(g_{a\beta}\Gamma^{\beta}_{qw}) + \Gamma^{\beta}_{qe}\Gamma_{a\beta w} - \Gamma^{\beta}_{qw}\Gamma_{a\beta e}\\ &= \partial_w \Gamma_{aqe} - \Gamma^{\beta}_{qe}\Gamma_{\beta aw} - \partial_e \Gamma_{aqw} + \Gamma^{\beta}_{qw}\Gamma_{\beta ae}.\end{aligned} \tag{14}$$

Definition 1.7. Ricci Tensor. In the special case when $\beta = e$, we have that so called Ricci tensor, that is

$$R^e_{qwe} = R_{qw}$$

Remark. Note, that if we apply the inverse of the metric we get,

$$g^{vp} R_{vp} = g^{vp} R^q_{vpq} = R^v_v \tag{15}$$

$R^v_v = R$ is called the Ricci scalar. In a sense what the Ricci tensor measures is the deviation of the volume of an Euclidean ball compared to a the volume in a Riemannian manifold.

1.4 Variations on Ricci Curvature

Before we have a bit more of work to do before we move on to the action pricipal.

1.4.1 Variation of Curvature Tensor

We want to compute,

$$\delta R^\beta_{qwe} = \partial_w \delta\Gamma^\beta_{qe} - \partial_e \delta\Gamma^\beta_{qw} + \delta\Gamma^\alpha_{qe}\Gamma^\beta_{\alpha w} + \Gamma^\alpha_{qe}\delta\Gamma^\beta_{\alpha w} - \delta\Gamma^\alpha_{qw}\Gamma^\beta_{\alpha e} - \Gamma^\alpha_{qw}\delta\Gamma^\beta_{\alpha e}.$$

While this does not give us the answer immediately, it does brings us closer. Since $\delta\Gamma^\beta_{\alpha e}$ is a tensor, we can compute its covariant derivative. That is,

$$(\delta\Gamma^p_{vu})_{:l} = \partial_l(\delta\Gamma^p_{vu}) + \Gamma^p_{sl}\delta\Gamma^s_{vu} - \Gamma^s_{vl}\delta\Gamma^p_{su} - \Gamma^s_{ul}\delta\Gamma^p_{vs}. \tag{16}$$

Now using (16) observe,

$$\begin{aligned}\delta R^\beta_{qwe} &= \partial_w \delta\Gamma^\beta_{qe} - \partial_e \delta\Gamma^\beta_{qw} + \delta\Gamma^\alpha_{qe}\Gamma^\beta_{\alpha w} + \Gamma^\alpha_{qe}\delta\Gamma^\beta_{\alpha w} - \delta\Gamma^\alpha_{qw}\Gamma^\beta_{\alpha e} - \Gamma^\alpha_{qw}\delta\Gamma^\beta_{\alpha e} \\ &= (\delta\Gamma^\beta_{qe})_{:w} - (\delta\Gamma^\beta_{qw})_{:e}.\end{aligned} \tag{17}$$

So the variation in variation on the curvature tensor depends on the variation of the Christoffel symbols.

1.4.2 Ricci Tensor

Now, using (17), we compute

$$\begin{aligned}\delta R_{qw} &= \delta R^e_{qwe} \\ &= (\delta\Gamma^e_{qe})_{:w} - (\delta\Gamma^e_{qw})_{:e}.\end{aligned} \tag{18}$$

1.4.3 Stokes Theorem

From Dirac (1996), we have for vector A^u

$$A^u_{:u} = \partial_u A^u + \Gamma^u_{vu} A^u \tag{19}$$

Gauβ theorem,

$$\int A^u_{:u}\sqrt{-g}d^4x = \int \partial_u(A^u\sqrt{-g})d^4x.$$

1.4.4 Ricci Scalar

The variation of the Ricci scalar is given by,

$$\begin{aligned} \delta R &= \delta(g^{uv} R_{uv}) \\ &= \delta g^{uv} R_{uv} + g^{uv} \delta R_{uv} \\ &= \delta g^{uv} R_{uv} + g^{uv}((\delta\Gamma^e_{ue})_{:v} - (\delta\Gamma^e_{uw})_{:e}) \\ &= \delta g^{uv} R_{uv} + (g^{uv}\delta\Gamma^s_{vu} - g^{us}\delta\Gamma^p_{pu})_{:s}. \end{aligned} \tag{20}$$

Now, if we multiply $(g^{uv}\delta\Gamma^s_{vu} - g^{us}\delta\Gamma^p_{pu})_{:s}$ by $\sqrt{-g}$, we get the total derivative. Following Dirac, we get that

$$\sqrt{-g}A^u_{:u} = \partial_u(\sqrt{-g}A^u),$$

and as an immediate consequence,

$$\frac{\delta R}{\delta g^{uv}} = R_{uv}. \tag{21}$$

1.4.5 Variation on Christoffel Symbol of the Second Kind

Consider a small variation

$$\delta\Gamma^l_{uv} = \frac{1}{2} g^{la}((\delta g_{av})_{:u} + (\delta g_{au})_{:v} - (\delta g_{uv})_{:a}). \tag{22}$$

With (22), we can write

$$\delta R = R_{uv}\delta g^{uv} + g_{uv}\Box\delta g^{uv} - (\delta g^{uv})_{:v:u} \tag{23}$$

1.4.6 $\delta\sqrt{-g}$

Lastly, recall that

$$\begin{aligned} \delta g &= g g^{uv} \delta g_{uv} \\ &= g(-g_{uv}\delta g^{uv}). \end{aligned}$$

Hence,

$$\begin{aligned} \delta\sqrt{-g} &= \frac{-1}{2\sqrt{-g}}\delta g \\ &= \frac{1}{2}\sqrt{-g}(g^{uv}\delta g_{uv}) \\ &= \frac{-1}{2}\sqrt{-g}(g_{uv}\delta g^{uv}) \end{aligned}$$

2 Action Principal

In the study of general relativity one encounters the following action integral that yields Einstein's field equations. We now have the necessary tools to explore the following equation:

$$I = \int \frac{c^4}{16\pi G} R\sqrt{-g} d^4 x. \tag{24}$$

By the principal of least action a small variation of (24) should be zero. That is,

$$\begin{aligned}
0 &= \delta I \\
&= \delta(\int \frac{c^4}{16\pi G} R\sqrt{-g} d^4 x) \\
&= \frac{c^4}{16\pi G} \int \delta(R\sqrt{-g}) d^4 x \\
&= \frac{c^4}{16\pi G} \int \delta(R\sqrt{-g}) d^4 x \\
&= \frac{c^4}{16\pi G} \int (\delta R + \frac{R}{\sqrt{-g}} \delta\sqrt{-g})\sqrt{-g} d^4 x \\
&= \frac{c^4}{16\pi G} \int (\frac{\delta R}{\delta g^{uv}} + \frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{uv}})\sqrt{-g}\delta g^{uv} d^4 x \\
&= \frac{c^4}{16\pi G} \int (R_{uv} + R\frac{-1}{2} g_{uv})\sqrt{-g}\delta g^{uv} d^4 x
\end{aligned} \tag{25}$$

and so

$$R_{uv} + R\frac{-1}{2} g_{uv} = 0.$$

2.1 Matter Fields

However, if we were to incorporate some matter field L, our action integral would become

$$I = \int (\frac{c^4}{16\pi G} R + L)\sqrt{-g} d^4 x \tag{26}$$

and so the variation is

$$\begin{aligned}
0 &= \delta I \\
&= \int (\frac{c^4}{16\pi G} \delta(R\sqrt{-g}) + \delta(L\sqrt{-g})) d^4 x \\
&= \int (\frac{c^4}{16\pi G} \frac{\delta(R\sqrt{-g})}{\delta g^{uv}} + \frac{\delta(L\sqrt{-g})}{\delta g^{uv}})\delta g^{uv} d^4 x \\
&= \int (\frac{c^4}{16\pi G}(R_{uv} + R\frac{-1}{2} g_{uv}) + \frac{1}{\sqrt{-g}} \frac{\delta(L\sqrt{-g})}{\delta g^{uv}})\sqrt{-g}\delta g^{uv} d^4 x.
\end{aligned} \tag{27}$$

Hence, (27) yields the following solution,

$$\frac{c^4}{16\pi G}(R_{uv}+R\frac{-1}{2}g_{uv})+\frac{1}{\sqrt{-g}}\frac{\delta(L\sqrt{-g})}{\delta g^{uv}}=0 \tag{28}$$

which implies that,

$$\begin{aligned}\frac{c^4}{16\pi G}(R_{uv}+R\frac{-1}{2}g_{uv}) &= \frac{-1}{\sqrt{-g}}\frac{\delta(L\sqrt{-g})}{\delta g^{uv}}\\ &= -\frac{\delta L}{\delta g^{uv}}+\frac{1}{2}g_{uv}L\\ &= \frac{1}{2}T_{uv}\end{aligned} \tag{29}$$

which we will call the density and flux tensor. So far we have dealt with cases that are common to a study of general relativity and widely accepted.

2.2 f(R)

Khoury and Weltman model comes from a more general action integral. Instead of the action on the Ricci scalar, we consider a family of functions of the Riic scalar. The action integral is then

$$I=\int(\frac{c^4}{16\pi G}f(R)+L)\sqrt{-g}d^4x \tag{30}$$

As before, we compute a small deviation and we get the following result (using (23)),

$$\begin{aligned}0 &= \delta I\\ &= \int(\frac{c^4}{16\pi G}(\delta f(R)\sqrt{-g}+f(R)\delta\sqrt{-g})+\delta(L\sqrt{-g}))d^4x\\ &= \int(\frac{c^4}{16\pi G}(\delta f(R)\sqrt{-g}+f(R)\delta\sqrt{-g})+\delta(L\sqrt{-g}))d^4x\\ &= \int(\frac{c^4}{16\pi G}(\frac{\partial f(R)}{\partial R}\delta R\sqrt{-g}+f(R)\delta\sqrt{-g}+\delta(L\sqrt{-g}))d^4x\end{aligned} \tag{31}$$

And so we have,

$$\frac{\partial f(R)}{\partial R}R_{uv}-\frac{1}{2}f(R)g_{uv}+g_{uv}\Box\frac{\partial f(R)}{\partial R}-(\frac{\partial f(R)}{\partial R})_{:u:v}=\frac{8\pi G}{c^4}T_{uv}. \tag{32}$$

3 Chameleon Particle

We investigated various action integral and their respective field equations. Now, to the main portion of this project. To be consistent with the Khoury and Weltman paper consider the following transformations,

$$I_a=\int\frac{c^4}{16\pi G}R\sqrt{-g}d^4x=\int\frac{M_{pl^2}}{2}R\sqrt{-g}d^4x. \tag{33}$$

Moreover, we introduce a scalar field ϕ with potential $V(\phi)$ with action

$$I_s = -\int (\frac{1}{2}(\partial\phi)^2 + V(\phi))\sqrt{-g}d^4x \tag{34}$$

and matter fields $\psi_m^{(i)}$ with actions

$$I_m = -\int L(\psi_m^{(i)}, g_{uv}^{(i)}) \tag{35}$$

which will be coupled to ϕ by the following expression,

$$g_{uv}^{(i)} = \exp(\frac{2b_i\phi}{M_{pl}})g_{uv}. \tag{36}$$

Furthermore, we want to impose the following restrictions on the potential,

$$V'(\phi) < 0,\ V''(\phi) > 0,\ V'''(\phi) < 0.$$

Thus, in this framework we have the following action,

$$\begin{aligned} I &= \int (\frac{M_{pl}^2}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi))\sqrt{-g} - L(\psi_m^{(i)}, g_{uv}^{(i)})d^4x \\ &= \int (\frac{M_{pl}^2}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi) - \frac{1}{\sqrt{-g}}L(\psi_m^{(i)}, g_{uv}^{(i)}))\sqrt{-g}d^4x. \end{aligned} \tag{37}$$

However, to find the variation we need find

$$\begin{aligned} \delta(-\frac{1}{2}(\delta\phi)^2) &= \delta(-\frac{1}{2}(\phi)_{:u}(\phi)_{:u}) \\ &= \frac{-1}{2}(\delta(\phi)_{:u}(\phi)_{:u} + (\phi)_{:u}\delta(\phi)_{:u}) \\ &= -(\phi)_{:u}\delta(\phi)_{:u} \\ &= -(\phi)_{:u}(\delta\phi)_{:u} \end{aligned} \tag{38}$$

Now we take a variation in the direction of ϕ,

$$
\begin{aligned}
0 &= \delta I \\
&= \delta \int (\frac{M_{pl}^2}{2} R - \frac{1}{2}(\phi)_{:u}(\phi)_{:u} - V(\phi) - \frac{1}{\sqrt{-g}} L(\psi_m^{(i)}, g_{uv}^{(i)}))\sqrt{-g} d^4x \\
&= \delta \int (\frac{M_{pl}^2}{2} R - \frac{1}{2}(\phi)_{:u}(\phi)_{:u} - V(\phi) - \frac{1}{\sqrt{-g}} L(\psi_m^{(i)}, g_{uv}^{(i)}))\sqrt{-g} d^4x \\
&= \int (-(\phi)_{:u}\delta(\phi)_{:u} - \partial_\phi V(\phi)\delta\phi - \frac{1}{\sqrt{-g}} \frac{\partial L}{\partial \phi} \delta\phi)\sqrt{-g} d^4x \\
&= \int (-(\phi)_{:u}\delta(\phi)_{:u} - \partial_\phi V(\phi)\delta\phi - \sum_i \frac{1}{\sqrt{-g}} \frac{\partial L}{\partial g_{uv}^{(i)}} \frac{\partial g_{uv}^{(i)}}{\partial \phi} \delta\phi)\sqrt{-g} d^4x \\
&= \int ((\phi)_{:u:u}\delta\phi - \partial_\phi V(\phi)\delta\phi - \sum_i \frac{1}{\sqrt{-g}} \frac{\partial L}{\partial g_{uv}^{(i)}} \frac{\partial g_{uv}^{(i)}}{\partial \phi} \delta\phi)\sqrt{-g} d^4x \\
&= \int (\Delta\phi - \partial_\phi V(\phi) - \sum_i \frac{1}{\sqrt{-g}} \frac{\partial L}{\partial g_{uv}^{(i)}} \frac{\partial g_{uv}^{(i)}}{\partial \phi})\delta\phi\sqrt{-g} d^4x
\end{aligned}
\tag{39}
$$

where we used the following formula from Riemannian geometry,

$$
\int_V (\phi)_{:u}\delta\phi\sqrt{-g} dV = \int_{\partial V} \phi\partial_u N\sqrt{-g} dS.
$$

Thus, from (39),

$$
\begin{aligned}
\Delta\phi &= \frac{\partial V(\phi)}{\partial \phi} + \sum_i \frac{1}{\sqrt{-g}} \frac{\partial L}{\partial g_{uv}^{(i)}} \frac{\partial g_{uv}^{(i)}}{\partial \phi} \\
&= \frac{\partial V(\phi)}{\partial \phi} + \sum_i \frac{1}{\sqrt{-g}} \frac{\partial L}{\partial g_{uv}^{(i)}} \frac{\partial \exp(\frac{2b_i\phi}{M_{pl}}) g_{uv}}{\partial \phi} \\
&= \frac{\partial V(\phi)}{\partial \phi} + \sum_i \frac{1}{\sqrt{-g}} \frac{\partial L}{\partial g_{uv}^{(i)}} \frac{2b_i}{M_{pl}} g_{uv} \\
&= \frac{\partial V(\phi)}{\partial} + \sum_i \frac{2}{\sqrt{-g}} \frac{\partial L}{\partial g_{uv}^{(i)}} \frac{b_i}{M_{pl}} g_{uv} \\
&= \frac{\partial V(\phi)}{\partial \phi} + \sum_i T^{uv} \frac{b_i}{M_{pl}} g_{uv}.
\end{aligned}
\tag{40}
$$

3.1 $V(\phi) = M^{4+n}\phi^{-n}$

We consider the case where the potential follows an inverse power-law, namely,

$$
V(\phi) = M^{4+n}\phi^{-n}
$$

with this we should have the 7th equation from Khoury and Weltman. Now Khoury and Weltman, assumed that $g_{uv}T^{uv} \approx \rho_i^*$ where ρ_i^* represent the energy density. More-so, the energy density has the form $-\rho_i = \rho_i^* \exp(\frac{3b_i\phi}{M_{pl}})^2$

$$\begin{aligned}
\Delta\phi &= \frac{\partial V(\phi)}{\partial\phi} + \sum_i T^{uv}\frac{b_i}{M_{pl}}g_{uv} \\
&= \frac{\partial V(\phi)}{\partial\phi} + \sum_i \frac{b_i}{M_{pl}}\rho_i \exp(\frac{b_i\phi}{M_{pl}}).
\end{aligned} \tag{41}$$

Now, we can express the dynamics of ϕ with a single potential,

$$V_{eff} = V(\phi) + \sum_i \rho_i \exp(\frac{b_i\phi}{M_{pl}}) \tag{42}$$

so that

$$\partial_\phi V_{eff} = \frac{\partial V(\phi)}{\partial\phi} + \sum_i \frac{b_i}{M_{pl}}\rho_i \exp(\frac{b_i\phi}{M_{pl}}) \tag{43}$$

We now plot the effective potential.

```
Manipulate[
 Plot[M^(4 + n)/x^n + p Exp[b x], {x, 0, 10}], {M, 0, 100}, {n, 0,
  10}, {p, 0, 10}, {b, 0, 10}]
```

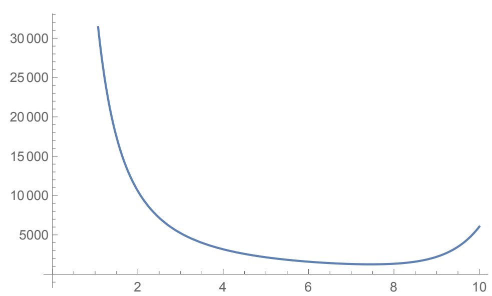

Figure 1: Small ρ

2. I do not know how to rationalize this step. As such I will assume it to hold.

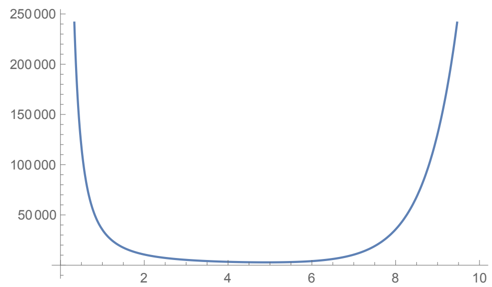

Figure 2: Large ρ

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