Homework 5: Graphical Models, MDPs

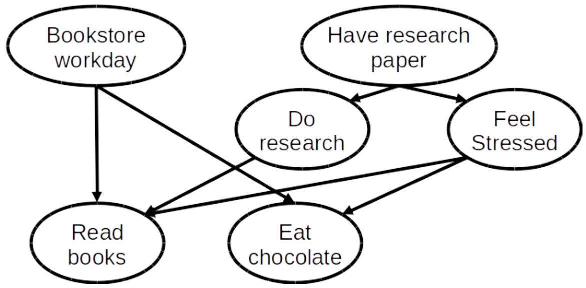
Introduction

There is a mathematical component and a programming component to this homework. Please submit your **tex, PDF, and Python files** to Canvas, and push all of your work to your GitHub repository. If a question requires you to make any plots, please include those in the writeup.

Bayesian Networks [7 pts]

Problem 1

In this problem we explore the conditional independence properties of a Bayesian Network. Consider the following Bayesian network representing a student's work. Each random variable is binary (true/false).



The random variables are:

- Have research paper: Does the student have a research paper?
- Do Research: Is the student doing research?
- Feel Stressed: Is the student feeling stressed?
- Bookstore Workday: Is the student working at the bookstore?
- Read Books: Is the student reading a book?
- Eat Chocolate: Is the student eating chocolate?

For the following questions, $A \perp B$ means that events A and B are independent and $A \perp B|C$ means that events A and B are independent conditioned on C. Use the concept of d-separation to answer the questions and show your work.

1. Is Bookstore Workday $\perp$ Have research paper? If NO, give intuition for why.
2. Is Bookstore Workday $\perp$ Have research paper | Read Books? If NO, give intuition for why.
3. Is Do Research $\perp$ Eat Chocolate? If NO, give intuition for why.
4. Is Do Research $\perp$ Eat Chocolate | Feel Stressed? If NO, give intuition for why.
5. Suppose the student has done some mindfulness exercises to avoid stress eating. Now, when they are stressed, they read (fun) books but don't eat chocolate. Draw the modified network.
6. For this modified network, is Do Research $\perp$ Eat Chocolate? If NO, give an intuition why. If YES, describe what observations (if any) would cause them to no longer be independent.

Solution

1.

Yes, Bookstore Workday $\perp$ Have research paper because Bookstore Workday and Have research paper are d-separated at every undirected graph separating them.

2.

No, Bookstore workday and Have research paper are not d-separated by Read books because the path Have research paper – Feel stressed – Read books – Bookstore workday is not blocked. The intuition is that if a student is reading a book, then having a research paper reduces the probability of working at the bookstore.

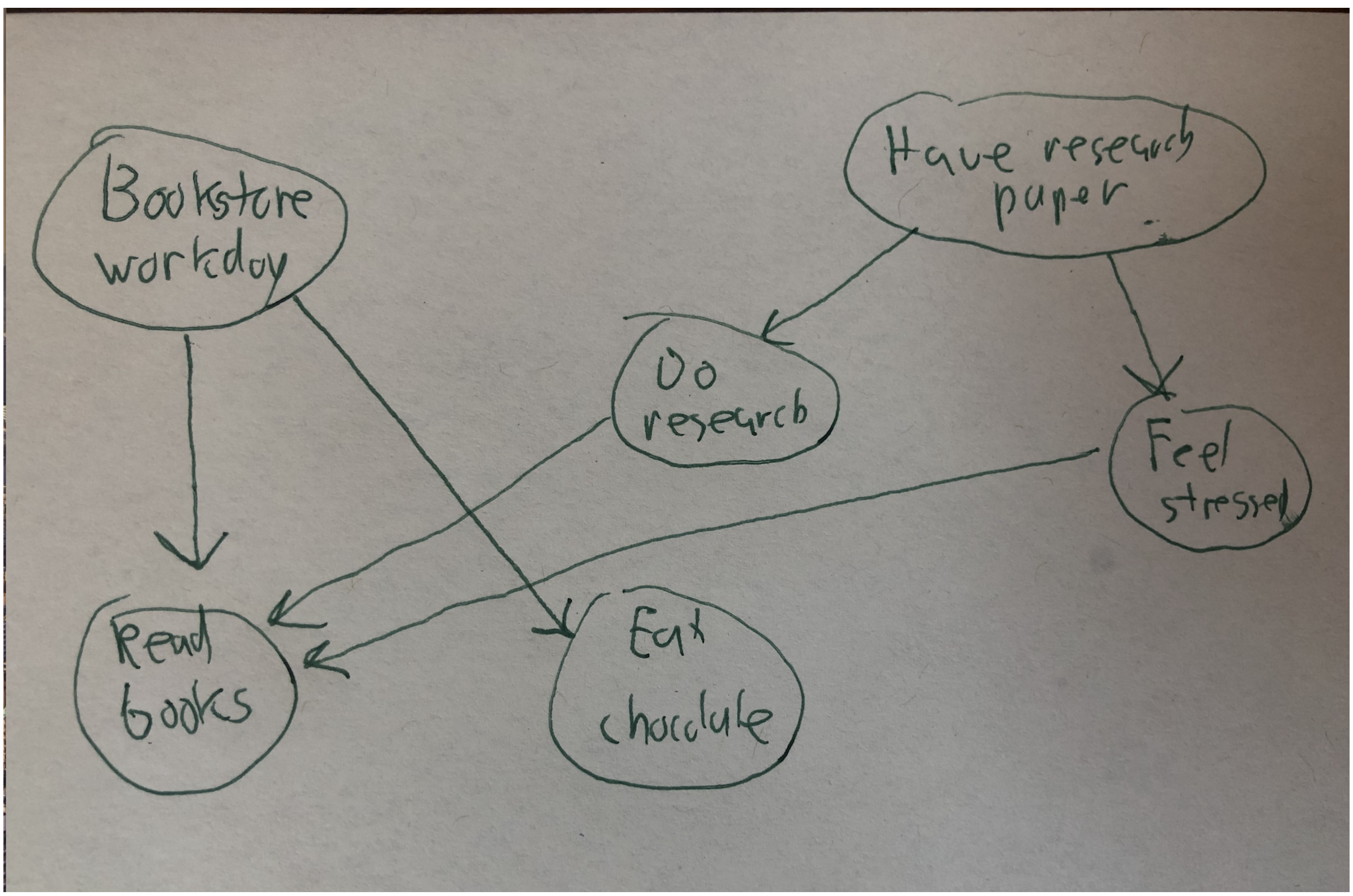
3.

No, Do research and Each chocolate are not d-separated because the path Do research – Have research paper – Feel stressed – Eat chocolate is not blocked. The intuition is that doing research increases the chance that they have a research paper, which increases the chance they feel stressed, which increases the chance that they are eating chocolate.

4.

Yes, every undirected graph between Do Research and Eat Chocolate is blocked. The intuition is that feeling stressed gives you all the information that you would get from knowing that you're eating chocolate.

5.



6.

Yes, Do Research $\perp$ Eat Chocolate because Do Research and Eat Chocolate are d-separated. Any path must path through Read books and is blocked.

Kalman Filters [7 pts]

Problem 2

In this problem, you will implement a one-dimensional Kalman filter. Assume the following dynamical system model:

$$\begin{aligned} z_{t+1} &= z_t + \epsilon_t \\ x_t &= z_t + \gamma_t \end{aligned}$$

where z are the hidden variables and x are the observed measurements. The random variables ϵ and γ are drawn from the following Normal distributions:

$$\begin{aligned} \epsilon_t &\sim N(\mu_\epsilon, \sigma_\epsilon) \\ \gamma_t &\sim N(\mu_\gamma, \sigma_\gamma) \end{aligned}$$

where $\mu_\epsilon = 0$, $\sigma_\epsilon = 0.05$, $\mu_\gamma = 0$ and $\sigma_\gamma = 1.0$

You are provided with the observed data x and the hidden data z in `kf-data.csv`, and the prior on the first hidden state is $p(z_0) = \mathrm{N}(\mu_p,\sigma_p)$ where $\mu_p = 5$ and $\sigma_p = 1$

1. The distribution $p(z_t|x_0...x_t)$ will be Gaussian $N(\mu_t, \sigma_t^2)$. Derive an iterative update for the mean μ_t and variance σ_t^2 given the mean and variance at the previous time step (μ_{t-1} and σ_{t-1}^2).
2. Implement this update and apply it to the observed data above (do not use the hidden data to find these updates). Provide a plot of μ_t over time as well as a $2\sigma_t$ interval around μ_t (that is $\mu_t \pm 2\sigma_t$). Does the Kalman filter "catch up" to the true hidden object?
3. Repeat the same process but change the observation at time $t = 10$ to $x_{t=10} = 10.2$ (an extreme outlier measurement). How does the Kalman filter respond to this outlier?
4. Comment on the misspecification of dynamical system model for these data. Based on the previous two parts, how does this misspecification affect the predictions?

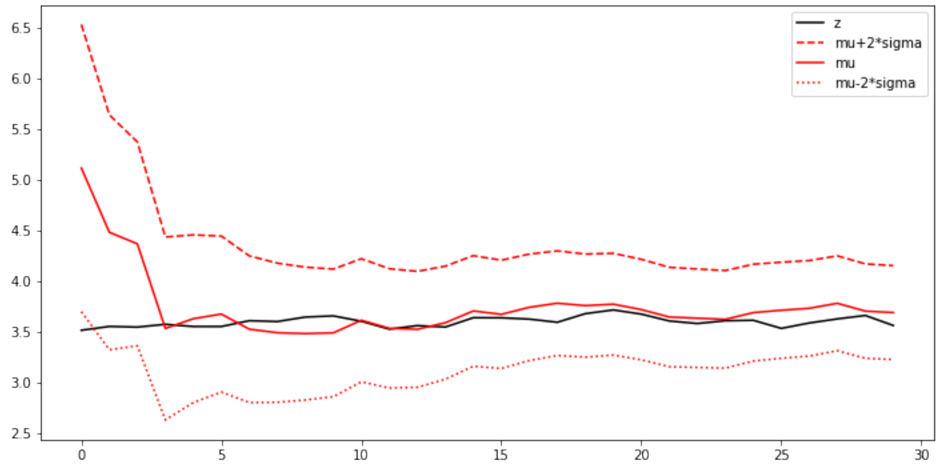
Solution

1.

$p(z_t|x_0...x_t) \propto p(x_t|z_t) \cdot p(z_t|x_0, ..., x_{t-1})$
$= N(z_t, \sigma_\gamma^2) \cdot \int p(z_t|z_{t-1}, x_0, ..., x_{t-1}) \cdot p(z_{t-1}|x_0, .., x_{t-1}) \cdot dz_{t-1}$
$= N(z_t, \sigma_\gamma^2) \cdot \int p(z_t|z_{t-1}) \cdot p(z_{t-1}|x_0, .., x_{t-1}) \cdot dz_{t-1}$
$= N(z_t, \sigma_\gamma^2) \cdot \int N(z_{t-1}, \sigma_\epsilon^2) \cdot N(\mu_{t-1}, \sigma_{t-1}^2) \cdot dz_{t-1}$
$= N(z_t, \sigma_\gamma^2) \cdot N(\mu_{(t-1)^*}, \sigma_{(t-1)^*}^2)$ where $\mu_{(t-1)^*} = \mu_{t-1}$ and $\sigma_{(t-1)^*}^2 = \sigma_{t-1}^2 + \sigma_\epsilon^2$
$= N(\mu_t, \sigma_t^2)$ where $\mu_t = \sigma_t^2 \cdot (\frac{x_t}{\sigma_\gamma^2} + \frac{\mu_{(t-1)^*}}{\sigma_{(t-1)^*}^2})$ and $\sigma_t^2 = (\frac{1}{\sigma_{(t-1)^*}^2} + \frac{1}{\sigma_\gamma^2})^{-1}$

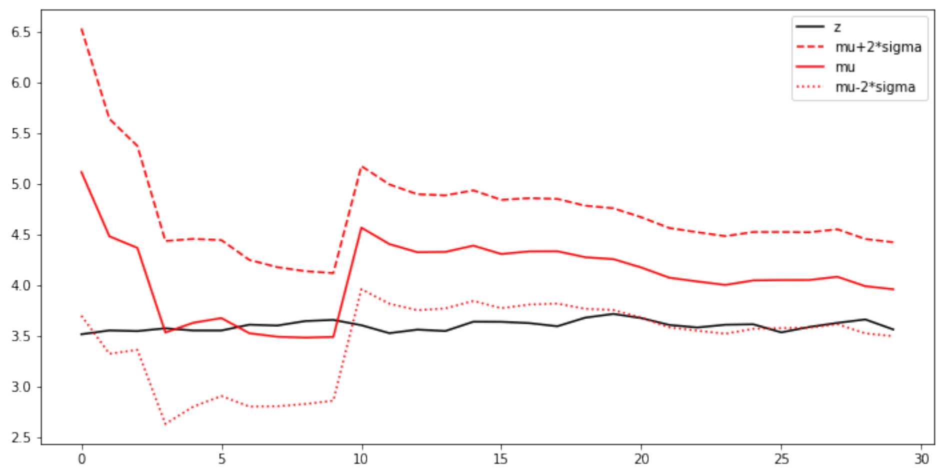
Initialization:
$p(z_0|x_0) = \frac{p(x_0|z_0) \cdot p(z_0)}{\int p(x_0|z_0) \cdot p(z_0) \cdot dz_0} = N(\mu_0, \sigma_0^2)$ where $\mu_0 = \sigma_0^2 \cdot (\frac{\mu_p}{\sigma_p^2} + \frac{x_0}{\sigma_\gamma^2})$ and $\sigma_0^2 = (\frac{1}{\sigma_p^2} + \frac{1}{\sigma_\gamma^2})^{-1}$

2.



From the above plot, we see that the Kalman filter does catch up to the true hidden object.

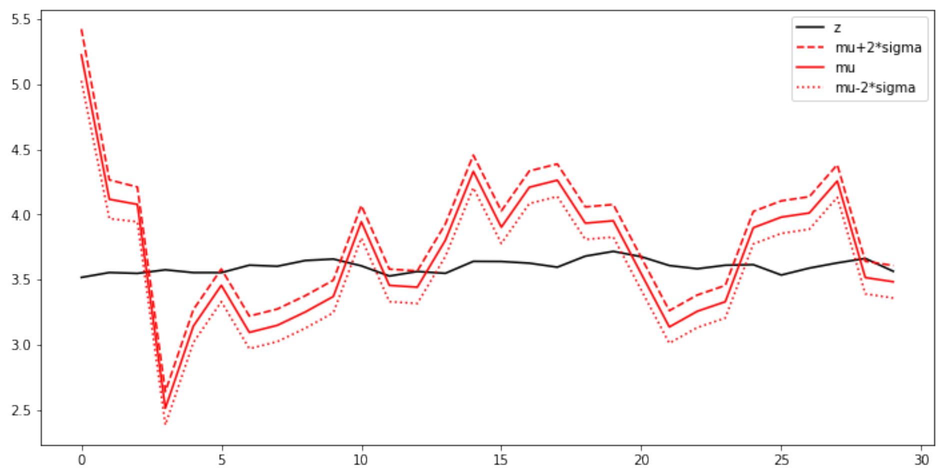
3.



From the above plot, we see that the Kalman filter is thrown off by the extreme outlier measurement but that it does appear to catch up to the true hidden object.

4.

Playing with the gamma parameters – $\mu_\gamma, \sigma_\gamma^2$ – we see that if we set the variance to be too low that the predictions vary wildly as seen in the below graph.


z
mu+2*sigma
mu
mu-2*sigma
5.5
5.0
4.5
4.0
3.5
3.0
2.5
0
5
10
15
20
25
30

Problem 3 (Explaining Away, 7 pts)

In this problem, you will carefully work out a basic example with the explaining away effect. There are many derivations of this problem available in textbooks. We emphasize that while you may refer to textbooks and other online resources for understanding how to do the computation, you should do the computation below from scratch, by hand.

We have three binary variables, rain r, grass-wet g, and sprinkler s. The conditional probability tables look like the following:

$$\begin{aligned} p(r=1) &= 0.25 \\ p(s=1) &= 0.5 \\ p(g=1|r=0,s=0) &= 0 \\ p(g=1|r=1,s=0) &= .75 \\ p(g=1|r=0,s=1) &= .75 \\ p(g=1|r=1,s=1) &= 1 \end{aligned}$$

1. You check on the sprinkler without checking on the rain or the grass. What is the probability that it is on?
2. You notice it is raining and check on the sprinkler without checking the grass. What is the probability that it is on?
3. You notice that the grass is wet and go to check on the sprinkler (without checking if it is raining). What is the probability that it is on?
4. You notice that it is raining and the grass is wet. You go check on the sprinkler. What is the probability that it is on?
5. What is the explaining away effect above?

Solution

1.

$p(s=1) = 0.5$

2.

$p(s=1|r=1) = p(s=1) = 0.5$

3.

$$p(s=1|g=1) = \frac{p(s=1,g=1)}{p(g=1)} = \frac{p(s=1,g=1|r=1)p(r=1) + p(s=1,g=1|r=0)p(r=0)}{p(g=1)}$$
$$\longrightarrow p(s=1,g=1|r=1) = \frac{p(s=1,g=1,r=1)}{p(r=1)} = \frac{p(s=1)\cdot p(r=1)\cdot p(g=1|s=1,r=1)}{p(r=1)}$$
$$\longrightarrow p(s=1,g=1|r=1) = p(s=1)\cdot p(g=1|s=1,r=1)$$
$$p(s=1|g=1) = \frac{p(s=1)p(r=1)p(g=1|s=1,r=1) + p(s=1)p(r=0)p(g=1|s=1,r=0)}{p(g=1)}$$

$$\longrightarrow p(g=1) = \sum_{s_i=(0,1)} \sum_{r_i=(0,1)} p(g=1|s=s_i, r=r_i)p(s=s_i)p(r=r_i) = 0.5$$
$$p(s=1|g=1) = 0.8125$$

4.

$$p(s=1|r=1,g=1) = \frac{p(s=1,r=1,g=1)}{p(r=1,g=1)} = \frac{p(s=1)\cdot p(r=1)\cdot p(g=1|s=1,r=1)}{p(r=1,g=1)}$$
$$= \frac{p(s=1)\cdot p(r=1)\cdot p(g=1|s=1,r=1)}{p(r=1)p(s=1)p(g=1|r=1,s=1)+p(r=1)p(s=0)p(g=1|r=1,s=0)}$$
$$= \frac{p(r=1)\cdot p(s=1)p(g=1|s=1,r=1)}{p(r=1)\cdot[p(s=1)p(g=1|r=1,s=1)+p(s=0)p(g=1|r=1,s=0)]}$$
$$= \frac{p(s=1)p(g=1|s=1,r=1)}{p(s=1)p(g=1|r=1,s=1)+p(s=0)p(g=1|r=1,s=0)}$$
$$p(s=1|r=1,g=1) = \frac{0.5}{0.5+0.375} \approx 0.571$$

5

From above, we see that $p(s=1|g=1) = 0.8125$ while $p(s=1|r=1,g=1) = 0.571$; that is, when we additionally condition on the rain observation, some of the probability that the sprinkler is on is “explained away”, or some probability is “lost” - the sprinkler becomes “less active” to our eyes.

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- Collaborators: Moh Osman
- Approximately how long did this homework take you to complete (in hours): Thursday night (about 6 hours), and most of Friday (about 14 hours)