Savings Bonds, Retractable Bonds and Callable Bonds

A pricing approach?

B500 Masterseminar on Finance

Eberhard Karls University Tübingen
Faculty of Economics
and Social Sciences
Department of Finance
Headed by Prof. Dr. Christian Koziol

Name: **Claudia Priester, 4069199**
Konstantin Smirnov, 3980253
Miriam Wagner, 3850710
Advisor: M. Sc. Sebastian Weitz

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CB Callable Bond

CAGR Compound Annual Growth Rate

NPV Net Present Value

SB Savings Bond

RB Retractable Bond

I investor

Contents

1. Motivation

While 22.8% of the issued dollar-amount in U.S. corporate bonds has been distributed via callable bonds in 1996, bonds with call options had a share of 68.0% in the U.S. bond market in 2016[1]. This significant boost of callable bonds points out the increased relevance of this debt-type.

The analysis of callable bonds is not only relevant with respect to bond market issues but also relating to real estate credits. §489 BGB contains the credit takers right to early terminate a real estate loan after ten years. Hence, this credit type is comparable to an ordinary credit with an embedded European-type call option exercisable in ten years.

To also get an idea of the general development and pricing theory of the counterpart of callable bonds - savings and retractable bonds - we investigate these debt instruments, too.

2. Claim Description

2.1. Callable Bonds

Callable Bond (CB) play a significant role on the bond market. This type of debt instrument allows the issuer to redeem the bond before maturity. Therefore a CB can be understood as a straight bond with an embedded American (early redemption possible at any point of time) or European (early redemption only possible at fixed date(s)) call option. This call option captures the issuer´s right to buy back the underlying bond at a prespecified price K prior to maturity date[2] and thus permits an evening-up transaction for the issuer.

The relationship between the investor and issuer can be summarized as follows:

Figure 1: Long - Short Relationship Callable Bond

2.1.1. Bond Value

From investor (I)´s perspective the value of a CB is composed of the value of a straight bond B_0 (with the same coupon payments and maturity) less the value of the embedded call option C_0[3]. The latter has to be paid by the issuer and hence reduces the overall value of the CB.

Another possibility of structuring a CB is to fix the issuing price equivalent to the straight bond´s issuing price but to choose a higher coupon in order to compensate I for the higher inherent risk.

[1] see SIFMA (2017)
[2] see Brennan and Schwartz (1977) p. 84
[3] see Fabozzi (2012) p. 191

The value of a CB containing an American-type-option can be duplicated as follows:

	${}_ty_T < y^*$	${}_ty_T > y^*$
straight bond *(long)*	PV(straight)	PV(straight)
- call option *(short)*	-max(PV(straight) - K ;	0)
= callable bond	**min(K** ;	**PV (straight))**

Figure 2: Duplication Portfolio Callable Bond [Investor´s perspective]

The critical spot rate is depicted by y^*. It represents the break-even rate, at which exercising the option gets worthwhile. As can be seen in figure 2 the optimal strategy, in order to minimize the bond value, is to exercise the call-option if the current spot rate ${}_ty_T$[4] undercuts the critical interest rate[5]. A more advanced analyze will be conducted in section 4.2.

2.1.2. Purpose vs. Risk

The main purpose for issuing CBs is a corporation´s aim to benefit from a potential decrease in the interest rate level during the lifetime of the bond. If this case arises the corporation is able to redeem the bond early and refinance its capital demand at cheaper conditions until maturity T. Simultaneously it can issue a long term bond and gain certainty about the upper limit of future cash flow liabilities.

Hence, the issuer has a limited downside risk, due to the call price that is reflected by the lower investment amount received from I. While simultaneously facing the whole upside potential of the call option. In a bond setting the upside potential can be understood as the possibility of minimizing the bond value due to maximizing the options value[6].

From Is point of view it is very important to visualize that (s)he as the writer of the call option, faces the risk to suffer (unlimited) severe losses as interest rates fall, captured by the term „downside risk of the option“. In case of decreasing interest rates, I will receive the guaranteed call price K, which is smaller than the true market value. Compared to the initial conditions provided by the CB, I is only able to invest the amount K at lower rates and hence losses interest gains. Now, (s)he faces a reinvestment risk[7] that generally does not occur for investing in straight bond. The difference $K - PV(straight) < 0$ represents the present value of all future losses due to lower reinvestment opportunities. The notional example of an already outstanding CB in the appendix illustrates the relationship between these values.

Albeit I is able to reap an extra interest yield if the call option is not exercised. This causes a limited upside potential for I.

[4]Spot rate for an investment until maturity T, valid at time t.
[5]see Fabozzi (2012) p. 860
[6]see Brennan and Schwartz (1977) p. 75
[7]see Brennan and Schwartz (1977) p. 84 and Fabozzi (2012) p. 57

2.1.3. Further motives

Among others, a CB can be used as an efficient tool for mitigating the debt overhang problem between equity and debt holders, which was first mentioned by Myers (1977). To shortly sketch the problem, we investigate an investment with a positive Net Present Value (NPV). So, it would be globally efficient to conduct the investment. Albeit additional funds, paid by equity holders, are needed to cover the investment´s amount. However the, already outstanding, debt value will firstly increase as a consequence of the better secured contract. The equity value will only increase if the appreciation in debt value is smaller than the investment´s NPV. Hence equity holders only gain if the NPV is large enough, caused by the free-rider position of debt holders[8].

This underinvestment problem can be solved by the issuance of CBs. If the corporation is able to call the bond before conducting the new investment the old debt holders will not participate at the investment´s positive NPV. The equity holders are able to issue new debt, but with the important distinction that the new debt holders anticipate the efficient investment and will price the debt value fairly. With this tool the equity holders are able to capture the full NPV[9].

Furthermore CBs can be used to skirt contract conditions[10]. At issuance date, debt holders may anticipate the equity holders´ action that are going to have a negative impact on the debt value. This could be for example the issuance of further debt with the same (or even higher) seniority or a merger. To prevent the equity holders from fooling the debt holders afterwards, the latter are going to submit such discriminating actions with appropriate clauses in the contract.

For still being able to enforce their visions in the future equity holders issue CBs. In this case a contract that for example forbids the issuance of new debt can easily be canceled. Hence, the call-option can be used to buy financial flexibility.

2.2. Savings Bonds

In contrast to CBs there as well exists structured debt instruments, which contain the contrary put right held by I. If the put option can be exercised at any point of time during the lifetime of the bond it is named Savings Bond (SB)[11]. The American-type put option captures Is right to sell back the underlying bond to the issuer at a prespecified price K, at any point of time. Often SBs contain a fixed holding period. During this period I is not allowed to exercise the put option or at least only if (s)he accepts certain interest rate losses[12].

[8]see Myers (1977) p. 147-155
[9]see Schulze (1996) p. 17
[10]see Vu (1986) p. 248
[11]see Brennan and Schwartz (1977) p. 83
[12]see Treasury Direct (2017)

The relationship between the investor and issuer can be summarized as follows:

Figure 3: Long - Short Relationship Savings Bond

2.2.1. Bond Value

From Is perspective the SB-value can be understood as the value of a straight bond B_0 with the same coupon payments and maturity, that (s)he has to pay as the investment amount to the company plus the price of the embedded put option P_0.

Another structuring possibility is to fix the issuing price equivalent to the straight bonds price, but to choose a lower coupon in order to compensate the issuer for granting the put right.

Furthermore some SBs exhibit an increase in coupon payments during the term to discourage I from redeeming the bond too early as well as compensating them for a general increase in the interest rate level, which they could not participate if the put option is not exercised[13].

The SBs value can be replicated as follows:

		$_ty_T < y^*$		$_ty_T > y^*$	
straight bond *(long)*		PV(straight)		PV(straight)	
+ put option *(long)*	+max(	0	;	K - PV(straight)	)
= savings bond	**max(**	**PV (straight)**	**;**	**K**	**)**

Figure 4: Duplication Portfolio Savings Bond [Investor´s perspective]

Contrary to the CB-optimal-exercise-strategy it holds for a SB that the embedded put option should be immediately exercised if the current spot rate $_ty_T$ exceeds the critical interest rate y^*.

2.2.2. Purpose vs. Risk

In the past the main intention for issuing SBs was the increased capital demand of national governments caused for example by world war II. [14]. The government as the writer of the option absorbs the interest rate risk for the investors, that normally inherent the investment in straight bonds. Hence SBs become especially attractive for risk-averse private persons. As a result national governments get cash from a broader base of investors and can use the capital to support for example an economic growth.

The purchase of SBs creates a comfortable situation for I. On the one hand (s)he can rely on the prime solvency of national governments. On the other hand, if I wants to liquidate the bond before maturity (s)he does not face the risk of a depreciation in the bond value caused by a rise

[13]see Brennan and Schwartz (1977) p. 68
[14]see Brennan and Schwartz (1977) p. 67

in interest rates[15]. The interest rate risk is eradicated. This secured position is created due to the additional wealth that comes from the put option, once the interest rates are going to rise. The positive put value completely compensates the capital loss of the straight bond.

Whenever the trade with options is conducted, the limited downside risk for I goes along with the risk that the option expires worthless and the price for the put option P_0 was paid in vain. This occurs when the interest rates are not going to rise during the lifetime of the option.

Moreover it is important to state that there is no secondary market for SBs[16]. To fully analyze the risk characteristics we should not neglect this additional potential risk source. Because SBs are not marketable I cannot sell the bond to other market participants. If I has to liquidate the bond before maturity (for example due to solvency reasons) this can have severe consequences. In case the interest rates are low and I exercise the put option (s)he does not get the fair bond value, but the lower value K. This negative characteristic could at least partly be compensated, if I does not put the SB before maturity and takes instead a loan at current, i.e. low, interest rates to satisfy her/his cash flow needs. The cash inflows from the initial SBs can be used to pay back the loan. Albeit due to transactions costs, I does not receive the full difference between $PV(Straight) - K$.

The issue of SBs bear a great risk for the issuer. If the bond is redeemed before maturity the issuer has to refinance at (generally) higher rates. Moreover this can force her/him into a vicious cycle. This risk is more severe in case of an American-type put option, which is the reason why only governments issue SB. For governments this risk is not relevant, because it is justifiable to assume that there is always a possibility to generate liquidity in order to redeem the putted SB.

2.3. Retractable Bonds

Similar to SBs, Retractable Bond (RB)s are structured as a straight bond with an embedded investor´s put right. However the option is an European-type put option[17], i.e. it can merely be exercised at prespecified date(s).Whereas SBs are solely issued by national governments, RBs are mainly issued by corporations[18].

Thanks to the parallel properties of RBs and SBs we are only going to elaborate the differences and important special features of RBs, while excluding the reconsideration of the bonds value, structure and duplication.

[15]see Brennan and Schwartz (1977) p. 68
[16]see The Balance (2017)
[17]see Brennan and Schwartz (1977) p. 69
[18]see Brennan and Schwartz (1977) p. 69

2.3.1. Additional Risks

From Is perspective it is rational to exercise the put option, either if the interest rate level rises and (s)he can reinvest her/his investment amount at higher rates, or if the economic prospect of the corporation diminishes and the current coupon does no longer reflect the needed risk adjusted fair coupon, which would compensate I for overtaking a higher default risk. Especially the latter can have severe consequences for the corporation, due to the high correlation between diminishing economic prospect and exercised put options.

If I exercises the put option the corporation is forced to refinance their capital demand at higher rates. For solvent corporations this merely causes additional costs. But corporations with an (even only temporary) capital shortage can get into a vicious cycle and at worst be finally forced to file for bankruptcy. This creates an additional risk source for I in turn, because they suffer in default case whereas the equity holders can go out without any capital loss.

2.3.2. Further motives

Why are there still corporation issuing RBs? Initially, as mentioned in section 2.2.1, the structure of RBs allows to issue a bond with a lower coupon compared to an otherwise identical straight bond. This is especially attractive for corporations, e.g. Startups, which cannot settle up the asked coupon rate for an equivalent straight bond. Albeit particularly for Startups the issuance of such bonds is highly risky. Because of the not yet consolidated market position, a temporary capital shortage in the future is very likely.

Not only focusing on the change in interest rate level, RBs can be used as a tool for mitigating the asset substitution problem, investigated as one of the first by Jensen and Meckling 1976.To shortly sketch the problem, we assume an investment which exhibits a negative NPV and additionally increases the risk of a corporation[19]. Hence, it would be globally inefficient to conduct the investment. Due to the higher default risk the debt value will decrease, because the upside potential is limited to the credit volume, whereas the downside risk is completely born by the debt holders. If the adverse NPV effect is not too severe the wealth transfer from debt to equity holders is positive. Thus, the equity value will increase through conducting the inefficient investment.

This overinvestment problem can be solved, if the issuance of RB is demanded by the investors. In this case I is able to sell the bond back to the issuer, whenever an unfavorable investment is conducted. (S)he no longer carries the higher default risk. In case the issuer is forced to refinance her/his capital needs with a new debt contract, the new debt holders are going to

[19]see Jensen and Meckling (1976) p. 312 ff.

increase their asked coupon rate in order to accommodate the higher default risk. Hence, RB can be used by corporations which needs to provide a security to I for compensating her/him for not participating at investment decisions.

3. Economic Intuition and Issuing Development

3.1. U.S. Interest Rates

Analyzing the development of issued and/or outstanding bonds of the introduced bond claims we are taking a look at the U.S. corporate and savings bonds market. Since we also interpret our findings with respect to the interest rate, the development of the discounted U.S. interest rate is depicted in figure 5. We see that the interest rate shows sections of heavily declining and increasing rates as well as periods of stagnation[20]. The recent policy of the FED shows again an upward trend of the interest rate, firstly ushered in Dec. 2015.

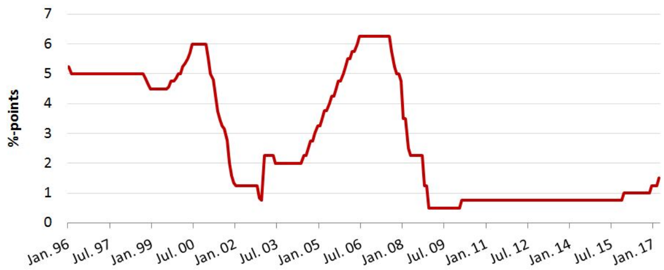

Figure 5: Discounted U.S. Interest Rate

3.2. Expected Issuing Behavior

In the following the relationship between the issuance of bonds with an embedded option and the current or expected interest rate level is elaborated.

If a corporation expects the interest rates to (further) decline, it makes sense to issue CBs. After interest rates have fallen, the corporation can redeem the CB early and refinance at cheaper rates. With respect to this we expect that the issuance of CBs decreases as the (expected) interests rate level increases or stagnates on a very low level (2004-2006; Dec.2008-today).

Due to Is incentive to exercise its put right, it is wisely for corporations and governments to only issue RBs and SBs if they can reasonably assume that interests will not increase during the bond´s term. If a bond containing a put option is issued and interests rose, the issuer would be forced to redeem the bond early and had to finance at higher rates. As described in 2.3.1, in the worst case this can force the issuer into a vicious cycle. Hence we expect that the issuance of bonds with an embedded put option will drop as (expected) interests increases or if they are on a very low level and hence a further decline gets unlikey(Dec. 2008-today).

[20]see Federal Reserve Bank of St. Louis (2017)

3.3. Empirical Evidence

Starting with the analysis of CBs - mostly issued by corporations - we see that the issued amount contained in CBs is continuously rising. While a total amount of $515 bn. (inflation adjusted[21]) in terms of non-convertible bonds has been issued in 1996, this value rose up to $1,409.6 bn. in 2016[22]. This reflects a Compound Annual Growth Rate (CAGR) of 5.2%. However in the same period, the fraction of non-convertible bonds that contains a call option increased from $125 bn. to $971.1 bn. with a CAGR of 10.8%. Hence the importance of CBs increased during the last two decades.

While the amount of issued non-callable bonds has been way higher than the one of CBs in 1996, the opposite is true in 2016. In 2016, 69% of the total amount of issued non-convertible bonds contained a call option (see figure 6).

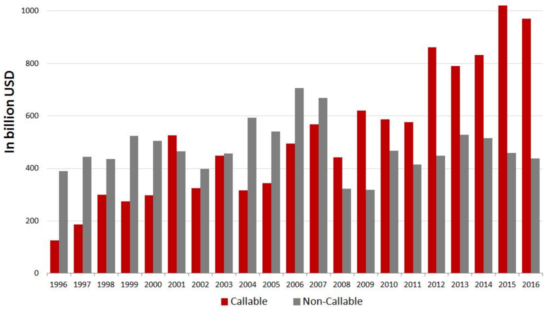

Figure 6: Inflation adjusted issuanced amount of U.S. corporate bonds

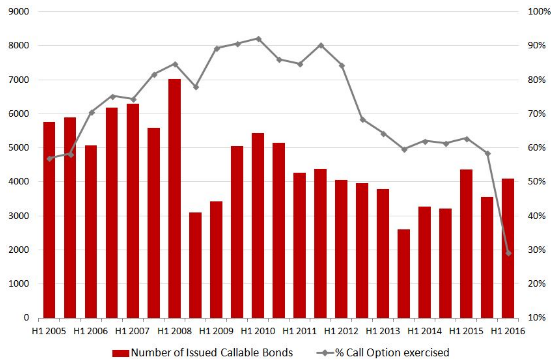

Figure 7: U.S. Corporate Bond Market, Callable Bonds

Due to the issuance incentives with respect to interest rates, this development is surprising and not in line with our expectations described above. Since 2009, once interest rates reached a historical low level, we would have expected a decreasing issuing amount. Indeed the opposite case occured. Especially since the FED declared in December 2015 to slowly, but continuously raise interests, corporations cannot have wrong expectations about declining interests and therefore „falsely“ issue CBs. Hence we see that also other factors (see 2.1.3) must influence a corporation´s decision whether to issue CBs.

Furthermore the second graph shows the number of issued CBs in each half year and how many percent of these bonds have been called yet. While more than 80% of the bonds issued between H1 2008 and H1 2012 have been called, the percentage of called bonds issued from H2 2012 onwards decreased to approx. 60%[23].

[21]see The World Bank (2017)
[22]see SIFMA (2017)
[23]see Bloomberg (2017)

SBs are solely issued by governments. Since SBs are meant to also gain individuals of the middle class as investors, the issued amount per person is capped. For example a Series EE bond is capped to a maximum amount per Social Security Number of $10,000[24]. The inflation adjusted[25] amount of outstanding SBs accounted for $2,223 bn. in 2000 and shrank to $2.025 bn. in 2016. While 75% of all outstanding Treasury bonds have been SBs in 2000, the share of SBs decreased to 15% in 2016[26].

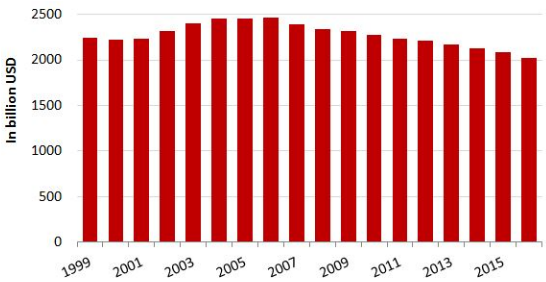

Figure 8: Total Amount of Outstanding savings bonds (inflation adjusted)

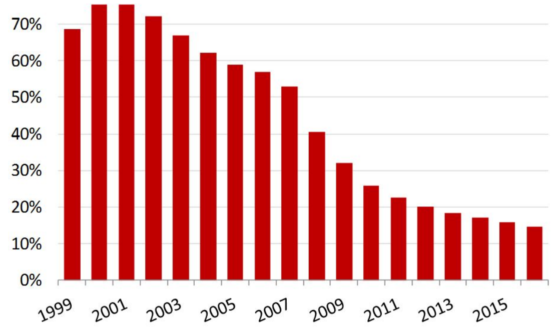

Figure 9: Share of U.S. savings bonds in Total Treasury Bonds

For governments the issuance of putable bonds in order to exploit different interest rates expectation between them and I plays only a minor role. Independent of the interest rate evolution, it is not surprising that the importance of SBs is decreasing since these bonds were firstly issued to overcome capital shortages during world war II[27]. The government can nowadays finance at better conditions in which it does not have to increase its coupon payments to discourage the investor from redeeming the bond too early.

RBs are mostly issued by corporations. Figure 10 on page 10 shows the number of issued bonds that feature a put option and the percentage of bonds for which the put option has been exercised yet. Comparing the issued number of CBs with the issued number of putable bonds we can see also the minor role of RBs. While 4.095 bonds with call option have been issued during H1 2016, 24 bonds with a put option have been issued during the same half of the year[28]. We can also see the relatively irrelevant role of RBs looking at the development of the total number of issued bonds with a put option. Starting with 254 issued putable bonds in H1 2000, the number of issued bonds with put option decreased intensely.

[24]see Treasury Direct (2017)
[25]see The World Bank (2017)
[26]see SIFMA (2017) and see Treasury Direct (2017)
[27]see Brennan and Schwarz (1977) p. 67
[28]see Bloomberg (2017)

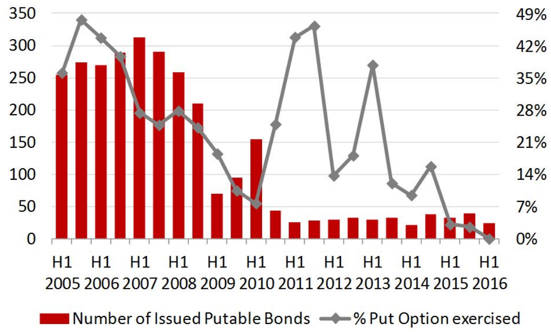

Figure 10: U.S. Corporate Bond Market, putable bonds

	Issued Putables	Option Exercised	% Option Exercised
Investment Grade	190	14	7.4%
High Yield	31	8	25.8%
Not Rated	442	73	16.5%
Total	663	95	14.3%

Figure 11: Rating of corporations issuing putable bonds from 01/2010 until 05/2017

This development is in line with our expectations. The issuance of RBs fell as interests reached its lowest level in 2009, rose slightly afterwards (need to issue RBs under more difficult financing conditions) and remained on a very low level after the crisis under very low interest rates. Still it is surprising that 190 out of all 663 issued putable bonds since January 2010 have an investment grade rating (according to S&P and Moody´s)[29]. Furthermore 84 corporations that issued putable bonds are rated with AA- or even higher. Therefore we see that not only companies with more difficult financing conditions have an incentive to issue retractable bonds. However since 473 out of 663 corporations (71%) do not have an investment grade rating, all in all this reflects our assumption made in section 3.2.

4. Pricing: Using an Explicit Finite Difference Method

In this section we price four instruments, namely a straight bond, a SB, a CB and a RB where the underlying instantaneous interest rate follows a stochastic process[30]. Specifically we use the Explicit Finite Difference Method in order to solve numerically for the partial differential equation given in the paper:[31]

$$\frac{1}{2}\sigma^2 r^2 G_{rr} - rG - G_\tau + c = 0 \tag{1}$$

Generally speaking the Finite-Difference Methods uses a Taylor series expansions in order to approximate the partial derivatives terms of the PDE [32].

To solve the PDE with an Explicit Finite Difference Method, we have to set up a grid first by discretizing our input variables r and T. Hence, we define our input variables as

$$x = r = ih, \text{ where } i \in [0, M]$$

[29] see Bloomberg (2017)
[30] see Brennan and Schwartz (1977) p. 73
[31] An derivation of the PDE is given in the Appendix
[32] see Brandimarte (2006) p. 475 - 483

and

$$\tau = T - t = nk, \text{ where } n \in [0, N]$$

where i and n are index numbers, and h and k are step sizes with respect to the interest rate and time. Hence one can define the rectangular grid as

$$x \in [0, x_{max}], \text{ with } x_{max} = M \cdot h$$

and

$$\tau \in [0, T], \text{ with } T = N \cdot k$$

This can be applied on (13):

$$\frac{1}{2}\sigma i^2 h^2 G_{rr} - ihG_r - G_\tau + ci = 0 \tag{2}$$

Further in the explicit scheme, we use following Taylor series expansions in order to approximate the partial derivatives in (2):[33]

The central approximation for the 2nd derivative with respect to x (or r):

$$[G_{xx}]_i^n = \frac{1}{h^2}(G_{i+1}^n - 2G_i^n + G_{i-1}^n) + \mathcal{O}(h^2) \tag{3}$$

The central approximation of for the first derivative with respect to x (or r):

$$[G_x]_i^n = \frac{1}{2h}(G_{i+1}^n - G_{i-1}^n) + \mathcal{O}(h^2) \tag{4}$$

and the forward approximation for the first derivative with respect to time:

$$[G_\tau]_i^n = \frac{1}{k}(G_i^{n+1} - G_i^n n) + \mathcal{O}(k) \tag{5}$$

Plugging (3), (4) and (5) into the discretized (2), this will lead to:

$$G_i^n = \delta a G_{i-1}^{n+1} + \delta(-2a + 1)G_i^{n+1} + \delta a G_{i-1}^{n+1} + \delta ck \tag{6}$$

where

$$a = \frac{1}{2}\sigma i^2 k$$
$$\delta = (\frac{1}{1 + ihk})$$

Brennan and Schwartz (1978) show that the explicit scheme representation(6) is basically just a representation of a trinomial tree:

$$G_i^n = d_1 G_{i-1}^{n+1} + d_2 G_i^{n+1} + d_3 G_{i-1}^{n+1} + \delta ck \tag{7}$$

[33]Note that the $\mathcal{O}$ terms are converging with the speed of h^2 or k to zero.

$$d_1 = \delta a$$
$$d_2 = \delta(-2a + 1)$$
$$d_3 = \delta a$$

where δ is the discount-factor and d_1, d_2 and d_3 are hence the risk-neutral probability for every node i in time-step n.

4.1. Boundary Conditions and Payoffs

One major aspect of the Finite Difference Method is the definition of the boundaries. These points are the only values which we assume to be known. Since we are in a default-free framework, we assume that the bond-value at maturity is always equal to unity(terminal condition):

$$G(r, 0) = 1 \tag{8a}$$

We assume non-negative riskless-rates. A sufficient condition for $r = 0$ can be described as:

$$-G_\tau + c = 0 \tag{8b}$$

Further, due to the inverse relationship between a bond and the riskless interest rate, the value of the bonds is zero if $r \to \infty$:

$$\lim_{r \to \infty} G(r, \tau) = 0 \tag{8c}$$

As in the trinomial tree method, the explicit scheme algorithm solves the equations with a backward induction starting at the terminal condition. In order to get the values of the option-adjusted bonds, the payoff functions, which were discussed in section 2, have to be respected at each state and time step i and k in the grid of the straight bond while. For a SB and a CB, following payoff functions has to be fulfilled at each node i and k:

Since a SBs can be described as an straight coupon bond plus an American-type put option at par, the payoff function is defined as:

$$G_i^{n+1} = max(G_i^n, 1) \tag{9a}$$

The CB is equal to a straight coupon bond less an American-type call option at par, the payoff function is defined as:

$$G_i^{n+1} = min(G_i^n, 1) \tag{9b}$$

Since a RB is equal to an European-type put option at time τ_p, the payoff function only strikes at redemption date τ_p:

$$G_i^{\tau_p+1} = max(G_i^{\tau_p}, 1) \tag{9c}$$

4.2. Results

Table 12 shows the input parameters which were used in the computation.[34]

Parameter	**Values**
Coupon	8%
Time to Maturity	5 Years
Time to Maturity - RB	20 Years, retractable after 5
σ	0.045
M	400
N	10000

Figure 12: Estimation Results for $\hat{\gamma}$

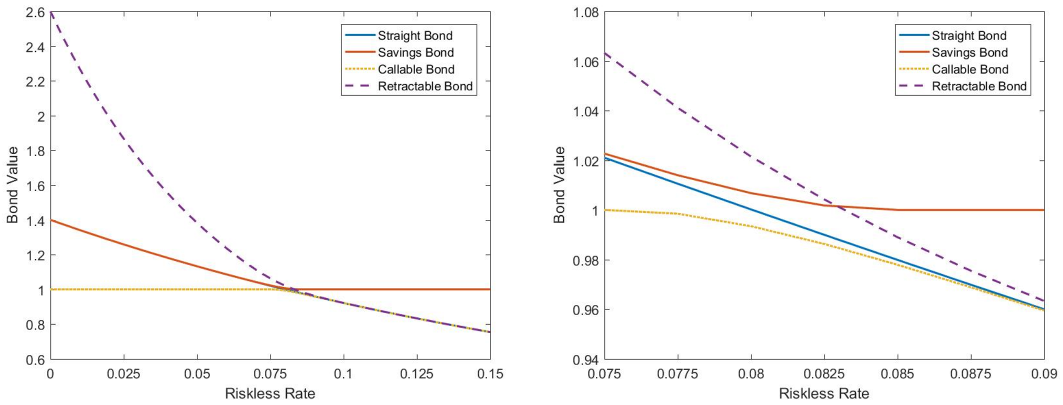

Figure 13: Bond values as a function of the riskless rate

Figure 13 depicts the value of a straight bond with a maturity of five years and an annual coupon amounting to eight units. It serves as a benchmark for the analyzed debt instruments.

The yellow (red) line represents the value of a CB (SB), which exhibits the same maturity and coupon payment as the straight bond. As already mentioned in section 2.3 a RB is the European-counterpart to a SB. In order to provide more illustrative material we priced an 20-years RB that can be putted in five years, however providing the same annual coupon payments. The RB´s underlying is therefore a 20-years straight bond. It is depicted as a dashed blue line.

In general, an options value is composed of two parts. The intrinsic value reflects the maximum out of zero and the value if the option is exercised immediately, i.e. the difference between $PV(straight)$ and K[35]. If the interest rate decrease is not too severe it is optimal for the issuer of an in-the-money American CB to not immediately exercise the option, and instead hoping for

[34] Note that due to numerical instability which results from the long time to maturity of the RB, N has to be set to a large value.

[35] see Hull (2012) p. 201

a further decrease. This additional value is represented by the second component of the options value, the time value of money [36]. The critical interest rates y^* represent the rate at which the time value of money is zero and the early redemption is throughout optimal.

Firstly we analyze the shape of the American-type CBs value function. As derived in section 2.1.1, the value of a CB can be replicated due to a straight bond and an embedded call option, held by the issuer. If the current spot rate until maturity T far exceeds the critical interest rate, i.e. ${}_ty_T \gg y^*$, the option is worthless and the CB coincides with the straight bonds value. By contrast, if the interest rate level falls the generated call option value adjust the CB value function downwards. In this scenario I is no longer willing to pay the full price of an equivalent straight bond, because the early redemption of the embedded option gets more and more likely. (S)he is only poised to invest into a CB if the price is lower. Is the spot rate lower than the critical rate, i.e. ${}_ty_T < y^*$, the option is deep in-the-money and diminish the value to $K = 1$. Anticipating that the issuer will immediately call the bond if the interest rates are low enough, I just pays the exercise price. Otherwise arbitrage opportunities for the issuer are created.

A mirror-inverted picture plots the SB value function. Whereas the SB value coincidences with the value for the straight bond when interest rates are low, the put option comes into place once the interest rate are rising. Now, I has to pay a higher investment amount to the issuer, in order to compensate her/him for the increased exercise likelihood. The put option is deep in-the-money, once the current spot rates exceeds the critical interest rates, i.e. ${}_ty_T > y^*$. Then the SB´s value approaches $K = 1$, to eradicate arbitrage opportunities.

The value function of a RB far exceeds the value of a SB in case the put option is out-of-the money. This is only caused by the fact that the underlying straight bonds display a maturity of 20 years. Apart from that the same interpretation valid for an SB in relation to a change in the interest rate can be applied for the RB.

In figure 14 on page 15 the change in the critical interest rate is plotted as a function of time (left-hand side) and as function of the volatility (right-hand side).[37]

Given the current spot rate undercuts the fixed coupon of a CB the issuer has to carefully outweigh the trade-off between the current coupon disadvantage and the possibility of a further decline in interest rates during the bond´s term. This value is captured by the term „time value of the option“ .When analyzing the development of the critical interest rate during the lifetime of a five-years CB this time value gets vivid. In $t = 0$ the critical interest rates amounts to 7.6%.

[36] see Hull (2012) p. 201

[37] In this computation, the grid parameters were set to M=1000 due to illustrative reasons

Hence, at the beginning the drop in interest rate level has to be pronounced in order to exercise the option early. As the call option can only be exercised once the issuer cannot participates at a further decline in interest level. So, the drop must be severe enough in order to renounce the call right. As time goes by the likelihood of a further severe decrease in interest rates diminishes. Thus the issuer is willing to exercise the call option even if the refinancing conditions are only slightly better. Otherwise the option would expires worthless at maturity. Hence, the option´s time value decreases as time goes by.

The mirror-inverted arguments apply for the development of SB-critical interest rate. At the beginning of the bond´s lifetime, I only exercise the put option if the reinvestment conditions are way higher than the coupon of the underlying putable bond.

As well it is interesting to analyze the development of the critical interest rate as a function of the volatility assumed in the pricing model. With increasing volatility σ the CB and SB values drift apart. Anticipating a certain volatility the issuer of a CB again outweigh the trade-off between the current coupon disadvantage (compared to market conditions) and the possibility of further gains, if the interest rate level additionally falls. If the volatility in interest rates is high, the future fluctuation in interest rates level is more pronounced and hence a sharp decrease is more likely. In this case the critical interest rate is lower in order to discourage the issuer of exercising the bond to early. In contrast, if the volatility is small, the required drop in interest rates is less pronounced, as the likelihood for an additional decrease gets very small. The issuer rather wants to secure the current gains than speculating on further declines.

As always the SB creates a mirror-inverted picture. Given a high volatility, I will only exercise the put option if the reinvestment conditions are advantageous enough.

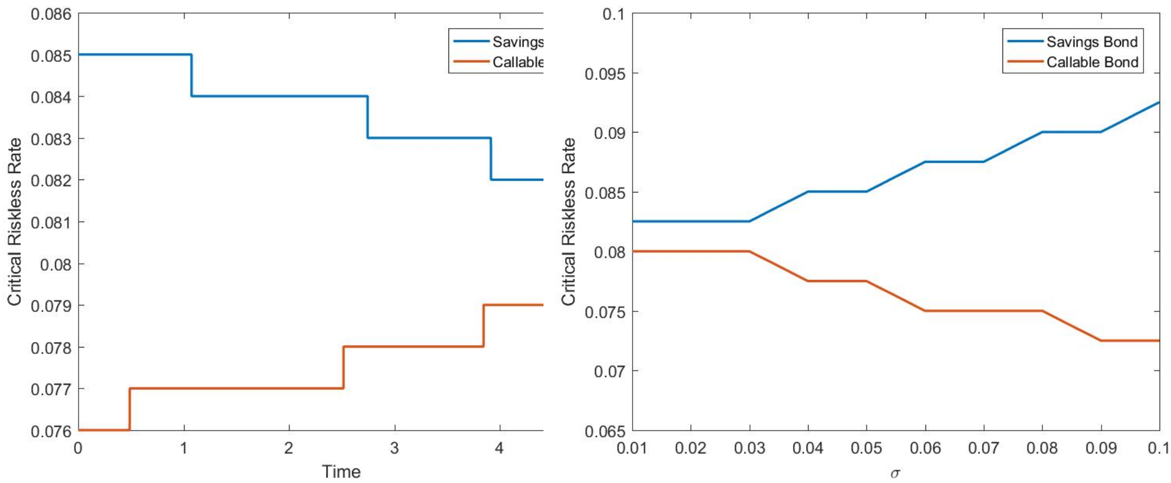

Figure 14: Critical interest rate as a function of time and σ

5. Conclusion

A. Appendix

A.1. Value composition

In this simple example we assume a callable bond that provides an annual coupon of eight units and a remaining time to maturity of two years. The coupon that has to be paid on the evaluation date $t = 2$ are sunk costs and does not influence the call decision any more.

	(logical second after) t = 2	t = 3	t = 4
Cash Flow - Straight Bond -	(+ 8)	+ 8	+ 108
PV (straight) ($_2y_3 = {}_2y_4 = 6\%$)	**+ 103,66**		
Cash Flow - Callable Bond -	(+ 8)	+ 8	+ 108
Call at K = 100 in t = 2	**+ 100** = min (100 ; 103.66)	-	-
Reinvestment (at fair terms: $_2y_3 = {}_2y_4 = 6\%$)	- 100	+ 6	+ 106
Loss (due Call short)	0	-2	- 2
PV (Loss) ($_2y_3 = {}_2y_4 = 6\%$)	**- 3,66**		

Figure 15: Relationship Value Straight Bond, Callable Bond and Call Option

As can be seen in figure 15 the interest rate level in $t = 2$ shrinks to six percentage points. Assuming that the critical interest rates is larger, the issuer is calling the bond before maturity and immediately pays the exercise price $K = 100$ to I. Hence, I receives the initial investment amount before her/his planning horizon ends and is therefore forced to reinvest the received capital at lower rates. Each period (s)he losses two units compared to the initial callable bond. The present value of those losses exactly coincides to the value difference between K and $PV(straight)$.
This also holds for savings bonds, respectively retractable bonds.

A.2. Derivation of the PDE

In this section we price four instruments, namely a straight bond, a savings bond, a callable bond and a retracable bond where the underlying instantaneous interest rate follows a stochastic process (Brennand and Schwartz, 1977):

$$dr = \sigma(r)dt + \sigma(r)dz \tag{10}$$

where dz is a Gaussian-Wiener process with $\mathbb{E}(dz)$=0 and $\mathbb{E}(dz^2)$=dt and σ is the volatility. It is assumed that the instantaneous expected rate of interest on default-free securities of all maturities equals the instantaneous risk-free rate of interest (Pure Expectations Hypothesis). Further, we assume perfect capital markets, with no transaction costs, taxes or restrictions on short-sales or other institutional frictions (Brennan and Schwartz, 1977)

The market value of all four mentioned bonds is then simply a function G (generic bond) which depends on the interest rate r and time to maturity τ. Applying Ito's lemma on the function of the genereic bond G(r,τ) leads to:

$$dG = G_r dr - G_\tau dt + \frac{1}{2} G_{rr} (dr)^2; \tag{11}$$

where G_r, G_τ and G_{rr} are the respective derivatives with respect to interest rate or time to maturity. Using (10) with respect to the pure expectation hypothesis and the continuous coupon rate c on (11), this leads to following partial differential equation (PDE):

$$\frac{1}{2}\sigma^2(r) G_{rr} + \mu(r) G_r - rG - G_\tau + c = 0 \tag{12}$$

Further we set $\mu(0)= 0$ and σ(r)=σr which corresponds to a driftless geometric Brownian motion. (12) is the final PDE of a straight bond which is the underlying of all tree other option adjusted bonds:

$$\frac{1}{2}\sigma^2 r^2 G_{rr} - rG - G_\tau + c = 0 \tag{13}$$

A.3. Brennan and Schwartz Recalculation

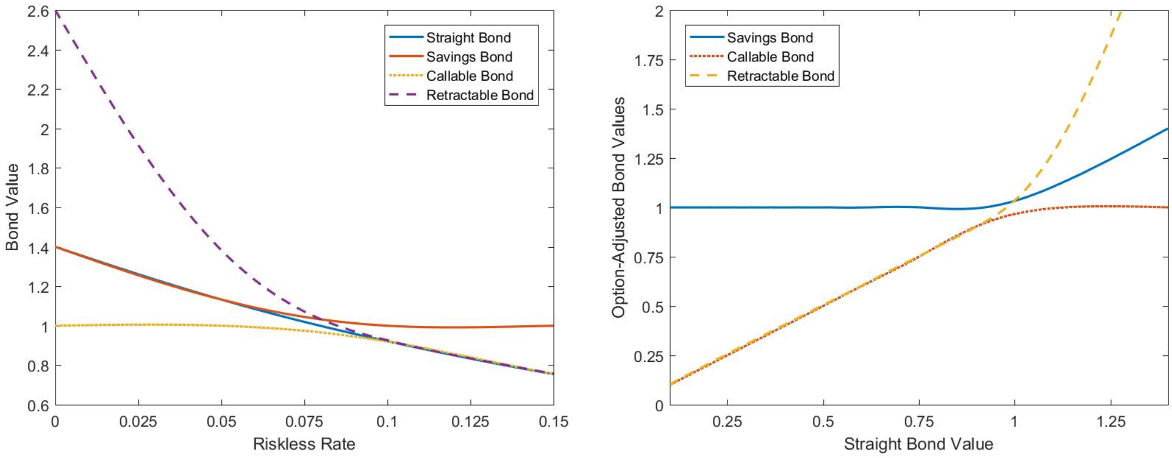

Figure 16: Bond values as a function of the riskless rate

This section contains a quick remark on the results Brennand and Schwartz (1977) computations. Since in their work they are not really precise about the technical implementation, in this section we try to explain the computational differences compared to our results. Figure 16) basically illustrates the exact computational results as in the paper of Brennan and Schwartz (1977) . In order to get these values, the grid-parameters were set equal to table 12, whereas the grid parameters were set to Mx=20 and N=100. In order to get a smooth function, the values were interpolated (low-pass interpolation). We hence assume that the Brennan Schwartz values were quite inprecise compared to our computation due to the low technological standards during the time their paper was written.

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