Now, we construct a mapping $\theta : I\!R^2 \rightarrow I\!R^2$ according to (II.9)

$$\begin{aligned}\theta(x) &= \beta(h(x)) + A^{-1}\beta(h(f(x))) \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} x_1 + \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (x_2 - \sin x_2) \\ &= \begin{pmatrix} x_1 \\ -x_1 - 2(x_2 - \sin x_2) \end{pmatrix}. \end{aligned} \tag{III.42}$$

Clearly, $\theta$ is invertible only if $x_2 \neq 2m\pi$, where $m$ is any integer. However, $(x_1, x_2) = (2m\pi, 2m\pi)$ are the fixed points of the system, and an observer is not necessary for these cases. Thus that $\theta(x)$ is not invertible at $x_2 = 2m\pi$ does not affect the proposed approach.

## IV. Final Remarks

In this note, we present a new and simple approach for the construction of a change of coordinates for the design of nonlinear discrete-time observer. Since any continuous-time systems have to be put into a certain discretized pattern for both numerical calculations and simulations, discretization is the only possibility for the computation of a continuous-time system. The proposed method can be applied to continuous-time systems once their discretized structures are determined, and a further discussion will be reported elsewhere.

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# A Sufficiently Smooth Projection Operator

Z. Cai, M. S. de Queiroz, and D. M. Dawson

***Abstract*—In this note, we introduce a new parameter projection operator for adaptation laws. The projection operator replaces the common Lipschitz continuity property with the stronger property of arbitrarily many times continuous differentiability. The proposed projection is useful for backstepping-based, robust adaptive controllers which require multiple differentiations of the adaptation law.**

***Index Terms*—Adaptation law, backstepping, disturbance, projection operator, robust adaptive control.**

## I. Introduction

We consider a general class of systems

$$\begin{aligned}\dot{x} &= f(x,\theta,d) + g(x)u \\ y &= h(x)\end{aligned} \tag{1}$$

affine in the uncertain constant parameter $\theta \in \mathbb{R}^p$, where $d(t)$ denotes unknown bounded external disturbances. The adaptive control problem consists of finding a dynamic state feedback controller

$$u = u(x,\hat{\theta}) \text{ (control law)} \tag{2}$$

$$\dot{\hat{\theta}} = \tau(x,\hat{\theta}) \text{ (adaption law)} \tag{3}$$

where $\hat{\theta}(t) \in \mathbb{R}^p$ denotes the estimate of $\theta$, that drives $y(t)$ to zero or a small residual set while keeping all closed-loop signals bounded. It is well known that the performance of such adaptive controllers can significantly deteriorate and even become unstable when $d(t) \neq 0$ [1]. In this case, $\hat{\theta}(t)$ cannot be proven bounded, leading to the unboundedness of other closed-loop signals. Common approaches for counteracting this problem include adding a robustifying (leakage) term to the adaptation law (3) (e.g., the $\sigma$-modification [1] and the $e_1$-modification [5]), or using a projection operator [1], [3], [7] to confine $\hat{\theta}(t)$ to a bounded convex set in the parameter space. Leakage modifications have the major disadvantage of not recovering the disturbance-free $(d(t) \equiv 0)$ stability performance of the unmodified adaptation law if $d(t) = 0 \; \forall \, t \geq T > 0$. On the other hand, projection operators preserve the ideal properties of the adaptive controller if the disturbance disappears, but require parameter bounds to be known a priori. Generally, projection-based adaptation laws are discontinuous, which violates the Lipschitz condition for existence of classical solutions to differential equations. Furthermore, the discontinuity is not desirable from an implementation standpoint. This shortcoming however was addressed in [6] by introducing a boundary layer around the convex set that resulted in a Lipschitz continuous projection operator.

Manuscript received June 7, 2004; revised October 28, 2004, July 11, 2005, and August 10, 2005. Recommended by Associate Editor S. S. Ge. This work was supported in part by the National Science Foundation under Grants DMS-0424011 and CMS-0447576, and by the Louisiana Board of Regents under Grant LEQSF(2002-05)-RD-A-13.

Z. Cai is with the Department of Electrical and Computer Engineering, University of Iowa, Iowa City, IA 52242-1527 USA (e-mail: zhijun-cai@uiowa.edu).

M. S. de Queiroz is with the Department of Mechanical Engineering, Louisiana State University, Baton Rouge, LA 70803-6413 USA (e-mail: zcail@lsu.edu; mdequel@lsu.edu).

D. M. Dawson is with the Department of Electrical and Computer Engineering, Clemson University, Clemson, SC 29634-0915 USA (e-mail: ddawson@ces.clemson.edu).

Digital Object Identifier 10.1109/TAC.2005.861704

In parallel to this, the class of systems to which adaptive control can be applied was vastly broaden with the advent of the integrator backstepping design [3]. This design procedure allows one to adaptively stabilize systems (1) that are in the so-called parametric strict-feedback form [3]. In the standard adaptive backstepping design, the recursive procedure generates at each step a new adaptation law. Therefore, an $n$th-order parametric strict-feedback system (where $n \geq 3$) requires the $(n-2)$th derivative of the first adaptation law, the $(n-3)$th derivative of the second adaptation law, and so on. In the disturbance-free case, the tuning functions method [3] avoids the differentiation of the adaptation laws as well as overparametrization. However, when $d(t) \neq 0$ and $\dot{\hat{\theta}} = \mathrm{Proj}(x, \hat{\theta})$ where $\mathrm{Proj}(\cdot)$ denotes some projection operator, the tuning functions in the time derivative of the Lyapunov function will not be cancelled by the adaptation law as in the disturbance-free (no projection) case nor can be proven nonpositive. The recourse left is the standard adaptive backstepping procedure. This however would require differentiation of $\mathrm{Proj}(\cdot)$, which at best is Lipschitz continuous. That is, the derivatives of $\mathrm{Proj}(\cdot)$ would be discontinuous or not well defined.

In this note, we introduce a sufficiently smooth projection operator that circumvents the problem just described. The projection operator is a smoothened version of the one proposed in [6], which replaces the Lipschitz continuity with the stronger property of arbitrarily many times continuous differentiability while introducing minor or no modifications to the other projection properties. The new projection operator is for example directly applicable to the robust adaptive controller proposed in, for example, [2], which relied on the "practical differentiability" of the Lipschitz continuous projection operator. The note is organized as follows. In Section II, we develop the new projection operator and prove its properties. In Section III, a simple motivating example is presented to illustrate its use. A numerical simulation of the motivating example is provided in Section IV.

## II. Projection Operator

Let $\theta$ in (1) belong to the compact convex set $\Omega := \{\theta : \|\theta\| \leq \theta_0\}$, where $\theta_0$ is a known positive constant, and let $\tilde{\theta} := \theta - \hat{\theta}$ be the parameter estimation error. The standard Lipschitz continuous (normalized) projection operator is given by [4], [6]

1st Algorithm

$$\dot{\hat{\theta}} = \mathrm{Proj}_l(\mu, \hat{\theta}) = \begin{cases} \mu, & \text{if } p_l(\hat{\theta}) \leq 0 \\ \mu, & \text{if } p_l(\hat{\theta}) \geq 0 \text{ and } \nabla p_l(\hat{\theta})^T \mu \leq 0 \\ \left(I - \frac{p_l(\hat{\theta}) \nabla p_l(\hat{\theta}) \nabla p_l(\hat{\theta})^T}{\nabla p_l(\hat{\theta})^T \nabla p_l(\hat{\theta})}\right)\mu, & \text{otherwise} \end{cases} \tag{4}$$

where $\mu(t) \in \mathbb{R}^p$ is a known, $n$ times continuously differentiable ($\mathcal{C}^n$) variable

$$p_l(\hat{\theta}) = \frac{\hat{\theta}^T \hat{\theta} - \theta_0^2}{\varepsilon^2 + 2\varepsilon\theta_0} \tag{5}$$

$\varepsilon$ is an arbitrary positive constant, and $\nabla$ is the gradient operator. If $\hat{\theta}(0) \in \Omega$, the above projection operator is known to have the following properties [4], [6].

- P1) $\left\|\hat{\theta}(t)\right\| \leq \theta_0 + \varepsilon \; \forall\, t \geq 0$.
- P2) $\tilde{\theta}^T \mathrm{Proj}_l(\mu, \hat{\theta}) \geq \tilde{\theta}^T \mu$.
- P3) $\left\|\mathrm{Proj}_l(\mu, \hat{\theta})\right\| \leq \|\mu\|$.
- P4) $\mathrm{Proj}_l(\mu, \hat{\theta})$ is Lipschitz continuous.

Using (4) and (5) as a starting point, our goal is to design a projection operator that replaces property P4 with the stronger property of $\mathcal{C}^n$ with minor or no modifications to properties P1)–P3). To this end, first note that (4) is equivalent to the single equation

$$\mathrm{Proj}_l(\mu, \hat{\theta}) = \mu - \frac{\left(\frac{p_l(\hat{\theta}) + |p_l(\hat{\theta})|}{2}\right)\left(\frac{\nabla p_l(\hat{\theta})^T \mu + |\nabla p_l(\hat{\theta})^T \mu|}{2}\right) \nabla p_l(\hat{\theta})}{\nabla p_l(\hat{\theta})^T \nabla p_l(\hat{\theta})}. \tag{6}$$

2nd Algorithm

We now replace (6) and (5) with the following smoothened (unnormalized) function:

$$\dot{\hat{\theta}} = \mathrm{Proj}_d(\mu, \hat{\theta}) = \mu - \frac{\eta_1 \eta_2}{4\left(\varepsilon^2 + 2\varepsilon\theta_0\right)^{n+1} \theta_0^2} \nabla p_d(\hat{\theta}) \tag{7}$$

where

$$p_d(\hat{\theta}) = \hat{\theta}^T \hat{\theta} - \theta_0^2 \tag{8}$$

$$\eta_1 = \begin{cases} p_d^{n+1}(\hat{\theta}), & \text{if } p_d(\hat{\theta}) > 0 \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

$$\eta_2 = \frac{1}{2} \nabla p_d(\hat{\theta})^T \mu + \sqrt{\left(\frac{1}{2} \nabla p_d(\hat{\theta})^T \mu\right)^2 + \delta^2} \tag{10}$$

and $\delta$ is an arbitrary positive constant. The characteristics of the proposed projection operator are delineated in the following theorem.

*Theorem 1:* If $\hat{\theta}(0) \in \Omega$, then the adaptation law with the projection given by (7)–(10) has the following properties (the bar on the number denotes a modified property).

- P1) $\left\|\hat{\theta}(t)\right\| \leq \theta_0 + \varepsilon \; \forall\, t \geq 0$.
- P2) $\tilde{\theta}^T \mathrm{Proj}_d(\mu, \hat{\theta}) \geq \tilde{\theta}^T \mu$.
- P$\bar{3}$)

$$\left\|\mathrm{Proj}_d(\mu, \hat{\theta})\right\| \leq \|\mu\| \left[1 + ((\theta_0 + \varepsilon)/\theta_0)^2\right] + \left((\theta_0 + \varepsilon)/(2\theta_0^2)\right)\delta.$$

- P$\bar{4}$) $\mathrm{Proj}_d(\mu, \hat{\theta})$ is $\mathcal{C}^n$.

*Proof:*

*Property P1:* When $p_d(\hat{\theta}) \leq 0$, we know directly from (8) that $\left\|\hat{\theta}\right\| \leq \theta_0 < \theta_0 + \varepsilon$. Now, let

$$V_\theta := \frac{1}{2} \hat{\theta}^T \hat{\theta} \tag{11}$$

whose derivative along (7) is given by

$$\dot{V}_\theta = \hat{\theta}^T \mu - \frac{\eta_1 \eta_2}{2\left(\varepsilon^2 + 2\varepsilon\theta_0\right)^{n+1} \theta_0^2} \hat{\theta}^T \hat{\theta}. \tag{12}$$

When $p_d(\hat{\theta}) > 0$ and $\nabla p_d(\hat{\theta})^T \mu = 2\hat{\theta}^T \mu \leq 0$, we have

$$\dot{V}_\theta \leq -\frac{\eta_1 \eta_2}{2(\varepsilon^2 + 2\varepsilon\theta_0)^{n+1} \theta_0^2} \hat{\theta}^T \hat{\theta} < 0 \tag{13}$$

since $\eta_1, \eta_2 > 0$. Thus, $\left\|\hat{\theta}\right\|$ is brought back into $\Omega$. When $p_d(\hat{\theta}) > 0$ and $\nabla p_d(\hat{\theta})^T \mu > 0$, we know from (10) that $\eta_2 > \nabla p_d(\hat{\theta})^T \mu$ and, therefore

$$\dot{V}_\theta \leq \left[1 - \frac{\left(\hat{\theta}^T \hat{\theta} - \theta_0^2\right)^{n+1}}{\left(\varepsilon^2 + 2\varepsilon\theta_0\right)^{n+1} \theta_0^2} \hat{\theta}^T \hat{\theta}\right] \hat{\theta}^T \mu \tag{14}$$

upon substitution from (8). Since $\hat{\theta}^T \mu > 0$, we need to prove the bracketed term in (14) is negative. Assume $\left\|\hat{\theta}\right\| = \theta_0 + \varepsilon$, then

$$1 - \frac{\left(\hat{\theta}^T \hat{\theta} - \theta_0^2\right)^{n+1}}{\left(\varepsilon^2 + 2\varepsilon\theta_0\right)^{n+1} \theta_0^2} \hat{\theta}^T \hat{\theta} = 1 - \frac{\hat{\theta}^T \hat{\theta}}{\theta_0^2} < 0. \tag{15}$$

Thus, whenever $\left\|\hat{\theta}\right\| = \theta_0 + \varepsilon, \dot{V}_\theta < 0$ and $\left\|\hat{\theta}\right\|$ is drawn toward $\Omega$.

*Property P2:* When $p_d(\hat{\theta}) \leq 0$, we have $\eta_1 = 0$ so $\tilde{\theta}^T \mathrm{Proj}_d(\mu, \hat{\theta}) = \tilde{\theta}^T \mu$. For the case when $p_d(\hat{\theta}) > 0$, we first calculate

$$\begin{aligned}\tilde{\theta}^T \mathrm{Proj}_d(\mu, \hat{\theta}) &\geq \tilde{\theta}^T \mu \\ &\Leftrightarrow \tilde{\theta}^T (\mathrm{Proj}_d(\mu, \hat{\theta}) - \mu) \geq 0 \\ &\Leftrightarrow \tilde{\theta}^T \hat{\theta} \left[\frac{1}{2(\varepsilon^2 + 2\varepsilon\theta_0)^{n+1}\theta_0^2}\eta_1\eta_2\right] \leq 0 \qquad (16)\end{aligned}$$

upon use of (7). Since the bracketed term in (16) is positive, we need to prove $\tilde{\theta}^T \hat{\theta} \leq 0$

$$\begin{aligned}\tilde{\theta}^T \hat{\theta} &= (\theta - \hat{\theta})^T \hat{\theta} \\ &= -\frac{1}{2}(\hat{\theta}^T\hat{\theta} - \theta^T\theta) - \frac{1}{2}(\hat{\theta} - \theta)^T(\hat{\theta} - \theta) \\ &= -\frac{1}{2}(\left\|\hat{\theta}\right\|^2 - \|\theta\|^2) - \frac{1}{2}\left\|(\hat{\theta} - \theta)\right\|^2 \\ &\leq -\frac{1}{2}(\left\|\hat{\theta}\right\|^2 - \|\theta\|^2) < 0 \qquad (17)\end{aligned}$$

since $\|\theta\| \leq \theta_0$ and $\left\|\hat{\theta}\right\| > \theta_0$.

*Property* $\bar{\mathrm{P3}}$*:* When $p_d(\hat{\theta}) \leq 0$, $\left\|\mathrm{Proj}_d(\mu, \hat{\theta})\right\| = \|\mu\|$. When $p_d(\hat{\theta}) > 0$, we know that due to property P1 $\theta_0^2 < \hat{\theta}^T\hat{\theta} \leq (\theta_0 + \varepsilon)^2$ and, therefore

$$\frac{\hat{\theta}^T\hat{\theta} - \theta_0^2}{\varepsilon^2 + 2\varepsilon\theta_0} \leq 1. \qquad (18)$$

Now, calculating the norm of $\mathrm{Proj}_d(\mu, \hat{\theta})$

$$\begin{aligned}\left\|\mathrm{Proj}_d\left(\mu, \hat{\theta}\right)\right\| &\leq \|\mu\| + \left\|\left(\frac{\hat{\theta}^T\hat{\theta} - \theta_0^2}{\varepsilon^2 + 2\varepsilon\theta_0}\right)^{n+1} \frac{\eta_2\hat{\theta}}{2\theta_0^2}\right\| \\ &\leq \|\mu\| + \left\|\frac{\hat{\theta}\eta_2}{2\theta_0^2}\right\| \qquad (19)\end{aligned}$$

where (18) was used. Since $\eta_2 = \hat{\theta}^T\mu + \sqrt{(\hat{\theta}^T\mu)^2 + \delta^2} < 2|\hat{\theta}^T\mu| + \delta$, the right-hand side of (19) can be upper bounded by

$$\begin{aligned}\left\|\mathrm{Proj}_d(\mu, \hat{\theta})\right\| &\leq \|\mu\| + \left\|\frac{\hat{\theta}}{2\theta_0^2}\right\| \left(2\left\|\hat{\theta}^T\mu\right\| + \delta\right) \\ &\leq \|\mu\| + \frac{\theta_0 + \varepsilon}{2\theta_0^2}\left(2\left\|\hat{\theta}\right\|\|\mu\| + \delta\right) \\ &\leq \|\mu\|\left(1 + \frac{(\theta_0 + \varepsilon)^2}{\varepsilon_0^2}\right) + \frac{\theta_0 + \varepsilon}{2\theta_0^2}\delta. \qquad (20)\end{aligned}$$

*Property* $\bar{\mathrm{P4}}$*:* Since the product of $\mathcal{C}^n$ functions is also $\mathcal{C}^n$, we require the individual functions contained in (7) to be $\mathcal{C}^n$. The $n$th-order continuous differentiability of the gradient operator and the function $\sqrt{x^2 + \delta^2}\ \forall\, x \in \mathbb{R}$ is obvious; thus, we know $\nabla p_d(\hat{\theta})$ and $\eta_2(\hat{\theta}, \mu)$ are $\mathcal{C}^n$. For the function

$$f(x) = \begin{cases} x^{n+1}, & \text{if } x > 0 \\ 0, & \text{otherwise}\end{cases} \qquad (21)$$

we have

$$\frac{d^i f(x)}{dx^i} = \begin{cases} \frac{(n+1)!}{(n+1-i)!}x^{n+1-i}, & \text{if } x > 0 \\ 0, & \text{otherwise}\end{cases} \qquad (22)$$

for $i = 1, \ldots, n$ since

$$\begin{aligned}\lim_{x\to 0^+}\frac{f^{(i-1)}(x) - f^{(i-1)}(0)}{x - 0} &= \lim_{x\to 0^+}\frac{\frac{(n+1)!}{(n+2-i)!}x^{n+2-i} - 0}{x} \\ &= \frac{(n+1)!}{(n+2-i)!}\lim_{x\to 0^+}x^{n+1-i} = 0 \\ \lim_{x\to 0^-}\frac{f^{(i-1)}(x) - f^{(i-1)}(0)}{x - 0} &= \lim_{x\to 0^-}\frac{0 - 0}{x} = \lim_{x\to 0^-} 0 = 0.\end{aligned} \qquad (23)$$

It then follows that $\eta_1(\hat{\theta})$ is $\mathcal{C}^n$. ■

## III. EXAMPLE

We consider the following system as a motivating example for the proposed projection:

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi(x_1)\theta + d \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \\ y &= x_1\end{aligned} \qquad (24)$$

where $x_i(t) \in \mathbb{R}, i = 1, 2, 3, 4$ are the measurable states, $\varphi(x_1) \in \mathbb{R}$ is a known $\mathcal{C}^3$ nonlinearity, and $u(t) \in \mathbb{R}$ is the control input. We assume the disturbance is bounded by $|d(t)| \leq \bar{d}$.

First, note that when $d(t) \equiv 0$, the adaptive controller given in [3, Sec. 4.2] ensures $y(t) \to 0$ as $t \to \infty$ and $x_i(t), \hat{\theta}(t), u(t) \in \mathcal{L}_\infty\ \forall\, x_i(0), i = 1, 2, 3, 4$. When $d(t) \neq 0$, if we modify the adaptation law in [3, eq. (4.138)] to $\dot{\hat{\theta}} = \Gamma\mathrm{Proj}_d\left(\tau_4, \hat{\theta}\right)$, the term

$$\sum_{k=1}^{3} z_{k+1}\frac{\partial\alpha_k}{\partial\hat{\theta}}\left(\Gamma\tau_4 - \dot{\hat{\theta}}\right) \qquad (25)$$

cannot be cancelled. Also, unlike the term $\tilde{\theta}\left(\tau_4 - \Gamma^{-1}\dot{\hat{\theta}}\right)$, the nonpositivity of (25) cannot be proven using property P2. In the following, we show that the problem can be easily solved by fusing the standard adaptive backstepping with the projection operator proposed in Section II.

Consider the continuous adaptive controller

$$u = \alpha_4\left(x_1, x_2, x_3, x_4, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4\right) \qquad (26)$$

$$\begin{aligned}\dot{\hat{\theta}}_i &= \gamma\mathrm{Proj}_d\left(\mu_i, \hat{\theta}_i\right) \\ \mu_i &= -\varphi(x_1)\frac{\partial\alpha_{i-1}}{\partial x_1}z_i, \qquad i = 1, 2, 3, 4\end{aligned} \qquad (27)$$

where

$$z_i = x_i - \alpha_{i-1} \qquad (28)$$

$$\begin{aligned}\alpha_i = &-c_i z_i - z_{i-1} + \varphi(x_1)\frac{\partial\alpha_{i-1}}{\partial x_1}\hat{\theta}_i \\ &+ \sum_{j=1}^{i-1}\left[\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1} + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\gamma\mathrm{Proj}_d\left(\mu_j, \hat{\theta}_j\right)\right] \\ &- k_n\left(\frac{\partial\alpha_{i-1}}{\partial x_1}\right)^2 z_i\end{aligned} \qquad (29)$$

$\gamma$, $c_i$, $k_n$ are positive constants, $\mathrm{Proj}_d(\cdot)$ is given by (7)–(10), and $\partial\alpha_0/\partial x_1 := -1$, $\alpha_0 = z_0 := 0$ for notational convenience as in [3]. Notice that $u$ will require $\mathrm{Proj}_d\left(\mu_j, \hat{\theta}_j\right), j = 1, 2, 3, 4$ to be $C^{3-j}$ so that $n = 3 - j$ in (9). The previous adaptive controller guarantees the ultimate boundedness of $y(t)$ and the boundedness of all closed-loop

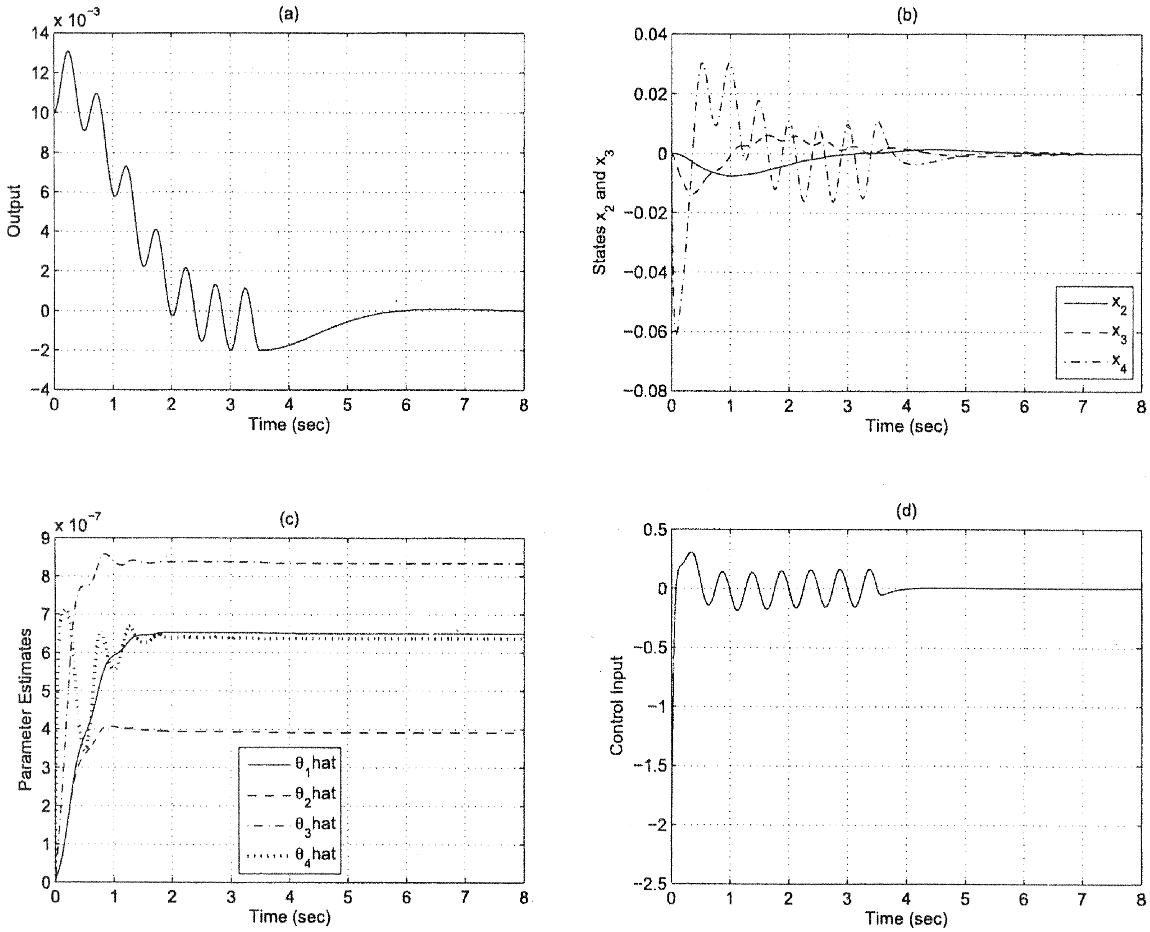


Fig. 1. (a) Output $y(t)$. (b) System states $x_i(t)$, $i = 2, 3, 4$. (c) Parameter estimates $\hat{\theta}_i(t)$, $i = 1, 2, 3, 4$. (d) Control input $u(t)$.

signals. The stability result can be proven by using the Lyapunov function

$$V := \frac{1}{2}\sum_{i=1}^{4}\left(z_i^2 + \gamma^{-1}\tilde{\theta}_i^2\right) \tag{30}$$

where $\tilde{\theta}_i := \theta - \hat{\theta}_i$. The derivative of (30) along the closed-loop system

$$\begin{aligned}\dot{z}_i =& -c_i z_i - z_{i-1} + z_{i+1} - \varphi(x_1)\frac{\partial \alpha_{i-1}}{\partial x_1}\tilde{\theta}_i \\ & - \frac{\partial \alpha_{i-1}}{\partial x_1} d - k_n\left(\frac{\partial \alpha_{i-1}}{\partial x_1}\right)^2 z_i \\ \dot{\hat{\theta}}_i =& -\gamma \mathrm{Proj}_d\left(\mu_i, \hat{\theta}_i\right)\end{aligned} \tag{31}$$

yields

$$\begin{aligned}\dot{V} \leq & -\sum_{i=1}^{4} c_i z_i^2 + \sum_{i=1}^{4}\tilde{\theta}_i\left[\mu_i - \mathrm{Proj}_d\left(\mu_i, \hat{\theta}_i\right)\right] \\ & + \sum_{i=1}^{4}\left[|z_i|\left|\frac{\partial \alpha_{i-1}}{\partial x_1}\right||d| - k_n\left(\frac{\partial \alpha_{i-1}}{\partial x_1}\right)^2 z_i^2\right]\end{aligned} \tag{32}$$

where (27) was used. After applying property P2 to the first bracketed term and completing the squares on the second bracketed term, we obtain

$$\dot{V} \leq -\sum_{i=1}^{4} c_i z_i^2 + \frac{d^2}{k_n} \leq -\min_i(c_i)\,\|\xi\|^2 + \frac{\bar{d}^2}{k_n} \tag{33}$$

where $\xi := [z_1\ z_2\ z_3\ z_4]^T$. Since $V$ is positive definite and $\dot{V}$ is negative semidefinite in the set $\left\{\xi \in \mathbb{R}^4 \mid \|\xi\| > \bar{d}\sqrt{1/(k_n \min_i(c_i))}\right\}$ and due to property P1, we have $z_i(t), \hat{\theta}_i(t) \in \mathcal{L}_\infty$ and $y(t)$ is ultimately bounded. It follows that (26), (27), (29), and (31) are bounded. It also follows from (33) and Barbalat's lemma that if $d(t) \to 0$ for $t \geq T > 0$, then $y(t) \to 0$ as $t \to \infty$.

*Remark 1:* Due to the exponent $n+1$ in (9), the use of the proposed projection operator in tandem with the backstepping design method may produce large control signals for high-order systems. In particular, the $j$th derivative of $\mathrm{Proj}_d\left(\mu_i, \hat{\theta}_i\right)$ with respect to $\hat{\theta}_i$ will have terms proportional to $(n+1)!/(n+1-j)!$.

## IV. Numerical Simulation

We performed a simulation of the motivating example (24) with

$$\varphi(x_1) = \frac{x_1^2}{4} \quad \theta = 2 \quad d(t) = \begin{cases} 0.02\sin 4\pi t, & 0 \leq t \leq 3.5 \\ 0, & \text{otherwise.}\end{cases} \tag{34}$$

The initial conditions of the system were set to $x_1(0) = 0.01$, $x_2(0) = x_3(0) = x_4(0) = 0$, and $\hat{\theta}_i(0) = 0$, $i = 1, 2, 3, 4$. The controller parameters were selected by trial-and-error until a good regulation performance was obtained. This tuning process resulted in the following values for the parameters: $c_1 = 0.01$, $c_2 = c_3 = 0.5$, $c_4 = k_n = 1$, $\gamma = 2$, $\theta_0 = 3$, $\varepsilon = 1$, and $\delta = 1$. Fig. 1 shows the simulation results for the output $y(t)$, states $x_i(t)$, $i = 2, 3, 4$, parameter estimates $\hat{\theta}_i(t)$, $i = 1, 2, 3, 4$, and control input $u(t)$. The simulation shows the adaptive controller effectively regulating the output to zero after the disturbance disappears. Despite Remark 1, we were easily able to tune the control gains in this example to yield a small control input. However, we noticed the control input is highly sensitive to the gains $c_1$ and $k_n$, which are the ones that get propagated most through the backstepping procedure. For example, increasing $k_n$ from 1 to 1.5 causes the transient magnitude of the control signal to increase to $3 \times 10^5$.

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# Delay Robustness of a Class of Nonlinear Systems and Applications to Communication Networks

Xingzhe Fan and Murat Arcak


***Abstract*—In this note, we first study a class of nonlinear interconnected systems and derive sufficient conditions for their global asymptotic stability in the presence of time delays. These conditions give estimates for the maximum admissible delays that the nominal delay-free model can tolerate without losing stability. We then study a subclass with a specific interconnection structure that appears, among other applications, in several communication networks. To characterize their robustness to delays in backward and forward channels, we combine our main result with a passivity-based stability analysis, applicable to this subclass. We conclude with application examples from Kelly's network flow model of the Internet, and code-division multiple access (CDMA) uplink power control of cellular networks.**


***Index Terms*—Code-division multiple access (CDMA), network flow control, passivity, time-delay systems.**

## I. INTRODUCTION AND PROBLEM FORMULATION

In this note, we consider the nonlinear time-delay system in Fig. 1, where the feedforward block is

$$\dot{x} = f(x) + g(x)u, \qquad x \in R^N \tag{1}$$


Manuscript received July 20, 2004; revised January 12, 2005. Recommended by Associate Editor R. Srikant. This work was supported by the National Science Foundation under Grant ECS-0238268.



The authors are with the Department of Electrical, Computer, and Systems Engineering, Rensselaer Polytechnic Institute, Troy, NY 12180 USA (e-mail: fanx@rpi.edu; arcak@ecse.rpi.edu).



Digital Object Identifier 10.1109/TAC.2005.861703


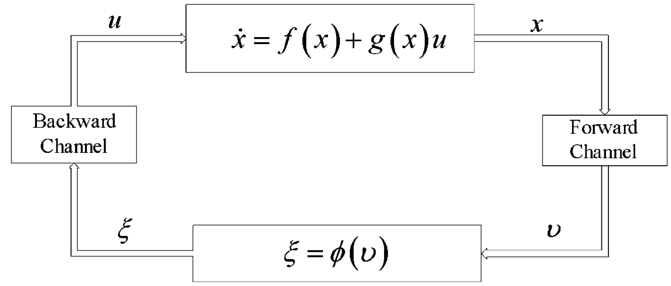


Fig. 1. Distributed feedback system with time-delays in forward and backward channels.

and the feedback block is a multivariable nonlinearity $\phi(\cdot) : \mathrm{R}^L \to \mathrm{R}^L$; that is

$$\xi = \phi(\upsilon), \qquad \xi \in R^L. \tag{2}$$

In the absence of time-delays, the outputs of the forward and backward channels are

$$\upsilon = \lambda(x) \quad \text{and} \quad u = \mu(\xi) \tag{3}$$

where $\lambda(\cdot) : \mathrm{R}^N \to \mathrm{R}^L$ and $\mu(\cdot) : \mathrm{R}^L \to \mathrm{R}^N$. In the presence of delays

$$\upsilon_j = \lambda_j\left(x_1\left(t - \tau_{1j}^f\right), x_2\left(t - \tau_{2j}^f\right), \ldots, x_N\left(t - \tau_{Nj}^f\right)\right), \quad j = 1, \ldots L \tag{4}$$

$$u_i = \mu_i\left(\xi_1\left(t - \tau_{i1}^b\right), \xi_2\left(t - \tau_{i2}^b\right), \ldots, \xi_L\left(t - \tau_{iL}^b\right)\right), \quad i = 1, \ldots N. \tag{5}$$

Our goal is to derive conditions under which global asymptotic stability of the delay-free model (1)–(3) is preserved, and to give estimates for the maximum admissible forward and backward delays

$$\bar{\tau}_f = \max_{i,j}\left\{\tau_{ij}^f\right\} \quad \bar{\tau}_b = \max_{i,j}\left\{\tau_{ij}^b\right\}. \tag{6}$$

Our main interest in this formulation is because it encompasses several classes of communication networks where the feedforward block represents control algorithms implemented by the users [e.g., rate control in TCP, and power control in code-division multiple access (CDMA)], and the feedback block represents the calculations carried out at remote locations (routers, base stations, etc.) Transmission and queuing delays can be significant in these networks, and threaten stability properties achievable for delay-free models. The class (1)–(5) encompasses other applications, such as control systems where feedback signals are generated at remote locations, and transmitted to the actuators via communication channels.

In Section II, we give conditions for system (1)–(2), (4)–(5) to remain globally asymptotically stable in the presence of small delays. The main idea is to treat the difference between the signals in (3) and their delayed variants in (4)–(5) as a disturbance acting on the nominal system. We then proceed to show that this disturbance satisfies an input-to-state stability (ISS) gain [2] with respect to the states of the nominal system, and derive conditions under which the nominal system satisfies a complementary ISS gain with respect to the disturbances. Stability is then established from the ISS Small-Gain Theorem [3], [4].

In Section III, we restrict our attention to a subclass of the system in Fig. 1, where the feedforward and feedback blocks consist of decentralized subsystems, and the forward and backward channels exhibit a "symmetric" structure that preserves passivity properties of the feedback loop. We then combine our main result with a passivity-based