EBERHARD KARLS
UNIVERSITÄT
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Binomial Option Pricing

B 570 PC-Lab in Finance
Exercise 4
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1 Binomial option valuation model

The binomial option pricing model, by means of a graphical representation, permits to show, in a clear and reliable manner, the evolution of price of different type of options. With the same graphical representation it can also be depicted the movements of the underlying stock prices, that are needed to find the actual price of derivatives.

1.1 Theoretical approach

The characteristic of the binomial option pricing model is the tracing of the evolution of the most important underlying variables in discrete-time. This is done with the calculation of the prices in each time steps between the starting date and the expiration date. The valuation with the binomial option pricing model is formed by three mainly steps (Hull, 2012):

1. *Generation of the evolution of stock prices, by means of a duplicating portfolio*
   The assets price are assumed to follow a random walk and to be such that no arbitrage opportunities exists. Also, there are two assumptions that define the duplicating portfolio: the first one tells that the duplicating portfolio generates the same return as the option price at time T and the second tells that, in equilibrium, the duplicating portfolio has the same price as the option because the law of one price holds.
   The binomial representation of the stock prices starts with the single value S_0 (today's price). With an iterative valuation, we can calculate the new stock prices at each step, that are given by the stock price of the previous steps multiplied by the parameters u and d, where u is given by one plus the rate of return for an up movement and d is given by one plus the rate of return for a down movement. Each node represents a possible value of the underlying at a given point represented in graph 1.

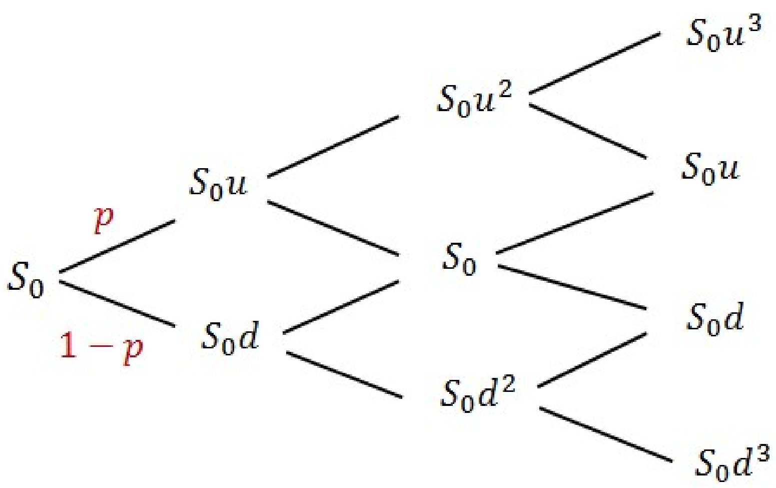

Figure 1: Stock price evolution

To be noticed that the corresponding time between each step of the evolution of the stock prices is determined by $t = \frac{T}{n}$, where T is time to maturity and n the total number of steps considered.
The term *p* and *(1-p)* design respectively the probabilities an up movement or a down movement to take place. These probabilities are risk-neutral probabilities, because it is assumed that the agents expect a return on the asset equal to the risk-free rate. For this reason, the derivatives are priced "as if" the market is risk-neutral, by means of the artificial probabilities *p* and *(1-p)* . Following this reasoning, it is used the risk-neutral evaluation in the binomial option pricing and the risk-free interest rate in the further calculations.
For the calculation of *u*, *d*, *p* and *(1-p)* two different methods can be used. The first one was introduced by Cox/Ross/Rubinstein (1979) where the the parameters for the binomial tree calculation are defined as follows:

$$p = \frac{exp((r-q)\Delta t) - d}{u - d}$$

$$u = exp(\sigma\sqrt{\Delta t})$$
$$d = exp(-\sigma\sqrt{\Delta t})$$
$$\Delta t = \frac{T-t}{n}$$

The second method is the one introduced by Jarrow/Rudd (1983). It artificially defines the value of the probabilities *p* and *(1-p)* to be equal:

$$p = \frac{1}{2}$$

$$u = exp((r - q - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t})$$
$$d = exp((r - q - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t})$$
$$\Delta t = \frac{T-t}{n}$$

2. *Calculation of the option value at each final node*
   At the expiration date T, the price of the option is calculated with the formula for the final payoffs of the call or of the put, both for American and European options:

$$Call : max(0; S_T - X)$$
$$Put : max(0; X - S_T)$$

3. *Iterative evaluation of the option prices*
At each step, a backward calculation is implemented in order to find the price in each step until today's date t_0. For the European call, the formula used is the following:
$$C_t^E = e^{-r\Delta t}[pC^+_{t+\Delta t} + (1-p)C^-_{t+\Delta t}]$$

where $e^{-r\Delta t}$ is discounting factor for each time, $C^+_{t+\Delta t}$ represents price of the call in the up move at the time step Δt after t and $C^-_{t+\Delta t}$- represents the price of the call in the down move at the time step Δt.
For the European Put we get

$$P_t^E = e^{-r\Delta t}[pP^+_{t+\Delta t} + (1-p)P^-_{t+\Delta t}]$$

The valuation of the American call or put, giving their possibilities to be exercised before maturity, requires some additional steps. In each step, in fact, it is necessary to check whether is more convenient to exercise the option or not. For the call we get:
$$max(C_t^{NEX}, C_t^{EX})$$
where C_t^{NEX} is the price of the call if it is not exercised early. Therefore we get:

$$C_t^{NEX} = e^{-r\Delta t}[pC^+_{t+\Delta t} + (1-p)C^-_{t+\Delta t}]$$

C_t^{EX} is the call price when the early exercise takes place. Therefore we get

$$C_t^{EX} = max(0, S_t^* - X)$$

where $S_t^* = S_t$ if there is no dividend payment at t or $S_t^* = S_t + D$ for the dividend payment at date t_D.

For the put same equations also hold respectively:

$$max(P_t^{NEX}, P_t^{EX})$$

where P_t^{NEX} is the price of the put if it is not exercised early. Hence:

$$P_t^{NEX} = e^{-r\Delta t}[pP^+_{t+\Delta t} + (1-p)P^-_{t+\Delta t}]$$

P_t^{EX} is the call price when the early exercise takes place. Therefore we get

$$P_t^{EX} = max(0, X - S_t)$$

To be noticed that for the call options, the early exercise may be optimal only at the dividend payment date, so Δt before t_D. For the put options, instead, the early exercise maybe optimal at any t. If there are dividends, it is convenient to exercise the put Δt after t_D.
Another important value that can be calculated is the early exercise premium. It is given by the difference, at t_0, of the price of the American option and of the price of the European option. The early exercise price is a positive value and tells of how much more an investor is willing to pay for an American option and

so for the possibility of early exercise, with respect to an European option. For the European put option it is given by:

$$E_0 = P_0^A - P_0^E$$

For the call option, the early exercise premium is positive if and only if there is a dividend payment. Otherwise it is not convenient to exercise the American call before maturity and the investors and not willing to pay more for an American call with respect to an European call.

In the next section, we describe the result of the Binomial option pricing valuation for both European and American put options, comparing the results for different values of the time steps, by means also of a graphical representation. We also compare the European put price with the corresponding BSM price. Finally, we also explain the movement of the early exercise premium for increasing time steps.

1.2 Discussion of results

Our analysis is based on a put option (JR) with the following data [1]

Table 1: Input data

	Definition	**Value**
S	Stock	45
X	Strike price	40
r	interest rate	0.02
q	dividend yield	0.06
$T-t$	time to maturity	1.50
σ	volatility	0.35

Applying the procedure described in the previous section, we were able to find, at first steps, the binomial tree for the underlying stock prices of the put option for n=*100* time steps. We calculate the evolution of stock prices with both the Cox/Ross/Rubinstein (1979) and the Jarrow/Rudd (1983) methods. The small differences in the stock prices' movements reflect the different calculation of the risk neutral probabilities that the two methods assume. For our further calculations we rely on the Jarrow/Rudd (1983) method.
Applying the backward iteration procedure we can also calculate the put option price at time t_0 according to the binomial tree approach. We obtain the following results for different valuation methods and different assumptions about the number of time steps. As it can be seen, for higher values of time steps n, the difference between the prices of the put calculated with the BSM method and with the Binomial method decrease. This because, as we increase the time steps, we get a better approximation of the put price.

[1]Data set 7

Table 2: BSM vs. Binomial

Number of steps	Binomial approach	BSM approach	Difference
n=100	5.7042	5.6898	0.0144
n=1000	5.6906	5.6898	0.0008

To better see how the two valuation methods behave, we draw a plot in which there is the comparison between the evolution of the two put prices, depending on the number of steps. The BSM method gives, independently from the value of n, the same put option price, while the number of steps deeply influences the Binomial calculation. For very low number of time steps, the put price is very volatile and the calculation not precise. However, once the number of steps reach the value of 100 the approximation starts to become quite reliable. The quality of the valuation is even higher as the value of n increases, as it can be seen in figure 2. For values of n equal to 15000, we can consider the resulting value to be the true value of the option (Broadie/Detemple, 1996).
Figure 3 shows the same movements of figure 2, but with a reduced view on the x-axis, considering the time steps only from 10 to 1000. From this it can be seen more accurately how the chaotic movement becomes more stable as n increases sufficiently.

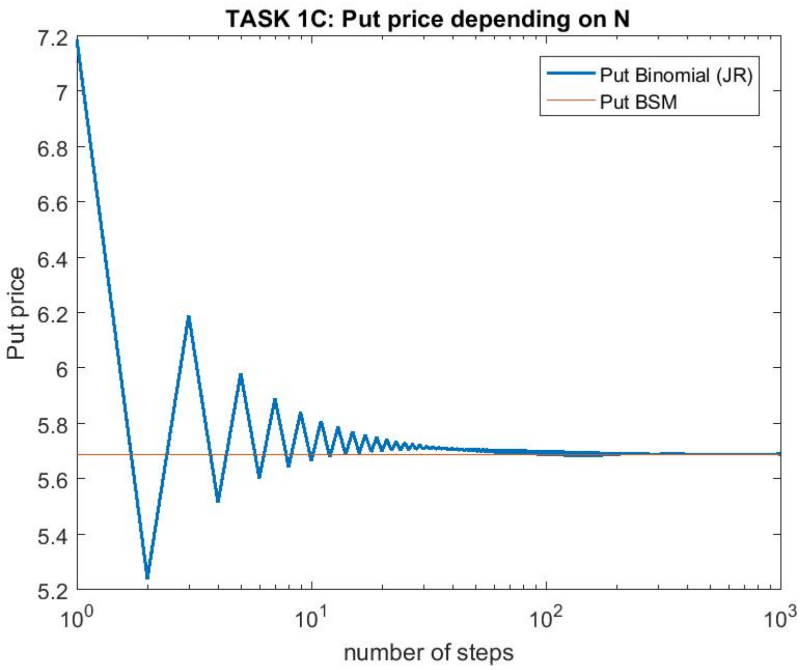

Figure 2: European put price evolution for n=1000

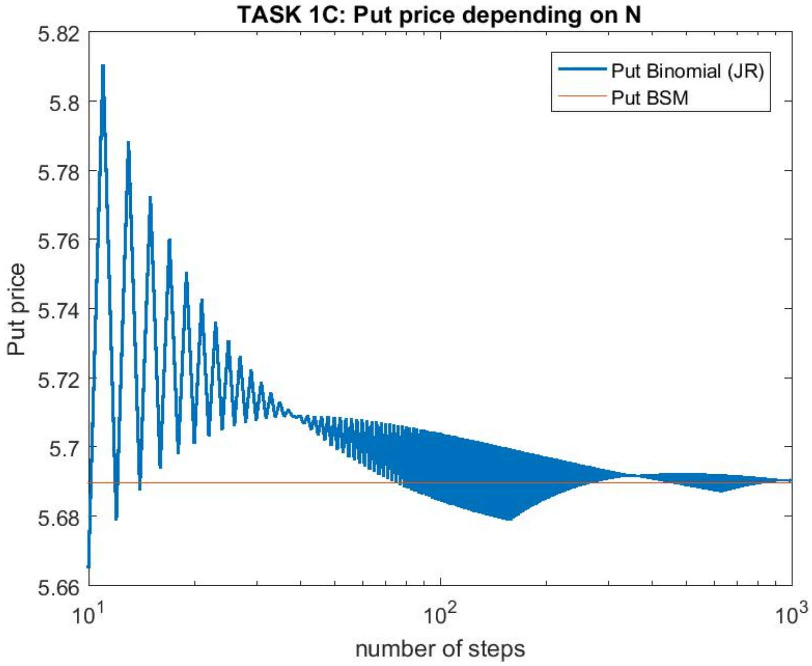

Figure 3: European put price evolution from n=10 to n=1000

The Binomial option pricing is a valuable tool also because it can overcome the impossibility of the BSM method to calculate the price of the American options. The Binomial valuation, with the comparison between the non-exercise price and the exercise price of an option in each step during the backward iteration, gives a reliable result also when the put option considered is an American type.

Applying the binomial tree valuation also to the American put option, we obtain the following results illustrated in table 3.

Table 3: Values for American Option

Number of steps	**Binomial approach**
n=100	5.7043
n=1000	5.6907

The evolution of the American put price is depicted in the following figure 4. As it can be seen, the movements follow approximately in the same manner the European put option calculation for the change in the time steps. An interesting comparison,

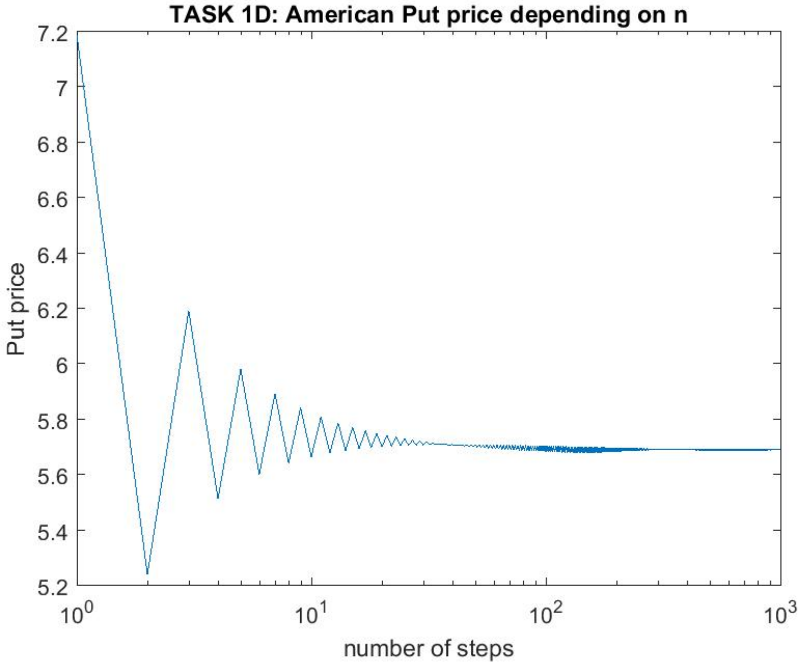

Figure 4: American put price evolution for n=1000

that can highlight in evidence the small difference between American and European put option is given in figures 5, 6, 7. Figure 5 shows the movement of the two put options prices for a time step range from 0 to 1000 with respect to the flat movement of the BSM price. Figure 6, instead, is a zoomed figure for tie steps from 150 to 1000. As in figure 5 is not so clear whether option price dominates the other, from figure 6 starts to become more noticeable the higher value of the American option. Finally, figure 7 depicts the zoom of the area in figure 6 where the areas are connecting and the movements are less chaotic. From this last figure it can be seen that the American put dominates the European one, even if slightly. Because the time steps are not so high, there is still a difference in the European put price with respect to the BSM price.

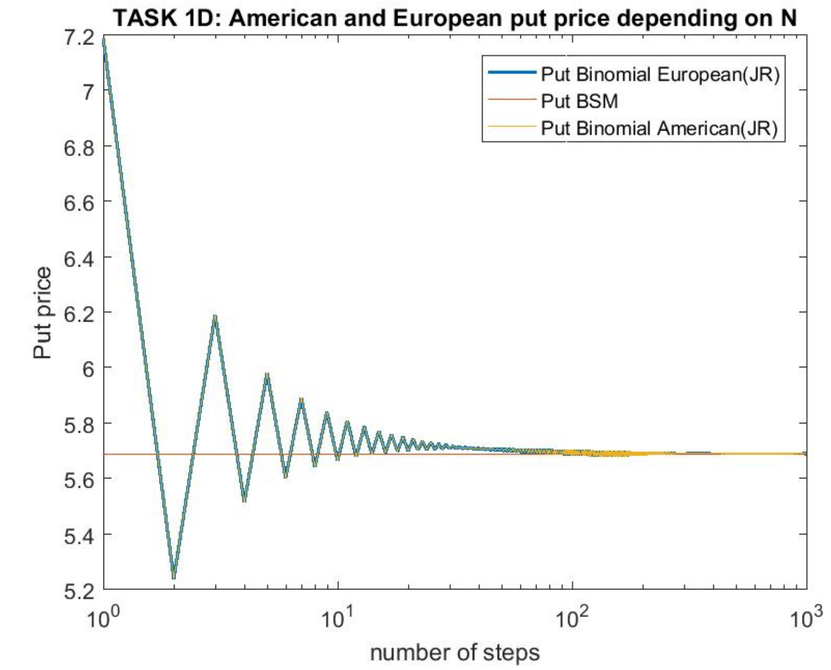

Figure 5: American vs. European binomial put price evolution for n=1000

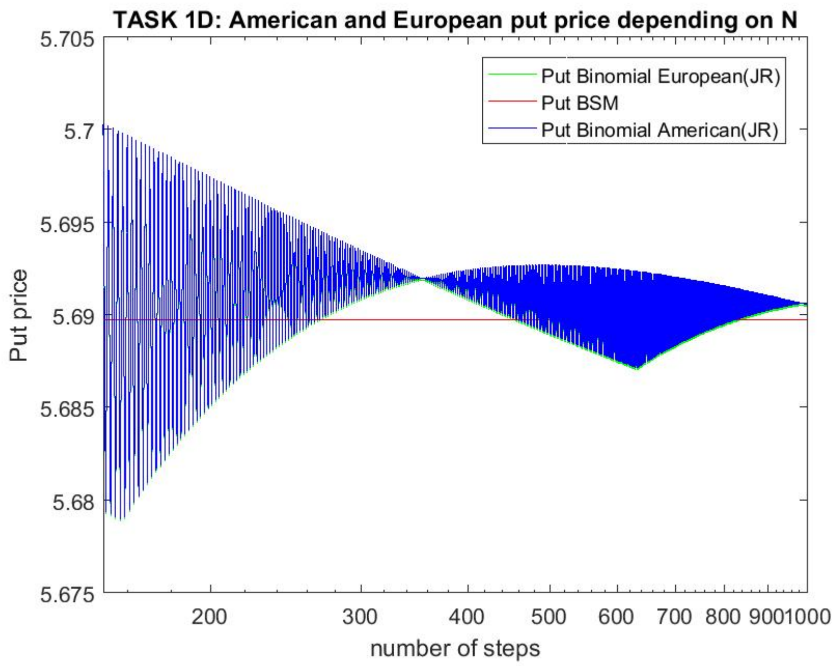

Figure 6: American vs. European zoomed in (150 to 1000)

The higher price of the American put, that both the graphs and the Binomial valuation depict, reflects the early exercise premium. In fact, an American put option is valued more if compared to an European option, because of its right to be exercised before maturity.

We calculate the early exercise premium in two ways: the first one finds it as the difference between the price of the American option at t_0 and the value of the European one at the same date t_0, both calculated with the Binomial approach. The second one,

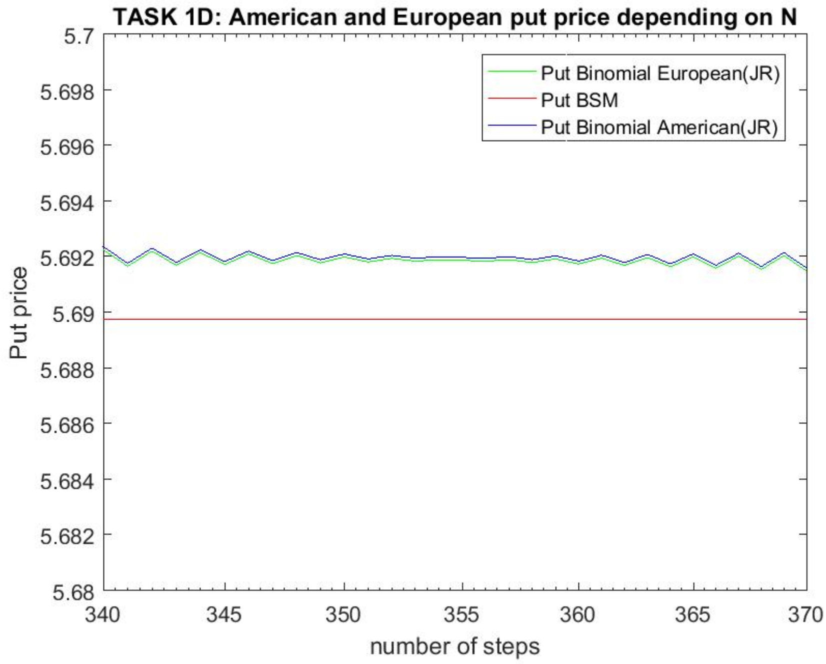

Figure 7: American vs. European zoomed in (340 to 370)

instead, subtract from the American put BBS price the BSM price.

For different time steps, we get the following results in this case.

Table 4: Early exercise for n

Number of steps	**Early exercise premium vs. Binomial**	**Early exercise premium vs. BSM**
n=100	1.0068e-04	0,01451
n=1000	1.0913e-04	9.0717e-04

The early exercise premium is very low for the put considered, explaining the small difference between the American and the European put prices, that also the graphs show. However, the early exercise premium is higher when the time steps are higher. This because, if the number of periods increases, it is also more probable that the American option can be exercised before t_0. This increasing probability reflects an increase in the difference between the European and the American put option and so, an increase in the early exercise premium. Figure 9 explains the behavior of the early exercise for the changing time steps: When n less than 10, the early exercise is equal to zero, because, given the very short number of time of steps is harder to gain profits from the early exercise of the American option. Instead, as n increases the possibility of an early exercise of the put option increases as well, leading to an higher early exercise premium. To be noticed that, for time steps between 10 and 100, the slope of the early exercise premium is deeper, while, for time steps between 100 and 1000 the slope is flatter. In fact, when the time steps are not so high, also a small increase in n can lead to an increase of the probability of profitable early exercise. When, instead, the time steps are already high, an increase in n is less significant because the early exercise, if profitable, could have already taken place. If, instead, we consider the early exercise

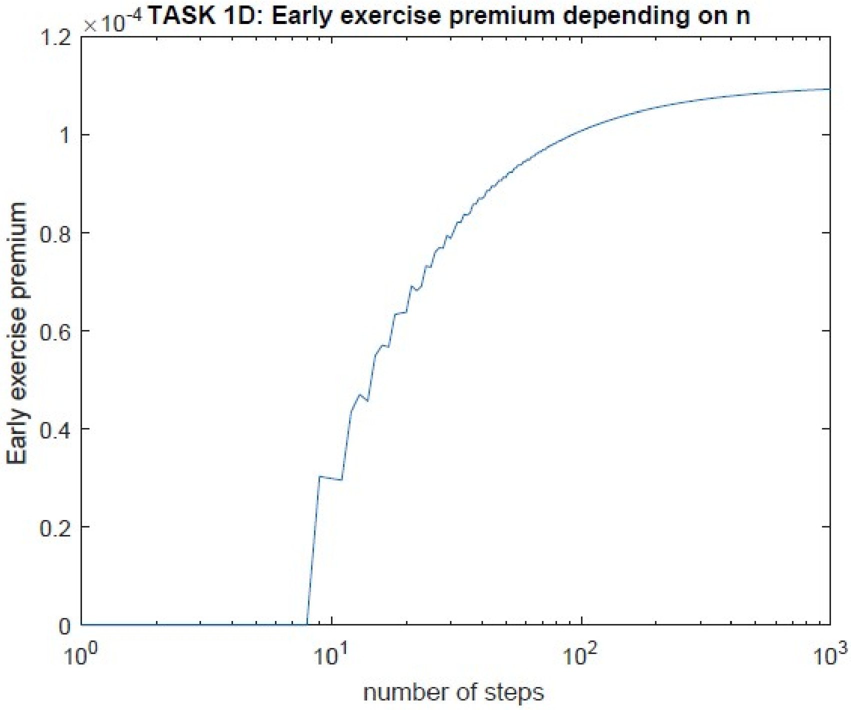

Figure 8: Early exercise premium (Binomial) for increasing n

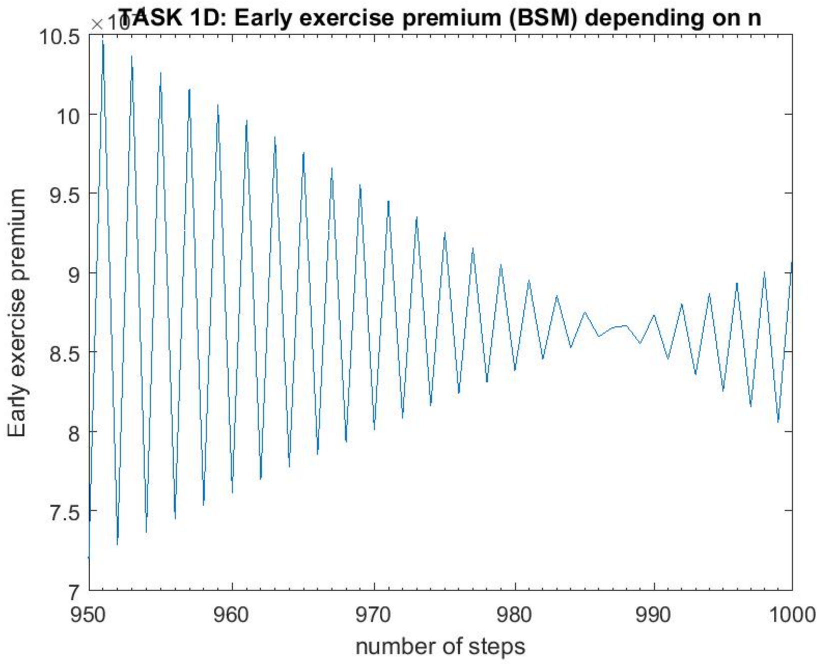

Figure 9: Early exercise premium (BSM) for increasing n, zoomed in from n=950 to n=1000

premium as a difference between the Binomial option pricing method for the American put and the value of the same put calculated with the BSM method, we end up with a different representation of the premium itself. Since the BSM gives the theoretical true value of an European option, this approach is more reliable. As you can see in the graph below, the early exercise premium using the BSM European put is higher if compared to the Binomial. Hence, even for n=1000 the early exercise premium calculated subtracting the European Binomial still contains estimation errors. The difference is even more remarkable for lower time steps.

1.3 Technical implementation

In the script *Task1.m* all computations of this task can be executed with respect to our data set. In the first part the stock price evolutions with the JR and CRR approach are simulated by calling the functions *S_JR.m* and *S_CRR.m* . In both functions we used a double for-loop to generate the simulated stock price matrix with respect to the formulas from the theoretical section for the given number of time steps. In order to compare the BSM-price with the binomial option price we set up a function called Binomial_JR.m which computes the the standard BSM price as well as the binomial price. Therefore, our data set was used as input parameters. To differentiate between a call or a put we additionally set a flag indicator with help of if and elseif statements where the input 1 stands for a call computation while -1 indicates a put computation. Additionally, we differentiate between an European and an American option by using a binary variable with different "if" and "elseif" statements as a further input argument where 0 indicates a European option while 1 indicates an American option.
To compute the binomial price the stock price evolution tree has to be called before applying the backward recursion process. The standard BSM option prices are computed by calling further functions which were set up. They contain the standard BSM closed-form equations. In order to calculate the American option price we used another "if" conditions to check whether the option should be exercised or not before maturity. To plot the graphs we first created for each illustration an own matrix by calling the functions with a for-loop for different inputs and then make use of the *semilogx* command to get a logscaled x-axis illustration. To compute the early exercise premium we simply subtracted the corresponding matrices.

2 Binomial Black-Scholes method (BBS)

Another discrete time valuation method for pricing options is given by the Binomial Black-Scholes method (BBS). This method starts, as the binomial model, from the valuation of the behavior of the stock prices and follows the same backward iterative procedure. The difference here is that. At the time step just before option maturity[2], the Black-Scholes value replaces the usual "continuation value".

2.1 Theoretical approach

In the case of an American option, we check at the n-th column of our binomial tree, whether

$$max(C_t^E, S_t - X)$$

for a call or for a put:

$$max(P_t^E, X - S_t)$$

[2]It is identified by the n-th column oft he binomial tree

where C_t^E and P_t^E are computed via the BSM-formula respectively. We test this maximum condition for every row of the n-th column of our binomial tree.
When instead we are dealing with an European option, given that the early exercise is not possible, we replace the value of the n-th column directly with the Black-Scholes-Merton formula for the put or for the call option.
The other steps of the BBS calculation follow the same procedure and the same reasoning of a standard Binomial option pricing valuation, as described in exercise 1.
In the following section we apply the BBS method for our dataset and we compare the result with the standard Binomial method. In the following step, we also calculate the early exercise premium for the BBS method in two ways, considering first the value at t_0 of the corresponding European put calculated using the BBS method and then with the price of the put using the Black-Scholes-Merton model. At the end, we depict also the calculation error of the BBS with respect to the one of the Binomial calculation and we draw also the computational time of the two methods.

2.2 Discussion of results

From the application of the Binomial Black-Scholes method we obtain the following results for the American put option calculation. The Binomial approach makes again use of the Jarrow/Rudd risk-neutral probabilities.

Table 5: Binomial vs. BBS

Number of steps	Binomial approach	BBS method	Difference
n=100	5.7043	5.6945	0.0098
n=1000	5.6907	5.6904	3.2894e-04

As it can be noticed, the difference between the two valuation methods becomes smaller as the number of time steps increases, giving a more reliable solution. The put prices calculated by the two methods for different number of time steps are depicted in the following figure 10. The binomial method shows higher fluctuations for low value of n, reflecting an imperfect and not so reliable calculation for time steps from 1 to approximately 30. Instead, the BBS method shows, already from the beginning a stable path, with values in line with the price to which the method converge for higher n. The behaviour of the BBS valuation is made stable by the substitution of the Black-Scholes-Merton formula values that eliminate the big fluctuations in prices due to the iterative backward evaluation.

The following figure 11 shows the zoom of the previous one, underlying the chaotic movement of the Binomial valuation method with respect to the BBS one. The x-axis is reduced from 100 to 1000.
An interesting data that can be derived from these two methods is the early exercise premium. It is found in two different ways. The first one calculated the premium as the difference between the American put option and the corresponding European one calculated both with the BBS method and with the Binomial method. In figure 12 there is shown the increase in the early exercise premium when n increases. This is in line with what explained in the exercise 1. We can see that there is the tendency of

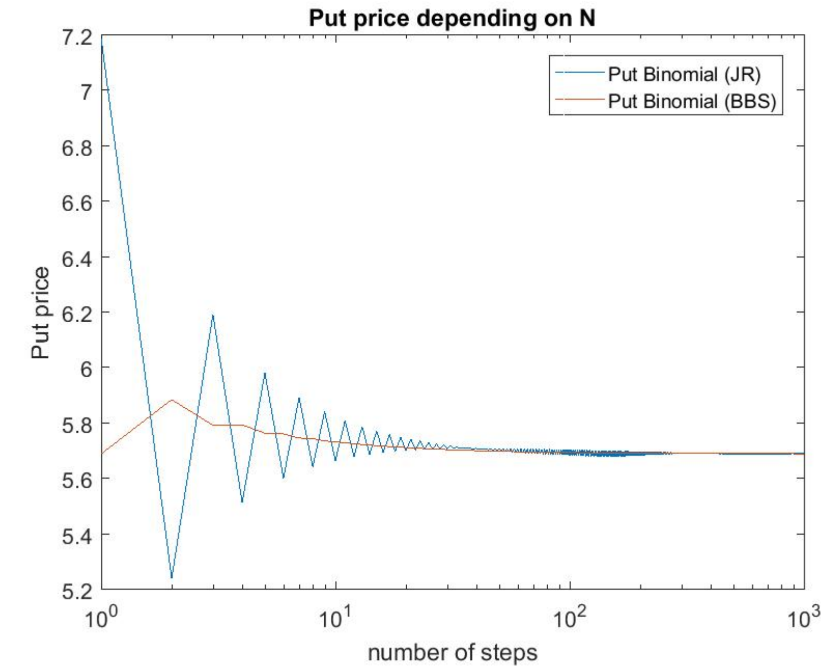

Figure 10: BBS vs. Binomial to n=1000

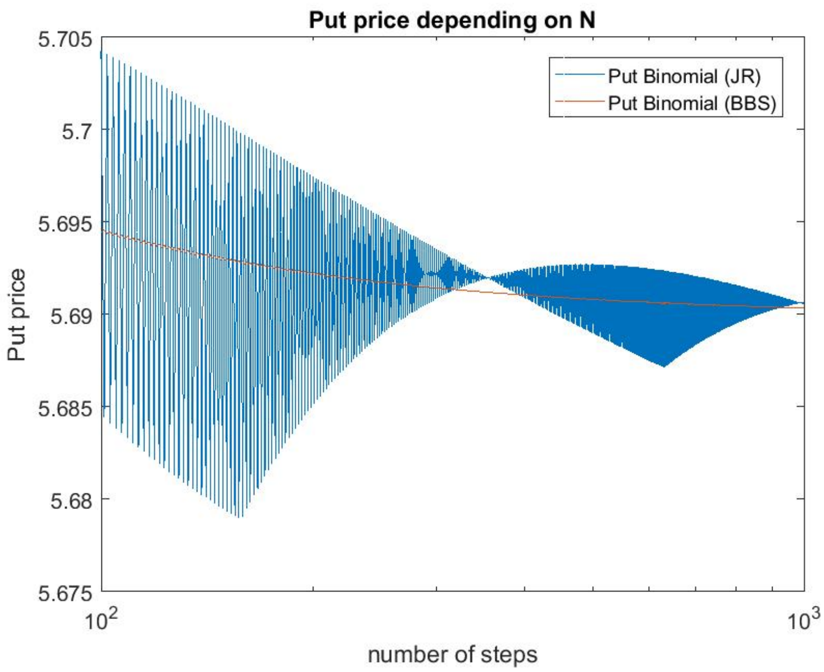

Figure 11: BBS vs. Binomial from n=100 to n=1000

the early exercise premium found with the Binomial method calculation to be worth less than the corresponding one for the BBS method.

However, this evaluation of the early exercise premia involves approximation errors, because the use of the same method includes a minor quality in the approximated results. If instead we subtract from the value of the American BBS and American Binomial put price the BSM put price, we end up with different early exercise premia. 13.

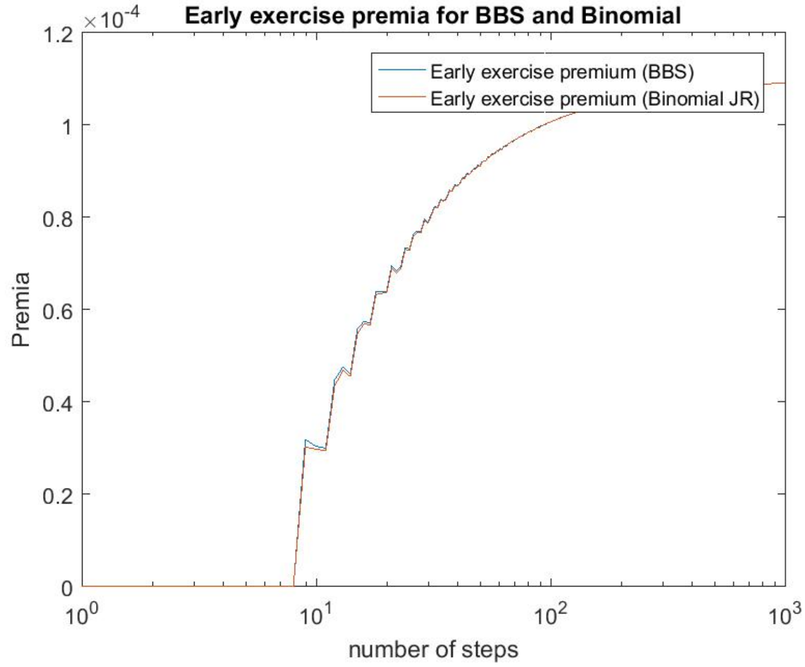

Figure 12: Early Exercise Premium of BBS vs. Binomial for increasing n

Given the high chaotic movement of the Binomial calculation, also the early exercise

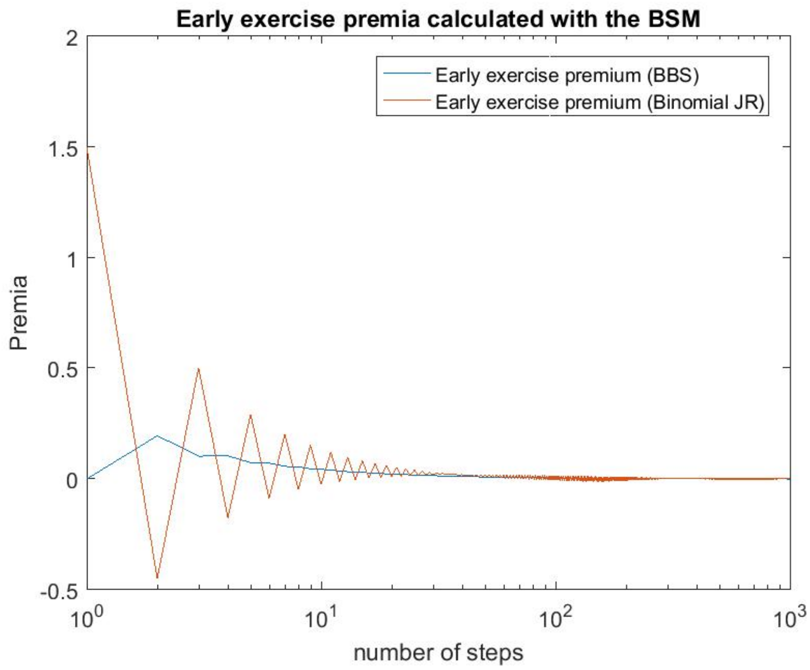

Figure 13: Early Exercise Premium of BBS and Binomial vs BSM for increasing n

premium for the American Binomial put is quite high at the beginning. At the end, when n is high an near to 1000, we have an early exercise of approximately zero. This because, as we have seen before, the price of the American put considered, with both the BBS and the Binomial approach, is quite similar to the one of the European put calculated with the BSM method. The corresponding difference in the early exercise premia calculated with the two methods is drawn in the following figure, where, from an higher difference, the early exercise premia become approximatively equal for n=1000.

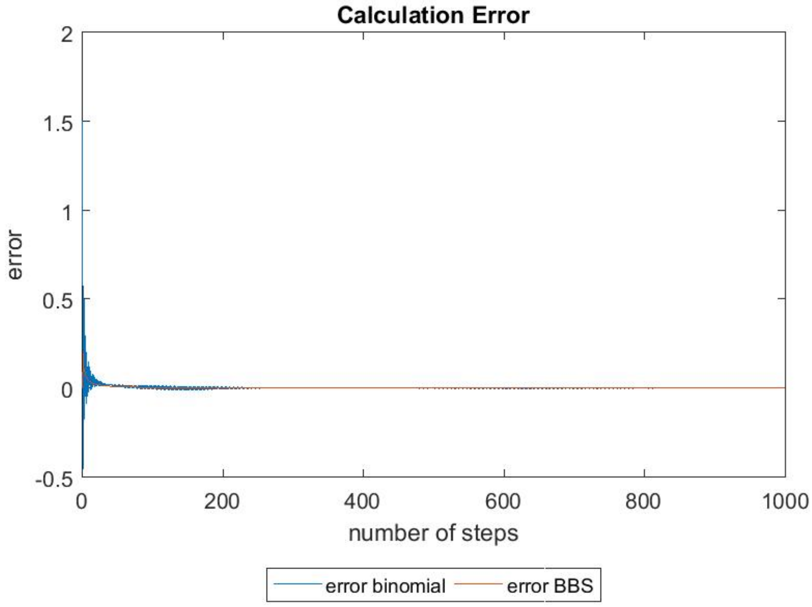

Figure 14: Calculation error for increasing n

As last step, we want to calculate the approximate error of the two methods. To do so, we calculate the value of the American put for n=15000 with the Binomial method and we take it as the true value of the American put option considered, following Broadie/Detemple (1996). The value obtained by the calculation is 5.6899. From the difference between the value calculated by the the Binomial and the BBS methods and the assumed true value, we draw a graph showing the error in the calculation for different values of n. The error of the Binomial method is very high for lower value of n, as the chaotic movements of Figure 14 suggest. But, once n increases considerably, the error start to become very limited, suggesting the reliability of the approximation of the two methods.

2.3 Technical implementation

All BBS calculations can be executed in the mainscript *task2.m Task2.m*. In order to calculate our data set via the BBS approach, it was necessary to extend our binomial function of the first task. The function is called *binomial_JR_BBS.m* . Basically we just added the new condition for the binomial tree by writing the computed BSM put value into column n for each simulated stock n by using a for-loop. Afterwards the same recursive procedure takes place which was already described in task 1. We again made use of the two dummy variables -1 and 0 to differentiate between call/put and European/American. Note here that to our understanding a BBS European calculation is not reasonable since this approach was actually designed to estimate American options but due to comparison reasons for the early exercise premia we decided to set it up.

All graphs can be computed in the same way already described in part 1. Note here again that the calculation take some time since we use a high n.

3 The BBS method with Richardson extrapolation

The BBSR extension is a simple and effective approach to improve the estimate for American option pricing.

3.1 Theoretical Approach

In this task the option price for our data set was calculated by extending the BBS where two-point Richardson extrapolation is added to the approach (Broadie and Detemple 1996, p. 1244). Therefore following formula was used:

$$C = 2C_2 - C_1$$

where
C_2 = option price for n
C_1 = option price for n/2

In the next section, we calculate the option prices with the BBSR approach to highlight the improvement compared to the BBS movement.

3.2 Discussion of results

Applying the calculation with the BBSR method to our American put option, we get the following results:
As n increases the difference between the values for the American put calculated with

Table 6: BBS vs. BBSR Values

Number of steps	**American put option with BBSR method**	**BBS method**	**Difference**
n=100	5.6898	5.6945	-0.0046
n=1000	5.6899	5.6904	-4.8311e-04

the BBS and with the BBSR methods tends to converge. In fact, the difference between the two prices is very small. Comparing both methods to the two methods, it can be noticed that the BBSR, for n=1000, gives approximatively the same result of the price that we assume as true value[3], while the BBS results is more distant from the true value.

Figure 15 illustrates how the BBS and BBSR methods differ for increasing time steps n. As one can see the BBSR approach starts from a higher level while dropping more rapidly with increasing n compared to to the simple BBS method. The BBS graph approaches smoothly while BBSR graph shows a high fluctuation but it converges faster to its true value.

[3] The true value is 5.6899 and is calculated for the American put with the binomial method for n=15000

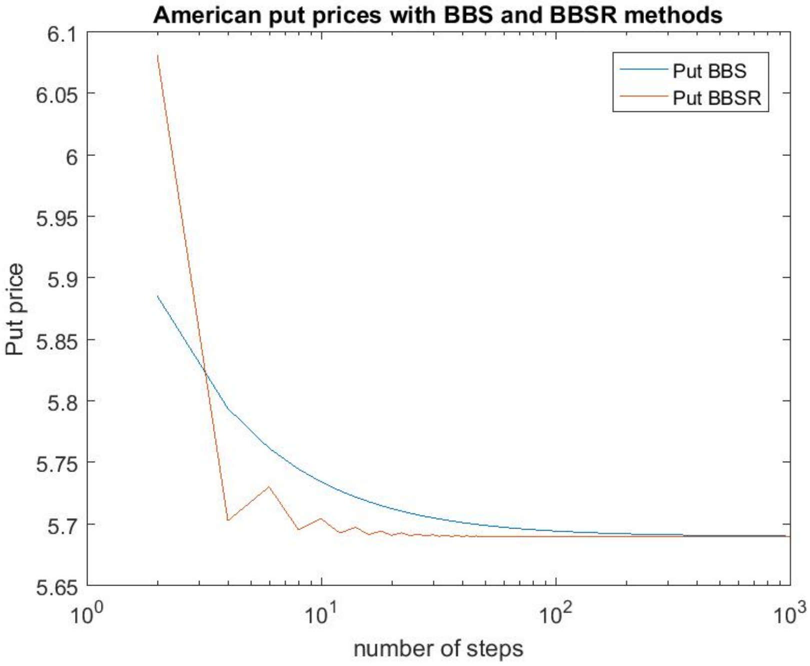

Figure 15: BBS vs. BBSR for increasing n

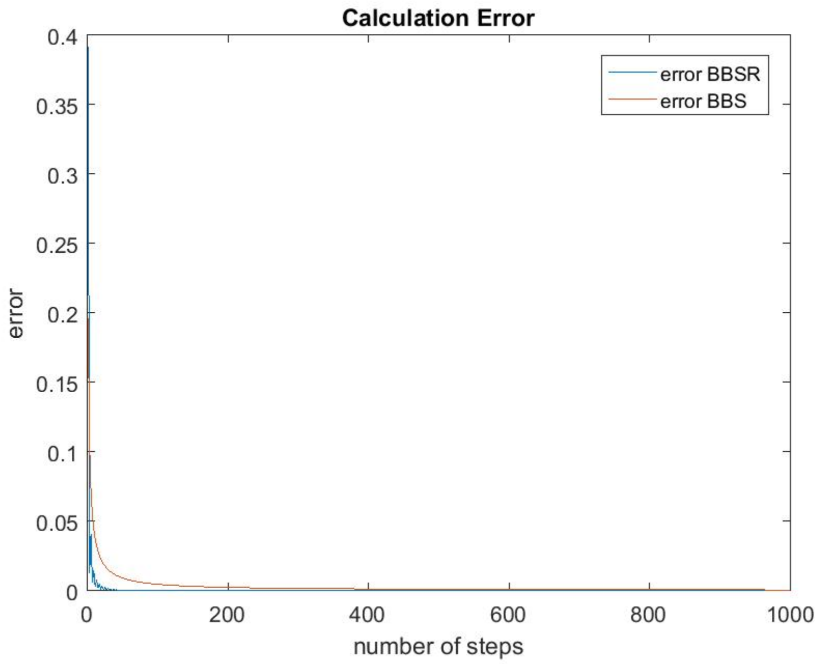

Figure 16: Calculation error BBS vs. BBSR for n=1000

This observation can also be found in figure 16 and 19. Both graphs again show the error term fluctuation to the true value for increasing n. Just as in section 2, both methods convergence to the true value 5.6899 represented in figure 15. But for $n < 100$ in figure 19 it gets clear that the BBSR method converges to its true value more rapidly just after 20 time steps while the BBS has still a relatively high and fluctuating calculation error.

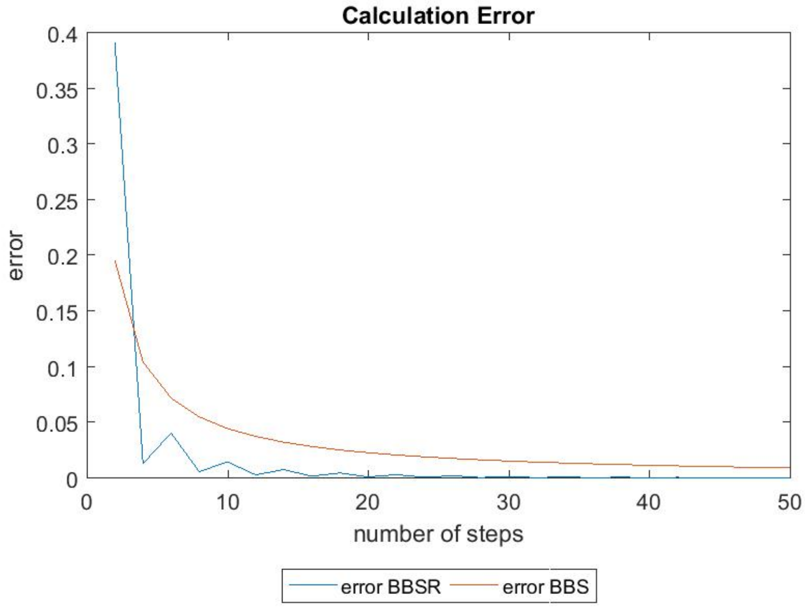

Figure 17: Calculation error BBS vs. BBSR for n 0 to 50

3.3 Technical implementation

All tasks of exercise 3 can again be executed in script *Task.3*. Herefore the main function Binomial_JR_BBSR.m was set up. To implement the BBSR approach we simply extended our Binomial_JR_BBS.m algorithm with respect to equation 3.1. As in previous tasks, an indicator for the European and American (0 or 1) option was implemented, as well as an indicator to differentiate between Call and Put (1 or -1) was implemented to differentiate between European and American option estimations. In order to plot the different graphs we created matrices with for-loops. Since we use 2 n- time steps, one has to delete the odd rows since they are filled with 0 zeros. In order to do that make use of the *any*-command.

4 Time interpretation

At the very last, we create also another Matlab script, named Time_test to take into account the comparison between the computational time needed by the BBS, the Binomial method and BBSR to calculate the approximated option price. We have a new script to avoid computational interferences that can be arise if some calculation were already been made before the timing test.[4]
To be consistent in our comparison, we create the Matlab function named "Binomial_JR_time.m", which gives back only one output[5], the put price, as the other function used.
First of all, we compare the time between the Binomial and the BBS method for an

[4] For this reasons, we use the "clear" command in each section of the Matlab file.

[5] If we had used the script "Binomial_JR", the comparison would have been not completely reliable, given that the "Binomial_JR" function calculates also the BSM price that it is not required for the Binomial put value calculation.

American put option, considering time steps from 0 to 1000.

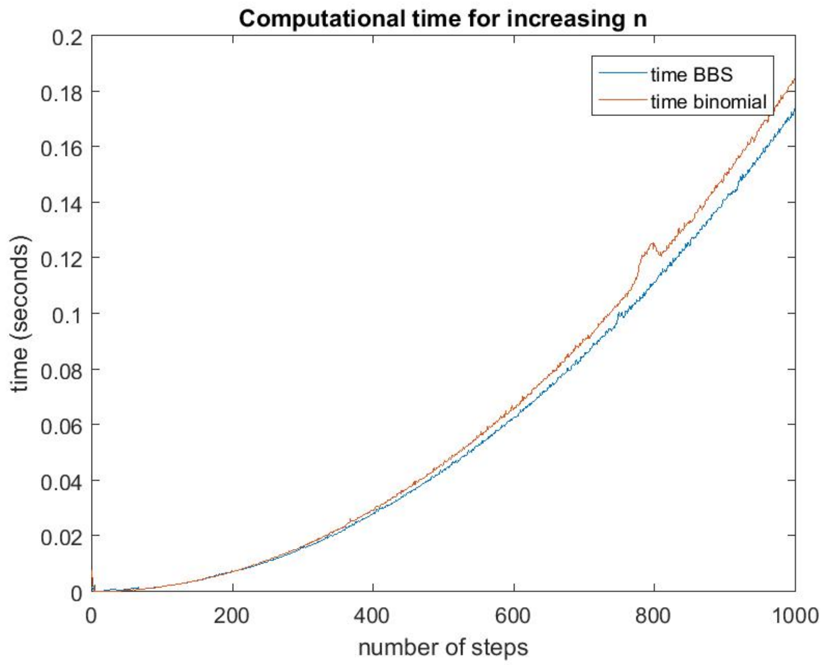

Figure 18: Computational time Binomial vs. BBS

Both the two methods show an increase in the time used when the number of step raises. This because, the better quality of the approximation is subject to a longer timing calculations. Even if, the time values can slightly change every time, due to the hardware of the computer and to the Matlab performances in a determined time, there are some effects that are always true in each proof we have made. The most important one is that the Binomial approach requires more time with respect to the BBS one. In fact, an explanation could be that for the Binomial calculation we have to calculate also the (n+1) row, while in the BBS method this is not necessary. Additionally, maybe the comparison in the n-th column of the BSM price and of the early exercise price, as for the BBS method, can take less than the comparison between the early exercise and the non-exercise price calculated with the iterative procedure as in the Binomial method.

In a second moment, we also compare the computational time of the BBS, of the BBSR and of the Binomial method with time steps n=2,4,6 up to n=1000. We can see that the BBS seems to take less time with respect to the other methods, that is in line with what we have already discovered in the previous figure. The more time consuming method is the BBSR, probably because of its necessity to recall the BBS computation for n and for n/2.

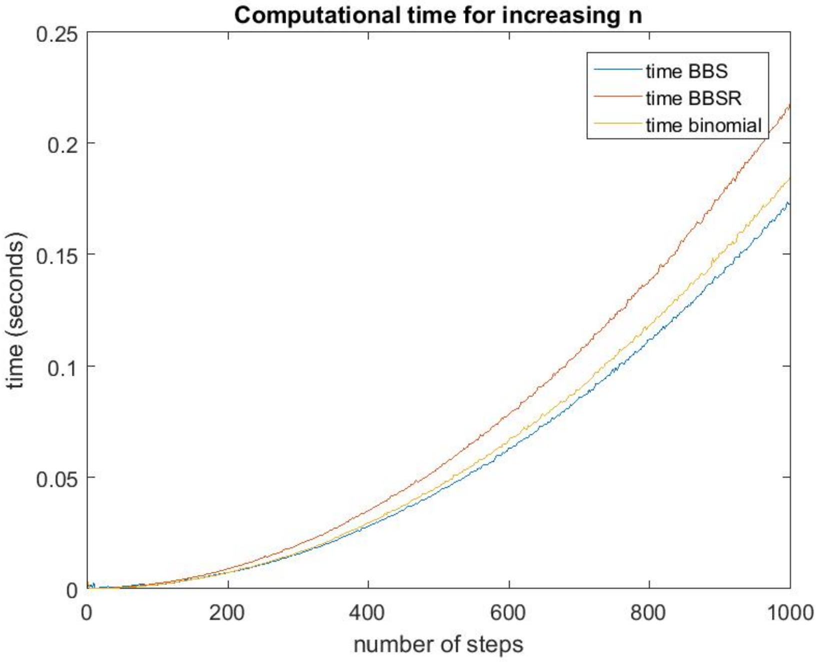

Figure 19: Computational time Binomial vs. BBS vs. BBSR

5 References

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