Stability of Minimal Surfaces

A Study on Scherk Surface

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Outline

Area

Definition

The Area $A(R)$ of the part $\boldsymbol{\sigma}(\mathrm{R})$ of the surface patch $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ corresponding to a region $R \subset U$ is

$$\begin{aligned} A(R) &= \int_R \|\boldsymbol{\sigma}_u \wedge \boldsymbol{\sigma}_v\| dudv \\ &= \int_R \sqrt{(\boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u)(\boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v) - (\boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v)^2} dudv \\ &= \int_R \sqrt{EG - F^2} dudv. \end{aligned}$$

Weingarten map

Definition

Let $\boldsymbol{\sigma}$ be a regular curve, i.e. $\boldsymbol{\sigma}_u \wedge \boldsymbol{\sigma}_v$ is non-vanishing. The Weingarten map $S = S(u, v)$ is a linear map of the tangent space T_p into itself defined as follows: if $a = a_1\boldsymbol{\sigma}_u + a_2\boldsymbol{\sigma}_v$ then,

$$Sa = -a_1 N_u - a_2 N_v.$$

S is also known as the shape operator.

Fundamental Forms

Definition

We can define the following three symmetric bilinear forms:

$$I(v,w) = v \cdot w$$
$$II(v,w) = Sv \cdot w$$
$$III(v,w) = Sv \cdot Sw.$$

We will use the following notation, when appropriate, for the coeffcients of the fundamental forms:

$$g_{ij} = \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$$
$$b_{ij} = -N_i \cdot \boldsymbol{\sigma}_j$$
$$c_{ij} = N_i \cdot N_j.$$

Notation

$$g_{ij} = (g_{ij}) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

$$b_{ij} = (b_{ij}) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

$$W = \sqrt{g} = \sqrt{EG - F^2} = \sqrt{|g_{ij}|}$$

Curvature

The Gaussian curvature K is given by

$$K = k_1 k_2 = |b_i^j| = |g^{jk} b_{ki}|$$

and the mean curvature H is given by

$$2H = k_1 + k_2 = b_{ij} g^{ij}.$$

First Variation

Theorem

The surface $\boldsymbol{\sigma} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a critical point of the area function A if and only if its mean curvature H is identically zero.

Proof outline. Choose a bounded domain $D \subset U$ and a differentiable function $\phi : \bar{D} \to \mathbb{R}$ where $\bar{D} = D \cup \partial D$ and $(u, v) \in D$. Define

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + t\phi \mathbf{N}$$

for some fixed $t \in (-\epsilon, \epsilon)$. Let $\tilde{g}$ and g be the metric tensor associated with $\tilde{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}$, respectively.

First Variation Proof

$$A(\tilde{\boldsymbol{\sigma}}) = \int_{\bar{D}} \sqrt{\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}^2}\, du\ dv.$$

where

$$\begin{aligned}
\tilde{g}_{11} &= \tilde{\boldsymbol{\sigma}}_u \cdot \tilde{\boldsymbol{\sigma}}_u \\
&= (\boldsymbol{\sigma}_u + t\phi_u \boldsymbol{N} + t\phi \boldsymbol{N}_u)^2 \\
&= \boldsymbol{\sigma}_u^2 + 2t\phi\langle \boldsymbol{N}_u | \boldsymbol{\sigma}_u \rangle + O(t^2) \\
&= g_{11} + 2t\phi b_{11} + O(t^2),
\end{aligned}$$

$$\tilde{g}_{12} = g_{12} + t\phi(\langle \boldsymbol{N}_u | \boldsymbol{\sigma}_v \rangle + \langle \boldsymbol{N}_v | \boldsymbol{\sigma}_u \rangle) + O(t^2).$$

and

$$\tilde{g}_{22} = g_{22} + 2t\phi b_{22} + O(t^2),$$

First Variation Proof

$$\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}^2 = g_{11}g_{22} - g_{12}^2 + 4t\phi W^2 H + O(t^2).$$

The are function is then,

$$A(\tilde{\boldsymbol{\sigma}}) = \int_{\bar{D}} \sqrt{W^2(1 + 4t\phi H)}\ du\ dv + O(t^2)$$

and using the expansion $\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2)$, then,

$$\begin{aligned} A(\tilde{\boldsymbol{\sigma}}) &= \int_{\bar{D}} W(1 + 2t\phi H)\ du\ dv + O(t^2) \\ &= \int_{\bar{D}} W\ du\ dv + 2t \int_{\bar{D}} HW\phi\ du\ dv + O(t^2) \end{aligned}$$

First Variation Proof

Taking the derivative with respect to t,

$$D[A(\boldsymbol{\sigma})](\phi) = \frac{d}{dt}A(\tilde{\boldsymbol{\sigma}}) = \int_{\bar{D}} 2\phi HW \; du \; dv.$$

If $D[A(\boldsymbol{\sigma})](\phi) = 0$ then $\int_{\bar{D}} 2\phi HW \; du \; dv = 0$. Since this is true for every ϕ, we can choose $\phi = H$ and so

$$\int_{\bar{D}} H^2 W \; du \; dv = 0.$$

However, since $H^2 \geq 0$ everywhere we must have $H = 0$.

Definition

We say $\boldsymbol{\sigma}$ is a minimal surface if it has zero mean curvature.

Holomorphic function

Definition

Holomorphic function. If a complex function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is holomorphic (or analytical), then u and v have first partial derivatives with respect to x and y, and satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Complex Analysis

Theorem

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a conformal surface patch. Consider the complex coordinates in the plane of which U is an open subset by setting $z = u + \mathrm{i}v$ for $(u, v) \in U$. We define $\phi(z) = \boldsymbol{\sigma}_u - \mathrm{i}\boldsymbol{\sigma}_v$ and we say $\boldsymbol{\sigma}$ is minimal if and only if the function ϕ is holomorphic on U.

Complex Analysis

Theorem

If $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ *is a conformal minimal surface, the vector-valued holomorphic function* $\phi = (\phi_1, \phi_2, \phi_3)$*, defined in the previous slide, satisfies the following:*

(a.) ϕ *is nowhere zero*

(b.) $\Sigma\phi_k^2 = 0$ *for* $k = 1, 2, 3.$

Conversely, if U *is simple-connected and if* ϕ_k *are holomorphic functions on* U *satisfying* (a.) *and* (b.)*, there is a conformally parametrized minimal surface* $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ *such that* ϕ *satisfies* $\phi(z) = \boldsymbol{\sigma}_u - i\boldsymbol{\sigma}_v$*. Moreover,* $\boldsymbol{\sigma}$ *is uniquely determined by* ϕ_k *up to a translation.*

Weierstrass-Enneper Representation

Theorem

Weierstrass-Enneper Representation. *Let $f(z)$ be a holomorphic function on an open set $U \subset \mathbb{C}$, not indetically zero, and let $g(z)$ be a meromorphic function on U such that if $z_0 \in U$ is a m^{th} pole of of g, for $m \geq 1$, then z_0 is also a zero of f of order greater thatn $2m$. Then,*

$$\phi = (\frac{1}{2}f(1-g^2),\ \frac{i}{2}f(1+g^2),\ fg)$$

and it satisfies the previous theorem. Conversely, every holomorphic function ϕ satisfying these conditions arises in this way.

Observe that if $\phi_1 - i\phi_2$ is not identically zero, we can define

$$f = \phi_1 - i\phi_2 \text{ and } g = \frac{\phi_3}{\phi_1 - i\phi_2}.$$

Parametrization

Scherek's surface can be given by the parametric equation

$$\boldsymbol{\sigma}(u,v) = (arg\frac{z+\mathrm{i}}{z-\mathrm{i}},\ arg\frac{z+1}{z-1},\ log|\frac{z^2+1}{z^2-1}|)$$

with $z \neq \{\pm 1, \pm \mathrm{i}\}$ and where $\arg z$ is the angle that the real axis makes with z.

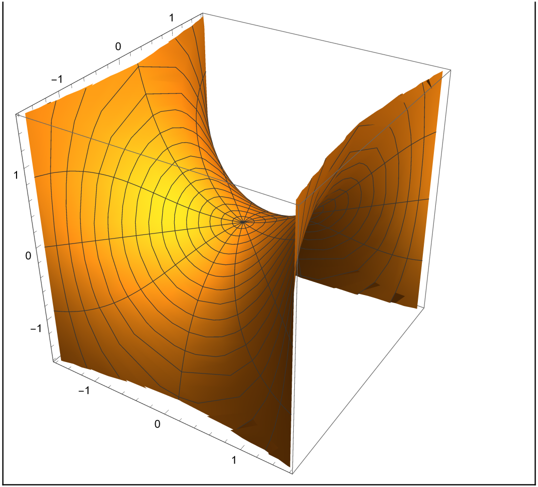
We can construct a complex function $\phi_k = (\sigma^k)_u - \mathrm{i}(\sigma^k)_v$

$$\phi_1 = -\frac{2}{1+z^2},\ \phi_2 = -\frac{2\mathrm{i}}{1-z^2},\ \text{and } \phi_3 = \frac{4z}{1-z^4}$$

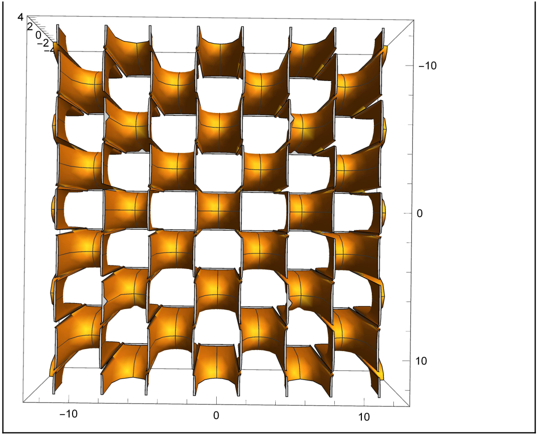
and from this we get the Weierstrass-Enneper data

$$f = \frac{4}{z^4-1} \text{ and } g = -z.$$

Plot



Plot



Second Variation

Theorem

Suppose $\boldsymbol{\sigma} = \boldsymbol{\sigma}(u, v)$ is regular in the closure of Ω, then

$$\begin{aligned} D^2[A(\boldsymbol{\sigma})](\phi) &= \frac{d^2}{dt^2} A(\tilde{\boldsymbol{\sigma}}) \\ &= \int_\Omega (|\nabla_u \phi|^2 + 2K\phi^2)\sqrt{|g_{ij}|}\; du\; dv \\ &= \int_\Omega (|\nabla_u \phi|^2 + 2K\phi^2)\; dA \\ &= \int_\Omega (|\nabla \phi|^2 + 2KW\phi^2)\; du\; dv. \end{aligned}$$

Code

We found that $\phi = K$ destabilizes the integral,

$$\int_{\Omega} (\phi \frac{1}{\sqrt{|(g_{jk})|}} \sum_{j,k} \partial_j (g^{jk} \sqrt{|(g_{jk})|} \partial_k \phi) + 2|g^{jk} b_{ki}| \sqrt{|(g_{ij})|} \phi^2) dudv$$

$$\approx -2.4439.$$