**Math 4B Final Exam Review Problems**

These are practice problems to help you prepare for the final exam, to be submitted via Gradescope (see instructions on Gauchospace). There is a forum on Gauchospace to discuss the problems, and your TAs and I will provide feedback. Some (but not all) of these problems are representative of exam problems. However, this collection of problems is much longer than your actual exam will be.

The final exam is cumulative and will cover material from sections 1.1-1.3, 2.1-2.5, 2.7, 2.9, 3.1-3.8, 7.1-7.9, 9.1-9.3 in your textbook. However, these review problems only cover material we discussed after exam 2 (7.1-7.9, 9.1-9.3). I recommend that you look over the exam 1 and 2 review problems to help prepare for the older material.

1. If $A$ is a $3 \times 3$ matrix with real coefficients, then which describe the system of DEs

$$\vec{\mathbf{x}}' = A\vec{\mathbf{x}}?$$

Circle all that apply.

A. Linear

B. First Order

C. Constant Coefficients

D. Homogeneous

2. Given the following $2 \times 2$ linear system with constant coefficients

$$\begin{aligned} \vec{\mathbf{x}}' &= A\vec{\mathbf{x}} && \text{(H)} \\ \vec{\mathbf{x}}' &= A\vec{\mathbf{x}} + \vec{\mathbf{g}}(t), && \text{(N)} \end{aligned}$$

where $\vec{\mathbf{g}}$ is not the zero vector. Which of the following statements are true? Justify your answers.

A. If $\vec{x}_h$ is a solution to (H) and $\vec{x}_p$ is a solution to (N), then $\vec{x}_h + 2\vec{x}_p$ is a solution to (N).

B. If $\vec{x}_1$ and $\vec{x}_2$ are both solutions to (N), then $\vec{x}_1 - \vec{x}_2$ is a solution to (H).

3. Find the general solution to the system. Is the equilibrium solution $\vec{\mathbf{0}}$ unstable or stable? Explain.

$$\vec{\mathbf{x}}' = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix} \vec{\mathbf{x}}.$$

4. Consider the system

$$\vec{\mathbf{x}}' = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \vec{\mathbf{x}}.$$

(a) Find the general solution for the system.

(b) Using your general solution, state a fundamental matrix $\mathbf{X}(t)$ for the system. Verify that $\mathbf{X}(t)$ is indeed a fundamental matrix.

(c) Is the equilibrium solution $\vec{\mathbf{0}}$ a source, sink or saddle? Is it unstable or stable? Explain.

(d) Sketch the phase portrait.

5. Consider the system

$$\vec{\mathbf{x}}' = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix} \vec{\mathbf{x}}.$$

(a) Find the general solution for the system.

(b) Is the equilibrium solution $\vec{\mathbf{0}}$ a source, sink or saddle? Is it unstable or stable? Explain.

(c) State the equations of the $v$- and $h$-nullclines for the system.

(d) Sketch the phase portrait, using nullclines as an aid.

6. Consider the system

$$\vec{\mathbf{x}}' = \begin{bmatrix} 0 & 1 \\ -9/4 & 0 \end{bmatrix} \vec{\mathbf{x}}.$$

(a) Find the general solution for the system.

(b) Classify the equilibrium solution $\vec{\mathbf{0}}$. Explain.

(c) State the equations of the $v$- and $h$-nullclines for the system.

(d) Sketch the phase portrait, using nullclines as an aid.

7. Consider the system

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 2t \\ 0 \end{bmatrix}.$$

(a) Solve the system using the method of undetermined coefficients.

(b) Solve the system using variation of parameters.

(c) Solve the system by decoupling.

8. Let $y'' + by' + 2y = 0$ be the equation of a damped vibrating spring with mass $m = 1$, damping coefficient $b > 0$, and spring constant $k = 2$.

   (a) Convert this second order equation into a system of two first order equations.

   (b) Express the eigenvalues for this system in terms of $b$.

   (c) Describe the stability of the equilibrium solution $\vec{\mathbf{0}}$ for $b > 2\sqrt{2}$. Justify your claim with information about the eigenvalues of the matrix for the system.

   (d) Connect the behavior of solutions near an equilibrium of this type with the spring mass system with damping coefficient $b > 2\sqrt{2}$ and explain why your answer for part (c) is (or is not) what one should expect.

9. Consider the nonlinear, autonomous systems of DEs (1) and (2) below.

$$\begin{aligned} x' &= (2+x)(y-x) \\ y' &= (4-x)(y+x) \end{aligned} \tag{1}$$

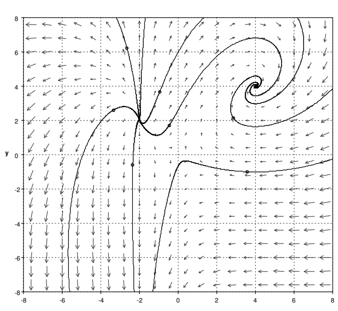
$$\begin{aligned} x' &= (2-x)(x-y) \\ y' &= (4-x)(y+x) \end{aligned} \tag{2}$$

(a) Find all equilibrium solutions of each system.

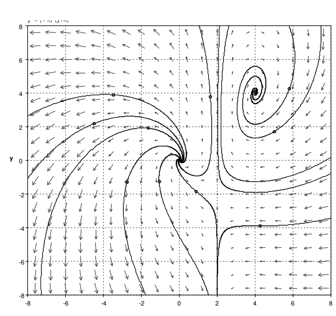
(b) Find the corresponding linear system near each equilibrium solution.

(c) Find the eigenvalues of each linear system to determine the stability of each equilibrium solution.

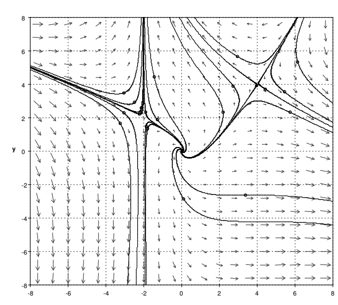
(d) Use this information to identify which of the phase portraits below corresponds to each system.



A



B



C

10. Find and characterize the equilibrium points/critical points of the nonlinear system

$$\begin{aligned} x' &= xy \\ y' &= 1 - x^2 - y^2 \end{aligned}$$