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Exercise #2: Monte Carlo Simulation in Finance

PC Lab in Finance

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1 European options

The focus of this assignment lays on the numerical valuation of different options by applying a Monte Carlo Simulation. Generally a Monte-Carlo simulation simulates a sample of N of stochastic processes. In our case we simulate random paths for different stock prices. Since our stock prices follow a standard Brownian Motion process,...

$$\frac{\partial S(t)}{\partial S(t)} = rdt + \sigma w(t) \tag{1}$$

where w is $\mathcal{N}(0, T)$ standard Brownian Motion
σ= volatility of stock price
r= average rate of return

Therefore we can use following equations to simulate a stock price in time T:
Step-by-step approach:

$$S_{t+\delta t} = S_t e^{(r-q-\frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t} * \epsilon} \tag{2}$$

One-step approach:

$$S_T = S_0 e^{(r-q-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T} * \epsilon} \tag{3}$$

where ϵ is $\mathcal{N}(0, 1)$ and δt=$\frac{T}{n}$

To evaluate a European Option in Task 1 we use data set #4:
S=40, X=35, T=1, r=0.02, q=0.05, σ=0.22, seed=127

1.A Simulate stock prices

In this subtask we simulated 7 evolution's of a stock price with 7 different σ with respect to our data set, shown in Figure 1 . Therefore we applied the formula 2 since our stock prices follow a geometric Brownian motion. We used only ten steps n=10 for illustrative reasons. It can be clearly seen that for increasing σ has logically a stronger fluctuation due to the higher standard deviation.

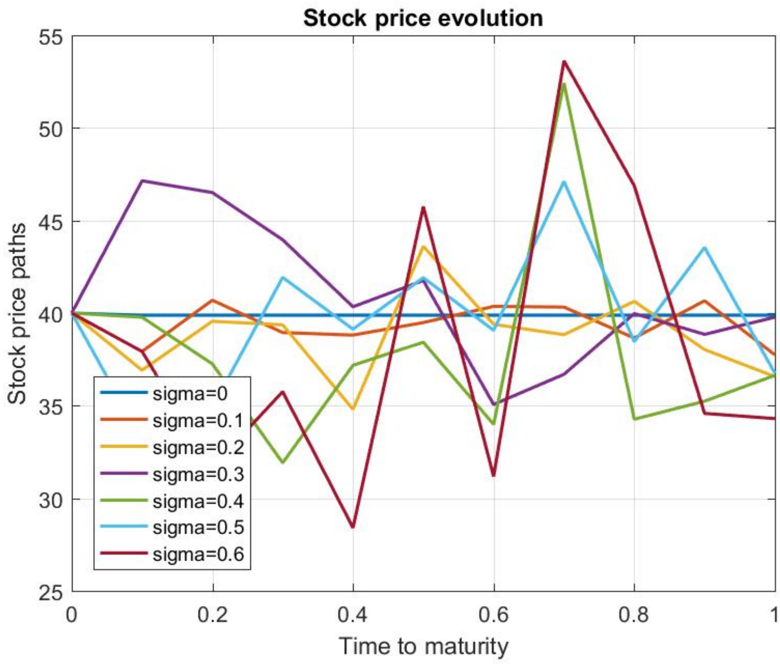

Figure 1: 7 realizations of a stock price depended on σ

1.B Option price valuation with Monte Carlo simulation

In this subtask we valuated a European option with the given data set by using the standard BSM approach and a Monte Carlo simulation. In order to achieve a Monte Carlo value we simulated up to 300,000 paths of the stock. Since a European option is not path dependent we used the more efficient one step-approach from equation 3 regarding computation time to simulate stock prices. In order to estimate the price of the option in general with this approach, one has to take the payoff with respect to:

$$Payoff_c = \max(0, S - X) \tag{4}$$

To calculate the final value, the mean of this vector was calculate and then discounted by with respect to r. In order to validate our estimation we compared our value with the value computed with the close-form solution, illustrated in figure 2. As you can clearly see the option price adjust to the theoretical BSM value with increasing N. Due to the Central Limit Theorem and the law of large numbers a standard normal distributed stochastic process converges to its population mean according to the theory, as in our case.

This can also be seen by taking the confidence interval into account. It was calculated with the following formula:

$$CI = c_{MC} \pm 1.96S \tag{5}$$

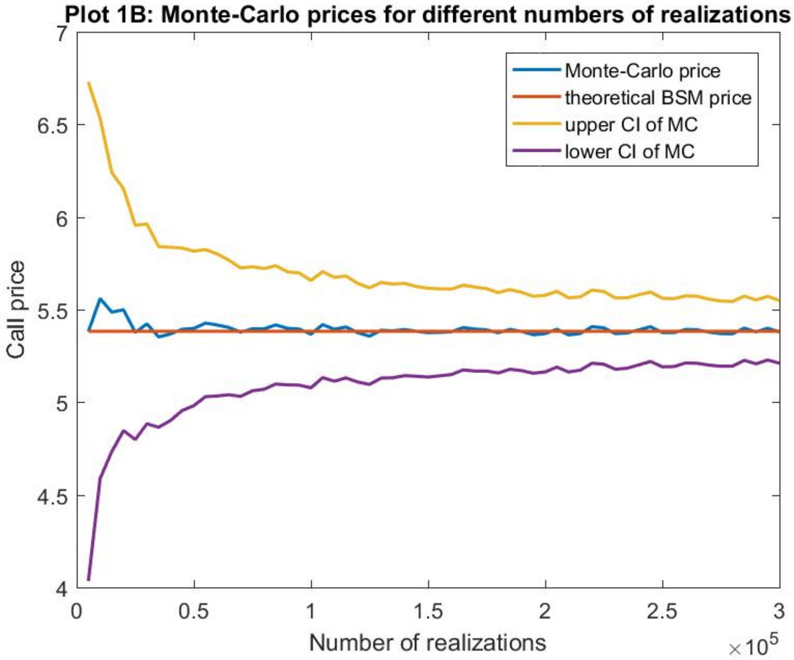

Figure 2: European Call Option value for different N

Note that if the number of samples N is large enough the estimator for a standard deviation is defined as(Ross, 2012):

$$S =$$

The interval that the true value of the estimation lays around the true value and shrinks with an increasing number of realizations which is also in line with theory.
Figure 3 again proves our results since the simulated stock prices follow a log-normal distribution. This kind of distribution is compared to a standard "bell form" positive right skewed and non-zero since stock prices can never be zero. Due to Geometric Brownian motion, the rate of return on a stock is continuously compounded and driven by a standard normal distribution as you can see in Formula 3 or 2. taking the logs of this formula this would imply a log-normal distribution for a future stock price in time T. (Hull, pp.43..)

1.C Antithetic Variates Technique

In this subtask we used the antithetic variates technique to reduce the variance of our estimation by applying following formula (Hull 2012, p. 453):

$$\hat{f} = \frac{f_1 + f_2}{2}$$

Where f1 is a function with a positive ϵ and f2 is a function with a negative ϵ.[1]

[1] The ϵ are of course standard-normal distributed.

Variance reduction techniques can be useful as it can be seen in Figure 4. It can improve your estimation results even if the sample size is small. If you compare the new estimated results with the results without variance reduction, the variance is clearly smaller for smaller sample sizes. Note that this technique could improve your computation efficiency, since you have to simulate less realizations to get good estimations.

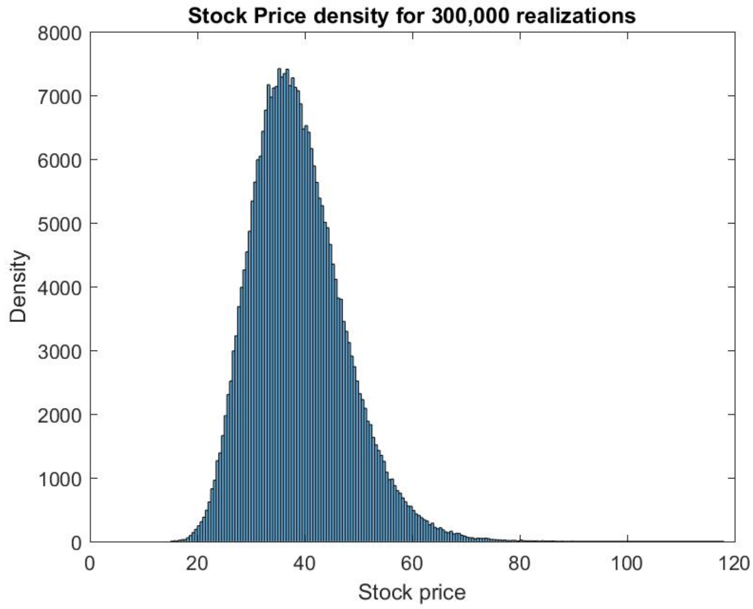

Figure 3: Stock price distribution with N=300,000

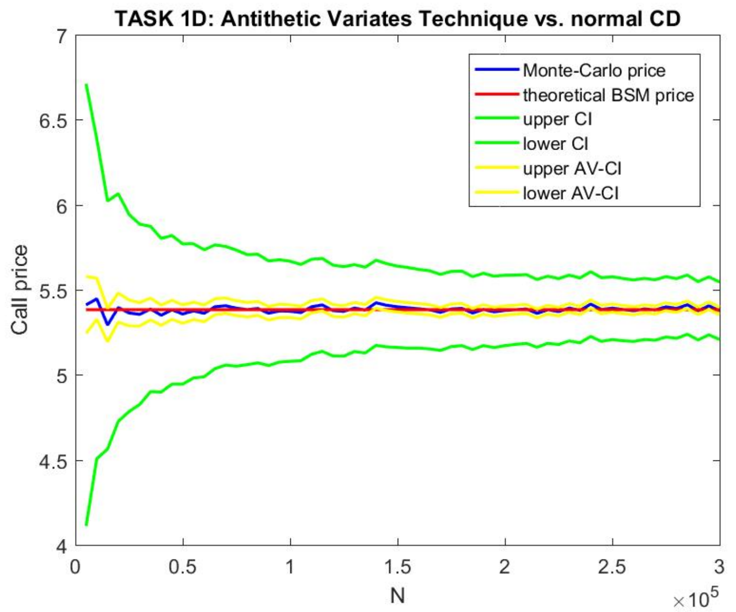

Figure 4: Normal approach vs. Antithetic Variates Technique

1.D Technical Notes on 1:

By running the file *TASK_1A.m* the different realizations from subtask 1A will be plotted. In order to do so we simply applied formula 2 to our data for a different σ-starting from 0 to 0.6.
All calculations and plots for subtask 1B to 1D can be called in the mainscript *TASK1_BCD.m*. In order to calculate and plot the values from subtask 1B we first set up a function called *BSMCall* which calculates the option price with the standard BSM approach. To calculate and plot the Monte Carlo value and its confidence-interval we created a function called *MC_price_euro* were we basically used the theoretical approach mentioned above

1. simulate N stock prices with respect to equation 3 and a standard normal distributed ϵ to get the *S-Vector*.
2. create a *Payoff-Vector* with respect to equation 4
3. calculate the mean of that vector
4. discount the mean with respect to r to get to your final value MC_price.

The confidence intervals were just basically calculated with equation 5. To plot the functions we again setup an own function called *plot_1B.m*. Therefore we created a for-loop for the number of realizations from N=5000, increasing by N=5000, up to 300,000 steps where for each step the calculations functions were called. To plot the histogram we simply used the Matlab command *histogram(S)*. To compare the reduced variance approach with the standard-approach extended our function from 1B as it can be seen in function *1D_plot.m*. Here we created a matrix with two different stocks where one contains a positive ϵ and the other a negative ϵ. In order to get the value and the plot for the reduced variance we just followed the same approach mentioned above.

2 Exchange Options

In this task we use a modified Monte-Carlo procedure as we did in Task 1 to value a European Exchange option. A European Exchange Option is defined as:

$$EEO_T = \max(S_{1,T} - S_{2,T}, 0). \tag{6}$$

with respect to the following data: $\begin{pmatrix} S_{1,t} \\ S_{2,t} \end{pmatrix} = \begin{pmatrix} 110 \\ 130 \end{pmatrix}$, q $= \begin{pmatrix} 0.06 \\ 0.02 \end{pmatrix}$, $\sigma = \begin{pmatrix} 0.2 \\ 0.4 \end{pmatrix}$, ρ= -0.5 and r= 0.04%.

2.A Monte-Carlo valuation

As you can see the value of the option depends on the behavior of two stock prices. Therefore we need to mind the correlation between both stocks. Therefore we use a slightly different formula for the stock price evolution with a constant standard deviation:

$$S_1 = S_{1,0} e^{(r-q-\frac{1}{2}\sigma_1^2)(T-t)+\sqrt{T-t}*\epsilon_1} \tag{7}$$

and vice versa for

$$S_2 = S_{2,0} e^{(r-q-\frac{1}{2}\sigma_2^2)(T-t)+\sqrt{T-t}*\epsilon_2} \tag{8}$$

Note that also here we use the one-step approach for the valuation of an European Exchange option since is not path-depend. Therefore this approach is applied since its more efficient with respect to computation time.
Due to the correlation between the stocks we have to set up a variance-covariance matrix first:

$$\begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

In order to get two correlated random variables for equation 7 and 8, we multiply two random variables and multiply it with our decomposed (Choletsky) variance-covariance matrix:

$$\begin{pmatrix} \sigma_1^2 & 0 \\ \sigma_1\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

where z is $\mathcal{N}(0, 1)$.

This correlated random variables now can be used to calculate the stock price evolution with respect to Formula 3. After simulating our correlated stock prices 500,000 times, we took the payoffs of our two S1 and S20 vectors, and then take the discounted mean in order to get our final price for a European Option. The confidence interval was calculated as in Task 1. We computed following results:

Table 1: Results of 2A

EEO_{MC}	Upper CI	Lower CI
13.94731	15.7049	12.189634

2.B Closed-form solution & computation for different ρ

Besides the simulated value of 2.A there also exists a closed form solution.[2] Since we used 500,000 realizations valuated price in the closed form solutions is almost equal to our estimated price.

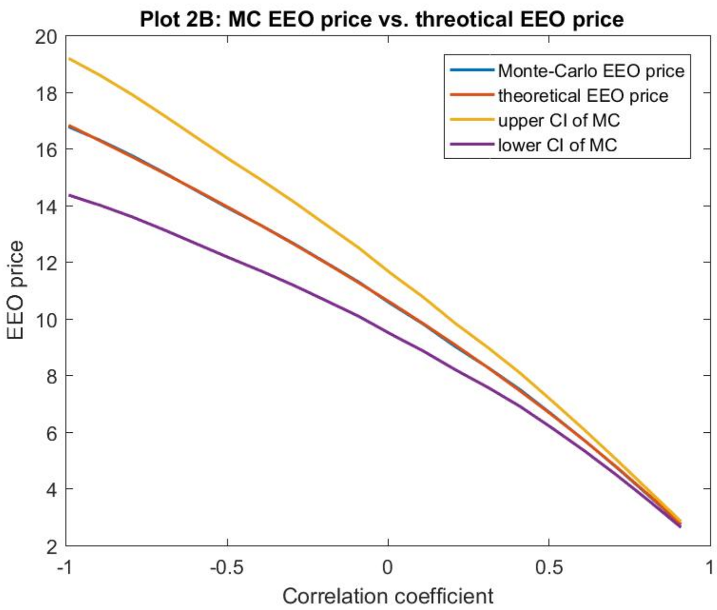

Figure 5: EEO price for different ρ

Figure 5 represents the behavior of our valuation of our option value for different correlation coefficient from -1 to 1.[3] By comparing the results one can clearly see that the more positively correlated the results are, the less the value of the option will be. Or said differently the price of the EEO is a decreasing function of the correlation factor. In figure it can be seen that the confidence interval converges to the true value with increasing correlation. This is again a result of the payoff structure of a European exchange option. If the difference between both stocks increases the price of the option will rise. This will be more likely the case if the correlation is negative which automatically implies a higher variation due to a bigger price interval. This behavior will be more clear for the other extrem value for a correlation of 1. Since both stocks are perfectly positively correlated, the value of the option will be either 0 or really small. Hence the variance of the values in this case is either low, compared to a high standard deviation and therefore the confidence interval will be either strait, as it is the case in Figure 5.

[2]Formulas for the closed-form solution can be found in the appendix of the exercise sheet # 2

[3]A Choletsky Decomposition only works for singular positve matricies, therefore we used -0.99 to 0.99 for computation

2.C 3D plots for different stock prices

First of all the two figures are based on the formulas given in the exercise. Precisely, we used the payoff function at maturity for figure 7 and the closed-form valuation formula for figure 7. Figure 6 shows the value of the European exchange option for given

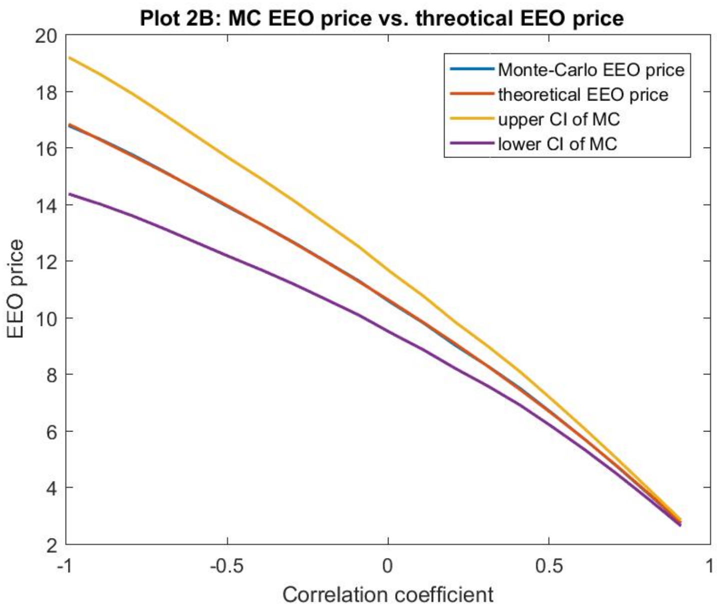

Figure 6: EEO price for different ρ

parameters at t=0. We can easily observe that the option is never zero independently of stock price evolution of the two assets. Since the stock price of asset 1 has a positive impact on the value of the option it becomes obvious that an increase in the stock price of asset 1 leads to an increase of the value of the option given the stock price of asset 2 does not change. Of course this effect is more pronounced if the price of asset 2 is low. Therefore the option reaches very high values for high stock prices in asset 1 and low stock prices in asset 2. It is also very important to understand how the stock prices of the two assets behave to each other. In other words we or an investor can look at the correlation coefficient to understand the dynamics of the stock prices. Since we deal with a negative correlation coefficient we expect that prices drift away from each other which can lead to a positive scenario (value increase) S1 goes up and S2 goes down or to a negative scenario (value decrease) S1 decreases and S2 increases. However in most cases correlation coefficients are determined of past stock prices which have drawbacks such as it is backward looking and it is subjective. Figure 7 illustrates the dynamic of the payoff function for European exchange options at maturity. The payoff depends on the difference between S1 and S2 at T. If S1 is larger than S2 we can observe a positive payoff. In the other case the payoff will be zero. Obviously the payoff will be greater if the price of asset 1 is high compared to the price of asset 2. We can see this result

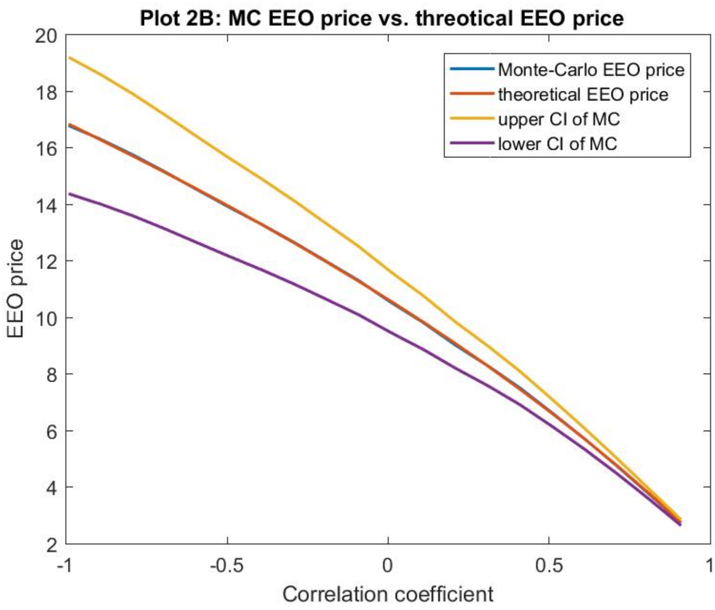

Figure 7: EEO price for different ρ

in figure 2 where the highest payoff is given when S2 is equal to zero.

2.D Technical Notes on Task 2

Technical notes on 2C: The two plots are implemented by using two for-loops for S1 and S2. We also used meshgrid command for the 3D coordination system.

3 Exotic Options

In this exercise we deal with barrier options. These options are path-depending which takes not only the option value at expiration into account but also the fact whether the price of the underlying hits a predefined barrier during the life of the option. In our case we analyse the features of Down-and-in Calls (DIC) and Down-and-out Calls (DOC). A DOC ceased to exist if the underlying hits the Barrier (H). In contrast the DIC only exist if the Barrier was hit during the life time of the option. Both options represent the same payoff at maturity as a standard call option. However the barrier options have an additional condition which makes them normally less expensive than a standard call.

3.A Monte-Carlo valuation

$$DOC(S,T) = \max(S_T - X, 0), \textit{if } S_t > H \textit{ for all } t \in [0,T] \tag{9}$$

$$DIC(S,T) = \max(S_T - X, 0), \textit{if } S_t \leq H \textit{ for at least one } t \in [0,T] \tag{10}$$

where H=140, X=150, S=160, q=0.04, r=0.03 and σ=0.3

3.B Closed-form solutions

3.C extended Monte Carlo valuation

Exit probabilities(Moon 2008)
DOC:

$$P_{n+1} = e^{-2\frac{(S_n-H)(S_{n+1}-H)}{\sigma^2 S_n^2 \delta t}} \tag{11}$$

where $S_\tau \leq H$ in $\tau \in (t_n, t_{n+1})$, if $P_n > U_n$

4 References

Hull, John C. *Options, Futures and other Derivatives*, 8th edition, Pearson Education Limited, Harlow 2012, pp. 414 - 425.