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Approximation of solutions of functional differential equations of the pointwise type by solutions of the induced optimization problem

Levon Beklaryan · **Armen Beklaryan**

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Abstract It is well known [7] that soliton solutions for finite difference analogs of equations of mathematical physics are in one-to-one correspondence with solutions of the induced functional differential equations of the pointwise type (FDEPT; also known as mixed-type functional differential equations), defined on the whole line. In the present paper, it is shown that solutions of FDEPT, defined on the whole line, are approximated by solutions of the corresponding initial-boundary value problems, defined on an expanding sequence of intervals of the line. In turn, the construction of a solution at each of the finite intervals reduces to their approximation by solutions of the induced optimization problem. The possibilities of this approach is demonstrated in examples of a problem from the theory of plastic deformation.

Keywords Traveling waves · Functional differential equations · Equations of mathematical physics · Splines · Forward-backward differential equation

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1 Introduction

The theory of differential equations with delay, which has developed rapidly in recent decades [10, 25, 37], has in many ways acquired a complete form and is now actively used in modeling various objects. Numerical algorithms for solving equations with delay of various types

Central Economics and Mathematics Institute RAS
Nachimovky prospect 47, 117418 Moscow, Russia
Tel.: +7-916-670-2867
E-mail: beklar@cemi.rssi.ru
ORCID: 0000-0001-5695-4450

National Research University Higher School of Economics
Kirpichnaya Ulitsa 33/5, 105187 Moscow, Russia
Tel.: +7-903-514-8597
E-mail: abeklaryan@hse.ru
ORCID: 0000-0001-5141-5140

were also developed [8,35]. At the same time, numerical methods for equations with advanced arguments (and, moreover, with mixed type deviations) are practically not studied, although references to them have been encountered for a long time, mainly in connection with the classification of equations with deviating argument [14]. As a rule, there is a parsing of equations of a particular kind with further obtaining the existence and uniqueness theorems based on the use of the properties of the right-hand side [2] and application of methods, such as the study of the roots of the characteristic quasi-polynomial [18], collocation methods and finite element scheme (expansion of a solution in terms of basis functions of some finite-dimensional space) [1,31,32] or through the construction of a Hilbert space of the reproducing kernels on the basis of boundary conditions [30].

Meanwhile, interest in problems with simultaneous presence of advance and delay is caused by problems in which such systems appear as a necessary object of study. We mention three such problems.

In the theory of motion control, if a system has a delay, under the necessary optimality conditions in the form of the maximum principle, the corresponding conjugate system has an advancing [39].

In some applied problems, for example from the field of electrical engineering, mathematical models lead to differential equations with advance and delay [13,26,27].

In numerical methods for solving boundary value problems, one of the main algorithms is the sweep method, reducing the boundary value problem to the initial one by reversal of time [34]. If the system was delayed, when the time reverses, there is an advance.

These and other model problems of differential systems with simultaneous delay and advance make it urgent to develop a methodology for such problems, and, due to the complexity of the tasks, the main tool for solving the problems should be numerical methods [38]. Within the framework of this paper, one of the approaches to the construction of such numerical methods is analyzed, based on the formulation of an induced optimization problem. As will become clear below, the initial-boundary conditions for FDEPT are not local, so the solution to any variational problem associated with them cannot be obtained by simple local improvements and requires global optimization. Note that for problems of finite-dimensional optimization there are many approaches to finding a global extremum, sufficiently efficient algorithms are built, there are a large number of publications that present precedents for successfully solved problems [15,45, for example]. After discretization of the continuous optimal control problem (OCP), we obtain a finite-dimensional extremal problem and, formally speaking, we can assume that efficient algorithms for searching for global extremum of functionals also exist. However, the dimensions of successfully solved non-convex finite-dimensional problems, as a rule, are small, and the complexity of the global search problem increases dramatically with dimensionality increasing (the effect of the "curse of dimensionality"). In order to construct adequate approximations of OCP, at least 100 discretization nodes (finite dimensional variables) are usually required, which does not allow in practice to use the achievements of the theory of mathematical programming in solving such problems. Thus, the task of finding global extremum for OCP remains one of the most acute tasks in the extreme problems theory. Among classical approaches to this task one cannot fail to mention the works [9] and [29]. Some works that apply search and genetic algorithms are also well-known [12,33]. But unfortunately these efforts have not yet brought about effective algorithms that would be able to solve a wide spectrum of practical problems [17].

The way out of the current situation is the idea of constructing specialized optimization algorithms that deeply take into account the specific features of OCP as an extremal problem. The most important specific feature of OCP as an extremal problem is the presence of special theory and a significant number of deep theoretical results that constitute the most important sections of the theory of optimal control. One of the most beautiful ideas in this area is the idea of reducing the optimal control problem to a low-dimensional problem in the terminal phase space, defined as a minimum of a function on the reachable set of the control system [28]. Unfortunately, the problem of approximating the reachable set in the numerical solution is not much simpler than the problem of finding a global extremum. Nevertheless, the properties of the reachable set known from theoretical studies can be used to construct specialized optimization algorithms for OCP [43]. In particular, the property of connectedness allows one to construct a scheme of continuous variation of the control, resulting in a form of a continuous phase trajectory on the reachable set. Note that the idea of using the property of connectedness in the construction of numerical optimization algorithms belongs to A.G. Chentsov [41]. In the majority of well-known approaches to the construction of non-convex optimization methods, the solution of the problem is divided into two stages: "global", where a wide scan of the variable space is performed, and "local", aimed on local refinement of the obtained solution [45]. The combination of different methods at each stage, as well as the order of the alternation of stages, determines the specific computational algorithm. We consider an newly idea of combining both stages in one algorithmic design and constructing a method that allows at each iteration both the "global scan" procedure and the local improvement of a known solution in the case when the scan was not successful.

2 Optimization Problem and Numerical Methods

Let us give a formal statement of the optimization problem for the search for numerical solutions of FDEPT. We consider the system

$$F_i(t,\dot{x}(t),x(t+n_1),\dots,x(t+n_s)) = 0, \quad i=\overline{1,k}, \quad t\in[t_l,t_r],$$

where $F:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{ns}\longrightarrow\mathbb{R}^n$ is a mapping of $C^{(0)}$ class; $n_j\in\mathbb{Z}$, $j=\overline{1,s}$; $t_l,t_r\in\mathbb{R}$; values of the derivatives of the phase variables are defined on the extended interval $t\in[t_{ll},t_{rr}]$, $t_{ll}=t_l+\min\{0,n_1,\dots,n_s\}$, $t_{rr}=t_r+\max\{0,n_1,\dots,n_s\}$

$$\begin{aligned}\dot{x}(t) &= h_l(t), \quad t\in[t_{ll},t_l],\\ \dot{x}(t) &= h_r(t), \quad t\in[t_r,t_{rr}],\end{aligned}$$

where $h_l,h_r:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is a mapping of $C^{(0)}$ class. The initial-boundary conditions are given by the functionals

$$K_m(\dot{x}(\tau),x(\tau_1),\dots,x(\tau_p)) = 0, \quad m=\overline{1,q}, \quad \tau,\tau_i\in[t_l,t_r], \quad i=\overline{1,p}.$$

The optimization problem is to find a trajectory $\hat{x}(t)$ that delivers the minimum of the residual functional

$$\begin{aligned}I(\hat{x}(t)) = \nu^{(N)}\Bigg(&\sum_{i=1}^{k}\int_{t_l}^{t_r}F_i^2(t,\dot{\hat{x}}(t),\hat{x}(t+n_1),\dots,\hat{x}(t+n_s))dt+\\ &\int_{t_{ll}}^{t_l}[\dot{\hat{x}}(t)-h_l(t)]^2dt+\int_{t_r}^{t_{rr}}[\dot{\hat{x}}(t)-h_r(t)]^2dt\Bigg)+\\ &\nu^{(K)}\sum_{m=1}^{q}K_m^2(\dot{\hat{x}}(\tau),\hat{x}(\tau_1),\dots,\hat{x}(\tau_p)),\end{aligned}$$

where $\nu^{(N)},\nu^{(K)}\in\mathbb{R}_+$ are weighting coefficients.

The proposed approach to the investigation of boundary value problems is based on the Ritz method and spline collocation constructions and was implemented in [24,44]. In order to solve the problem of the class under consideration, the trajectories of the system are discretized on a grid with a constant step, and a generalized residual functional is formulated that includes both the weighted residual of the original differential equation and the residual of the boundary conditions. A spline differentiation technique is used to evaluate the derivatives of the desired trajectories of the system, based on two spline approximation designs: using cubic natural splines and using a special type of spline whose second derivatives at the edges are also controlled using optimized parameters.

A set of algorithms for local and global optimization was implemented for solving the stated finite-dimensional problems. The used technology includes: an algorithm for sequentially increasing the accuracy of approximation by multiplying the number of nodes in the grid of the discretization; algorithms for the difference evaluation of the derivatives of the functional from the first to the sixth degree of accuracy inclusive; method of successively increasing the precision of spline differentiation.

The corresponding software complex (SC) *OPTCON-F* was implemented in the language *C* under the control of operating systems *OS Windows*, *OS Linux* and *Mac OS* using compilers *BCC 5.5* and *GCC*. SC was designed to obtain a numerical solution of boundary value problems, parametric identification problems and optimal control for dynamical systems described by FDEPT [21].

Among the local algorithms included in the SC *OPTCON-F* there are: 1) PARTAN method; 2) Powell-Brent's method; 3) gradient method of confidence intervals; 4) Barzilai-Borwein method; 5) Newton's method with the difference estimate of the Hessian matrix; 6) generalized quasi-Newtonian method; 7) direct-dual method of gradient descent; 8) differential Euler optimization method of the 2nd order; 9) differential Adams optimization method of the 4th order and others.

As algorithms of "closers" there are: 1) adaptive modification of the Hooke-Jeeves method; 2) stochastic search methods in random subspaces of the indicated (2, 3, 4, or 5) dimension; 3) local version of the curvilinear search method.

Non-local algorithms that form the basis of the SC are: 1) "parabolic" method – a combination of coordinate-wise descent with a periodic multistart and a non-local one-dimensional search by the parabola algorithm; 2) non-local method of curvilinear search; 3) Luus-Jaakola method; 4) "forest" method – multivariant adaptive method of random multistart and others.

As part of the work on the SC, taking into account the specifics related to the qualitative properties of FDEPT, the following key results were obtained:

- Algorithm of construction of "controllable splines" has been developed. The problem of obtaining a high-precision approximation to the derivative of a function of one variable over a set of values of the function itself, defined on a fixed grid, was investigated. The performed computational experiments showed that in order to achieve good accuracy in estimating derived trajectories for FDEPT systems, it is not possible to use known types of splines ("natural", with additional boundary conditions, Akima splines, etc.). The proposed algorithm is based on the use of derivatives of cubic spline functions.
- A technology has been developed for approximation of general FDEPT systems using a finite-dimensional unconditional minimization problem. By analyzing the behavior of homeomorphisms on the initial (main) time interval, an extended time interval is calculated, for which the discretized grid is constructed. To approximate the initial continuous problem on a fixed grid over time, the Ritz method is used: the trajectories are

approximated using controlled spline functions, the coefficients of which are selected by searching for the minimum of the residual functional.

- A specialized global optimization algorithm was developed, based on the idea of curvilinear search. To search for the minimum of the non-convex residual functional, a specialized global optimization algorithm has been developed, based on the idea of curvilinear search on the reachable set of a control system produced using a pair of randomly generated "support" controls. Using the property of connectedness, variations in the space of control functions that do not violate the existing constraints are constructed at iterations of the algorithm. This is achieved by direct projection on a parallelepiped. To improve the current approximation, a modification of a non-local one-dimensional search algorithm based on the "parabolic" method is applied [23]. Also, as a globalizing mechanism, a non-local search in random directions is used, repeated many times at each iteration of the algorithm. To solve auxiliary non-convex problems of one-dimensional search, a modification of the stochastic *P*-algorithm, proposed by A. Zhigljavsky and A. Žilinskas [45], is implemented [22]. To enhance reliability of the proposed method, a periodic random multi-start was provided in the algorithm's construction.
- An algorithm for the numerical integration of FDEPT systems based on the sequential discretization technique has been developed. To improve the efficiency of calculations, tools have been implemented to build a sequence of approximative finite-dimensional optimization problems with a growing number of variables. At the same time, solutions obtained at the previous stages of calculations performed on the current discretization grid are projected onto a new grid with an increased number of nodes while preserving the qualitative and quantitative characteristics of the trajectories.
- The proposed numerical algorithms were tested on a wide range of tasks [21] with using the principle of "the best of known solutions" [16]. In all considered problems, the proposed algorithm allowed us to find the best known solution. The calculation experiments that have been carried out demonstrate considerably high fidelity of the proposed algorithms.

The above heuristic search algorithm for the solution $\hat{x}(t)$ can be justified on the basis of the existence and uniqueness theorems for initial-boundary value problems for the investigated FDEPT, as well as theorems on approximating solutions of such equations on the whole line by solutions of the initial-boundary value problem on a sequence of expanding intervals. A description of such equations and corresponding results are presented in the following sections.

3 Solutions of the Traveling Wave Type

For equations of mathematical physics, which are the Euler-Lagrange equation of the corresponding variational problem, an important class of solutions are traveling wave solutions (soliton solutions) [36,42]. In a number of models, such solutions are well approximated by traveling wave solutions for finite difference analogs of the original equations, which, in place of a continuous environment, describe the interaction of clumps of a environment placed at lattice sites [19,42]. Emerging systems belong to the class of infinite-dimensional dynamical systems. The most widely considered classes of such problems are infinite systems with Frenkel-Kontorova potentials (periodic and slowly growing potentials) and Fermi-Pasta-Ulam (potentials of exponential growth), a broad survey of which is given in the paper [40].

In the theory of plastic deformation, the following infinite-dimensional dynamical system is studied

$$m\ddot{y}_i = y_{i+1} - 2y_i + y_{i-1} + \phi(y_i), \quad i \in \mathbb{Z}, \quad y_i \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{1}$$

where potential $\phi(\cdot)$ is given by a smooth periodic function. The equation (1) is a system with the Frenkel-Kontorova potential [19]. Such a system is a finite difference analog of the non-linear wave equation. It simulates the behavior of a countable number of balls of mass m placed at integer points of the numerical line, where each pair of adjacent balls is connected by an elastic spring, and describes the propagation of longitudinal waves in an infinite homogeneous absolutely elastic rod. The study of such systems with various potentials is one of the intensively developing directions in the theory of dynamical systems. For these systems, the central task is to study solutions of the traveling wave type as one of the observed wave classes.

Definition 1 We say that the solution $\{y_i(\cdot)\}_{-\infty}^{+\infty}$ of the system (1), defined for all $t \in \mathbb{R}$, has a traveling wave type, if there is $\tau > 0$, independent of t and i, that for all $i \in \mathbb{Z}$ and $t \in \mathbb{R}$ the following equality holds

$$y_i(t+\tau) = y_{i+1}(t).$$

The constant τ will be called a *characteristic* of the traveling wave. ■

The proposed approach is based on the existence of a one-to-one correspondence of solutions of the traveling wave type for infinite-dimensional dynamical systems with solutions of induced FDEPT [4]. In order to study the existence and uniqueness of traveling wave solutions, it is proposed to localize solutions of the induced FDEPT in the function spaces, majorized by functions with a given exponential growth. This approach is particularly successful for systems with Frenkel-Kontorova potentials. In this manner, it is possible to obtain a “correct” extension of the concept of a traveling wave in the form of solutions of the quasi-traveling wave type, which is related to the description of processes in inhomogeneous environments, for which the set of traveling wave solutions is trivial [5,6].

For the infinite-dimensional dynamical system under consideration, the study of solutions of the traveling wave type with the characteristic τ, i.e. solutions of the system

$$\ddot{y}_i = m^{-1}(y_{i+1} - 2y_i + y_{i-1} + \phi(y_i)), \quad i \in \mathbb{Z}, \quad t \in \mathbb{R},$$
$$y_i(t+\tau) = y_{i+1}(t)$$

turns out to be equivalent to the study of a solution space of the induced FDEPT

$$\ddot{x}(t) = m^{-1}(x(t+\tau) - 2x(t) + x(t-\tau) + \phi(x(t))), \quad t \in \mathbb{R}. \tag{2}$$

In this case, the corresponding solutions are related as follows: for any $t \in \mathbb{R}$

$$x(t) = y_{[t\tau^{-1}]}(t - \tau[t\tau^{-1}]),$$

where $[\cdot]$ means the integer part of a number. The corresponding solution of the traveling wave type is determined by the rule

$$y_0(t) = x(t), \quad y_i(t) = y_0(t + i\tau), \quad i \in \mathbb{Z}, \quad t \in \mathbb{R}.$$

In fact, the described connection between solutions of the traveling wave type of the infinite-dimensional dynamical system and solutions of the induced functional-differential equation is a fragment of a more general scheme that goes beyond the scope of this article.

4 Existence and Uniqueness Theorem of Traveling Waves

Investigation of traveling wave solutions refers to one of the intensively developing directions. For the equations of mathematical physics, as well as for various application problems, the class of bounded solutions such as a traveling wave and, in particular, noted solutions with asymptotic at infinity with respect to time is of interest. Different approaches are used for this. One of them is based on the method of linearization and localization of the spectrum for the operator generating the equation in variations, with the subsequent provision of the condition for the existence of an attractive solution. In particular, this approach is implemented in the works [11,20].

In the framework of our approach, solutions of the traveling wave type for dynamical systems, described by infinite-dimensional ordinary differential equations with phase space in the form of infinite sequences, are investigated. In particular, such dynamical systems are finite difference analogs of the equations of mathematical physics. Such dynamical systems induce FDEPT and vice versa. In this case, there is a one-to-one correspondence between solutions of the traveling wave type for an infinite-dimensional differential equation and solutions of an induced FDEPT. On this path, conditions for the existence and uniqueness of solutions of the traveling wave type, with the growth restrictions both in time and in space, arise. It is very important that the conditions for the existence of a traveling wave solution are formed in terms of the right-hand side of the equation and the characteristics of the traveling wave, without using either the linearization and spectral properties of the corresponding equation in variations.

We assume that the non-linear potential ϕ satisfies the Lipschitz condition with constant L. Thus, we should study solutions of the functional-differential equation (2) with a quasilinear right-hand side. A solution of the FDEPT with a quasilinear right-hand side will be sought in a one-parameter family of Banach spaces of functions that have at most exponential growth. The exponent is the parameter of the selected family of functions, which is defined as follows

$$\mathscr{L}_\mu^n C^{(k)}(\mathbb{R}) = \left\{ x(\cdot) : x(\cdot) \in C^{(k)}(\mathbb{R}, \mathbb{R}^n), \max_{0 \le r \le k} \sup_{t \in \mathbb{R}} \|x^{(r)}(t)\mu^{|t|}\|_{\mathbb{R}^n} < +\infty \right\} \tag{3}$$

with the norm

$$\|x(\cdot)\|_\mu^k = \max_{0 \le r \le k} \sup_{t \in \mathbb{R}} \|x^{(r)}(t)\mu^{|t|}\|_{\mathbb{R}^n}.$$

In our approach for the initial infinite-dimensional dynamical system (1) we will study traveling wave solutions having a given growth (exponential) both in time and space. To this end, we define a vector space

$$K^n = \overline{\prod_{i \in \mathbb{Z}} R_i^n}, \quad R_i^n = \mathbb{R}^n, \quad i \in \mathbb{Z}$$
$$(\varkappa \in K^n, \quad \varkappa = \{x_i\}_{-\infty}^{+\infty})$$

with the standard topology of the complete direct product (metrizable space).

In particular, the elements of the space K^2 are infinite sequences

$$\varkappa = \{(u_i, v_i)'\}_{-\infty}^{+\infty}, \quad u_i, v_i \in \mathbb{R}$$

(prime means transposition).

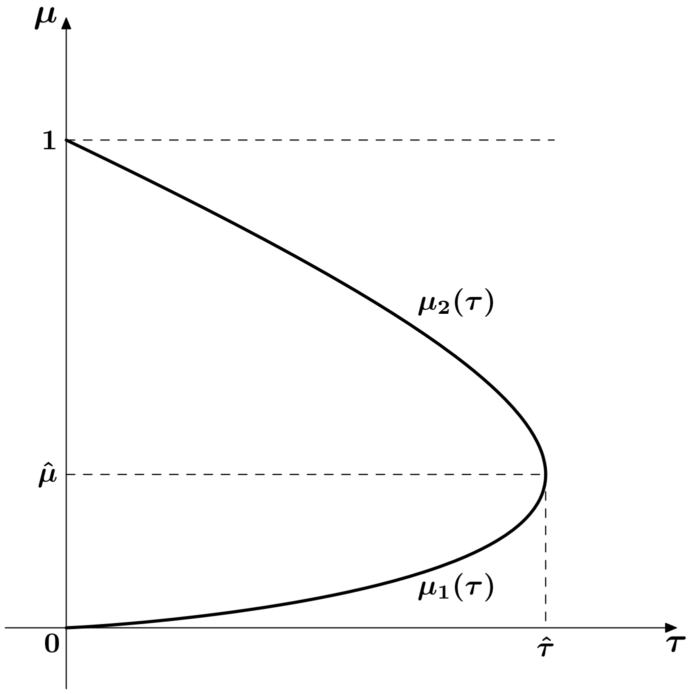


Fig. 1: Graphs of functions $\mu_1(\tau), \mu_2(\tau)$

In the space K^n we define a family of Hilbert subspaces $K^n_{2\mu}, \mu \in (0,1)$

$$K^n_{2\mu} = \left\{ \varkappa : \varkappa \in K^n; \quad \sum_{i=-\infty}^{+\infty} \|x_i\|^2_{\mathbb{R}^n} \mu^{2|i|} < +\infty \right\}$$

with the norm

$$\|\varkappa\|_{2\mu} = \left(\sum_{i=-\infty}^{+\infty} \|x_i\|^2_{\mathbb{R}^n} \mu^{2|i|} \right)^{\frac{1}{2}}.$$

Here μ is a free parameter, due to which the solution space will be selected.

We consider a transcendental equation with respect to two variables $\tau \in (0,+\infty)$ and $\mu \in (0,1)$

$$C\tau \left(2\mu^{-1} + 1\right) = \ln \mu^{-1}, \tag{4}$$

where

$$C = \max\{1; 2m^{-1}\sqrt{L^2+2}\}.$$

The set of solutions of the equation (4) is described by functions $\mu_1(\tau), \mu_2(\tau)$ given in Fig. 1.

We formulate the theorem of existence and uniqueness of a solution of the traveling wave type.

Theorem 1 **([5])** *For any initial values $\bar{i} \in \mathbb{Z}, a, b \in \mathbb{R}, \bar{t} \in \mathbb{R}$ and characteristics $\tau > 0$ satisfying the condition*

$$0 < \tau < \hat{\tau},$$

for the initial system of differential equations (1) *there exists a unique solution of the traveling wave type $\{y_i(\cdot)\}_{-\infty}^{+\infty}$ with characteristic τ such that it satisfies the initial conditions $y_{\bar{i}}(\bar{t}) = a, \dot{y}_{\bar{i}}(\bar{t}) = b$. For any parameter $\mu \in (\mu_1(\tau), \mu_2(\tau))$ the vector function $\omega(t) = \{(y_i(t), \dot{y}_i(t))^T\}_{-\infty}^{+\infty}$ belongs to the space $K^2_{2\mu}$ for any $t \in \mathbb{R}$, and the function $\rho(t) = \|\omega(t)\|_{2\mu}$ belongs to the space $\mathscr{L}^1_{\sqrt[\tau]{\mu}} C^{(1)}(\mathbb{R})$. Such a solution depends continuously on the initial values $a, b \in \mathbb{R}$, as well as on the mass m.* ■

Theorem 1 not only guarantees the existence of a solution but also determines the limitation of its possible growth both in time t and in coordinates $i \in \mathbb{Z}$ (over space). It is obvious that for each $0 < \tau < \hat{\tau}$ the space $K^2_{2\mu}$, at $\mu < \mu_2(\tau)$ but close to $\mu_2(\tau)$, is much narrower than the space $K^2_{2\mu}$, at $\mu > \mu_1(\tau)$ but close to $\mu_1(\tau)$. The theorem guarantees the existence of a solution in narrower spaces and uniqueness in wider spaces.

The full text of the proof of Theorem 1, as well as a detailed description of the proposed approach, is given in the papers [4–6].

5 Functional Differential Equations

In Section 4 we noted that the study of solutions of the traveling wave type is equivalent to the study of solutions of an induced FDEPT. We now turn to the study of such general equations. The most important goal in the study of FDEPT is the study of the *basic initial-boundary value problem*

$$\dot{x}(t) = f(t, x(q_1(t)), \ldots, x(q_s(t))), \quad t \in B_R, \tag{5}$$

$$\dot{x}(t) = \varphi(t), \quad t \in \mathbb{R} \backslash B_R, \quad \varphi(\cdot) \in L_\infty(\mathbb{R}, \mathbb{R}^n), \tag{6}$$

$$x(\bar{t}) = \bar{x}, \quad \bar{t} \in \mathbb{R}, \quad \bar{x} \in \mathbb{R}^n, \tag{7}$$

where: $f : \mathbb{R} \times \mathbb{R}^{ns} \longrightarrow \mathbb{R}^n$ – mapping of the $C^{(0)}$ class; $q_j(\cdot),\ j = 1, \ldots, s$ – diffeomorphisms of the line preserving orientation; B_R is either closed interval $[t_0, t_1]$ or closed half-line $[t_0, +\infty)$ or line $\mathbb{R}$. $L_\infty(\mathbb{R}, \mathbb{R}^n)$ – the space of measurable functions that are essentially bounded on each finite interval.

The solution of the equation (5) is any absolutely continuous function $x(t)$, $t \in \mathbb{R}$ that satisfies this equation almost everywhere. *The solution of the basic initial-boundary value problem* is any solution of the equation (5) that satisfies the boundary condition (6) and the initial condition (7). Denote by Q the group generated by the deviation functions, i.e. $Q = < q_1, \ldots, q_s >$.

On the right-hand side of the equation (5) the mapping $f : \mathbb{R} \times \mathbb{R}^{n \cdot s} \longmapsto \mathbb{R}^n$ satisfies the following conditions:

(a) $f(\cdot) \in C^{(0)}(\mathbb{R} \times \mathbb{R}^{n \cdot s}, \mathbb{R}^n)$;

(b) for any $t, z_j, \bar{z}_j, j = \overline{1, s}$ there is a quasilinear growth

$$\|f(t, z_1, \ldots, z_s)\|_{\mathbb{R}^n} \leq M_0(t) + M_1 \sum_{j=1}^{s} \|z_j\|_{\mathbb{R}^n}, \quad M_0(\cdot) \in C^{(0)}(\mathbb{R}, \mathbb{R}),$$

and a Lipschitz condition

$$\|f(t, z_1, \ldots, z_s) - f(t, \bar{z}_1, \ldots, \bar{z}_s)\|_{\mathbb{R}^n} \leq M_2 \sum_{j=1}^{s} \|z_j - \bar{z}_j\|_{\mathbb{R}^n};$$

(c) there exists $\mu^* \in \mathbb{R}_+$ such that

$$M_0(\cdot) \in \mathscr{L}^n_{\mu^*} C^{(0)}(\mathbb{R});$$

(d) the quantities

$$h_j = \sup_{t \in \mathbb{R}} |t - q_j(t)|, \quad j = 1, \ldots, s$$

are finite;

(e) the family of functions

$$\tilde{f}_{q,z_1,\ldots,z_s}(t) = f(q(t), z_1, \ldots, z_s)(\mu^\star)^{|q(t)|}, \quad q \in Q, \quad z_1, \ldots, z_s \in \mathbb{R}^n,$$

is equicontinuous on any finite interval.

The continuity condition, the growth conditions with respect to the phase variables and time variable, as well as the Lipschitz condition, (conditions (a)–(b)) are standard conditions in the theory of ordinary differential equations. In fact, in item (b), the first inequality in the form of the growth condition with respect to the phase variables and the time variable is a consequence of the second condition in the form of the Lipschitz condition. Under the Lipschitz constant M_2 should understand the minimum value among the possible values of these constants. Accordingly, we can assume that $M_1 = M_2$. At the same time, we separately wrote out the first inequality in order to formulate condition (c) for the function $M_0(\cdot)$. The condition (c) for the function g is related to the study of solutions on the half-line and the line, which requires certain restrictions on the time growth of the right-hand side. Note that we can always satisfy condition (d) by making a time change but we can break condition (c). The last condition (e) is necessary for the right-hand side of the induced infinite-dimensional ordinary differential equation, with the phase space in a suitable Banach space, to easily establish the fact of its Bochner integrability. Actually, condition (e) can be removed, but this leads to more technical complications, since the right-hand side of the induced infinite-dimensional ordinary differential equation will be only measurable and it will be necessary to establish the fact of its Bochner integrability [4]. We have the following theorem on the existence and uniqueness of solution.

Theorem 2 ([3], p.570) *If for $\mu \in (0, \mu^\star) \cap (0,1)$ the inequality*

$$M_2 \sum_{j=1}^{s} \mu^{-|h_j|} < \ln \mu^{-1}, \tag{8}$$

is satisfied then for any $\bar{x} \in \mathbb{R}^n$, $\varphi(\cdot) \in L_\infty(\mathbb{R}, \mathbb{R}^n)$ there exists a solution $x(\cdot) \in \mathscr{L}^n_\mu C^{(0)}(\mathbb{R})$ of the initial-boundary value problem (5)–(7). *This is the only solution. Moreover, the solution $x(\cdot) \in \mathscr{L}^n_\mu C^{(0)}(\mathbb{R})$ depends continuously both on the initial-boundary conditions $\bar{x} \in \mathbb{R}^n$, $\varphi(\cdot) \in L_\infty(\mathbb{R}, \mathbb{R}^n)$ and on the right-hand side $f(\cdot)$ (as an element of the space with Lipschitz norm).* ■

If the inequality (8) holds for some $\mu \in (0, \mu^\star) \cap (0,1)$ then there are $\mu_1, \mu_2 \in (0, \mu^\star) \cap (0,1)$ that define the maximal interval (μ_1, μ_2) for whose points the inequality (8) is valid. Obviously, the space $\mathscr{L}^n_\mu C^{(0)}(\mathbb{R})$ with the parameter $\mu \in (\mu_1, \mu_2)$ close to μ_2 is narrower than with the parameter $\mu \in (\mu_1, \mu_2)$ close to μ_1. It follows from the theorem that the solution exists in the distinguished narrow spaces and is unique in the wider spaces.

Along with the initial-boundary value problem (5)–(7), we consider some of its reformulation. To this end, we define a normed function space

$$\mathscr{L}_\mu^n L_\infty(\mathbb{R}) = \left\{ x(\cdot) : x(\cdot) \in L_\infty(\mathbb{R}, \mathbb{R}^n), \sup_{t\in\mathbb{R}} vrai \|x(t)\mu^{|t|}\|_{\mathbb{R}^n} < +\infty \right\}, \quad \mu \in (0,1]$$

with a norm

$$\|x(\cdot)\|_\mu = \sup_{t\in\mathbb{R}} vrai \|x(t)\mu^{|t|}\|_{\mathbb{R}^n}.$$

Reformulation of the initial-boundary value problem

$$\begin{aligned} &\dot{x}(t) = f(t, x(q_1(t)), \ldots, x(q_s(t))), \quad t \in B_R, \\ &\dot{x}(t) = \varphi(t), \quad t \in \mathbb{R} \backslash B_R, \quad \varphi(\cdot) \in \mathscr{L}_\mu^n L_\infty(\mathbb{R}), \\ &x(\bar{t}) = \bar{x}, \quad \bar{t} \in \mathbb{R}, \quad \bar{x} \in \mathbb{R}^n. \end{aligned}$$

It is obvious that the solvability of the original initial-boundary value problem and its reformulation are equivalent. Theorem 2 is also valid for the reformulation of the initial-boundary value problem, only with the replacement of the boundary condition $\varphi(\cdot) \in L_\infty(\mathbb{R}, \mathbb{R}^n)$ by the boundary condition $\varphi(\cdot) \in \mathscr{L}_\mu^n L_\infty(\mathbb{R})$.

In what follows, we confine ourselves to considering the case of constant commensurable deviations. Without loss of generality, in this case we can assume that the deviations are integer, i.e. $q_j(t) = t + n_j$, $n_j \in \mathbb{Z}$, $j = 1, \ldots, s$. Similarly, we can assume that the initial moment $\bar{t}$ and the ends t_0, t_1 of the interval of the definition of the equation are also integer.

The second-order FDEPT (2) can be reduced to the system

$$\begin{cases} \dot{z}_1(t) = \tau z_2(t) \\ \dot{z}_2(t) = \tau m^{-1} \left(z_1(t+1) - 2z_1(t) + z_1(t-1) + \phi(z_1(t)) \right). \end{cases}$$

For such a system the Lipschitz constant M_2 has the form

$$M_2 = C\tau, \quad C = \max\{1; 2m^{-1}\sqrt{L^2+2}\}$$

and inequality (8) takes the form

$$C\tau \left(2\mu^{-1} + 1\right) < \ln \mu^{-1}$$

that explains the meaning of the transcendental equation (4).

Theorem 3 **([4])** *If for $\mu \in (0, \mu^\star) \cap (0,1)$ the inequality*

$$M_2 \sum_{j=1}^{s} \mu^{-|n_j|} < \ln \mu^{-1},$$

is satisfied then for any $\bar{x} \in \mathbb{R}^n, \varphi(\cdot) \in \mathscr{L}_\mu^n L_\infty(\mathbb{R})$ there exists a solution $x(\cdot) \in \mathscr{L}_\mu^n C^{(0)}(\mathbb{R})$ of the initial-boundary value problem (5)–(7). *Such a solution is unique and continuously depends on the right-hand side $f(\cdot)$ (as an element of the space with Lipschitz norm). Moreover, for any $\bar{x}_1, \bar{x}_2 \in \mathbb{R}^n, \varphi_1(\cdot), \varphi_2(\cdot) \in \mathscr{L}_\mu^n L_\infty(\mathbb{R})$ the corresponding solutions $x_1(\cdot), x_2(\cdot)$ of the initial-boundary value problem* (5)–(7) *satisfy the estimate*

$$\|x_1(\cdot) - x_2(\cdot)\|_\mu^{(0)} \leq C_f \|[\varphi_1(\cdot) - \varphi_2(\cdot)][1 - \chi_{[t_0,t_1]}(\cdot)]\|_\mu + \|\bar{x}_1 - \bar{x}_2\|_{\mathbb{R}^n}, \tag{9}$$

where $\chi_{[t_0,t_1]}(\cdot)$ is the characteristic function of the interval $[t_0, t_1]$ and the constant C_f depends only on the function f. ■

On the basis of the Theorem 3 we are going to formulate a proposition on the approximation of solutions of an initial-boundary value problem defined on the whole line by solutions of the initial-boundary value problem defined on the interval $[-k,k]$ as $k \to +\infty$. We consider the initial-boundary value problem on the whole line $B_R = \mathbb{R}$

$$\dot{x}(t) = f(t, x(t+n_1), \ldots, x(t+n_s)), \quad t \in \mathbb{R}, \tag{10}$$
$$x(\bar{t}) = \bar{x}, \quad \bar{t} \in \mathbb{R}, \quad \bar{x} \in \mathbb{R}^n \tag{11}$$

and for each $k \in \mathbb{Z}$ the initial-boundary value problem on a finite interval $B_R = [-k,k]$

$$\dot{x}(t) = f(t, x(t+n_1), \ldots, x(t+n_s)), \quad t \in [-k,k], \tag{12}$$
$$\dot{x}(t) = \varphi(t), \quad t \in \mathbb{R} \backslash [-k,k], \quad \varphi(\cdot) \in \mathscr{L}_1^n L_\infty(\mathbb{R}), \tag{13}$$
$$x(\bar{t}) = \bar{x}, \quad \bar{t} \in \mathbb{R}, \quad \bar{x} \in \mathbb{R}^n. \tag{14}$$

Theorem 4 *If for $\mu \in (0, \mu^\star) \cap (0,1)$ the inequality*

$$M_2 \sum_{j=1}^{s} \mu^{-|n_j|} < \ln \mu^{-1}, \tag{15}$$

is satisfied then for any $\bar{x} \in \mathbb{R}^n, \varphi(\cdot) \in \mathscr{L}_1^n L_\infty(\mathbb{R})$ the solution $\hat{x}(\cdot)$ of the initial-boundary value problem (10)–(11)*, as an element of the space $\mathscr{L}_\mu^n C^{(0)}(\mathbb{R})$, is approximated by solutions $\hat{x}_k(\cdot)$ of the initial-boundary value problem* (12)–(14) *as $k \to +\infty$ and we have the estimate*

$$\|\hat{x}(\cdot) - \hat{x}_k(\cdot)\|_\mu^{(0)} \leq C_f \|[\dot{\hat{x}}(\cdot) - \varphi(\cdot)][1 - \chi_{[-k,k]}(\cdot)]\|_\mu. \tag{16}$$

Proof We note that the inequality (15) will also hold for some $\bar{\mu} > \mu$. Then, by the Theorem 3, for the solution $\hat{x}(\cdot)$ the inclusion $\hat{x}(\cdot) \in \mathscr{L}_{\bar{\mu}}^n C^{(0)}(\mathbb{R})$ also holds, and, taking into account condition (b), the validity of the other inclusion $\dot{\hat{x}}(\cdot) \in \mathscr{L}_{\bar{\mu}}^n C^{(0)}(\mathbb{R})$ will follow. If we denote $\dot{\hat{x}}(\cdot) = \psi(\cdot)$ then for any $k \in \mathbb{Z}$ the function $\hat{x}(\cdot)$ is a solution of the initial-boundary value problem

$$\begin{aligned}
&\dot{x}(t) = f(t, x(t+n_1), \ldots, x(t+n_s)), \quad t \in [-k,k] \\
&\dot{x}(t) = \psi(t), \quad t \in \mathbb{R} \backslash [-k,k], \quad \psi(\cdot) \in \mathscr{L}_{\bar{\mu}}^n L_\infty(\mathbb{R}), \\
&x(\bar{t}) = \bar{x}, \quad \bar{t} \in \mathbb{R}, \quad \bar{x} \in \mathbb{R}^n.
\end{aligned}$$

Applying the estimate (9) to the solutions $\hat{x}(\cdot), \hat{x}_k(\cdot)$, we obtain

$$\|\hat{x}(\cdot) - \hat{x}_k(\cdot)\|_\mu^{(0)} \leq C_f \|[\psi(\cdot) - \varphi(\cdot)][1 - \chi_{[-k,k]}(\cdot)]\|_\mu.$$

Since $\psi(\cdot) \in \mathscr{L}_{\bar{\mu}}^n L_\infty(\mathbb{R}), \varphi(\cdot) \in \mathscr{L}_1^n L_\infty(\mathbb{R})$ then $C_f \|[\psi(\cdot) - \varphi(\cdot)][1 - \chi_{[-k,k]}(\cdot)]\|_\mu \to 0$ as $k \to +\infty$ from which the assertion of the theorem follows. □

We can indicate the rate of approximation of the solution $\hat{x}(\cdot)$ by the solutions $\hat{x}_k(\cdot)$.

Corollary 1 *Let the conditions of Theorem 4 be satisfied, and (μ_1, μ_2) is the maximal interval on which the inequality* (15) *holds. Then for any arbitrarily small number ε, $0 < \varepsilon < \mu_2 - \mu_1$ there is $C_{f,\varepsilon}$ such that the following estimate is true*

$$\|\hat{x}(\cdot) - \hat{x}_k(\cdot)\|_\mu^{(0)} \leq C_{f,\varepsilon} \left(\frac{\mu_1}{\mu_2 - \varepsilon} \right)^k.$$

Proof Repeating the arguments from the proof of Theorem 4, we obtain that for any $\bar{\mu} \in (\mu_1, \mu_2)$ we have the inclusion $\hat{x}(\cdot), \dot{\hat{x}}(\cdot) \in \mathscr{L}^n_{\bar{\mu}} C^{(0)}(\mathbb{R})$. Using the estimate (16), for any $\mu, \bar{\mu} \in (\mu_1, \mu_2), \mu < \bar{\mu}$ we obtain a chain of estimates

$$\|\hat{x}(\cdot) - \hat{x}_k(\cdot)\|^{(0)}_{\mu} \leq C_f \|[\dot{\hat{x}}(\cdot) - \dot{\hat{x}}_k(\cdot)][1 - \chi_{[-k,k]}(\cdot)]\|_{\mu} = C_f \sup_{t \in \mathbb{R} \setminus [-k,k]} vrai \|[\dot{\hat{x}}(t) - \varphi(t)]\mu^{|t|}\|_{\mathbb{R}^n}.$$

Then there is a constant $C_{f,\varepsilon} > 0$ such that for any $\mu, \bar{\mu}, \mu < \bar{\mu} \leq \mu_2 - \varepsilon$ the following estimates hold

$$C_f \sup_{t \in \mathbb{R} \setminus [-k,k]} vrai \|[\dot{\hat{x}}(t) - \varphi(t)]\mu^{|t|}\|_{\mathbb{R}^n} \leq C_{f,\varepsilon} \sup_{t \in \mathbb{R} \setminus [-k,k]} vrai \bar{\mu}^{-|t|}\mu^{|t|} \leq$$
$$C_{f,\varepsilon} \sup_{t \in \mathbb{R} \setminus [-k,k]} vrai \bar{\mu}^{-|t|}\mu_1^{|t|} \leq C_{f,\varepsilon} \left(\frac{\mu_1}{\bar{\mu}}\right)^k,$$

that is, for all $\bar{\mu} \in (\mu_1, \mu_2 - \varepsilon)$ we have the estimate

$$\|\hat{x}(\cdot) - \hat{x}_k(\cdot)\|^{(0)}_{\mu} \leq C_{f,\varepsilon} \left(\frac{\mu_1}{\bar{\mu}}\right)^k.$$

In particular

$$\|\hat{x}(\cdot) - \hat{x}_k(\cdot)\|^{(0)}_{\mu} \leq C_{f,\varepsilon} \left(\frac{\mu_1}{\mu_2 - \varepsilon}\right)^k.$$

□

We note that in the sequence of initial-boundary value problems on intervals $[-k, k]$ the boundary function can be an arbitrary fixed essentially bounded function $\varphi(\cdot) \in \mathscr{L}^n_1 L_\infty(\mathbb{R})$.

6 Numerical Experiments

Next, the results of the computational experiments on the study of initial-boundary value problems for systems of FDEPT using *OPTCON-F* software will be presented. Before demonstrating the examples, it is necessary to make a number of significant observations:

(a) in the SC *OPTCON-F*, the possibility of satisfying the condition of uniform boundedness of the derivative is realized. The maximum deviation from zero is determined by the Lipschitz constant of the right side of the equation, by the parameter μ, and also by the deviations of the argument;
(b) numerical integration on an interval with initial-boundary conditions is realized as an integration process with given boundary conditions at the left end and procedures for minimizing deviations from the boundary conditions at the right end for the solution constructed with observance of the restrictions from the preceding paragraph;
(c) in the obtained theorem on approximating solutions of the original equation on the whole line by solutions on expanding finite intervals $[-k, k]$, the only restriction on the boundary conditions themselves is the condition for their uniform boundedness. Moreover, according to Corollary 1, the approximation rate does not depend on the type of boundary conditions. Example 2 is shown to demonstrate this statement;

(d) we note that the obtained condition for the existence of a solution of the traveling wave type is just a sufficient condition. Therefore, solutions of the traveling wave type can also be numerically constructed for $\tau > \hat{\tau}$, as demonstrated in Example 3. Nevertheless, many of the central conditions of the presented theory are "exact", that is, there are examples of equations for which violation of the indicated conditions leads to the absence of solutions;
(e) From the point of view of setting the problem, the SC, in this case, constructs a solution on the interval $[-k-1, k+1]$. Accordingly, the larger the value of k, the less obvious is the fulfillment of the boundary conditions. Nevertheless, it was possible to extend the constructed numerical solutions to the interval $[-1.1k, 1.1k]$. Also note that the scale on the axes is different and the graphs are highly compressed, which leads to a strong variation effect, which is not really the case, but is only the effect of scale;
(f) in all the examples below $\hat{\tau} \simeq 0.15718495$, weighting coefficients are $v^{(N)} = v^{(K)} = 1$ (the question of the optimal choice, in a certain sense, of the weighting coefficients depending on the type of the equation, the algorithms used, and a number of other factors is the subject of a separate study and is beyond the scope of this article);
(g) in the SC *OPTCON-F* there is the possibility of sequential application of various algorithms within the framework of constructing a solution for one task. Thus, the constructed intermediate solution in the previous step becomes the starting solution ("baseline") for the following algorithm. In this case, such an implementation does not prevent the global search algorithms from "popping out" of the local solution. Separately, we note the presence of a programming module that allows predetermining the order of application of algorithms, as well as the construction of complex chain of steps (conditional statements, loops, etc.) depending on the current or historical values of a number parameters (for example, error estimation or number of iterations). For the examples presented below, in addition to the basic global optimization algorithm described in Section 2, the following scheme was used in cycle: the generalized quasi-Newtonian and Powell-Brent's methods (with bi-directional line search along each dimension) were used alternately, and after the error changed by less than 10^{-h} the adaptive modification of the Hooke-Jeeves method was used l times (h and l are computable functions on the basis of the loop iteration number, as well as the Lipschitz constants of the equation itself and a number of other technical characteristics). The stopping criterion depended on the number of iterations in the first part of the cycle, as well as the current error estimate and its dynamics;
(h) in view of all the points listed above, as well as stochastic elements in the applied algorithms, the presented value of the residual functional (RF, i.e. error of a solution) can not be used to estimate the theoretical rate of convergence.

Let us consider the FDEPT of the following form

$$\ddot{x}(t) = m^{-1}\left(x(t+\tau) - 2x(t) + x(t-\tau) + \frac{A_3x^3(t) + A_1x(t)}{x^4(t) + A_2x^2(t) + A_0}\right), \quad t \in \mathbb{R}, \qquad (17)$$

where $A_0, A_1, A_2, A_3 \in \mathbb{R} \setminus \{0\}, m, \tau \in \mathbb{R}_+$. Using a time-variable transformation the equation (17) can be rewritten in the form of the following system of equations of the first order

$$\begin{cases} \dot{z}_1(t) = \tau z_2(t), \\ \dot{z}_2(t) = \tau m^{-1}\left(z_1(t+1) - 2z_1(t) + z_1(t-1) + \dfrac{A_3z_1^3(t) + A_1z_1(t)}{z_1^4(t) + A_2z_1^2(t) + A_0}\right). \end{cases} \qquad (18)$$

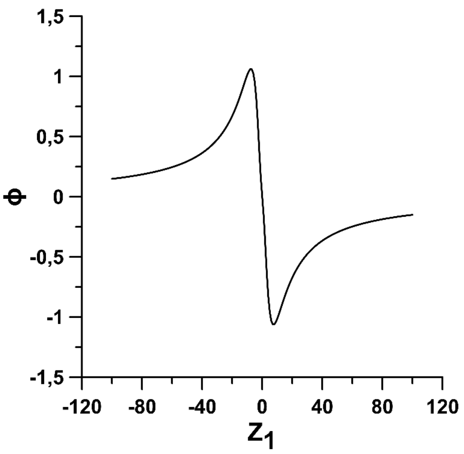


Fig. 2: Graph of the perturbation function

Under this system, we have the following real parameters: $A_0, A_1, A_2, A_3, \tau, m$. The motivation for specifying such a perturbation function is the desire to obtain an odd function that fades out at infinity and stabilizes the deviation around zero. The diagram of the perturbation function (for the parameters in the example below) is shown in Fig. 2.

6.1 Example 1

We consider dynamical system in the following form:

$$\begin{cases} \dot{z}_1(t) = 0.1 z_2(t), \\ \dot{z}_2(t) = 0.1 \times 100^{-1} \left(z_1(t+1) - 2z_1(t) + z_1(t-1) + \dfrac{-15 z_1^3(t) - 60 z_1(t)}{z_1^4(t) + 50 z_1^2(t) + 400} \right), \end{cases} \quad t \in \mathbb{R},$$

initial conditions

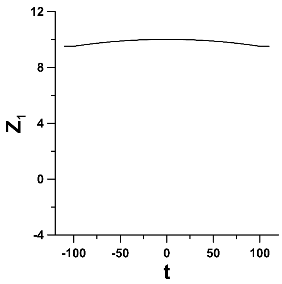
$$\begin{cases} z_1(0) = 10, \\ z_2(0) = 0. \end{cases} \tag{19}$$

Here, with respect to the system (18), we have $A_0 = 400, A_1 = -60, A_2 = 50, A_3 = -15, \tau = 0.1, m = 100$. Thus we obtain that $L \simeq 0.2445, C = 1$. Hence in this case the equality (4) takes the form

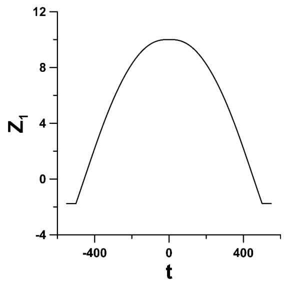
$$0.1(2\mu^{-1} + 1) = \ln \mu^{-1}$$

that has on the interval $(0,1)$ two solutions with approximate values $\mu_1(0.1) = 0.084$ and $\mu_2(0.1) = 0.672$ (the exact values are expressed in terms of the Lambert W-function and can't be written out in quadratures).

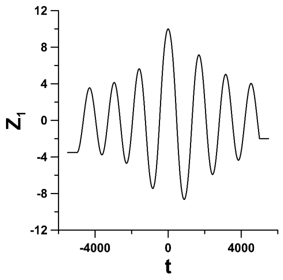
Taking into account the impossibility of considering the numerical solution of the system on an infinite interval, we introduce the parameter k and the corresponding family of



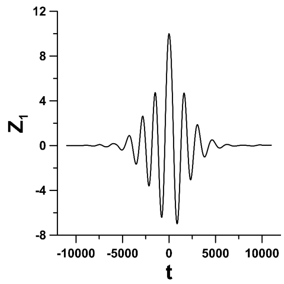
(a) $k = 100$, $RF \simeq 8.3 \times 10^{-9}$



(b) $k = 500$, $RF \simeq 1.1 \times 10^{-7}$



(c) $k = 5000$, $RF \simeq 6.8 \times 10^{-9}$



(d) $k = 10000$, $RF \simeq 9.3 \times 10^{-9}$

Fig. 3: Trajectories of the system (20) at different k.

expanding initial-boundary value problems

$$
\begin{aligned}
&\begin{cases}
\dot{z}_1(t) = 0.1 z_2(t), \\
\dot{z}_2(t) = 0.1 \times 100^{-1} \left(z_1(t+1) - 2z_1(t) + z_1(t-1) + \dfrac{-15 z_1^3(t) - 60 z_1(t)}{z_1^4(t) + 50 z_1^2(t) + 400} \right),
\end{cases} \\
&t \in [-k, k], \\
&\textit{boundary conditions} \\
&\begin{cases}
\dot{z}_1(t) = 0, \\
\dot{z}_2(t) = 0,
\end{cases} \quad t \in (-\infty, -k] \cup [k, +\infty), \\
&\textit{initial conditions} \\
&\begin{cases}
z_1(0) = 10, \\
z_2(0) = 0.
\end{cases}
\end{aligned} \tag{20}
$$

According to the Theorem 4, the solution of the system (20) converges (according to the metric of the space $\mathscr{L}_\mu^n C^{(0)}(\mathbb{R})$ with $\mu \in (\mu_1(\tau), \mu_2(\tau))$) to the solution of the system (19) as $k \to \infty$. The graphs of the solution of the system (20) at different values of k are shown in Fig. 3.

Since the equation (17) is autonomous, the solution space of such equation is invariant with respect to time-variable shifts. So far as the right-hand side of the equation is an odd function of its arguments, the solution space of such equation can withstand the reflection transformation with respect to the axis t. Therefore, it suffices to consider a family of so-

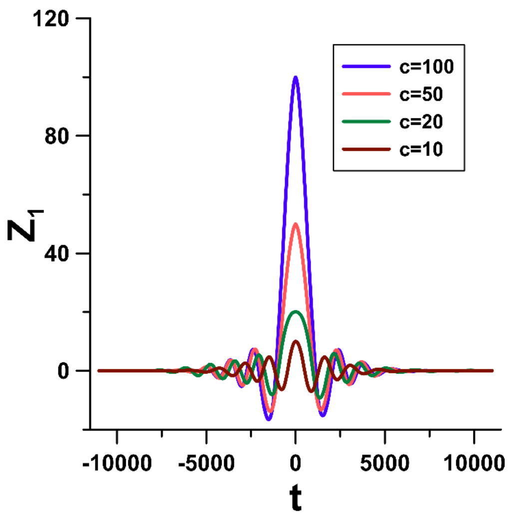


Fig. 4: Trajectories of the system (20) at different c

lutions of the initial problem (19) with a value of $z_1(0)$ from zero to the infinity. Figure 4 shows the integral curves for different values of the parameter $c = z_1(0)$ for the system (20).

6.2 Example 2

As mentioned earlier, in this examples we change the boundary conditions from constants to bounded functions. The graphs of the solution of the systems (21) and (22) at different values of k are shown in Fig. 5 and Fig. 6 respectively. As we see from the graphs, after damping of the oscillations from the effect of the boundary conditions, solutions stabilize to the former form, which is asserted in Theorem 4.

6.2.1 Example 2.1

$$
\begin{aligned}
&\begin{cases}
\dot z_1(t) = 0.1 z_2(t),\\
\dot z_2(t) = 0.1\times 100^{-1}\left(z_1(t+1) - 2z_1(t) + z_1(t-1) + \dfrac{-15z_1^3(t) - 60z_1(t)}{z_1^4(t) + 50z_1^2(t) + 400}\right),
\end{cases}\\
&t \in [-k, k],\\
&\textit{boundary conditions}\\
&\begin{cases}
\dot z_1(t) = 0.1\sin\left(\dfrac{\pi t}{100}\right),\\
\dot z_2(t) = \dfrac{\pi}{100}\cos\left(\dfrac{\pi t}{100}\right),
\end{cases}\quad t \in (-\infty, -k] \cup [k, +\infty),\\
&\textit{initial conditions}\\
&\begin{cases}
z_1(0) = 10,\\
z_2(0) = 0.
\end{cases}
\end{aligned}
\tag{21}
$$

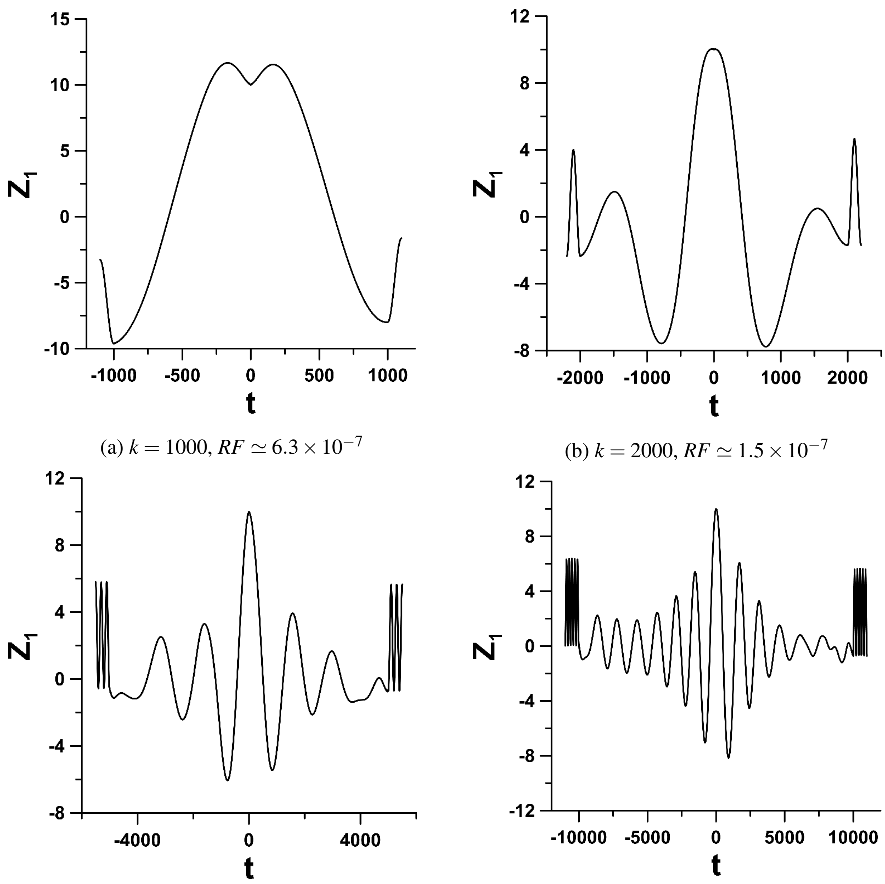


Fig. 5: Trajectories of the system (21) at different k.

6.2.2 Example 2.2

$$
\begin{cases}
\dot{z}_1(t) = 0.1 z_2(t), \\
\dot{z}_2(t) = 0.1 \times 100^{-1} \left(z_1(t+1) - 2z_1(t) + z_1(t-1) + \dfrac{-15z_1^3(t) - 60z_1(t)}{z_1^4(t) + 50z_1^2(t) + 400} \right),
\end{cases}
$$

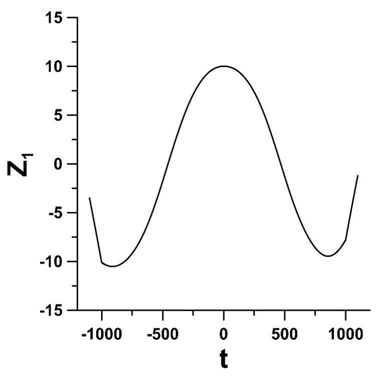
$t \in [-k, k]$,

boundary conditions

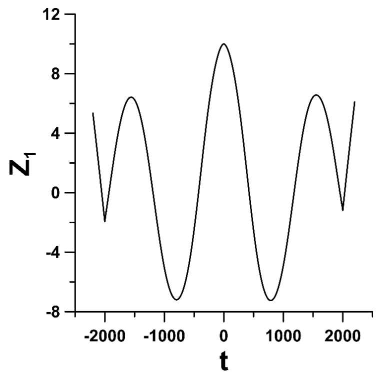
$$
\begin{cases}
\dot{z}_1(t) = 0.1 \times \dfrac{100 \ln(|t|)}{t}, \\
\dot{z}_2(t) = \dfrac{100(1 - \ln(|t|))}{t^2},
\end{cases}
\quad t \in (-\infty, -k] \cup [k, +\infty), \tag{22}
$$

initial conditions

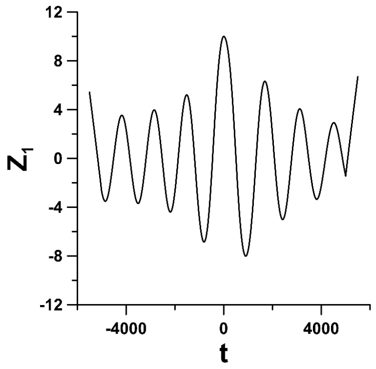
$$
\begin{cases}
z_1(0) = 10, \\
z_2(0) = 0.
\end{cases}
$$



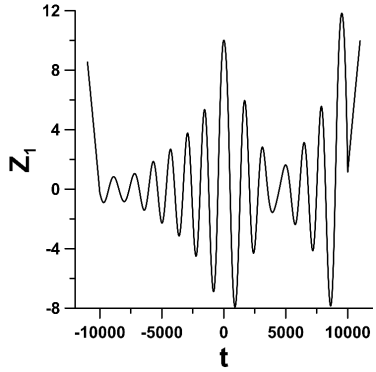
(a) $k = 1000$, $RF \simeq 4.2 \times 10^{-8}$



(b) $k = 2000$, $RF \simeq 4.9 \times 10^{-8}$



(c) $k = 5000$, $RF \simeq 1.4 \times 10^{-8}$



(d) $k = 10000$, $RF \simeq 3.2 \times 10^{-8}$

Fig. 6: Trajectories of the system (22) at different k.

6.3 Example 3

In this example, we consider the dependence of the dynamics of the solution from Example 1 as a function of the parameter τ (see Fig. 8). Equations (17) and (18) are equivalent for the same characteristics $\tau > 0$. The solutions of equation (18) are obtained from the solutions of equation (17) with the help of time extension by the inverse of τ. Therefore, the solutions of equation (18) for $\tau = 0$ are constants (10 in this case). On the other hand, after compressing the solutions of equation (18) by the value of τ, for τ tending to zero, solutions of equation (17) for $\tau = 0$ are obtained (see Fig. 7).

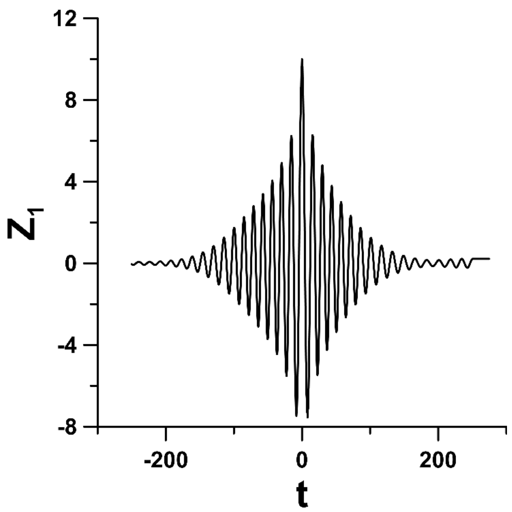
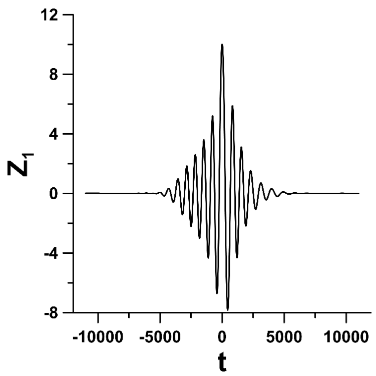
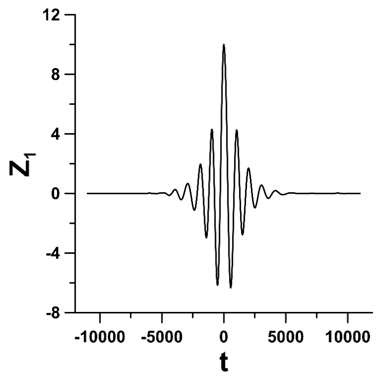


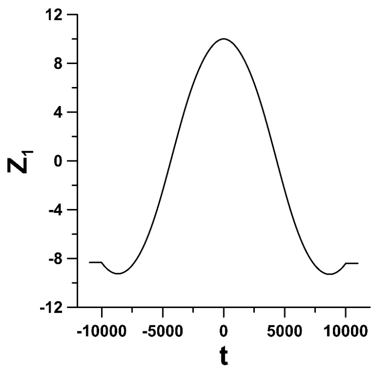
Fig. 7: Solution of equation (17) at $\tau = 0$



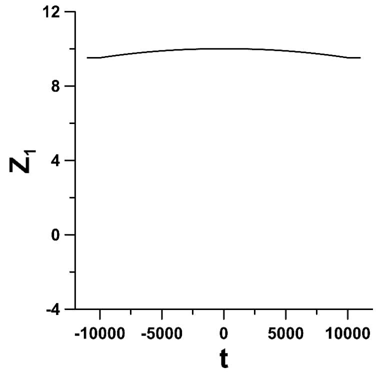
(a) $\tau = 0.2$, $RF \simeq 6.9 \times 10^{-8}$



(b) $\tau = 0.15$, $RF \simeq 1.1 \times 10^{-8}$



(c) $\tau = 0.01$, $RF \simeq 1.4 \times 10^{-8}$



(d) $\tau = 0.001$, $RF \simeq 8.6 \times 10^{-10}$

Fig. 8: Trajectories of the system (20) at different τ with $k = 10000$.

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