I. TOPOLOGICAL FIELD OF COMPLEX NUMBERS

We assume that all the basics of $\mathbb{R}$ is known; namely its algebraic operations, basic calculus and topology of reals. We construct complex numbers from scratch on top of this.

As a set: $\mathbb{C} = \{(a,b) : a,b \in \mathbb{R}\}$; so it can be thought as $\mathbb{R}^2$, but that notation suggests the vector space structure and $\mathbb{C}$ will have much more than that.

Algebraic operations: let $(a,b), (c,d) \in \mathbb{C}$.

$(a,b) + (c,d) := (a+c, b+d)$; so addition is done componentwise.

$(a,b) \cdot (c,d) := (ac-bd, ad+bc)$; why is it defined this way?

Equipped with these operations, $\mathbb{C}$ becomes a "field":

- Both + & · are associative and commutative (easy to check.)
- Additive identity element is $(0,0)$. (easy to check.)
- Additive inverse of (a,b) is $(-a,-b)$. (" " ")
- Multiplication distributes over addition (" " ")
- Multiplicative identity element is $(1,0)$ (" " ")
- Multiplicative inverse of $(a,b) \neq (0,0)$:

  We need to solve $(a,b) \cdot (c,d) = (1,0)$ for c and d.

  S. $(ac-bd, ad+bc) = (1,0)$:

$$\begin{array}{rl} a / & ac - bd = 1 \\ b / & ad + bc = 0 \\ \hline & a^2 c - a\cancel{bd} = a \\ & a\cancel{bd} + b^2 c = 0 \\ \hline & \underbrace{(a^2+b^2)}_{\neq 0} c = a \implies c = \dfrac{a}{a^2+b^2} \end{array}$$

$\Uparrow$ after some calc.

$d = \dfrac{-b}{a^2+b^2}$.

As mentioned above, $\mathbb{C}$ becomes a real vector space with $+$ from above and the scaler multiplication given as $\alpha(a,b) = (\alpha a, \alpha b)$. So it is nothing other than $\mathbb{R}^2$ and $\{(1,0), (0,1)\}$ is a basis.

It has a canonical copy of the real field in it; namely $\{(a,0) : a \in \mathbb{R}\}$. This means that this set is subfield of $\mathbb{C}$, and it is isomorphic to $\mathbb{R}$. || FROM NOW ON, WE IDENTIFY $\mathbb{R}$ WITH THIS!

In particular, we write 1 in the place of $(1,0)$. The other basis element has a name as well: $i := (0,1)$.

So any element (a,b) of $\mathbb{C}$ can be written as

$$(a,b) = a(1,0) + b(0,1) = a \cdot 1 + b \cdot i = \underline{\underline{a+bi}}$$

WE'LL USE THIS RATHER THAN (a,b) FROM NOW ON.

Note that $i \cdot i = (0,1)\cdot(0,1) = (0\cdot 0 - 1\cdot 1, 0\cdot 1 + 1\cdot 0)$
$= (-1,0) = -1$.

Hence i is one of the square roots of -1, which did <u>not</u> exist in $\mathbb{R}$.

As a matter of fact, any complex number has a square-root: $a+bi = (c+di)^2$ (solve for c & d in terms of a & b.)

$(c+di)^2 = c^2 - d^2 + 2cd\,i$. So $c^2 - d^2 = a$ & $2cd = b$

Taking squares: $c^4 - 2c^2d^2 + d^4 = a^2$ & $4c^2d^2 = b^2$
Adding them: $c^2 + d^2 = \sqrt{a^2+b^2} \Rightarrow d^2 = \sqrt{a^2+b^2} - c^2$.

Then $a = c^2 - \sqrt{a^2+b^2} + c^2$ and hence $c = \pm\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}}$

Continuing this we also get $d = \pm\sqrt{\dfrac{-a+\sqrt{a^2+b^2}}{2}} \cdot \operatorname{sgn} b$.

these give both square-roots of $a+bi$.

HW: Find the complex roots of any quadratic equation $az^2+bz+c=0$.

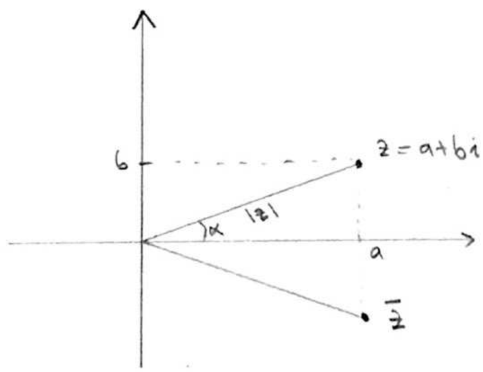
We've seen above $\mathbb{C}$ is $\mathbb{R}^2$ as a vector space. So we imagine $\mathbb{C}$ as the real plane. Then addition corresponds to vector addition. We'll see the geometric interpretation of multiplication in a bit.

Def: Let $z = a+bi$ be a complex number. We define the following:

$Re(z) = a$ (real part of z), $Im(z) = b$ (imaginary part of z),

$\overline{z} = a - bi$ (complex conjugate of z), $|z| = \sqrt{a^2+b^2}$ (abs. value of z)

Note that $|z|^2 = z \cdot \overline{z}$.



Note that $\cos \alpha = \frac{b}{|z|} = \frac{b}{\sqrt{a^2+b^2}}$ &

(if $z \neq 0$) $\sin \alpha = \frac{a}{|z|} = \frac{a}{\sqrt{a^2+b^2}}$

We define the argument, $Arg(z)$, of z to be the angle $\alpha \in [0, 2\pi)$ such that $\cos \alpha = \frac{b}{|z|}$. (Defined for $z \neq 0$.)

Note that if $z \in \mathbb{C} \setminus \{0\}$ with $r = |z|$ and $\alpha = Arg(z)$, then we may write $z = r(\cos\alpha + i\sin\alpha)$. This pair (r, α) actually determines z; meaning that there is only one complex number whose absolute value is r and argument is α.

This pair is called the polar coordinates of z.

(So we may as well identify $\mathbb{C} \setminus \{0\}$ with $\mathbb{R} \times [0, 2\pi)$.)

let's consider complex multiplication in terms of polar coordinates:

let $z_1 = r_1(\cos\alpha_1 + i\sin\alpha_1) \neq 0$ & $z_2 = r_2(\cos\alpha_2 + i\sin\alpha_2) \neq 0$.

$$z_1 \cdot z_2 = r_1 r_2 \Big(\underbrace{(\cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2)}_{\cos(\alpha_1+\alpha_2)} + i\,\underbrace{(\cos\alpha_1\sin\alpha_2 + \sin\alpha_1\cos\alpha_2)}_{\sin(\alpha_1+\alpha_2)}\Big).$$

So we multiply the lengths and add the angles when we multiply.

(Think about division yourselves.)

Whenever we have an absolute value, we have a metric and hence a topology.

$d(z_1, z_2) := |z_2 - z_1|$. (Distance of two complex numbers z_1 & z_2.)

So we may define $z = \lim_{n\to\infty} z_n$ if $\lim_{n\to\infty} |z_n - z| = 0$.

(Triangle inequality: $d(z_1, z_2) + d(z_2, z_3) \geq d(z_1, z_3)$.)

We may also define Cauchy sequences: A sequence $(z_n)_n$ is called Cauchy if for every $\varepsilon > 0$, there is $N > 0$ s.t for every $m, n > N$ we have $|z_n - z_m| < \varepsilon$.

Note that $z_n \to z$ if and only if $\operatorname{Re} z_n \to \operatorname{Re} z$ and $\operatorname{Im} z_n \to \operatorname{Im} z$. (The latter limits are real limits.)

Using this we may prove the following: let $(z_n)_n$ be a sequence of complex numbers. Then

(z_n) is Cauchy $\iff$ there is $z \in \mathbb{C}$ s.t $z_n \to z$.

(This means that $\mathbb{C}$ is a <u>complete</u> metric space.)

We may also talk about <u>series</u> and their convergence:

We say that a series $\sum_{k=1}^{\infty} z_k$ <u>converges</u> if the sequence $\left(s_n = \sum_{k=1}^{n} z_k\right)_n$ converges. If $L = \lim_{n\to\infty} s_n$, then we write $\sum_{k=1}^{\infty} z_k = L$.

We have the usual properties for series. In particular, if $\sum_{k=1}^{\infty} |z_k|$ converges, then so does $\sum_{k=1}^{\infty} z_k$. (We say $\sum_{k=1}^{\infty} z_k$ <u>converges absolutely</u> if $\sum_{k=1}^{\infty} |z_k|$ converges.)

<u>Proof</u>: Show $s_n = \sum_{k=1}^{n} z_k$ is a Cauchy sequence, using the triangle inequality.

$\boxed{1^{\underline{st}}\ \text{Lecture}}$

<u>Back to topology</u>: . $D(z_0, r) = \{ z \in \mathbb{C} : d(z, z_0) < r \}$. (open disc around z_0 of radius r.)
$(r > 0)$

- $U \subseteq \mathbb{C}$ is called <u>open</u> if for any $z_0 \in U$, there is $r = r(z_0) > 0$ such that $D(z_0, r) \subseteq U$.

$C^c = \tilde{C}$

- $C \subseteq \mathbb{C}$ is called <u>closed</u> if $\mathbb{C} \setminus C$ is open.

<u>HW</u>: Show that C is closed iff the following holds:

* If $(z_n)_n$ is a convergent sequence of elements of C, then $\lim z_n \in C$.

closure of X

- Let $X \subseteq \mathbb{C}$. Then $\overline{X}$ is the smallest closed set containing X.

boundary of X

- Let $X \subseteq \mathbb{C}$. Then ∂X is the set of $z_0 \in \mathbb{C}$ s.t for any $r > 0$ we have $D(z_0, r) \cap X = \emptyset$ and $D(z_0, r) \cap (\mathbb{C} \setminus X) \neq$

HW: Show that $\overline{X} = X \cup \partial X$ for any $X \subseteq \mathbb{C}$.

- $X \subseteq \mathbb{C}$ is <u>bounded</u> if there is $R \in \mathbb{R}^{>0}$ s.t $X \subseteq D(0,R)$.

- X is <u>compact</u> if X is closed & bounded.

HW: Show that X is compact if and only if the following happens:

* Let $U_i \subseteq \mathbb{C}$ be an open set for $i \in I$ s.t $\bigcup_{i \in I} U_i \supseteq X$.
Then there is a finite $I_0 \subseteq I$ s.t $\bigcup_{i \in I_0} U_i \supseteq X$.

- X is <u>connected</u> if for any non-empty $U, V \subseteq \mathbb{C}$ with $U \cap V = \emptyset$ and $U \cup V \supseteq X$, we have either $U \supseteq X$ or $V \supseteq X$.

- A <u>region</u> is a connected open set.

- Let $z_1, z_2 \in \mathbb{C}$. Then $[z_1, z_2] = \{ tz + (1-t)z : 0 \leq t \leq 1 \}$.
(<u>Line</u> segment connecting z_1 & z_2).

- A <u>polygonal line</u> is $[z_1, z_2] \cup [z_2, z_3] \cup \cdots \cup [z_n, z_{n+1}]$.

<u>HW</u>: Let $X \subseteq \mathbb{C}$. Suppose that for any $z_1, z_2 \in X$, there is a polygonal line, inside X connecting z_1 & z_2. Show that X is connected.

<u>Proposition</u>: For any region S and $s_1, s_2 \in S$, there is a polygonal line inside S connecting s_1 and s_2.

Proof: If $S = \emptyset$, then there is nothing to prove.

Suppose $S \neq \emptyset$ and fix $s_0 \in S$.

Let $U = \{ s \in S :$ there is a poly. line connecting s and $s_0 \}$ and

let $V = S \setminus U$.

It's easy to see that U & V are open. (This uses that U is open.)

So U, V open, $U \cap V = \emptyset$, and $U \cup V = S$.

Then either $U \supseteq S$ or $V \supseteq S$. Then $U = S$ (why?)

Therefore any $s_1, s_2 \in S$ can be connected to s_0 by a poly. line,

and hence to each other. ▨

[Aside: Actually, we may consider "paths" rather than "poly lines".

- A path from x to y is a continuous map $\varphi : [0,1] \to \mathbb{C}$
  such that $\varphi(0) = x$ and $\varphi(1) = y$.

- $X \subseteq \mathbb{C}$ is path connected if ...

- Let $S \subseteq \mathbb{C}$ be open. Then S is connected $\iff$ S is path connected]

Definition: Let $X \subseteq \mathbb{C}$, $z_0 \in X$ and $D(z_0, \varepsilon) \subseteq X$ for some $\varepsilon > 0$.

A function $f : X \to \mathbb{C}$ is continuous at z_0 if for every sequence

(z_n) with $\lim_n z_n = z_0$ we have $\lim_n f(z_n) = f(z_0)$.

HW: Show that f (as above) is continuous at z_0 if and only if

for any $\varepsilon > 0$, there is $\delta > 0$ such that $f(D(z_0, \delta)) \subseteq D(f(z_0), \varepsilon)$.

Let $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\tilde{f}(x,y) = (\mathrm{Re}(f(x+iy)), \mathrm{Im}(f(x+iy)))$

Write $u(x,y)$ and $v(x,y)$ for $\mathrm{Re}(f(x+iy))$ & $\mathrm{Im}(f(x+iy))$.

It's easy to see that f is continuous at $z_0 = x_0 + iy_0$ iff both u & v are continuous at (x_0, y_0) (as fuctions $\mathbb{R}^2 \to \mathbb{R}$.)

Let $S \subseteq \mathbb{C}$ be a region ("open" is enough) & $f: S \to \mathbb{C}$. We say f is <u>continuous</u> if it's continuous at every $z_0 \in S$.

We say $f \in C^n(S)$ if both u & v have continuos partial derivatives of order n.

Definition: Suppose that for each $n > 0$, a function $f_n: S \to \mathbb{C}$ is given, and $f: S \to \mathbb{C}$. (S region)

We say $(f_n)_n$ <u>converges to f uniformly in S</u> if for every $\varepsilon > 0$, there is $N > 0$ s.t for every $n > N$ and $z \in S$ we have $|f_n(z) - f(z)| < \varepsilon$.

[HW: If each f_n is continuous & f is the uniform limit of f_n (on S), then f is continuous as well.]

<u>Theorem</u> (Weierstrass M-test): Suppose each f_n is continuous in S. Suppose also that M_n are non-negative real numbers with

- $\sum_{n=1}^{\infty} M_n$ converges &
- $|f_n(z)| \leq M_n$ for all $z \in S$.

Then $\sum_{n=1}^{\infty} f_n(z)$ converges to a continuous function.
(The convergence is uniform. Hence by HW about the limit is continuous.)

<u>Proof</u>: It's clear that $f(z) := \sum_{n=1}^{\infty} f_n(z)$ converges for any $z \in S$; so we actually have function $f: S \to \mathbb{C}$.

Note that $\left| f(z) - \sum_{n=1}^{n_0} f_n(z) \right| = \left| \sum_{n=n_0+1}^{\infty} f_n(z) \right| \leq \sum_{n=n_0+1}^{\infty} M_n$

Given $\varepsilon > 0$, there is $N > 0$ s.t for $n_0 > N$, this is $< \varepsilon$. ∎

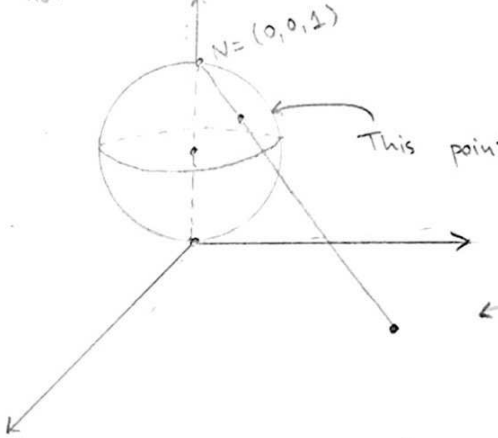
$\boxed{\text{Lecture 2}}$

Definition: • A sequence $(z_n)_n$ diverges to ∞ if $(|z_n|)_n$ diverges to ∞.
(We write $z_n \longrightarrow \infty$ or $\lim_n z_n = \infty$)

• $f: \mathbb{C} \to \mathbb{C}$ a function, $z_0 \in S$. We write $\lim_{z \to z_0} f(z) = \infty$ if

$\lim_{z \to z_0} |f(z)| = \infty$.

There is a way to make the concept of "infinity" more precise. We'll just sketch the idea and let you take care of the details. Let Σ be the sphere in $\mathbb{R}^3$ whose center is at $(0, 0, 1/2)$ and whose radius is $1/2$. So it looks like below:



This map is a homeomorphism (?) between $\Sigma \setminus \{N\}$ and $\mathbb{C}$.

So in a way, N is "the point at infinity".

Note that "circles" of Σ passing from N are lines in $\mathbb{C}$.
(See the book for more details.)

— o —

$f: S \to \mathbb{C}$ a function $S \subseteq \mathbb{C}$, $z_0 \in \overline{S}$.

We define $\lim_{z \to z_0} f(z) = L$ if for every sequence $(z_n)_n$ from S with $\lim_n z_n = z_0$, we have $\lim_n f(z_n) = L$.

(This definition should have appeared earlier. (It never appears in the book.))

II. ANALYTIC FUNCTIONS & POWER SERIES

Definition: • $f : S \to \mathbb{C}$ ($S \subseteq \mathbb{C}$ open), $z_0 \in S$. We say f is differentiable at z_0 if $\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ exists; if that is the case, then that limit is the derivative of f at z_0, denoted by $f'(z_0)$.

• If f as above is differentiable at every point of S, then we say that f is differentiable (on S).

• Let $f : S \to \mathbb{C}$, $X \subseteq S$ is not necessarily open. We say f is differentiable on X if there is an open subset S_1 of S containing X s.t. f is diff. on S_1.

HW. Let f, g be two functions on open $S \subseteq \mathbb{C}$ & $z_0 \in S$. Suppose that f and g are differentiable at z_0. Then so are $f+g$ and $f \cdot g$.

We may also consider $f : S \to \mathbb{C}$ as a function of two real-variables x, y by letting $f(x,y)$ to be $f(x+iy)$, and we may separate real and imaginary parts of f as $u(x,y)$ and $v(x,y)$. So both u & v are functions $\mathbb{R}^2 \to \mathbb{R}$. Therefore we may talk about their partial derivatives: $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $v_x = \cdot$, $v_y \cdots$ (of course, they might not exist.)

Suppose both u_x and v_x exist, then we define the partial derivative of f (in the x-direction) to be $f_x = u_x + i v_x$.

Similarly, if both u_y & v_y exists, then we define $f_y = u_y + i v_y$.

Proposition: Let f be differentiable at z_0. Then both f_x and f_y exist at z_0 and $f_y(z_0) = i f_x(z_0)$.

Proof: We know that $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists; call it L.

Letting $h \to 0$ through real values, we get:

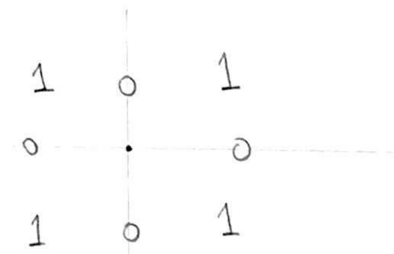
$$L = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = f_x(z_0). \qquad (z_0 = x_0 + iy_0).$$

Now letting $h \to 0$ through totally imaginary values, we get

$$L = \lim_{\eta\to 0} \frac{f(x_0, y_0+\eta) - f(x_0, y_0)}{i\eta} = \frac{f_y(z_0)}{i}.$$

So putting these together, we get $f_y(z_0) = i f_x(z_0)$. $\boxed{}$

Example: $f(x,y) = \begin{cases} 0 & : \text{if } x=0 \text{ or } y=0 \\ 1 & : \text{if } x\neq 0 \ \& \ y \neq 0 \end{cases}$



Now $f_x(0,0) = f_y(0,0) = 0$. But f is obviously not differentiable at $z_0 = 0$.

So converse is not correct.

The equation $f_y(z_0) = i f_x(z_0)$ can be written in terms of partials of u & v as follows: $u_y(z_0) + i v_y(z_0) = i(u_x(z_0) + i v_x(z_0))$.

So $u_y = -v_x$ and $v_y = u_x$ (at (x_0, y_0)).

These equation are called Cauchy-Riemann equations.

Proposition: Suppose that f_x, f_y exist and are continuous in a neighborhood of $z_0 = x_0 + iy_0$, and that $f_y = i f_x$ on that neighborhood. Then f is differentiable at z_0.

Proof: We'll show that $\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = f_x(z_0)$.

Now writing $h = \xi + i\eta$:

$$\frac{u(z_0+h) - u(z_0)}{h} = \frac{u(x_0+\xi, y_0+\eta) - u(x_0, y_0)}{\xi + i\eta}$$

$$= \underbrace{\frac{u(x_0+\xi, y_0+\eta) - u(x_0+\xi, y_0)}{\xi + i\eta}} + \underbrace{\frac{u(x_0+\xi, y_0) - u(x_0, y_0)}{\xi + i\eta}}$$

continuity of u_x & u_y is used here.

By the Mean Value theorem applied to $u(x_0+\xi, -)$, this equals

$$\frac{\eta}{\xi+i\eta} u_y(x_0+\xi, y_0+\theta_1\eta)$$

for some $\theta_1 \in (0,1)$

Similarly this equals

$$\frac{\xi}{\xi+i\eta} u_x(x_0+\theta_2\xi, y_0)$$

for some $\theta_2 \in (0,1)$.

Similarly, $$\frac{v(z_0+h) - v(z_0)}{h} = \frac{\eta}{\xi+i\eta} v_y(x_0+\xi, y_0+\theta_3\eta) + \frac{\xi}{\xi+i\eta} v_x(x_0+\theta_4\xi, y_0)$$

for some $\theta_3, \theta_4 \in (0,1)$.

On the other hand, we may write $f_x(z_0) = \frac{\eta}{\xi+i\eta} f_y(z_0) + \frac{\xi}{\xi+i\eta} f_x(z_0)$ by CR equations let's put these together, $\frac{f(z_0+h)-f(z_0)}{h} - f_x(z_0)$ equals::

$$\frac{\eta}{\xi+i\eta}\Big(\big(\underbrace{u_y(x_0+\xi,y_0+\theta_1\eta)-u_y(x_0,y_0)}_{\downarrow\, 0}\big)+i\big(\underbrace{v_y(x_0+\xi,y_0+\theta_3\eta)-v_y(x_0,y_0)}_{\downarrow\, 0}\big)\Big)+$$

$$\frac{\xi}{\xi+i\eta}\Big(\big(\underbrace{u_x(x_0+\theta_2\xi,y_0)-u_x(x_0,y_0)}_{\downarrow\, 0}\big)+i\big(\underbrace{v_x(x_0+\theta_4\xi,y_0)-v_x(x_0,y_0)}_{\downarrow\, 0}\big)\Big)$$

(as $h\to 0$)

& $\left|\frac{\eta}{\xi+i\eta}\right|, \left|\frac{\xi}{\xi+i\eta}\right| < 1$.

So $\lim\limits_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = f_x(z_0)$ as desired. ▨

<u>Definition</u>: let $f: S\to\mathbb{C}$, $S\subseteq\mathbb{C}$ open, $z_0\in S$. We say f is <u>analytic at z_0</u> if f is differentiable on an open neighborhood of z_0.

For $X\subseteq S$, we say f is <u>analytic on X</u> if f is differentiable on an open subset of S containing X.

Before analyzing analytic functions in more detail, we'll present a class of analytic functions: Power series.

$\boxed{\text{Lecture 3}}$

<u>Power Series</u>

As in the real case a power series is an infinite series of the form $\sum\limits_{k=0}^{\infty} C_k z^k$. We would like to consider this as a function of z, but first we have to investigate convergence of it.

We begin with the complex version of the root test.

Theorem: Consider the power series $\sum_{k=0}^{\infty} c_k z^k$, and let $L = \limsup_k \sqrt[k]{|c_k|}$.

① If $L=0$, then the power series absolutely converges for all $z \in \mathbb{C}$.

② If $L=\infty$, then the power series converges only for $z=0$.

③ Suppose $L \neq 0, \infty$ & let $R = \frac{1}{L}$. Then the series converges absolutely on $D(0,R)$ and diverges on $\mathbb{C} \setminus \overline{D}(0,R)$. (What happens on the boundary $\overline{D}(0,R) \setminus D(0,R)$ depends on the series.)

Proof: Suppose $L=0$. Then $\limsup_k |\sqrt[k]{c_k}||z| = 0$ for all $z \in \mathbb{C}$. So actually $\lim_k |\sqrt[k]{c_k}\, z| = 0$, and hence there is $N>0$ such that for every $k>N$ we have $|\sqrt[k]{c_k}\, z| < \frac{1}{2}$. Then $|c_k z^k| < \frac{1}{2^k}$ for $k>N$. By comparison test $\sum_{k=0}^{\infty} |c_k z^k|$ converges, and $\sum_{k=0}^{\infty} c_k z^k$ as well.

If $L=\infty$, then given $z \in \mathbb{C}\setminus\{0\}$, there is $N>0$ such that for every $k>N$ we have $|\sqrt[k]{c_k}| \geq \frac{1}{|z|}$. Then $|c_k z^k| \geq 1$ for $k>N$. Thus $|c_k z^k| \not\to 0$ (as $k \to \infty$) and the series can't converge.

Now let $L \neq 0, \infty$ & let $R = 1/L$.
Suppose $|z| < R$, say $|z| = R(1-2\delta)$. Then $\limsup_k |\sqrt[k]{c_k}||z| = 1-2\delta$. Then there is $N>0$ such that for every $k>N$ we have $|\sqrt[k]{c_k}\, z| < 1-\delta$. So $\sum_{k=0}^{\infty} c_k z^k$ is absolutely convergent.

If $|z|>R$, then $\limsup_k |c_k z^k| > 1$, and hence $|c_k z^k| \not\to 0$ and the series diverges. ∎

Given power series $\sum_{k=0}^{\infty} c_k z^k$, the quantity $R = \begin{cases} \infty & : \text{ if } \limsup |\sqrt[k]{c_k}| = 0 \\ 0 & : \text{ if } \quad '' \quad = \infty \\ \dfrac{1}{\limsup \sqrt[k]{c_k}} & : \text{ if } \quad \text{———} \quad \neq 0, \infty \end{cases}$ (8)

is called the radius of convergence of the power series.

Let $\delta > 0$. Then $\sum_{k=0}^{\infty} |c_k z^k| \leq \sum |c_k| (R-\delta)^k$ for $z \in \overline{D}(0, R-\delta)$.

So using the M-test, we get that $\sum_{k=0}^{\infty} c_k z^k$ is continuous on $D(0,R)$. (We generally call $D(0,R)$, the domain of convergence)

Examples: ① $\sum_{k=0}^{\infty} z^k$. $L = \limsup_k |1| = 1$. So $R = 1$.

Clearly, for $|z| = 1$, we have $|z^k| \to 1 \neq 0$. So the series diverges on the boundary.

② $\sum_{k=0}^{\infty} k z^k$ $L = \limsup_k |\sqrt[k]{k}| = 1$. So $R = 1$.

Again for $|z| = 1$, we have $|kz^k| \longrightarrow \infty \neq 0$.

③ $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ $L = \limsup |\sqrt[k]{1/k^2}| = 1$.

Suppose $|z| = 1$. Then $\sum_{k=1}^{\infty} \left|\frac{z^k}{k^2}\right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

④ $\sum_{k=1}^{\infty} \frac{z^k}{k!}$ $L = \limsup_k \left|\sqrt[k]{\frac{1}{k!}}\right| = 0$. So $R = \infty$.

⑤ $\sum_{k=0}^{\infty} z^{k^2}$. In this case $C_n = 1$ for square n & $C_n = 0$ for non-square n. Thus $L = 1$. Hence $R = 1$.

Theorem: Suppose $f(z) = \sum_{k=0}^{\infty} C_k z^k$ converges on $D(0,R)$. Then f is differentiable on $D(0,R)$ and $f'(z) = \sum_{k=1}^{\infty} k C_k z^{k-1}$

Proof: First note that $\limsup_k \sqrt[k-1]{|k C_k|} = \limsup_k \left(\sqrt[k-1]{\underbrace{\sqrt[k]{k}}_{\downarrow 1} \; \underbrace{\sqrt[k]{|C_k|}}_{\downarrow L}} \right)^k = L$

So $\sum_{k=1}^{\infty} k C_k z^{k-1}$ converges on $D(0,R)$.

Let $|z_0| < R$. We need $l = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) = 0$, where

$$g(z) = \sum_{k=1}^{\infty} k C_k z^{k-1}.$$

Using usual rules for infinite series we get

$$\frac{f(z) - f(z_0)}{z - z_0} - g(z_0) = \sum_{k=1}^{\infty} \left(C_k \frac{z^k - z_0^k}{z - z_0} - k C_k z_0^{k-1} \right)$$

$$= \sum_{k=1}^{\infty} C_k \left(z^{k-1} + z^{k-2} z_0 + \cdots + z z_0^{k-2} + (1-k) z_0^{k-1} \right)$$

$$= \underbrace{\sum_{k=1}^{N} \bigcirc}_{= A_N} + \underbrace{\sum_{k=N+1}^{\infty} \bigcirc}_{= B_N}$$

Let $r > 0$ be s.t. $|z_0| < r$. Then $\sum_{k=1}^{\infty} |k a_k r^{k-1}|$ converges and hence given $\varepsilon > 0$, there is $N > 0$ s.t. $\sum_{k=1}^{N} |k a_k r^{k-1}| < \varepsilon/4$

So $B_N(z) \leq \sum_{k=N+1}^{\infty} 2k|C_k| r^{k-1} < 2\frac{\varepsilon}{4} = \frac{\varepsilon}{2}$.

Note that $A_N(z) \to 0$ as $z \to z_0$. (since $A_N(z)$ is a polynomial, there is no problem with this.)

So given $\varepsilon > 0$ taking N large enough and z close enough to z_0, we get $A_N(z) + B_N(z) < \varepsilon$; showing that $l = 0$. ■

Corollary: Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ converge (absolutely) in $D(0,R)$. Then for any $n>0$, $f^{(n)}(z)$ exists in $D(0,R)$ and it's given by $f^{(n)}(z) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} C_k z^{n-k}$.

Note that $f^{(n)}(0) = n!\, C_n$. Hence $C_n = \frac{f^{(n)}(0)}{n!}$.

So $f(z)$ is given by the Taylor expansion $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$.

Lecture 4

Theorem: Suppose $f(z) = \sum_{k=0}^{\infty} C_k z^k$ is 0 at a nonzero sequence $(z_n)_n$ that converges to 0. Then $f \equiv 0$.

Proof: The radius R of convergence of f is non-zero, since z_n's are nonzero and f is assumed to converge at them. In particular, f is continuous on $D(0,R)$.

We prove that $C_N = 0$ by induction on N.

$N=0$: $C_0 = f(0) = \lim_n f(z_n) = 0$. (Continuity.)

Suppose $C_0 = C_1 = \cdots = C_{N-1} = 0$. Let $g(z) = \frac{f(z)}{z^N} = \sum_{k=N}^{\infty} c_k z^{k-N}$

Then g is cont. at 0, as well and $C_N = g(0) = \lim_n g(z_n)$

& $g(z_n) \to 0$

$\| \quad 0 = \lim_n 0 = \lim_n \frac{f(z_n)}{z_n}$ ∎

Corollary: Suppose that two power series $\sum_{k=0}^{\infty} a_k z^k$ and $\sum_{k=0}^{\infty} b_k z^k$ agree on a set for which 0 is a limit point. Then $a_k = b_k$ for all k.

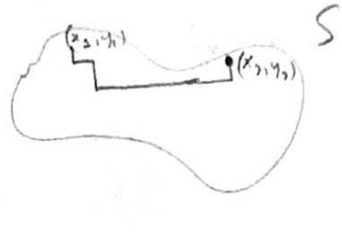
Until now we worked with power series around 0, but we may consider $\sum_{k=0}^{\infty} c_k (z - z_0)^k$; power series around z_0. The theory goes through without much change. The disc of convergence is $D(z_0, R)$ for some $R \in [0, \infty]$.

— o —

Back to study of general analytic functions.

Lemma: Let $u: \mathbb{R}^2 \to \mathbb{R}$ be such that u_x and u_y exist and equals zero on a region $S \subseteq \mathbb{R}^2$. Then $u \equiv$ constant on D.

Proof: Let $(x_1, y_1), (x_2, y_2) \in S$. Then we may connect those two points by a polygonal line whose parts are parallel to the axes.



So we may assume $x_1 = x_2$ or $y_1 = y_2$.

Suppose $x_1 = x_2$ & consider $L = \frac{u(x_1, y_1) - u(x_1, y_2)}{y_1 - y_2}$.

By MVT, we have $L = u_x(x_1, y_3)$ for some $y_3 \in [y_1, y_2]$

Then $L=0$. Hence $u(x_1,y_1) = u(x_1,y_2) = u(x_2,y_2)$.

The case $y_1=y_2$ can be handled in a similar way using $u_y \equiv 0$ on S. ■

<u>Proposition</u>: Let f be analytic on a region D. Suppose that $Re(f)=u$ is constant. Then f is constant.

<u>Proof</u>: By Cauchy-Riemann equations, we have $v_y = u_x = 0$ and $v_x = -u_y = 0$ on D. So by the lemma above, both u & v are constant. Hence f is constant. ■

Note that this proposition is still correct when u is replaced by v.

<u>Proposition</u>: Let f be analytic on a region D. Suppose that $|f|$ is constant on D. Then f is constant.

<u>Proof</u>: If $|f|=0$, then $f\equiv 0$. So assume $|f| \equiv C \neq 0$ on D. $|f| = u^2+v^2 \equiv C$.

$$(|f|)_x = 2u_x u + 2v_x v \equiv 0 \quad \& \quad (|f|)_y = 2u_y u + 2v_y v \equiv 0$$

$$2u_x u + 2v_x v = 2(u_x u - u_y v) \qquad 2u_y u + 2v_y v = 2(u_y u + u_x v)$$

$$\left.\begin{aligned} u(u_x u - u_y v) &\equiv 0 \\ v(u_y u + u_x v) &\equiv 0 \end{aligned}\right\} \Rightarrow u_x(u^2+v^2) \equiv 0 \Rightarrow u_x \equiv 0 \text{ on } D.$$

Similarly, we may show that $u_y \equiv 0$ as well. So using the above proposition, we get that f is constant. ■

Theorem: f analytic on D such that $f' \equiv 0$ on D. Then $f \equiv c$ on D.

Proof: Recall that we have shown $f'(z) = u_x(z) + i v_x(z)$. So $u_x \equiv 0$ and $u_y = v_x \equiv 0$ on D. Hence f is constant on D. ∎

— ∘ —

We have the following, which is proved exactly as in the real case:

Theorem: Suppose f is an injective analytic function on an open set S and let $T := f(S)$ (T has to be open as well). Assume also that f has an inverse g around $z_0 \in T$: z_0 has a neighborhood $U \subseteq T$ such that for every $z \in U$ we have $f(g(z)) = z$. Then g is also analytic around z_0 and

$$g'(z_0) = \frac{1}{f'(g(z_0))}.$$

(why isn't $f'(g(z_0)) \neq 0$?)

Exponential Function

We have already given the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ above and shown that its radius of convergence is ∞. We define this to be the (complex) exponential function: $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

① Now $\exp'(z) = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \exp(z)$, just as in the real case.

② Let $c \in \mathbb{C}$ & consider $f_c(z) = \exp(z) \cdot \exp(c-z)$.

Then $f_c'(z) = \exp(z)\exp(c-z) - \exp(z)\exp(c-z) = 0$. So $f_c(z) \equiv$ constant & that constant is $\exp(c)$.

Hence $\exp(z)\cdot\exp(c-z)=\exp(c)$ for every $c\in\mathbb{C}$.

So let $z=a\in\mathbb{C}$ and $c=a+b$ for some $b\in\mathbb{C}$. We get

$$\exp(a)\cdot\exp(b)=\exp(a+b)$$ (As in the real case.)

(3) How about when $z\in\mathbb{R}$? Is $\exp(z)=e^z$ where e is the Euler number? $(e=\exp(1))$.

Since exp is continuous, it suffices to investigate $z\in\mathbb{Q}$.

So: we'd like to show $\exp(\frac{m}{n})=e^{m/n}$ for $m,n\in\mathbb{Z}, n>0$.

(But what does $e^{m/n}$ mean? $e^{m/n}:=\sqrt[n]{e^m}$.)

Note that for $m\in\mathbb{N}$ we have $\exp(m)=\exp(1)^m=e^m$.

Also $\exp(m)\cdot\exp(-m)=1$. So $\exp(-m)=\exp(m)^{-1}$.

Now $\left(\exp(\frac{m}{n})\right)^n=\exp(m)=e^m$. Then $\exp(\frac{m}{n})=\sqrt[n]{e^m}=e^{m/n}$.

<u>Hence</u>: $\exp(z)=e^z$ for every real z.

<u>Trigonometric Functions</u>

We define complex versions of sine and cosine function by power series:

$$\sin z=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} \quad \& \quad \cos z=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}$$

$$L=\limsup_n\left\{\sqrt[n]{\frac{1}{(2n+1)!}}\right\}=0=\limsup_n\left\{\sqrt[n]{\frac{1}{(2n)!}}\right\}$$

So both sin and cos converge absolutely for each $z\in\mathbb{C}$.

Clearly, restrictions of sin and cos correspond to the real versions.

Properties:

- $\cos(-z) = \cos(z)$, $\sin(-z) = -\sin z$.

- $$\cos'(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \cdot 2n \, z^{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} z^{2n-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} z^{2n+1} = -\sin(z).$$

- $\sin'(z) = \cdots = \cos(z)$.

- $$\cos z + i \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + \sum \frac{(-1)^n}{(2n+1)!} i \, z^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(i)^{2n}}{(2n)!} z^{2n} + \sum \frac{(i)^{2n}}{(2n+1)!} i \, z^{2n+1}$$

$$= \exp(iz).$$

- $$\cos z = \frac{\exp(iz) + \exp(-iz)}{2} , \quad \sin z = \frac{\exp(iz) - \exp(-iz)}{2i} .$$

- $\cos^2 z + \sin^2 z = |\exp(iz)| = 1$

- $\exp(z) = \exp(x+iy) = e^x \cdot \exp(iy) = e^x \cdot (\cos y + i \sin y)$

(We could have defined exp this way.)

$\boxed{\text{Lecture 5}}$

III. INTEGRATION

For a function $f: S \to \mathbb{C}$ and $a,b \in S$, we'd like to define $\int_a^b f(z)\,dz$. However, in contrary to the real case, there are many "ways" from a to b (in S).

First let $f: [a,b] \to \mathbb{C}$ be continuous, write $f = u + iv$. We define

$$\int_a^b f(t)\,dt = \int_a^b u\,dt + i\int_a^b v\,dt .$$

<u>Definition</u>: let $z(t) = x(t) + iy(t)$ defined on $[a,b]$; z determines a "curve". If there is a partition $a = a_0 < a_1 < \cdots < a_{n-1} = a_n = b$ s.t both x and y are continuous on $[a,b]$, and continuously differentiable on each $[a_i, a_{i+1}]$ $(i = 0, \ldots, n-1)$, then we say z is <u>piecewise differentiable</u> (on $[a,b]$). In that case we write $\dot{z}(t)$ or $z'(t)$ for $x'(t) + iy'(t)$.

. If $\dot{z}(t) \neq 0$ except finitely many t, then z is called <u>smooth</u>.

. let $z_1 : [a,b] \to \mathbb{C}$, $z_2 : [a,b] \to \mathbb{C}$ be smooth curves. We say z_1 and z_2 are <u>smoothly equivalent</u> if there is a continuously differentiable injective $\lambda : [c,d] \to [a,b]$ such that $\lambda(c) = a$, $\lambda(d) = b$, $\lambda'(t) > 0$ for all t and $z_2(t) = z_1(\lambda(t))$.

This gives an equivalence relation. We'll call a class of this equivalence relation as a <u>smooth curve</u>. (or simply a curve.)

<u>Definition</u>: let $z: [a,b] \to \mathbb{C}$ be smooth and let $f: S \to \mathbb{C}$ be continuous at each $z(t)$ $(t \in [a,b])$. Then we define

$$\int_z f := \int_a^b f(z(t))\,\dot{z}(t)\,dt .$$

Proposition: let C be a smooth curve with representatives $z_1 : [a,b] \to \mathbb{C}$ and $z_2 : [a,b] \to \mathbb{C}$, and f cont. at each $z_1(t)$. Then

$$\int_{z_1} f = \int_{z_2} f .$$

Proof: Write $f = u + iv$; and $\lambda : [c,d] \to [a,b]$ be a smooth. eq. of z_1 & z_2. $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$$\int_{z_1} f = \int_a^b f(z(t))\, \dot{z}_1(t)\, dt = \int_a^b \left(u(z_1(t)) + i v(z_1(t)) \right) \left(x_1'(t) + i y_1'(t) \right) dt$$

$$= \int_a^b \left(u(z_1(t))\, x_1'(t) - v(z_1(t))\, y_1'(t) \right) dt + i \int_a^b \left(u(z_1(t))\, y_1'(t) + v(z_1(t))\, x_1'(t) \right) dt$$

and

$$\int_{z_2} f = \cdots = \int_c^d \left(u(z_1(\lambda(t)))\, x_1'(t) - v(z_1(\lambda(t)))\, y_1'(t) \right) \lambda'(t)\, dt + i \cdots$$

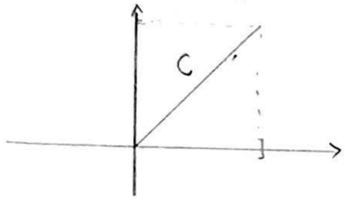
Since $\lambda(c) = a$ & $\lambda(d) = b$, this is just substituting $\lambda(t)$ in the place of t (in a real integral.) So they're equal. ☒

So we may define $\int_C f(z)\, dz$ for a smooth curve C and $f : S \to \mathbb{C}$ continuous at each point of C.

Given C with representative $z : [a,b] \to \mathbb{C}$, we may define $-C$ to be the curve given by
$$z_2 : [a,b] \to \mathbb{C}$$
$$t \longmapsto z(b+a-t)$$

Proposition: $\int_{-C} f = - \int_C f$. Proof: HW.

Examples: 1) $f(x+iy) = x^2 + iy^2$ $\quad C: z(t) = t+it \quad , t \in [0,1]$.



$$\int_C f(z)\,dz = \int_0^1 f(t+it)\,(1+i)\,dt$$

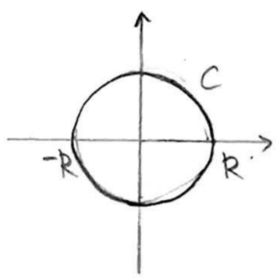
$$= \int_0^1 (t^2+it^2)(1+i)\,dt$$

We took out the scaler. We haven't proven this, but of course it is correct.

$$= (1+i)^2 \int_0^1 t^2\,dt$$

$$= (1+2i-1)\ \frac{1}{3}t^3 \Big|_0^1 = \frac{2i}{3}.$$

2) $f(z) = \frac{1}{z}$, $\quad C: z(t) = R\cos t + iR\sin t \quad , t \in [0, 2\pi]$.



$$\int_C f(z)\,dz = \int_0^{2\pi} f(R\cos t + iR\sin t)\,(-R\sin t + iR\cos t)\,dt$$

$$= i\int_0^{2\pi} \frac{1}{\cancel{R\cos t + iR\sin t}}\,(\cancel{R\cos t + iR\sin t})\,dt$$

$$= i\int_0^{2\pi} dt = 2\pi i.$$

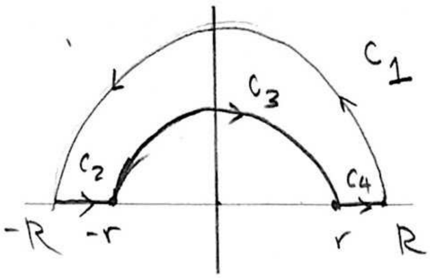
3) $f(z) = \frac{1}{z}$. Consider C as the "union" of the following four curves: $\quad (0 < r < R)$

$C_1: z_1(t) = R\cos t + iR\sin t \qquad t \in [0, \pi]$

$C_2: z_2(t) = t \qquad t \in [-R, -r]$

$C_3: z_3(t) = -r\cos t + ir\sin t \qquad t \in [0, \pi]$

$C_4: z_4(t) = t \qquad t \in [r, R]$



$$\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \int_{C_4} f(z)\,dz$$

$$= i\int_0^{2\pi} dt + \int_{-R}^{-r} \frac{1}{t}\,dt + (-i)\int_0^{2\pi} dt + \int_r^R \frac{1}{t}\,dt$$

$$= 0.$$

In examples 2 and 3, we integrate the same function on two diffcent "closed" curves and obtained two diffrent results. This seems to be against the fundamental theorem of calculus, but it's not. It's just that the statement needs to be a little bit more complicated.

Let's investigate taking the union of finitely many curves as in example 3. Say we have two smooth curves $C_1 : z_1(t)$ $t \in [a,b]$ and $C_2 : z_2(s)$, $s \in [c,d]$ with $z_1(b) = z_2(c)$; so they are back-to-back. Let's assume also that they don't intersect: $z_1(t) \neq z_2(s)$ for any $t \in (a,b)$ and $s \in (c,d)$. (This is not very necessary, we may allow them to intersect finitely many times.) Consider $z_2'(s) = z_2(s+c-b)$ for $s \in [b, b+d-c]$. It's clear that z_2 & z_2' are smoothly equivalent. So we may take z_2' to be representative of C_2. Now consider the curve

$$C_1 + C_2 : z(t) = \begin{cases} z_1(t) : t \in [a,b] \\ z_2'(t) : t \in [b, b+d-c] \end{cases} \quad (\text{defined on } [a, b+d-c])$$

Now this is again a smooth curve, with our definition above.

So we may define $\int_{C_1+C_2} f(z)\,dz$ as long as f is cont. on a region containing C_1 and C_2.

Proposition: let C_1 & C_2 be smooth curves that are back-to-back. Suppose that f is a continuous function defined on a region containing C_1 and C_2. Then $\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$.

Proof: HW.

$\boxed{\text{Lecture 6}}$

Lemma: $G : [a,b] \to \mathbb{C}$ continuous. Then $\left| \int_a^b G(t)\,dt \right| \leq \int_a^b |G(t)|\,dt$.

Proof: let $\int_a^b G(t)\,dt = R \cdot e^{i\theta}$ $R \geq 0$, $\theta \in [0, 2\pi)$.

So $R = \left| \int_a^b G(t)\,dt \right|$ and $\int_a^b e^{-i\theta} G(t)\,dt = R$.

Write $e^{-i\theta} G(t) = A(t) + i B(t)$. So $\int_a^b B(t)\,dt = 0$. (why?)

Then $R = \int_a^b e^{-i\theta} G(t)\,dt = \int_a^b A(t)\,dt = \int_a^b \operatorname{Re}(e^{-i\theta} G(t))\,dt$

Since $\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$, we get $R \leq \int_a^b |e^{-i\theta} G(t)|\,dt = \int_a^b |G(t)|\,dt$. ∎

Definition: let C be a smooth curve; let $z(t)$ $t \in [a,b]$ a representative of C. Then the length of C is

$$L(C) := \int_a^b |\dot{z}(t)|\,dt.$$

(This is a positive real number as long as C is smooth)

Theorem (ML-formula) Suppose $f : S \to \mathbb{C}$ is cont., $C \subseteq S$ a smooth curve and $|f(z)| \leq M$ for every $z \in C$. Then $\left|\int_C f(z)\,dz\right| \leq M \cdot L(C)$.

Proof: Let $C : z(t)$, $t \in [a,b]$.

Then
$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\,\dot{z}(t)\,dt\right| \leq \int_a^b |f(z(t))\,\dot{z}(t)|\,dt$$

By MVT (integral version):
$$\left|\int_C f(z)\,dz\right| \leq \underbrace{\max_{z\in C}|f(z)|}_{\leq M} \underbrace{\int_a^b |\dot{z}(t)|\,dt}_{= L(C)}$$
$\blacksquare$

Proposition: Suppose $(f_n)_n$ is a sequence of continuous functions converging uniformly to f on C. Then $\int_C f_n(z)\,dz \to \int_C f(z)\,dz$.

Proof:
$$\left|\int_C (f(z) - f_n(z))\,dz\right| \leq \varepsilon \cdot L(C)$$
(Given $\varepsilon > 0$, there is $N > 0$ such that this happens for $n \geq N$.) $\blacksquare$

(Fundamental Th.)

Proposition: $f : S \to \mathbb{C}$ cont., $C \subseteq S$ smooth, $F : S \to \mathbb{C}$ analytic. Suppose $F'(z) = f(z)$ on S. Then $\int_C f(z)\,dz = F(z(b)) - F(z(a))$

Proof: We'll define a curve D by $z_1(t) = F(z(t))$ $t \in [a,b]$.

We claim that $\dot{z}_1(t) = f(z(t))\,\dot{z}(t)$ for almost all $t \in [a,b]$.

$$\dot{z}_1(t) = \lim_{h\to 0} \frac{F(z(t+h)) - F(z(t))}{h} = \lim_{h\to 0} \frac{F(z(t+h)) - F(z(t))}{z(t+h) - z(t)} \cdot \frac{z(t+h) - z(t)}{h}$$

(Since $\dot{z}(t) \neq 0$ (except at fin. many t), we get $\delta > 0$ so that $|h| < \delta$ implies $z(t+h) - z(t) \neq 0$.)

So $\dot{z}_1(t) = f(z(t)) \cdot \dot{z}(t)$ using continuity of f and z.

Then $\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt = \int_a^b \dot{z}_1(t)\,dt = z_1(b) - z_1(a)$

$= F(z(b)) - F(z(a))$ ∎

Example: let $f(z)$ be a polynomial function, say $f(z) = z^2 + z - 1$.
Then $F(z)$ can be taken to be $\frac{1}{3} z^3 + \frac{1}{2} z^2 - z$.
Say $C : z(t) = t^2 + it \quad t \in [0,1]$.
We know that
$$\int_C f(z)\,dz = F(z(1)) - F(z(0)) = F(1+i) - F(0)$$
$$= \frac{1}{3}(1+i)^3 + \frac{1}{2}(1+i)^2 - (1+i)$$
$$= \frac{-2+2i}{3} + \frac{\cancel{2}i}{\cancel{2}} - 1 - \cancel{i} = \frac{-5+2i}{3}.$$

Let's check this using the definition of $\int_C f(z)$.

$$\int_C f(z)\,dz = \int_0^1 \left((t^2+it)^2 + (t^2+it) - 1 \right)(2t+i)\,dt$$

$$= \int_0^1 \left(t^4 + i2t^3 - \cancel{t^2} + \cancel{t^2} + it - 1 \right)(2t+i)\,dt$$

$$= \int_0^1 \left((t^4-1) + i(2t^3+t) \right)(2t+i)\,dt$$

$$= \int_0^1 (2t^5 - 2t - 2t^3 - t)\,dt + i \int_0^1 (t^4 - 1 + 4t^4 + 2t^2)\,dt$$

$$= \frac{1}{3}t^6 - \frac{1}{2}t^4 - \frac{3}{2}t^2 \Big|_0^1 + i\left(t^5 + \frac{2}{3}t^3 - t \Big|_0^1 \right) = -\frac{5}{3} + i\frac{2}{3}.$$ (Same as above.)

In general, we may consider $f(z) = \sum_{k=0}^{\infty} a_k z^k$ on $D(0,R)$. Then we may define $F(z) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1}$. Note that $\lim_{k\to\infty} \frac{\left| \frac{a_k}{k+1} z^{k+1} \right|}{|a_k z^k|} = \lim_{k\to\infty} \frac{|z|}{k+1} = 0$ for any $z \in \mathbb{C}$. So $F(z)$ converges whenever $f(z)$ does. Therefore, for any smooth curve C inside $D(0,R)$, we have

$\int_C f(z) = F(z(b)) - F(z(a))$ for any representation $z:[a,b] \to \mathbb{C}$ of C.

$$\int_C f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1}.$$

We have the following converse of the Fundamental Theorem:

<u>Proposition</u>: $f: S \to \mathbb{C}$ continuous, S region. Suppose that for any smooth curve $C \subseteq S$, the integral $\int_C f(z)\,dz$ depends only on the end points of C. Then f has an antiderivative in S.

<u>Proof</u>: Fix $z_0 \in S$ and for $z_1 \in S$ define $F(z_1) = \int_C f(z)\,dz$ where C is any curve connecting z_0 to z_1 (we use path connectivity of S).

We claim that $F'(z_1) = f(z_1)$ for any $z_1 \in D$.

Suppose $\varepsilon > 0$ is such that $D(z_1, \varepsilon) \subseteq S$. Given $|h| < \varepsilon$, let $w: [0,1] \to \mathbb{C}$ be the straight line from z_1 to $z_1 + h$: $w(t) = z_1 + th$.

Now $F(z_1+h) = \int_C f + \int_w f$ and $\frac{F(z_1+h) - F(z_1)}{h} - f(z_1) = \frac{\int_C f + \int_w f - \int_C f}{h} - f(z_1)$

Also $f(z_1) = h\,\frac{f(z_1)}{h} = (z_1 + h - z_1)\,\frac{f(z_1)}{h} \underset{\text{(why?)}}{=} \frac{f(z_1)}{h} \int_w dz = \int_w \frac{f(z_1)}{h}\,dz$

So $\dfrac{F(z_1+h) - F(z_1)}{h} - f(z_1) = \dfrac{\int_w f(z)\,dz - \int f(z_1)\,dz}{h} = \int_w \dfrac{f(z) - f(z_1)}{h}\,dz.$

Given $\varepsilon > 0$, we may choose $\delta > 0$ s.t for $|h| < \delta$ we have

$$\left| \frac{f(z) - f(z_1)}{h} \right| < \frac{\varepsilon}{|h|} \quad \text{for } z = w(t) \qquad (w \text{ depends on } h.)$$

So by ML formula $\left| \int_w \dfrac{f(z) - f(z_1)}{h}\,dz \right| \leq \dfrac{\varepsilon}{|h|} \cdot L(w) = \varepsilon.$

$(L(w) = |h|.)$

So $\left| \dfrac{F(z_1+h) - F(z_1)}{h} - f(z_1) \right| < \varepsilon$; and hence $F'(z_1) = f(z_1)$. ∎

Example: $f(z) = |z|$ defined on $S = \mathbb{C}$. But note that f is not continuous around 0. Hence it cannot have an antiderivative.

So $\int_C f$ should depend on C and not only the end pts.

Let $C_1 : z_1 : [0,1] \longrightarrow \mathbb{C}$, $t \longmapsto it$, $C_2 : z_2 : [0,1] \to \mathbb{C}$, $t \mapsto t$, and $C_3 : z_3 : [0, \frac{\pi}{2}] \to \mathbb{C}$, $t \mapsto e^{it}$.

Note that $C_2 + C_3$ starts from 0 and ends at i, just as C_1 does.

$$\int_{C_1} f = \int_0^1 |it|\, i\, dt = i \int_0^1 t\, dt = i \left. \frac{t^2}{2} \right|_0^1 = \frac{i}{2}.$$

$$\int_{C_2} f = \int_0^1 |t|\, dt = \left. \frac{t^2}{2} \right|_0^1 = \frac{1}{2}, \quad \int_{C_3} |e^{it}|\, i\, e^{it}\, dt = \left. ie^{it} \right|_0^{\pi/2} = i - 1.$$

So $\displaystyle\int_{C_2 + C_3} f = -\frac{1}{2} + i \neq \frac{i}{2}.$

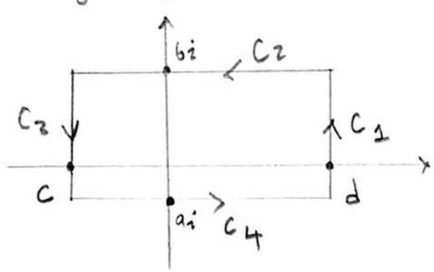
Definition: A curve C is called <u>closed</u> if it ends where it starts. More precisely if $z:[a,b] \to \mathbb{C}$ is a representative, then $z(b)=z(a)$.
A curve C is <u>simple</u> if it has no self-intersection except possibly at the end points. More precisely $z(s) \neq z(t)$ for any $s \neq t \in (a,b)$.

So if C is a closed curve and f has an antiderivative, then $\int_C f = 0$.

Now let's when C is a rectangle: $C = C_1 + C_2 + C_3 + C_4$ where

$C_1: z_1(t) = d + it \quad t \in [a,b]$, $\quad C_2 = z_2(t) = c+d-t+bi, \ t \in [c,d]$

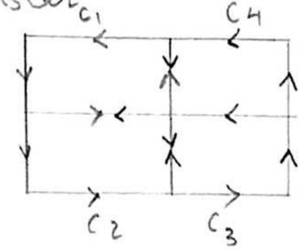
$C_3: z_3(t) = c + i(a+b-t) \quad, t \in [a,b]$, $\quad C_4: z_4(t) = t + ai \ , t \in [c,d]$.



<u>Theorem</u>: Let f be an entire function and C a rectangle.
Then $\int_C f = 0$. (Note that we don't know whether f has an antider.)

<u>Proof</u>: Let $I = \int_C f$. We'd like to show that $I = 0$.

Consider a subdivision of the interior of C into 4 parts:



So $I = \int_{C_1} f + \int_{C_2} f + \int_{C_3} f + \int_{C_4} f$

So for one $i \in \{1,2,3,4\}$, we have $\left| \int_{C_i} f \right| > \frac{|I|}{4}$.

Say $i=1$. Then we may subdivide C_1 into four rectangles, and (17)
again one of them have $\left|\int_{C^{(2)}} f\right| > \left|\int_{C^{(1)}} f\right| / 4 > I/4^2$.

Continuing this way, for each $i \in \mathbb{N}$ we have a rectangle $C^{(i)}$ with $\left|\int_{C^{(i)}} f\right| > I/4^i$. Let z_0 be in the "interior" of each $C^{(i)}$ (why is there such a z_0?)
(could be on a $C^{(i)}$.)

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \quad (f \text{ is analytic everywhere!})$$

Let's write $f(z) = f(z_0) + f'(z_0)(z - z_0) + \varepsilon(z) \cdot (z - z_0)$, where $\varepsilon(z) \to 0$ as $z \to z_0$.

$$\int_{C^{(i)}} f = \int_{C^{(i)}} \left(f(z_0) + f'(z_0)(z - z_0) + \varepsilon(z)\,(z - z_0) \right) dz$$

$$= \int_{C^{(i)}} \varepsilon(z)(z - z_0)\, dz$$ (This is because $f(z_0) + f'(z_0)(z - z_0)$ has an antiderivative.)

Let s be the length of the larger side of C, then

$$L(C^{(i)}) \leq \frac{4s}{2^i}.$$

Also for z on $C^{(i)}$ we have $|z - z_0| \leq \frac{\sqrt{2}\, s}{2^i}$.

Now $\varepsilon > 0$ be given. Take i_0 large enough that $|z - z_0| \leq \frac{\sqrt{2}\, s}{2^{i_0}}$ implies $|\varepsilon(z)| < \varepsilon$. Hence $|\varepsilon(z) \cdot (z - z_0)| < \varepsilon \cdot \frac{\sqrt{2}\, s}{2^{i_0}}$.

Then by ML-formula: $\left|\int_{C^{(i_0)}} f\right| < \varepsilon \cdot \frac{\sqrt{2}\, s}{2^{i_0}} \cdot \frac{4s}{2^{i_0}} = \varepsilon \cdot \frac{4\sqrt{2}\, s^2}{4^{i_0}}$

Then $|I| < \frac{\varepsilon \cdot 4\sqrt{2}\, s^2}{4^{i_0}} \cdot 4^{i_0} = \varepsilon \cdot 4\sqrt{2}\, s^2$ for arbitrary ε (no i_0!).

Therefore $I = 0$. ∎

Lecture 7

Actually, we do not need f to be entire for this theorem. If we go through the proof carefully, we see that all we need is that f is analytic on a region containing the rectangle C.

It follows that if f is entire, then f has an antiderivative F on the whole complex plane. Then for any curve C, the integral $\int_C f$ depends only on the end points of C. In particular $\int_C f = 0$ for a closed curve C.

<u>Theorem</u>: Let f be entire, and define $g(z) = \begin{cases} \frac{f(z) - f(a)}{z - a} & : z \neq a \\ f'(a) & : z = a. \end{cases}$

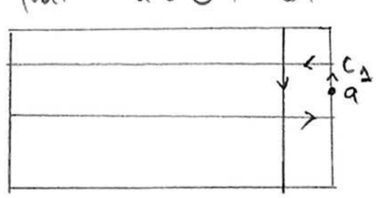
Then $\int_C g = 0$ for every curve C that bounds a rectangle.

<u>Proof</u>: First note that g is continuous everywhere. Let R be the rectangle that C bounds. So g is bounded on the compact set $R \cup C$, say $|g(z)| < M$ for every $z \in R \cup C$.

If $a \notin R \cup C$, then g is analytic on an open set containing $R \cup C$.

So $\int_C g = 0$.

Now suppose that $a \in C$. Let $\varepsilon > 0$. Subdivide R into 6 parts as follows:

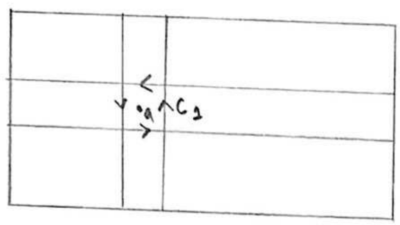


Also assume that $L(C_1) < \varepsilon / M$.

Then $\int_C g \underset{\text{why?}}{=} \int_{C_1} g$ & $\left|\int_{C_1} g\right| < M \frac{\varepsilon}{M} = \varepsilon$. By ML-formula

So $\left|\int_C g\right| < \varepsilon$ & hence $\int_C g = 0$.

Now let $a \in R$ & let $\varepsilon > 0$ be given. Subdivide R into the following 9 parts:



$\int_C g = \int_{C_1} g$ is clear

Assume that $L(C_1) < \frac{\varepsilon}{M}$

Then again $\left|\int_{C_1} g\right| < M \cdot \frac{\varepsilon}{M} = \varepsilon$, and

$\int_C g = 0$. ∎

It follows that g has an antiderivative and that $\int_C g = 0$ for any closed curve C.

Lemma: Let C be a circle centered at α and with radius R. Suppose $a \in D(\alpha, R)$. Then $\int_C \frac{dz}{z-a} = 2\pi i$.

Proof: $C : z(t) = \alpha + R \cdot e^{i\theta}$ $\quad \theta \in [0, 2\pi]$.

So $z'(t) = iRe^{i\theta}$ and $\frac{1}{z-\alpha} = \frac{1}{Re^{i\theta}}$. Then $\int_C \frac{dz}{z-\alpha} dz = 2\pi i$.
(not a!)

Also $\int_C \frac{dz}{(z-\alpha)^{k+1}} = 0$ for each $k > 0$. (Fundamental th.)

Now write $\frac{1}{z-a} = \frac{1}{(z-\alpha)-(a-\alpha)} = \frac{1}{(z-\alpha)\left(1 - \frac{a-\alpha}{z-\alpha}\right)} = \frac{1}{(z-\alpha)(1-w)}$ where $w = \frac{a-\alpha}{z-\alpha}$.

Note that $|w| = \frac{|a-\alpha|}{R}$ doesn't depend on z, for $z \in C$; and $|w| < 1$.

Write $\dfrac{1}{1-w} = \sum_{k=0}^{\infty} w^k$ (geometric series.)

So $$\frac{1}{z-a} = \frac{1}{z-\alpha} \cdot \sum_{k=0}^{\infty} w^k = \sum_{k=0}^{\infty} \frac{w^k}{z-\alpha} = \sum_{k=0}^{\infty} \frac{(a-\alpha)^k}{(z-\alpha)^{k+1}}.$$

$$\int_C \frac{dz}{z-a} = \sum_{k=0}^{\infty} \int_C \frac{(a-\alpha)^k}{z-\alpha}\, dz = \int_C \frac{dz}{z-\alpha} + \sum (a-\alpha)^k \int_C \frac{dz}{(z-\alpha)^{k+1}}$$

[convergence is uniform]

$$= 2\pi i + 0 = 2\pi i. \quad \blacksquare$$

<u>Theorem</u> (Cauchy Integral Formula): Let f be an entire function; $a \in \mathbb{C}$ and C is a circle centered at 0 and with radius $R > |a|$. Then $2\pi i \displaystyle\int_C \frac{f(z)}{z-a} = f(a)$.

<u>Proof</u>: We know that $\displaystyle\int_C \frac{f(z)-f(a)}{z-a}\, dz = 0$

So $f(a) \underbrace{\displaystyle\int_C \frac{dz}{z-a}}_{= 2\pi i} = \displaystyle\int_C \frac{f(z)}{z-a}\, dz$. Then $f(a) = \dfrac{1}{2\pi i} \displaystyle\int \frac{f(z)}{z-a}\, dz$. $\blacksquare$

by the lemma above

Now we're ready for Taylor expansion of entire functions.

<u>Theorem</u>: Let f be an entire function. Then $f^{(k)}(0)$ exists for $k > 0$ and $f(z) = \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$.

Proof: Given $a \in \mathbb{C} \setminus \{0\}$, we'll show that $f(z)$ is given by a power series on the closed disc $\overline{D}(0,|a|)$. Apriori this power series depends on a, however we know that if $g(z) = \sum_{k=0}^{\infty} c_k z^k$, then $c_k = \frac{g^{(k)}(0)}{k!}$ as long as g converges on $D(0,R)$ for some $R > 0$.

So given $a \in \mathbb{C} \setminus \{0\}$ finding $c_0, c_1, \ldots$ with $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for $z \in \overline{D}(0,|a|)$ is sufficient.

Let $R = |a| + 1$ and C the circle centered at 0 and with radius R; so $w \in C$ iff $|w| = R$. Below w always denotes an element of C, and $z \in \overline{D}(0,|a|)$.

Consider $\frac{1}{w-z} = \frac{1}{w(1-\frac{z}{w})} = \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}}$. This convergence is uniform in z.

So $$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw$$

$$= \frac{1}{2\pi i} \int_C \left(f(w) \sum_{k=0}^{\infty} \frac{z^k}{w^{k+1}} \right) dw$$

$$= \frac{1}{2\pi i} \sum_{k=0}^{\infty} z^k \int_C \frac{f(w)}{w^{k+1}}\, dw$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(w)}{w^{k+1}}\, dw \right) z^k$$

Let $c_k = \frac{1}{2\pi i} \int_C \frac{f(w)}{w^{k+1}}\, dw$. ▨

Corollary: If f is entire, then $f^{(k)}(z)$ exists for all $k>0$ and $z \in \mathbb{C}$.

Corollary: f entire, $a \in \mathbb{C}$. Then $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k$ for all $z \in \mathbb{C}$.

Proof: Consider $g(w) := f(w+a)$ .... ∎

Corollary: f entire, $g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & : z \neq a \\ f'(a) & : z = a \end{cases}$. Then g is entire as well.

Proof: $f(z) = f(a) + f'(a)(z-a) + \cdots$

$$f(z) - f(a) = (z-a)\left(f'(a) + f''(a)(z-a) + \cdots\right)$$

If $z \neq a$, then

$$g(z) = \frac{f(z)-f(a)}{z-a} = f'(a) + f''(a)(z-a) + \cdots$$

But if $z = a$, then the right hand side again converges to $f'(a) = g(a)$

So g is given by a converging power series and hence is entire. ∎

Corollary: f entire with finitely many zeros at $a_1, \ldots, a_N \in \mathbb{C}$. Define

$$g(z) = \frac{f(z)}{(z-a_1)\cdots(z-a_N)}$$ for $z \notin \{a_1, \ldots, a_N\}$, and suppose $\lim_{z \to a_i} g(z)$ exists

for each a_i. Then g is entire.

Proof: let $f_0(z) = f(z)$ and $f_i(z) = \frac{f_{i-1}(z) - f_i(z)}{z - a_i}$ for $i \geq 1$.

By induction, each f_i is entire, and $f_N(z) = g(z)$. ∎
for each i

$\boxed{\text{Lecture 8}}$

Theorem (Liouville): A bounded entire function is constant.

Proof: let $a, b \in \mathbb{C}$ and $R > 0$ s.t. $a, b \in D(0,R)$ and let C be $|w| = R$.
By Cauchy integral formula $f(b) - f(a) = \frac{1}{2\pi i}\left(\int_C \frac{f(z)}{z-b}\,dz - \int_C \frac{f(z)}{z-a}\,dz\right)$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)(b-a)}{(z-a)(z-b)} \, dz .$$

$$\left| \frac{f(z)\,(b-a)}{(z-a)\,(z-b)} \right| \leq \frac{M \cdot |b-a|}{(R-|a|)(R-|b|)}$$; where M is a bound on f.

Then $$|f(b) - f(a)| \leq \left| \frac{1}{2\pi i} \; \frac{M\,|b-a|}{(R-|a|)(R-|b|)} \; 2\pi R \right| \longrightarrow 0$$ as $R \to \infty$.

So $|f(b) - f(a)| = 0$ and $f(b) = f(a)$. ▨

Corollary: f entire, $k \geq 0$, $A, B \in \mathbb{R}^{>0}$ such that $|f(z)| \leq A + B|z|^k$ for all z. Then f is a polynomial of degree at most k.

Proof: We proceed by induction on k. Liouville theorem is $k=0$.

let $$g(z) = \begin{cases} \frac{f(z) - f(0)}{z} & z \neq 0 \\ f'(0) & : z = 0 \end{cases}$$

Then g is entire and $|g(z)| \leq C + D|z|^{k-1}$ for some $C, D \in \mathbb{R}^{>0}$. Then by induction g is a polynomial of degree at most $k-1$. Then f is a pol. of degree at most k. ▨

Theorem (Fundamental Theorem of Algebra): let $P(x) \in \mathbb{C}[x]$ be non-constant. Then $P(a) = 0$ for some $a \in \mathbb{C}$.

Proof: Suppose $P(z) \neq 0$ for any $z \in \mathbb{C}$. Then $f(z) = \frac{1}{P(z)}$ is an entire function. Since $P \not\equiv$ constant, we have $|P(z)| \to \infty$ as $|z| \to \infty$. So f is bounded. Then f is constant and hence P needs to be constant. ⨳ ▨

Theorem (Gauss-Lucas): P a polynomial with zeros at $a_1, \ldots, a_n$. Then the zeros of P' are in the convex hull of $a_1, \ldots, a_n$.

Proof: HW.

Now we start to study functions that are analytic in a region that is not necessarily the whole $\mathbb{C}$. We'll see that the "shape" of the region is important. We start with discs. Many things follow from earlier work.

Theorem: f analytic on $D = D(\alpha, r)$; R a rectangle contained in D with boundary curve C, and $a \in D$. Then $\int_C f = \int_C \frac{f(z) - f(a)}{z-a} dz = 0$.

Proof: Done before. ☒

Theorem: f analytic on $D = D(\alpha, r)$, $a \in D$. Then both $f(z)$ and $g(z) = \frac{f(z) - f(a)}{z-a}$ have anti-derivatives on D.

Proof: Follows from the previous theorem and the converse of the fundamental theorem. ☒

Theorem: f analytic on $D = D(\alpha, r)$, C curve contained in D, $a \in D$. Then $\int_C f \, dz = \int_C \frac{f(z) - f(a)}{z-a} dz = 0$.

Proof: Done earlier. ☒

Theorem (Cauchy Integral Formula): f analytic on $D = D(\alpha, R)$, $r < R$, $a \in D(\alpha, r)$. Then $f(a) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z-a} dz$ (C_r = circle cent. at α & radius r.)

Proof: Same as before. ☒

<u>Theorem</u>: f analytic on $D = D(\alpha, R)$. Then there are $C_0, C_1, \ldots$ such that $f(z) = \sum_{k=0}^{\infty} C_k z^k$ for all $z \in D$.

<u>Proof</u>: Again, this is very similar to the proof when f is entire let's go through it once again.

let $a \in D$ and take $r > 0$ with $a \in D(\alpha, r)$.

By Cauchy int. formula $f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w-z}\, dw$ for $z \in D(\alpha, |a\alpha|$

Then again $\frac{1}{w-z} = \sum_{k=0}^{\infty} \frac{(z-\alpha)^k}{(w-\alpha)^{k+1}}$ (uniformly in z.)

So $f(z) = \cdots = \sum_{k=0}^{\infty} C_k (z-\alpha)^k$ where $C_k = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{(w-\alpha)^{k+1}}\, dw$.

As before $C_k = \frac{f^{(k)}(\alpha)}{k!}$ on $D(\alpha, r) \ldots$ ▨

Now we start to work with on arbitrary open sets rather than the whole $\mathbb{C}$ or a disc.

<u>Theorem</u>: $f: D \to \mathbb{C}$ analytic, D open, $\alpha \in D$. Then there are $C_0, C_1, \ldots$ such that $f(z) = \sum_{k=0}^{\infty} C_k (z-\alpha)^k$ for $z \in D(\alpha, R)$ where $R > 0$ is such that $D(\alpha, R) \subseteq D$.

<u>Proof</u>: f is still analytic on $D(\alpha, R)$, as it's contained in D. ▨

<u>Example</u>: $f: \mathbb{C} \setminus \{1\} \to \mathbb{C}$, $f(z) = \frac{1}{z-1}$, let $\alpha = 2$.

$\frac{1}{z-1} = \frac{1}{1+(z-2)} = \sum_{k=0}^{\infty} (-1)^k (z-2)^k$ on $D(2,1)$. Not that this series diverges outside of $D(2,1)$

Example: $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = \frac{1}{z^2}$, $\alpha = 3$.

$$\frac{1}{z^2} = \left(\frac{1}{3 + (z-3)} \right)^2 = \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{9} \frac{k+1}{3^k} (z-3)^k .$$

$$R = \cdots = 3 .$$

So this series conv. in $D(3,3)$ and diverges outside.

Proposition: f analytic at α, $g(z) = \begin{cases} \frac{f(z) - f(\alpha)}{z - \alpha} & : z \neq \alpha \\ f'(\alpha) & : z = \alpha \end{cases}$

Then g is also analytic at α.

Proof: $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!} (z-\alpha)^k$ for $z \in D(\alpha, r)$, for some $r > 0$.

$$= f(\alpha) + f'(\alpha)(z-\alpha) + \cdots$$

$$f(z) - f(\alpha) = (z-\alpha) \left(f'(\alpha) + \frac{f''(\alpha)}{2} (z-\alpha) + \cdots \right)$$

So $g(z) = \sum_{k=0}^{\infty} \frac{f^{(k+1)}(\alpha)}{(k+1)!} (z-\alpha)^k$; on $D(\alpha, r)$ (even at $z = \alpha$).

So g is analytic at α. ∎

Proposition: If f is analytic at α, then f is infinitely diff. at α.

Proof: ....

Theorem: $f : S \to \mathbb{C}$ analytic, S region. Suppose that there is a sequence (in S) $(z_n)_n$ of zeros of f with $z_n \neq z_m$ for all $m \neq n$ and $z_0 = \lim_{n \to \infty} z_n \in S$. Then $f \equiv 0$ on D.

Proof: f has a power series expansion on $D(z_0, R)$ for some r and we've seen before that that power series is 0 on $D(z_0, R)$. So $f \equiv 0$ on $D(z_0, R)$.

Let $U = \{ z \in D : z$ is a limit pt of distinct zeros of $f \}$

$V = D \setminus U$

Clearly, U is open and $z_0 \in U$.
We claim that V is also open. Let $z \in V$, then there are $r>0$ $w \in D(z, r)$ with $f(w) \neq 0$, otherwise $z \in U$. But then $D(z, r) \subseteq V$.
Now $D = U \cup V$, $U \cap V = \emptyset$, and both U & V are open. Then either $U = \emptyset$ or $V = \emptyset$. But $U \neq \emptyset$, so $V = \emptyset$. Therefore $f \equiv 0$. ∎

$\boxed{\text{Lecture 9}}$

Corollary: f, g analytic on a region S. Suppose that there is a sequence (z_n) of distinct pts of S with $\lim_{n \to \infty} z_n \in S$, and $f(z_n) = g(z_n)$ for all z. Then $f = g$.

We may have that zeros of an analytic function accumulate to a point. However, f can't be analytic at that point. For instance $\sin\left(\frac{1}{z}\right)$.

Theorem: f entire and $\lim_{z \to \infty} f(z) = \infty$. Then f is a polynomial.

Proof: There is $R > 0$ such that $|f(z)| > 1$ for $z \notin D(0, R)$. So all possible zeros of f are in $D(0, R)$. So there are only finitely many of them; say $a_1, \ldots, a_N$. Then $g(z) = \frac{f(z)}{(z-a_1)\cdots(z-a_N)}$ is also entire and $g(z) \neq 0$ for any $z \in \mathbb{C}$.

So let $h(z) = \frac{1}{g(z)} = \frac{(z-a_1)\cdots(z-a_N)}{f(z)}$. It is also entire.

Since $\lim_{z\to\infty} f(z) = \infty$ we have $|h(z)| \leq A + |z|^N$ for some A and all $z \in \mathbb{C}$. Then by Liouville theorem $h(z)$ is a polynomial of degree at most N. But h has no zeros; so $h \equiv c \neq 0$

Then $f(z) = \frac{1}{c}(z-a_1)\cdots(z-a_N)$. ∎

<u>Question</u>: How about $f(z) = e^z$? Why isn't it against this theorem?

We may consider Cauchy Integral Formula as a Mean Value Theorem: $f(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\alpha}\,dz$ for any circle C centred at α. So C is given by $z(t) = \alpha + re^{it}$ $t \in [0, 2\pi]$

Then $f(\alpha) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\alpha+e^{it})}{\alpha+re^{it}-\alpha}\, ire^{it}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha+re^{it})\,dt$.

We "sum up" all values of f on a circle and divide by the length of $[0, 2\pi]$.

<u>Definition</u>: $f: D \to \mathbb{C}$ a function, $z_0 \in D$. z_0 is called a <u>relative/local maximum</u> if $|f(z)| \geq |f(w)|$ for all $w \in D(z_0, R)$ $(R>0)$.
Similarly we define z_0 to be a <u>relative/local minimum</u> if $|f(z)| \leq |f(w)|$ for all $w \in D(z_0, R)$.

<u>Theorem</u> (Maximum Modulus Principle): $f: S \to \mathbb{C}$ analytic, non-constant, S region.
Then for every $z \in S$ and $\delta > 0$, there is $w \in S \cap D(z, \delta)$ with $|f(w)| > |f(z)|$. (f has no relative maximum in S.)

Proof: $f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it})\, dt$ for any $r > 0$ with $D(z, r) \subseteq S$.

So $|f(z)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} |f(z + re^{it})|\, dt \leqslant \max\{|f(z+re^{it})| : t \in [0, 2\pi]\}$

So there is $w \in C$ with $|f(w)| \geq |f(z)|$. If this was equality for all $w \in C$ with $r \to 0$, then f would be constant in $D(z, r)$, hence on S. ◻

Corollary: f analytic on a bounded region S, and continuous on $\overline{S}$. Then f has a maximum on the boundary $\overline{S} \setminus S$.

Corollary (Minimum Modulus Principle) f analytic on a region S. If $f(z) \neq 0$ for $z \in S$, then there is $w \in D(z, r)$ with $|f(w)| < |f(z)|$.

Theorem: f analytic & non-constant on a closed disc D (so f is analytic on an open $U \supseteq D$). Suppose that $z_0 \in D$ is maximum of f on D. Then $f'(z_0) \neq 0$.

Proof: let's consider the power series expansion of f around z_0:

$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} z^k$. So for ξ around 0, we have

$$f(z_0 + \xi) = f(z_0) + \frac{f^{(k)}(z_0)}{k!} \xi^k + \frac{f^{(k+1)}(z_0)}{(k+1)!} \xi^{k+1} + \cdots$$

where $k > 1$ and $f^{(k)}(z_0) \neq 0$. Note that $f(z_0) = 0$ since z_0 is a max & f is not constant.

$$\overline{f(z_0 + \xi)} = \overline{f(z_0)} + \overline{\frac{f^{(k)}(z_0)}{k!}}\, \overline{\xi}^k + \cdots$$

So $\left|f(z_0+\xi)\right|^2 = \left|f(z_0)\right|^2 + f(z_0)\,\bar{\xi}^k \,\frac{\overline{f^{(k)}(z_0)}}{k!} + \overline{f(z_0)}\,\xi^k\,\frac{f^{(k)}(z_0)}{k!} + \dots$

Writing $\overline{f(z_0)}\, f^{(k)}(z_0) = Ae^{i\alpha}$ and $\xi = B\cdot e^{i\theta}$ with $A, B > 0$ and $\alpha, \theta \in [0, 2\pi)$ we have:

$$\underbrace{|f(z_0+\xi)|^2 - |f(z_0)|^2}_{=\ \underbrace{\left(|f(z_0+\xi)| + |f(z_0)|\right)}_{>0}\left(|f(z_0+\xi)| - |f(z_0)|\right)} = \underbrace{\frac{2A}{k!}}_{>0} B^k \cos(k\theta+\alpha) + \dots$$

For $|\xi|$ small enough the higher order terms do not contribute and the signs of $|f(z_0+\xi)| - |f(z_0)|$ and $\cos(k\theta+\alpha)$ are the same.

So $|f(z_0+\xi)| > |f(z_0)|$ for $\xi \in \bigcup_{i=0}^{k-1} \underbrace{\left\{re^{i\theta} : r\in(0,\varepsilon),\ \theta \in \left(\frac{-\pi+4\pi i-2\alpha}{2k}, \frac{\pi+4\pi i-2\alpha}{2k}\right)\right\}}_{U_i}$

(ε is fixed)

Since $k>1$, there is $i\in\{0,\dots,k-1\}$ s.t $D\cap z_0+U_i \neq \emptyset$. This finishes the proof, since z_0 can't be a max. of f. $\blacksquare$

$\boxed{\text{Lecture 10}}$

<u>Theorem</u> (Open Mapping Theorem) $f: S\to\mathbb{C}$ analytic, non-constant, $U\subseteq S$ open. Then $f(U)$ is also open.

<u>Proof</u>: let $\alpha\in U$. We'll find a disc $D\subseteq U$ around α and $\varepsilon>0$ s.t $D(f(\alpha),\varepsilon)\subseteq f(D)$. (Why does this finish the proof?)

We may assume $f(\alpha)=0$. Then there is a closed disc $\overline{D} = D\cup C$ around α s.t $f(z)\neq 0$ for all $z\in\overline{D}\setminus\{\alpha\}$. Let $\varepsilon' = \min\{|f(z)| : z\in C\}$

We claim that $D(0, \underbrace{\tfrac{\varepsilon'}{2}}_{=\varepsilon}) \subseteq f(D)$. So let $|w|<\varepsilon$.

For each $z \in C$ we have $|f(z)-w| \geq |f(z)| - |w| \geq 2\varepsilon - \varepsilon = \varepsilon$, 24
and $|f(\alpha) - w| = |-w| = |w| < \varepsilon$. Then $g(z) := f(z) - w$ gets a minimum on $z \in \overline{D}$. But then by Minimum modulus theorem, that minimum is 0. So there is $z_0 \in D$ with $f(z_0) - w = 0$.
So $w = f(z_0)$ for some $z_0 \in D$ and $w \in f(D)$. ∎

<u>Theorem</u> (Schwarz lemma) $D = D(0,1)$, f analytic on D, $f(0)=0$, and $f(D) \subseteq D$.
Then $|f(z)| \leq |z|$ for all $z \in D$ and $|f'(0)| \leq 1$. The equalities hold iff $f(z) = e^{i\theta} z$ for some fixed θ.

Proof: Define $g(z) = \begin{cases} \frac{f(z)}{z} & : z \neq 0 \\ f'(0) & : z = 0 \end{cases}$. Then g is also analytic on D.

Note that $|g(z)| \leq \frac{1}{r}$ for $z \in C_r$ (circle cent. at 0 and of rad. r)
So by Maximum Modulus Principle, $|g(z)| \leq 1$ for all z and hence
$|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.
Now suppose $|g(z_0)| = 1$ for some $z_0 \in D$. Then $|g|$ is constant by MMP, and hence $f(z) = z \cdot c$ for some c. But also $|c| = 1$.
Then $c = e^{i\theta}$ for some θ. ∎

What if $f(0) \neq 0$ in Schwarz lemma?
Given $\alpha \in D(0,1)$ consider the function $B_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z}$. It's analytic on a neighbourhood of $\overline{D} = \overline{D}(0,1)$; in particular on C (circle cent. at 0 and radius 1).
Note that $|B_\alpha(z)|^2 = \frac{z - \alpha}{1 - \bar{\alpha} z} \cdot \frac{\bar{z} - \bar{\alpha}}{1 - \alpha \bar{z}} = \frac{|z|^2 - \alpha\bar{z} - \bar{\alpha} z + |\alpha|^2}{1 - \alpha \bar{z} - \bar{\alpha} z + |\alpha|^2 |z|^2}$
So if $z \in C$, then $|B_\alpha(z)| = 1$.

Example: Suppose $f: D \to \bar{D}$ analytic, where $D = D(0,1)$, say $f(\frac{1}{2}) = 0$

Consider $g(z) = \begin{cases} f(z)/B_{1/2}(z) & : z \neq \frac{1}{2} \\ \frac{3}{4} f'(\frac{1}{2}) & : z = \frac{1}{2} \end{cases}$.

Note that $\dfrac{f(z)}{B_{\frac{1}{2}}(z)} = \dfrac{f(z)(1 - \frac{1}{2}z)}{z - \frac{1}{2}}$ & $\frac{1}{2}$ is a zero of f.

Then we know that g is analytic on D, and $|g(z)| \leq 1$.

This means $|f(z)| \leq |B_{1/2}(z)|$ for $z \in D$.

For instance, $|f(\frac{3}{4})| \leq |B_{1/2}(\frac{3}{4})| = \left| \dfrac{\frac{3}{4} - \frac{1}{2}}{1 - \frac{1}{2}\frac{3}{4}} \right| = \dfrac{1/4}{5/8} = \dfrac{2}{5}$.

This is a general strategy to approximate values of analytic functions at certain points.

Suppose $f(\frac{1}{2}) \neq 0$. This time consider $h(z) = \dfrac{f(z) - f(\frac{1}{2})}{1 - \overline{f(\frac{1}{2})} f(z)}$.

For $w \in \mathbb{C}$, we have $\left| \dfrac{w - f(\frac{1}{2})}{1 - \overline{f(\frac{1}{2})} w} \right| = \cdots = 1$ $(|f(z)| \leq 1)$.

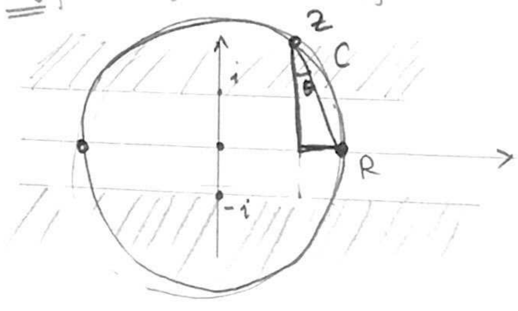
Then $h(z)$ is also bounded by 1 on D.

Now $h'(\frac{1}{2}) = \dfrac{f'(1/2)}{1 - |f(\frac{1}{2})|^2}$ & $|h'(\frac{1}{2})| > f'(\frac{1}{2})$.

So if we look at $\max \{ f'(\frac{1}{2}) : f: D \to \bar{D} \text{ analytic} \}$, then this max cannot be attained for f with $f(\frac{1}{2}) \neq 0$. (So we need $f(\frac{1}{2}) = 0$ to get this maximum. Actually, $B_{1/2}(z)$ gets that max.)

Proposition: Let f be entire with $f(z) \leq \frac{1}{|\operatorname{Im} z|}$ for all $z \neq 0$.
Then $f \equiv 0$.

Proof: Note that $|f(z)| \leq 1$ for $\operatorname{Im}(z) > 1$.



Let $g(z) = (z^2 - R^2) f(z)$.

For $z \in C$ with $\operatorname{Re} z \geq 0$, we have

$$|(z-R)f(z)| \leq |z-R| / \operatorname{Im}(z) = \sec\theta$$

where $\theta \in [0, \pi/4]$.

Then $|(z-R)f(z)| \leq \sqrt{2}$.

If $z \in C$ with $\operatorname{Re} z \leq 0$, then

$|(z+R)f(z)| \leq \sqrt{2}$.

Then $|g(z)| = |z+R||z-R||f(z)| \leq 3R$ for $z \in C$.

So by MMP, $|g(z)| \leq 3R$ for $z \in D(0,R)$.

But then $|f(z)| \leq \dfrac{3R}{|z^2 - R^2|}$ for $|z| \leq R$.

So $f(z) = 0$ by sending R to ∞. So $f \equiv 0$. ∎

— o —

Theorem (Morera's Th.) $f : S \to \mathbb{C}$ continuous, S open. Suppose that for every closed curve $C \subseteq S$ we have $\int_C f = 0$. Then f is analytic.

Proof: Let $a \in S$ and let $D \subseteq S$ be an open neighbourhood of a. Then f having an antiderivative is equivalent to the condition that $\int_C f = 0$ for every closed curve $C \subseteq D$. So f has an antiderivative F on D. But then F is analytic and $F'' = f'$ exists on D. So f is analytic ∎

Remark: It's enough to assume $\int_C f = 0$ for C bounding a rectangle in S.

Example: $f(z) = \int_0^\infty \frac{e^{zt}}{t+1}\,dt$. Let $S = \{z \in \mathbb{C} : Re(z) < 0\}$.

Let $z \in S$ & let $x = Re(z)$. Then $\int_0^\infty \frac{|e^{zt}|}{t+1}\,dt < \int_0^\infty e^{xt}\,dt = -\frac{1}{x}$.

$x < 0$

So $f(z)$ is really defined for $z \in S$.

Let C be the boundary of a rectangle $R \subseteq S$.

$$\int_C f = \int_C \left(\int_0^\infty \frac{e^{zt}}{t+1}\,dt \right) dz = \int_0^\infty \left(\int_C \frac{e^{zt}}{t+1}\,dz \right) dt$$

$$= \int_0^\infty \frac{1}{t+1} \underbrace{\int_C e^{zt}\,dz}_{=0} \, dt = 0.$$

(because e^{zt} has an antiderivative)

Hence f is analytic on S.

Definition: $D \subseteq \mathbb{C}$, $f_n : D \to \mathbb{C}$ for each $n > 0$, $f : D \to \mathbb{C}$. We say $(f_n)_n$ converges to f uniformly on compacta if $(f_n)_n$ converges to f uniformly on every compact $K \subseteq D$.

$\boxed{\text{Lecture 11}}$

Theorem: $D \subseteq \mathbb{C}$ open, f_n, f as above. Suppose $f_n \to f$ unif. on comps and that each f_n is analytic on D. Then f is also analytic.

Proof: f is clearly continuous (uniform convergence around a point is enough.)

Let C bound a rectangle in D.

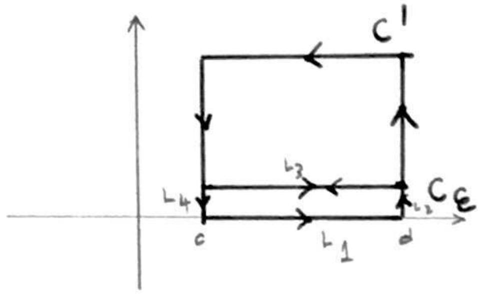
$$\int_C f = \int_C \lim_n f_n \underset{\text{uniformity}}{=} \lim_n \int_C f_n = \lim_n 0 = 0. \qquad \blacksquare$$

Theorem: D an open set, $L \subseteq D$ line segment. Suppose f is cont. on D and analytic on $D \setminus L$. Then f is analytic on D.

Proof: We may assume D is a disc and $L = [a,b] \subseteq \mathbb{R}$. (WHY?)

Let C be the boundary of a rectangle $R \subseteq D$. If $(R \cup C) \cap L = \emptyset$, then $\int_C f = 0$, since f is analytic on $D \setminus L$.

If $R \cap L = \emptyset$, but $C \cap L \neq \emptyset$, then L intersect a side of C infinitely. Then write $C = C' + C_\varepsilon$ as below:



$$\text{So } \int_C f = \int_{C'} f + \int_{C_\varepsilon} f \overset{f \text{ anal. on } D \setminus L}{=} 0 + \int_{C_\varepsilon} f$$

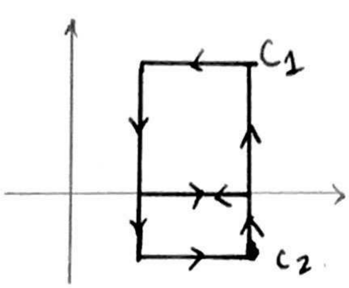
$$= \int_{L_1} f + \underbrace{\int_{L_2} f}_{\to 0} + \int_{L_3} f + \underbrace{\int_{L_4} f}_{\to 0} \quad (\text{as } \varepsilon \to 0)$$

L_1 is t for $t \in [c,d]$ and $-L_3$ is $t + i\varepsilon$ $t \in [c,d]$.

So $\int_{-L_3} f = \int_c^d f(t+i\varepsilon)\,dt$, and hence $\int_{L_3} f = -\int_c^d f(t+i\varepsilon)\,dt \xrightarrow{(\text{as } \varepsilon \to 0)} -\int_c^d f(t)\,dt = -\int_{L_1} f$.

Then $\int_C f = 0$.

Now assume $R \cap L = \emptyset$. Then $C = C_1 + C_2$ as below:



So $\int_C f = \int_{C_1} f + \int_{C_2} f = 0 + 0 = 0$.

So by Morera's Theorem, f is analytic ∎

Theorem (Schwarz Reflection Principle): Let D be a region contained in $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and suppose $\overline{D} \cap \mathbb{R} = L$ (a line segment). Let $D^* = \{z \in \mathbb{C} : \bar{z} \in D\}$ is the symmetric copy of D on the lower half plane. Let f be continuous on $\overline{D}$ and analytic on D & f is real on L. Then there is a function g analytic on $D \cup L \cup D^*$ with $g \restriction \overline{D} = f$.

Proof: Define $g(z) = \begin{cases} f(z) & : z \in \overline{D} \\ \overline{f(\bar{z})} & : z \in D^* \end{cases}$

Clearly g is analytic at each point of D.

Now let $z \in D^*$ and consider $\dfrac{g(z+h) - g(z)}{h}$.

For small enough h we have $z + h \in D^*$. So for such h we have

$$\frac{g(z+h) - g(z)}{h} = \frac{\overline{f(\bar{z}+\bar{h})} - \overline{f(\bar{z})}}{h} = \overline{\frac{f(\bar{z}+\bar{h}) - f(\bar{z})}{\bar{h}}}.$$

Now $\bar{z} + \bar{h} \in D$ and hence $\lim_{\bar{h} \to 0} \dfrac{f(\bar{z}+\bar{h}) - f(\bar{z})}{\bar{h}}$ exists, and g is analytic on D^*.

Then using the previous theorem g is analytic on $\overline{D} \cup D^*$. ∎

Corollary: Let D be a region that is symmetric with respect to the real axis. Suppose f is analytic on D and f is real on $D \cap \mathbb{R}$. Then $f(z) = \overline{f(\bar{z})}$.

Proof: Consider $D \cap \mathbb{H}$. Then conditions of Schwarz R.P. holds for this set & there is g on D defined as $\overline{f(\bar{z})}$ for $z \in D \setminus (D \cap \mathbb{H})$.

So $f(z) = \overline{f(\bar{z})}$ by uniqueness. ∎

IV. CAUCHY'S THEOREM

Definition: A region S is simply connected if the following happens:

(*) For every $z_0 \in \mathbb{C} \setminus S$ and $\varepsilon > 0$, there is a continuous curve $\gamma : [0,\infty) \to \mathbb{C}$ such that $\gamma(0) = z_0$, $\lim_{t\to\infty} \gamma(t) = \infty$, and for every $t \in [0,\infty)$ we have $d(\gamma(t), \mathbb{C}\setminus S) < \varepsilon$.

This definition is equivalent with the following condition:

(**) For any $z_0 \in \mathbb{C}\setminus S$ and closed curve $C \subseteq S$ we have $\int_C \frac{dz}{z-z_0} = 0$.

(We won't prove that (*) and (**) are equivalent, but we might assume it.)

For any $z_0 \in \mathbb{C}$ and closed C not passing through z_0 the integral

$\int_C \frac{dz}{z-z_0}$ is an integer multiple of $2\pi i$; so we call the number $\frac{1}{2\pi i}\int_C \frac{dz}{z-z_0}$ as the winding number of C around z_0. (We'll come back to this notion later.)

So (**) says that "no closed curve in S winds around any point of $\mathbb{C}\setminus S$".

Examples ① $S = D \setminus \{z_0\}$ where D is an open disc and $z_0 \in D$. S is a region. However, there is no (cont.) curve γ connecting z_0 to ∞ as in the definition.

② $S = D(0,2) \setminus \overline{D}(0,1) = \{ z \in \mathbb{C} : 1 < |z| < 2 \}$.
(This is an example of an annulus.)
Again S is connected, but not simply connected.

③ $S = D(0,1) \setminus \mathbb{R}_{\geq 0}$. Simply connected.

④ $S = \{z : -1 < \operatorname{Im} z < 1\}$ (infinite strip)

Simply connected ...

⑤ $T = $ graph of $\left(f : [0,1] \to \mathbb{R} ; f(x) = \sin\frac{1}{x}\right) \cup i\mathbb{R}_{\geq 0}$; and let $S = \mathbb{C} \setminus T$.

We won't prove it in detail, but S is simply connected. The difference of this example from previous ones is that we really need to go outside of T to connect i to infinity.

$\boxed{\text{Lecture 12}}$

Lemma: Let S be simply connected and C a simple closed polygonal path in S. Suppose that $y_1 \in \mathbb{R}$ is the max. such that $Z_1 = \{x + iy_1 : x \in \mathbb{R}\} \cap C$ is infinite, and that $y_2 \in \mathbb{R}$ is the second largest such real. Let $X_1 = \{x \in \mathbb{R} : x + iy_1 \in Z_1\}$. Then the set $R = \{x + iy : x \in X, y_2 \leq y \leq y_1\}$ is contained in S.

Proof: We'll show that for every $z_0 \in R$ and for any curve $\gamma : [0,\infty) \to \mathbb{C}$ with $\gamma(0) = z_0$, $\lim_{t\to\infty} \gamma(t) = \infty$, we have $\gamma \cap C \neq \emptyset$. This will show that $z_0 \in D$ (WHY? Distance of C and $\mathbb{C} \setminus S$ is positive.)

We prove this by induction on the number n of $y \in \mathbb{R}$ s.t. $\{x+iy : x \in \mathbb{R}\} \cap C$ is infinite. If there are two, then it's clear. (WHY?)

Let $n > 2$ and consider $X_2 = \{x \in \mathbb{R} : x + iy_2 \in C\}$ and $L = \{x + iy_2 : x \in X_1 \setminus X_2\}$.

If γ doesn't intersect $C \cap R$, then $\gamma \cap L \neq \emptyset$. So let $t_0 = \sup\{t : \gamma(t) \in R\}$

Let $C^* = (C \cap (\mathbb{C} \setminus R)) \cup \overline{L}$; it's a polygonal path in S, and by induction, there is $\gamma(t) \in C^*$. But $t > t_0$, and hence $\gamma(t) \notin R$.

So $\gamma \cap C \neq \emptyset$. ∎

Theorem (Cauchy's Theorem) Let f be an analytic function on a simply connected domain S. Then $\int_C f = 0$ for any closed curve $C \subseteq S$.

Proof: We'll show that $\int_C f = 0$ for closed simple polygonal path.
(This will finish the proof, because then $F(z) := \int_\Gamma f(w)\,dw$ where Γ is a polygonal path in S connecting a fixed z_0 to z will be well-defined and $F'(z) = f(z)$. (F is an antider. of f.))

We'll do this by induction on the number n of levels of C.

If $n=2$ then C is the boundary of the rectangle and that rectangle is in S. So $\int_C f = 0$; as shown before.

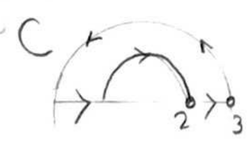
So let $n>2$, and let R, L, C^* be as in the lemma. Then

$\int_C f = \int_{\partial R} f + \int_{C^*} f$. Now $\int_{\partial R} f = 0$ as in the previous step,

and $\int_{C^*} f = 0$ by the induction hypothesis. Hence $\int_C f = 0$. ▨

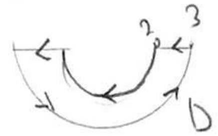
Let f be an analytic function on the annulus $S = \{z : 1 < |z| < 4\}$.
Consider C_2 and C_3 circles centered at 0 & of radiuses 2 and 3.
We claim that $\int_{C_2} f = \int_{C_3} f$ using Cauchy's Theorem.

Consider $S_1 := S \cap \{z : \mathrm{Im}(z) > -1\}$; this is simply connected. The curve C below is contained in S_1:



Then $\int_C f = 0$.

Similarly, let $S_2 = S \cap \{z : \mathrm{Im}(z) < 1\}$ & D the curve below.



Again $\int_D f = 0$. It's also clear that

$\int_C f + \int_D f = \int_{C_2} f + \int_{-C_2} f$.

Complex Logarithm

We'd like to define an analytic function f on S that is the (functional) inverse of exp. That is $\exp(f(z)) = z$.

First of all $\exp(w)$ never gets the value 0; so S can't contain 0.

The actual problem is that exp is not 1-1; so if f is an inverse then so is $g(z) = f(z) + 2\pi i k$ (for some $k \in \mathbb{Z}$). Thus there is at least one choice at this step.

Here is another problem: Suppose $f(z)$ exists and u & v are its real and imaginary parts. Then $z = \exp(z) = e(u(z)) \cdot e(i v(z))$. So if we write $z = Re^{i\theta}$, then $e(u(z)) = R$ and $v(z) = \theta + 2\pi k$ for some $k \in \mathbb{Z}$. So we have a choice for single z. More importantly if we go on a circle centered at 0 in order to get a continuous function v, we need to increase k at some point. (WHAT DOES THIS MEAN?)

In a nutshell: We need to be careful about the domain S of f.

<u>Definition</u>: If f is an analytic function on S and $\exp(f(z)) = z$ for all $z \in S$, then we say that f is an <u>analytic branch of $\log z$</u>.

<u>Theorem</u>: Let S be simply connected region not containing 0, and let $z_0 \in S$. Take $w \in \mathbb{C}$ such that $\exp(w) = z_0$ and call it $\log z_0$. Then the function $f(z) = \int_{C_z} \frac{dw}{w} + \log z_0$ is an analytic branch of $\log z$ on S, where C_z is any curve connecting z_0 to z.

(f is well-defined by Cauchy's Theorem.)

Proof: Note that $f'(z) = \frac{1}{z}$ for $z \in S$ (WHY?) So f is analytic on S. All we need is to show that $\exp(f'(z)) = z$ for all $z \in S$. So consider $\frac{z}{\exp(f(z))} = z \cdot \exp(-f(z))$, the derivative of this (analytic) function is $\exp(-f(z)) + z \cdot \exp(-f(z))(-f'(z)) = 0$. So this function is constant. ∎

What are good choices of S? Any disc not containing 0 would do. A more frequent choice is $S = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. (or $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$)

$\boxed{\text{Lecture 13}}$

Singularities

Def: We say f has an isolated singularity at z_0 if f is analytic on $D(z_0, r) \setminus \{z_0\}$ for some $r > 0$, but f is not analytic at z_0.

(sets of the form $D(z_0, r) \setminus \{z_0\}$ will be called "deleted neighbourhood of z_0".)

Note that if f has an isolated sing. at z_0, then f is not cont. at z_0. (WHY?)

There are three different kinds of isolated singularities:

① There is a function g that is analytic at z_0 and $f = g$ on a deleted neighbourhood of z_0.

Ex: $f(z) = \begin{cases} e^z & : z \neq 0 \\ 3 & z = 0 \end{cases}$ Take $g(z) = e^z$.

This kind is called a removable singularity.

② There are functions $A(z)$, $B(z)$ analytic at z_0 such that $A(z_0) \neq 0$, $B(z_0) = 0$, and $f(z) \equiv \frac{A(z)}{B(z)}$ on a deleted neigh. of z_0.

Ex: $f(z) = \frac{1}{z}$ or $f(z) = \frac{e^z}{z-2}$.

This kind is called a pole; and if z_0 is a zero of B of order k, then we say that z_0 is a pole of order k.

③ z_0 is neither removable nor a pole.

<u>Ex</u>: $f(z) = \exp\left(\frac{1}{z}\right)$

This kind is called as an <u>essential singularity</u>.

<u>Theorem</u>: Suppose that z_0 is an isolated singularity of f, and that $\lim_{z \to z_0} (z - z_0) f(z) = 0$. Then z_0 is removable.

<u>Proof</u>: Let $g(z) = \begin{cases} (z - z_0) f(z) & : z \neq z_0 \\ 0 & : z = z_0 \end{cases}$.

g is cont. at z_0 and is analytic on a deleted neigh. of f.

So g is analytic at z_0.

Then $h(z) := \frac{g(z)}{z - z_0}$ is also analytic at z_0. (WHY?) It's clear that $f \equiv h$ on a del. neigh. of z_0. ■

<u>Corollary</u>: Suppose that z_0 is an isolated singularity of f and f is bounded on a deleted neigh. of z_0. Then z_0 is a removable singularity.

<u>Proof</u>: $\lim_{z \to z_0} |(z - z_0) f(z)| \leq \underbrace{M \cdot \lim_{z \to z_0} z - z_0}_{M \cdot 0 = 0}$, where M is a bound for f around z_0. ■

<u>Theorem</u>: Suppose that f is analytic on a deleted neighbourhood of z_0 and suppose that $\lim_{z \to z_0} (z - z_0)^k f(z) \neq 0$, but $\lim_{z \to z_0} (z - z_0)^{k+1} f(z) = 0$ for some $k \in \mathbb{N}_{>0}$. Then z_0 is a pole of f of order k.

<u>Proof</u>: Define $g(z) = \begin{cases} (z - z_0)^{k+1} f(z) & : z \neq z_0 \\ 0 & : z = z_0 \end{cases}$

As above g is continuous at z_0, hence analytic at z_0.

Also $A(z) := \frac{g(z)}{z-z_0} = (z-z_0)^k f(z)$ is analytic at z_0. (Because $g(z_0)=0$.)

Note that $A(z_0) \neq 0$, and $f(z) = \frac{A(z)}{(z-z_0)^k}$, So z_0 is a pole of f of order k. ▨

As a result of this, if f has an essential singularity at z_0, then $\lim_{z \to z_0} (z-z_0)^k f(z) \neq 0$ for any $k \in \mathbb{N}$.

<u>Theorem</u> (Casorati-Weierstrass): Suppose that f has an essential singularity at z_0, and let D be a deleted neigh. of z_0 on which f is analytic. Then $R := f(D)$ is dense in $\mathbb{C}$.

<u>Proof</u>: Suppose that there is a disc $S = D(w, \delta)$ such that for every $z \in D$ we have $f(z) \notin S$. So for any $z \in D$ we have $\frac{1}{|f(z)-w|} < \frac{1}{\delta}$. So $\frac{1}{f(z)-w}$ has a removable sing. at z_0. Thus there is a function g analytic at z_0 s.t. $g \equiv \frac{1}{f(z)-w}$ on D.

So $f(z) = w + \frac{1}{g(z)}$ on D. But then f doesn't have an essential singularity at z_0. ▨

<u>Definition</u>: Let $(\alpha_k)_{k \in \mathbb{Z}}$ be a sequence indexed by integers. We say $\sum_{k=-\infty}^{\infty} \alpha_k$ converges if both $\sum_{k=0}^{\infty} \alpha_k$ and $\sum_{k=1}^{\infty} \alpha_{-k}$ converge; and in that case we say it equals the sum of these two series.

We consider series of the form $\sum_{k=\infty}^{\infty} \alpha_k z^k$; they are called <u>Laurent series</u> (around 0). We may determine their "domain of convergence" as follows.

Theorem: let $(a_k)_{k\in\mathbb{Z}}$ be given and let $R_1 = \dfrac{1}{\limsup_{k\to\infty} |a_k|^{1/k}}$; and $R_2 = \limsup_{k\to\infty} |a_{-k}|^{1/k}$. Then $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ converges in the annulus

$\{z : R_2 < |z| < R_1\}$. (If $R_1 \le R_2$, then it is the emptyset.)
Moreover f is analytic on this domain.
Proof: let $f_1(z) = \sum_{k=0}^{\infty} a_k z^k$ and $f_2(z) = \sum_{k=1}^{\infty} a_{-k} z^{-k}$. Then we know that

f_1 converges and analytic on $D(0, R_1)$, and f_2 converges and analytic for $\left|\frac{1}{z}\right| < \frac{1}{R_2}$. So f_1 & f_2 are both convergent and analytic on

$D(0, R_1) \cap \{z : |z| > R_2\} = \{z : R_2 < |z| < R_1\}$. ∎

Theorem: Suppose that f is analytic on the annulus $A = \{z : R_1 < |z| < R_2\}$. Then $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ in A; we say f has a Laurent expansion in A.

Moreover, this representation is unique.

Proof: let r_1, r_2 with $R_1 < r_1 < r_2 < R_2$ and take z with $r_1 < z < r_2$.
Also let C_1 & C_2 be circles centered at 0 and of radiuses r_1 & r_2.
Let $g(w) = \dfrac{f(w) - f(z)}{w - z}$; an analytic function on A.

Then $\int_{C_1} g\, dw = \int_{C_2} g\, dw$. (WHY?)

So $f(z)\int_{C_2 - C_1} \frac{dw}{w-z} = \int_{C_2 - C_1} \frac{f(w)}{w-z}\, dw$ & $f(z) = \frac{1}{2\pi i}\int_{C_2 - C_1} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}$

For $|w| > |z|$, we have $\frac{1}{w-z} = \cdots = \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}$, and for $|w| < |z|$, we have $\frac{1}{w-z} = \cdots = \sum_{n=1}^{\infty} -\frac{w^{n-1}}{z^n}$ Then:

$$f(z) = \frac{1}{2\pi i} \int_{C_2} \sum_{k=0}^{\infty} \frac{f(w) z^k}{w^{k+1}} dw + \frac{1}{2\pi i} \int_{C_1} \sum_{k=1}^{\infty} f\frac{(w) \; z^{-k}}{w^{-k+1}} dw \underset{(\text{why?})}{=} \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(w)}{w^{k+1}} dw \right) z^k,$$

where C is any circle in A. (How?)

Now we are done by taking $a_k = \frac{1}{2\pi i} \int_C \frac{f(w)}{w^{k+1}} dw$.

Uniqueness ... (Exercise) ☒

$\boxed{\text{Lecture 14}}$

We may, of course, carry Laurent expansions to any z_0:

If f is analytic on $A = \{ z : R_1 < |z - z_0| < R_2 \}$, then $f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$ on A, where $a_k = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{k+1}} dw$, and C any circle centered at z_0 and of radius between R_1 & R_2.

By taking z_0 an isolated singularity and $R_1 = 0$, we get a Laurent expansion of f on $D(z_0, \delta) \setminus \{z_0\}$ for some $\delta > 0$.

Example: $f(z) = \frac{(z+1)^2}{z}$ on $\mathbb{C} \setminus \{0\}$. Then $f(z) = \frac{1}{z} + 2 + z$. So the Laurent expansion is finite. Note that $f(z) \cdot z = 1 + 2z + z^2$, a polynomial.

(Compare with results on poles.)

Example: $f(z) = \dfrac{1}{z^2(1-z)}$ on $\{z : 0<|z|<1\}$

$$= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots$$

OK: $f(z) = \dfrac{-1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 - \cdots$

on $\{z : 0<|z-1|<1\}$

Example: $f(z) = \exp\left(\frac{1}{z}\right)$ on $\mathbb{C}\setminus\{0\}$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{z}\right)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!\, z^k} = \sum_{k=-\infty}^{0} \frac{z^k}{k!} \quad \text{for } z \neq 0.$$

Let's compare the types of isolated singularities and Laurent expansions at those points. If z_0 is an isol. singularity of $f(z)$ and $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ at $z \neq z_0$, then $\sum_{k=-\infty}^{-1} a_k(z-z_0)^k$ is called the principal part of f and $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ is called the analytic part of f.

Proposition: $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$ as above.

(i) z_0 is a removable singularity iff $C_k = 0$ for $k<0$.

(ii) z_0 is a pole of order d iff $C_k = 0$ for $k<-d$ and $C_{-d} \neq 0$.

(iii) z_0 is an essential singularity iff $C_k \neq 0$ for infinitely many $k<0$.

Proof: (i) f agrees with an analytic function around z_0 except (possibly) at z_0. So f has a power series expansion & so on...

(ii) $f(z) = \dfrac{Q(z)}{(z-z_0)^d}$ & $Q(z)$ is analytic & nonzero at z_0 & so on...

(iii) Similar ... ☒

Proposition: let P, Q be polynomials of z with $\deg P < \deg Q$. Define $R(z) = \frac{P(z)}{Q(z)}$ except at zeros of Q. Then R is a (finite) sum of polynomials of $\frac{1}{z-\alpha}$ where α varies in zeros of Q.

Proof: let $\alpha_1, \dots, \alpha_k$ be roots of $Q(z)$, say of multiplicities $m_1, \dots, m_k > 0$. Consider $R(z)$ around α_1: $R(z) = P_1\left(\frac{1}{z-\alpha_1}\right) + A_1(z)$ where P_1 is a polynomial because α_1 is a pole of order k_1 of R, and $A_1(z)$ is analytic (around α_1).

Then A_1 has removable sing. at α_1 the same principal parts at $\alpha_2, \dots, \alpha_k$ as R. So we may write $A_1 = P_2\left(\frac{1}{z-\alpha_2}\right) + A_2(z)$, where

P_2 is a polynomial & A_2 is analytic around α_2.

By continuing this way, we get $A_k(z) = R(z) - P_1\left(\frac{1}{z-\alpha_1}\right) - \dots - P_k\left(\frac{1}{z-\alpha_k}\right)$ where $P_1, \dots, P_k$ are polynomials and A_k analytic around $\alpha_1, \dots, \alpha_k$.

Actually, A_k has only removable singularities at $\alpha_1, \dots, \alpha_k$. So by defining A_k at $\alpha_1, \dots, \alpha_k$ accordingly, it is an entire function. Moreover it is bounded (Why?) So A_k is constant by Liouville's theorem. It's actually 0. (Why?)

Then $R(z) = P_1\left(\frac{1}{z-\alpha_1}\right) + \dots + P_k\left(\frac{1}{z-\alpha_k}\right)$. ■

— o —

Def: let γ be a closed curve and $a \in \mathbb{C}$ not on γ. Then we define the winding number of γ around a as

$$n(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}.$$

Theorem: For any γ & a as above, $n(\gamma, a) \in \mathbb{Z}$.

Proof: let γ be $z(t)$ with $t \in [0,1]$.

Define $F : [0,1] \longrightarrow \mathbb{C}$

$$s \longmapsto \int_0^s \frac{z'(t)}{z(t)-a}\, dt$$

Then $F'(s) = \dfrac{z'(s)}{z(s)-a}$ and by differentiating $(z(s)-a)\exp(-F(z))$ (with respect to s) we get that $(z(s)-a)\exp(-F(z))$ is constant.

Plugging in $s=0$, we get $(z(0)-a)\exp(-F(0)) = z(0)-a$ is that constant. So $\exp(F(z)) = \dfrac{z(s)-a}{z(0)-a}$. Now plug in $s=1$ to get

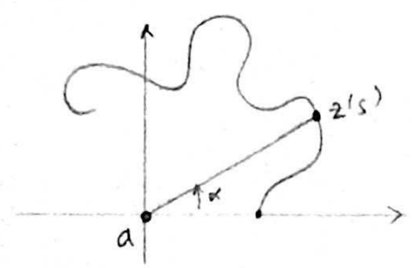
$\exp(F(1)) = \dfrac{z(1)-a}{z(0)-a} = 1$. So $F(1) \in \mathbb{Z}\, 2\pi i$ and $n(\gamma, a) = \dfrac{1}{2\pi i} F(1) \in \mathbb{Z}$. ■

assume $z(0) = 1$

let a be the origin & write $F(s) = u(s) + i v(s)$. Then

$e^{u(s)} \cdot e^{i v(s)} = z(s)$. Then $|z(s)| = e^{u(s)}$.

So $v(s)$ is a "continuous choice of argument" for $\dfrac{z(s)}{|z(s)|}$



Thus, every time γ wraps around a we get another 2π for $v(s)$.

let γ be a curve and let a vary in $\mathbb{C} \setminus \gamma$. Then it's easy to see that $n(\gamma, a)$ is a continuous function of a. But it is integer valued! So it has to be constant on connected components of $\mathbb{C} \setminus \gamma$. (WHY?)

Jordan Curve Theorem: Let γ be a (continuous) simple closed curve. Then $\mathbb{C} \setminus \gamma$ has exactly two connected components.

(This is intuitively very clear, but writing down a proof is very hard work.)

As a result of this, if γ is simple, closed then $n(\gamma, a) = 0$ for a in the unbounded component (why?) and $n(\gamma, a) = k \in \mathbb{Z}$ for the "inside" points. One way actually show that $k = \pm 1$ (depending on the orientation.)

Back to isolated singularities:

Def: Let z_0 be an isolated singularity of $f(z)$ & let $f(z) = \sum_{k=-\infty}^{\infty} C_k (z-z_0)^k$ in a deleted neighbourhood of z_0. Then C_{-1} is called the residue of f at z_0; notation $\mathrm{Res}(f, z_0)$.

Lecture 15

Note that $\mathrm{Res}(f, z_0) = \frac{1}{2\pi i} \int_\gamma f$ for any circle γ in a deleted neighbourhood of z_0. (As a matter of fact, we'll see in a bit that γ can be any curve in a deleted neigh. of z_0 on which f is analytic with $n(\gamma, z_0) = 1$.)

Proposition: Let z_0 be a pole of f of order k. Then

$$\mathrm{Res}(f, z_0) = \frac{1}{(k-1)!} \left((z-z_0)^k f(z) \right)^{(k-1)} (z_0).$$

(Note that z_0 is a removable singularity of $(z-z_0)^k f(z)$. So we fill in the gap and hence may take derivatives of the resulting function.)

Proof: Write $f(z) = C_{-k}(z-z_0)^{-k} + C_{-k+1}(z-z_0)^{-k+1} + \dots + C_{-1}(z-z_0)^{-1} + P(z-z_0)$

(where P is the principal part.)

Then $(z-z_0)^k f(z) = C_{-k} + C_{-k+1}(z-z_0) + \dots + C_{-1}(z-z_0)^{k-1} + (z-z_0)^k P(z-z_0)$

By differentiating $k-1$ many times we get:

$$\left((z-z_0)^k f(z) \right)^{(k-1)} = (k-1)!\ c_{-1} + k!\,(z-z_0)\ P(z-z_0).$$

Now plug in $z=z_0$ to get the desired result. ■

For $k=1$, this means that $\text{Res}(f, z_0) = (z-z_0)f(z)$, and if we write $f(z) = \frac{A(z)}{B(z)}$ where z_0 is a zero of $B(z)$ once, we get $\text{Res}(f, z_0) = \frac{A(z_0)}{B'(z_0)}$ (why?)

In particular, if we take $f(z) = \frac{1}{z}$ and $z_0 = 0$, then $\text{Res}(\frac{1}{z}, 0) = 1$.

Examples: ① $\text{Res}(\frac{1}{z^3}, 0) = 0$.

② $\text{Res}(\frac{1}{z^4-1}, i) = ?$ $\quad z^4 - 1 = (z^2-1)(z+i)(z-i)$. So i is a pole of order 1. Then $\text{Res}(\frac{1}{z^4-1}, i) = \frac{1}{4z^3}\Big|_i = \frac{1}{4i^3} = \frac{i}{4}$.

③ $\text{Res}(\sin(\frac{1}{z-1}), 1) = 1$ (Laurent series.)

<u>Theorem</u> (Cauchy's Residue Theorem): D simply connected domain, $z_1, \dots, z_m \in D$, f analytic on $D \setminus \{z_1, \dots, z_m\}$, γ a closed curve in D, not passing through $z_1, \dots, z_m$. Then

$$\int_\gamma f = 2\pi i \sum_{i=1}^{m} n(\gamma, z_i)\ \text{Res}(f, z_i).$$

Proof: let $P_i\left(\frac{1}{z-z_i}\right)$ be the principal part of f around z_i for $i=1,\dots,m$.

So $P_i\left(\frac{1}{z-z_i}\right) = \sum_{-k=1}^{\infty} C_{-k}(z-z_0)^{-k}$. Note that $\int_\gamma C_{-k}(z-z_0)^{-k} = 0$

for $k>1$ (WHY?) Then $\int_\gamma P_i\left(\frac{1}{z-z_i}\right)dz = C_{-i}\int_\gamma \frac{dz}{z-z_0} = \text{Res}(f,z_i)\,2\pi i\, n(\gamma,z_i)$

Now let $g(z) = f(z) - \left(P_1\left(\frac{1}{z-z_1}\right) + \dots + P_m\left(\frac{1}{z-z_m}\right)\right)$, with appropriate values at $z_1,\dots,z_m$. Then g is analytic on D and hence

$$\int_\gamma g = 0 \quad \& \quad \int_\gamma f = \sum_{i=1}^{m} \int_\gamma P_i\left(\frac{1}{z-z_i}\right) dz = 2\pi i \sum_{i=1}^{m} n(\gamma, z_i)\,\text{Res}(f,z_i).$$

$\blacksquare$

Definition: A function f is meromorphic on D if f has poles at $P \subseteq D$ and analytic on $D \setminus P$.

Theorem: let γ be a closed curve such that for every $a \in \mathbb{C}\setminus\gamma$ we have $n(\gamma,a) \in \{0,1\}$; $\{a \in \mathbb{C}\setminus\gamma : n(\gamma,a)=1\}$ will be referred to as the "inside" of γ. Suppose f is meromorphic inside and on γ, and f has no zeros or poles on γ. Then the integral $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}$ equals the difference of the number of zeros of f inside γ and the number of poles of inside γ. (counted with multiplicity.)

Proof: Let α be a zero of order k of f. So $f(z) = (z-\alpha)^k g(z)$ where g is analytic at α and $g(\alpha) \neq 0$.

Then $f'(z) = k(z-\alpha)^{k-1} g(z) + (z-\alpha)^k g'(z)$ & hence $\frac{f'(z)}{f(z)} = \frac{k}{z-\alpha} + \frac{g'(z)}{g(z)}$.

Since $\frac{g'(z)}{g(z)}$ is analytic at α, we have $\text{Res}\left(\frac{f'}{f}, \alpha\right) = k$.

Similarly, if β is a pole of f order k, then $\mathrm{Res}\left(\frac{f'}{f}; \beta\right) = -k$.

By the Residue Theorem, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \sum_{\substack{\alpha \text{ zero} \\ \text{of } f}} n(\gamma;\alpha)\,\mathrm{Res}\left(\frac{f'}{f}, \alpha\right) + \sum_{\substack{\beta \text{ pole} \\ \text{of } f}} n(\gamma,\beta)\,\mathrm{Res}\left(\frac{f'}{f}, \beta\right)$$

$$= \sum_{\substack{\alpha \text{ zero} \\ \text{of } f \text{ in } \gamma}} \mathrm{Res}\left(\frac{f'}{f}, \alpha\right) + \sum_{\substack{\beta \text{ pole} \\ \text{of } f \\ \text{in } \gamma}} \mathrm{Res}\left(\frac{f'}{f}, \beta\right)$$

This number is exactly what we need. $\blacksquare$

In particular, if f is analytic inside and on γ as above, and f has no zeros on γ, then $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}$ equals the number of zeros inside γ. This is called "Argument principal". (WHY?)

<u>Theorem</u> (Rouché): γ as above (closed + $n(\gamma, a) \in \{0,1\}$) let f and g be functions that are analytic inside and on γ. Suppose that $|f(z)| > |g(z)|$ on γ. Then f and $f+g$ have the same number of zeros inside γ.

<u>Proof</u>: Number of zeros of $f+g$ inside γ $= \frac{1}{2\pi i}\int_\gamma \frac{(f+g)'}{f+g} = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} + \frac{1}{2\pi i}\int_\gamma \frac{\left(1+\frac{g}{f}\right)'}{1+\frac{g}{f}}$.

$f+g = f\cdot\left(1+\frac{g}{f}\right)$
(note that f has no zeros on γ)

Note that for z on γ we have $\left|1-\left(1+\frac{g(z)}{f(z)}\right)\right| = \frac{|g(z)|}{|f(z)|} < 1$.

So $1+\frac{g(z)}{f(z)}$ has no zeros inside γ (WHY?) Then we get the desired result.

Example: For $z \in D(0,1)$ we have $|4z^2| > |2z^{10}+1|$ and clearly $z=0$ is the only zero of $4z^2$ and its multiplicity is 2. Then

$2z^{10}+1+4z^2$ has at most 2 zeros in $D(0,1)$. (Can you determine zeros of $2z^{10}+4z^2+1$?)

Lecture 16

Theorem: (Generalized Cauchy Integral Formula) γ as above, f analytic on simply connected domain D containing γ. Then for each z inside γ and $k \in \mathbb{N}$ we have

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{k+1}} \, dw .$$

(Note that we know this when γ is a circle.)

Proof: We have $f(w) = f(z) + f'(z)(w-z) + \frac{f''(z)}{2}(w-z)^2 + \dots$ for w in a neigh. of z. Hence $\operatorname{Res}\left(\frac{f(w)}{(w-z)^{k+1}}, z\right) = \frac{f^{(k)}(z)}{k!}$.

Also z is the only singularity of $\frac{f(w)}{(w-z)^{k+1}}$ in D.

Hence $\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{k+1}} \, dw = \operatorname{Res}\left(\frac{f(w)}{(w-z)^{k+1}}, z\right) = \frac{f^{(k)}(z)}{k!}$. ∎

Recall that if f_n's are analytic on region D that converge to f uniformly on compacta (in D), then f is also analytic on D. We may now show that $f_n' \longrightarrow f'$ uniformly on compacta:

$$f_n'(z) - f'(z) = \frac{1}{2\pi i} \int_C \frac{f_n(w) - f(w)}{(w-z)^2} \, dw$$ where C is the boundary of $D(z_0, r)$ for some fixed z_0 & $0<r<1$.

Given $\varepsilon > 0$, take $N > 0$ s.t. for all $n \geq N$ we have $|f_n(z) - f(z)| < \frac{\varepsilon r^2}{4}$ for $z \in \overline{D(z_0, r)}$. Then $|f_n'(z) - f'(z)| < \varepsilon$ by ML-inequality.

Now just recall that any compact set covered by finitely many closed discs.

<u>Theorem</u> (Hurwitz): $(f_n)_n$ analytic function on a region D, converging uniformly on compacta to f. Suppose that no f_n has a zero in D. Then either f is constantly 0 on D or f doesn't have a zero on D.

<u>Proof</u>: Let $z_0 \in D$ be a zero of f on D. If $f \not\equiv 0$, then there is a circle C centered at z_0 with $f(z) \neq 0$ for any $z \in C$.

So $\frac{f_n'}{f_n} \to \frac{f'}{f}$ uniformly on C, and hence $\int_C \frac{f_n'}{f_n} \to \int_C \frac{f'}{f}$.

But $\underbrace{\frac{1}{2\pi i} \int \frac{f'}{f}}_{\#\text{ of zeros of } f \text{ inside } C} > 0$ and $\frac{1}{2\pi i} \int \frac{f_n'}{f_n} = 0$. ▨

<u>Theorem</u>: $(f_n)_n$ analytic on a region D, $f_n \to f$ uniformly on compacta. If f_n's are 1-1, then either f is constant or f is 1-1.

<u>Proof</u>: Suppose that f is not 1-1 and take $z_0, z_1 \in D$ with $f(z_0) = f(z_1)$. Also let $D_0, D_1 \subseteq D$ discs around z_0 and z_1 with $D_1 \cap D_2 = \emptyset$. For large enough n, there is $z \in D_0$ with $f_n(z) = f(z_0)$. But f_n is 1-1, so $f_n(z_1) \neq f(z_1)$ $(= f(z_0))$. But then $f(z_1) \neq f(z_1)$ by Hurwitz's Theorem (applied to $f(z) - f(z_1)$). ▨

VI. EVALUATING INTEGRALS

① We'll calculate some indefinite integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$.

Recall that $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{r_1\to-\infty \\ r_2\to\infty}} \int_{r_1}^{r_2} f(x)\,dx$. If this limit exists, then it equals $\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$, which is somewhat easier to calculate.

However it might happen that the latter limit exists, but the former doesn't. So we need to be careful about this.

__Theorem__: Let f be a complex function that is analytic on the upper half plane and on the real line except for finitely many poles in the upper half plane. Assume also that for large enough $R>0$ $|f(z)| < \frac{c}{R^2}$ for $z \in C_R$ where C_R is the curve given by $z(t) = Re^{it}$ $t \in [0,\pi]$ and $c \in \mathbb{R}$. Then $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{i=1}^{s} \mathrm{Res}(f, z_i)$, where $z_1, \dots, z_s$ are the poles of f in the upper half plane.

__Proof__: Let C_R^* be $C_R \cup [-R,R]$; it's a simple closed curve.

So $\int_{C_R^*} f(z)\,dz = 2\pi i \sum_{i=1}^{s} \mathrm{Res}(f, z_i)$ for R large enough that all of the poles $z_1, \dots, z_s$ are inside C_R^*. We may also assume that $|f(z)| \leq \frac{c}{R^2}$ for $z \in C_R$. Then

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum_{i=1}^{s} \mathrm{Res}(f, z_i) \quad \& \quad \left| \int_{C_R} f \right| \leq \frac{c}{R^2}\pi R = \frac{\pi c}{R}$$

by ML-ineq.

So $\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum_{i=1}^{s} \mathrm{Res}(f, z_i)$. But the condition $|f(x)| < \frac{c}{x^2}$ for $x = \pm R$ gives that $\int_{-\infty}^{\infty} f(x)\,dx$ actually converges, hence equals $\lim_{R} \int_{-R}^{R} f(x)\,dx$. ∎

(basic calculus knowledge)

<u>Corollary</u>: $f(z) = \frac{P(z)}{Q(z)}$ where $P(z), Q(z)$ are polynomials with $\deg Q - \deg P \geq 2$ and $Q(x) \neq 0$ for $x \in \mathbb{R}$. Then $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{i=1}^{s} \mathrm{Res}(f, z_i)$ where $z_1, \dots, z_s$ are the zeros of Q in the upper half-plane.

<u>Example</u>: let $a, b \in \mathbb{R}_{>0}$ be distinct. Let $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i \left(\mathrm{Res}(f, ia) + \mathrm{Res}(f, ib) \right)$$

$$\mathrm{Res}(f, ia) = \lim_{z\to ia} \frac{z-ia}{(z^2+a^2)(z^2+b^2)} = \lim_{z\to ia} \frac{1}{(z+ia)(z^2+b^2)} = \frac{1}{2ia(-a^2+b^2)}$$

& similarly $\mathrm{Res}(f, ib) = \frac{1}{2ib(-b^2+a^2)}$

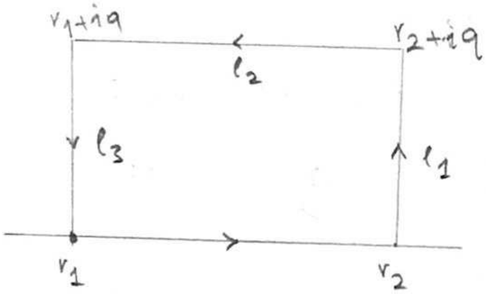
$$\text{Then } \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i \left(\frac{1}{2i} \left(\frac{1}{a(b^2-a^2)} - \frac{1}{b(b^2-a^2)} \right) \right)$$

$$= \pi \left(\frac{b-a}{ab(b^2-a^2)} \right) = \frac{\pi}{ab(a+b)}.$$

Next we consider $\int_{-\infty}^{\infty} f(x) e^{ix}\,dx$ for $f(x)$ similar to above.

Theorem: Suppose f is analytic on the upper half plane and on the real line except for finitely many poles on the upper half plane. Assume that for large enough $|z|$, we have $|f(z)| \leq \frac{c}{|z|}$ for a fixed constant c. Then $\int_{-\infty}^{\infty} f(x)e^{ix} = 2\pi i \sum_{i=1}^{s} \operatorname{Res}(f(z)e^{iz}, z_i)$ where $z_1, \dots, z_s$ are the poles of f in the upper half plane.

Proof: Given r_1, r_2, q let $C_{r_1,r_2,q}$ be the rectangle below:



So for large enough r_1, r_2, q we have

$$\int_{C_{r_1,r_2,q}} f(z)e^{iz} = 2\pi i \sum_{i=1}^{s} \operatorname{Res}(f(z)e^{iz}, z_i).$$

On the other hand:

$$\int_{C_{r_1,r_2,q}} f(z)e^{iz}\,dz = \int_{r_1}^{r_2} f(x)e^{ix}\,dx + \int_{l_1+l_2+l_3} f(z)e^{iz}\,dz.$$

$$\left|\int_{l_1} f(z)e^{iz}\,dz\right| = \left|\int_0^q f(r_2+it)\,e^{ir_2-t}\,i\,dt\right| \leq \int_0^q \frac{c}{r_2} e^{-t}\,dt \leq \frac{c}{r_2}$$

& similarly $\left|\int_{l_3} f(z)e^{iz}\,dz\right| \leqslant \frac{c}{r_1}$ for large enough r_1, r_2

$$\left|\int_{l_2} f(z)e^{iz}\,dz\right| = \left|-\int_{r_1}^{r_2} f(t+iq)\,e^{it-q}\,dt\right| \leq \left|\int_{r_1}^{r_2} \frac{c}{q} e^{-q}\,dt\right| = c\,\frac{r_2-r_1}{q e^q}$$

So $\lim_{q\to\infty} \int_{l_2} f(z) e^{iz} dz = 0$ & $\lim_{r_1\to-\infty} \lim_{r_2\to\infty} \int_{l_i} f(z) e^{iz} dz = 0$.
(for $i = 1, 3$)

Then $$\int_{-\infty}^{\infty} f(x) e^{ix} dx = 2\pi i \sum_{i=1}^{s} \operatorname{Res}(f(z)e^{iz}, z_i). \quad \blacksquare$$

Corollary: let $f(z) = \frac{P(z)}{Q(z)}$ where $P(z), Q(z)$ are polynomials with $\deg P(z) < \deg Q(z)$ and $Q(x) \neq 0$ for real x. Then $\int_{-\infty}^{\infty} f(x) e^{ix} dx = 2\pi i \sum \ldots$ $\blacksquare$

Lecture 17

$$\int_{-\infty}^{\infty} f(x) e^{ix} dx = \underbrace{\int_{-\infty}^{\infty} f(x) \cos(x) dx}_{\text{real if } f(x)\in\mathbb{R} \text{ for } x\in\mathbb{R}} + i \underbrace{\int_{-\infty}^{\infty} f(x) \sin(x) dx}_{\text{So is this}}.$$

Then $$\int_{-\infty}^{\infty} f(x)\cos x \, dx = \operatorname{Re}\left(2\pi i \sum_{i=1}^{s} \operatorname{Res}(f(z)e^{iz}, z_i)\right) \quad \& \quad \int_{-\infty}^{\infty} f(x)\sin x = \operatorname{Im}(-)$$

Examples: ① $$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+a^2)(x^2+b^2)} dx = \cdots = \frac{\pi\,(b^2 e^{-b} - a^2 e^{-a})}{b^2 - a^2}$$

② let's calculate $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$ we may not apply the theorem above directly, since $\frac{1}{x}$ has a pole on the real line. So we consider $\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x} dx$ instead

Note that $\operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x} dx\right) = \int_{-\infty}^{\infty} \frac{\sin x}{x}$.

Again let C_R be given by $z(t) = Re^{it}$ $t \in [0, \pi]$.

Then $\displaystyle\int_{C_R + [-R,R]} \frac{e^{iz}-1}{z}\,dz = 0$ since $\dfrac{e^{iz}-1}{z}$ is entire.

Therefore $\displaystyle\int_{-R}^{R} \frac{e^{ix}-1}{x}\,dx = \int_{C_R} \frac{1-e^{iz}}{z}\,dz = \int_{C_R} \frac{dz}{z} - \int_{C_R} \frac{e^{iz}}{z}\,dz$

$$= \pi i - \underbrace{\int_{C_R} \frac{e^{iz}}{z}\,dz}_{\to 0 \text{ as } R \to \infty}$$

So $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x} = \pi$.

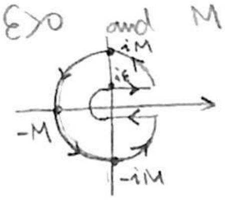
Next, we consider cases when f has a pole on the real line.

<u>Theorem</u>: Let P, Q be (complex) polynomial such that $\deg Q \geq \deg P + 2$ and Q has no zeros on the non-negative real line; and let $f(z) = \dfrac{P(z)}{Q(z)}$.

Then $\displaystyle\int_0^{\infty} f(x)\,dx = -\sum_{i=1}^{s} \operatorname{Res}\left(f(z)\log z,\, z_i\right)$, where $z_1, \dots, z_s$ are zeros of Q.

<u>Proof</u>: As before, the degree condition guarantees the convergence.

Let $\varepsilon > 0$ and $M > \varepsilon$ be given. We consider the curve $K_{\varepsilon, M}$ as below



So it has 4 parts:

I_1: Line segment from $i\varepsilon$ to $\sqrt{M^2-\varepsilon^2} + i\varepsilon$

C_M : circular arc of radius M from $\sqrt{M^2-\varepsilon^2}+i\varepsilon$ to $\sqrt{M^2-\varepsilon^2}-i\varepsilon$.

I_2 : line segment from $\sqrt{M^2-\varepsilon^2}-i\varepsilon$ to $-i\varepsilon$.

C_ε : (clockwise) circular arc of radius ε from $-i\varepsilon$ to $i\varepsilon$.

Let $S_{\varepsilon,M}$ be inside of $K_{\varepsilon,M}$; it is a simply connected domain not containing 0. So $\log z$ is an analytic function in $S_{\varepsilon,M}$ by choosing $\operatorname{Arg} z \in (0,2\pi)$.

Then for large enough M, and small enough ε we have

$$\int_{K_{\varepsilon,M}} f(z)\log z\, dz = 2\pi i \sum_{i=1}^{S} \operatorname{Res}\left(f(z)\log z,\, z_i\right).$$

$$\int_{I_1} f(z)\log z\, dz = \int_0^{\sqrt{M^2-\varepsilon^2}} f(i\varepsilon+t)\log(i\varepsilon+t)\, dt$$

So $\displaystyle\lim_{\varepsilon\to 0}\int_{I_1} f(z)\log z\, dz = \int_0^M f(t)\log(t)\, dt$ & hence $\displaystyle\lim_{\substack{\varepsilon\to 0\\ M\to\infty}}\int_{I_1} f(z)\log z\, dz = \int_0^\infty f(x)\log \, dx$

Similarly, $\displaystyle\lim_{\substack{\varepsilon\to 0\\ M\to\infty}}\int_{I_2} f(z)\log z\, dz = -\int_0^\infty f(x)(\log x + 2\pi i)\, dx$

Now consider $\displaystyle\int_{C_M} f(z)\log z\, dz$. By ML ineq. we have $\displaystyle\left|\int_{C_M} f(z)\log z\, dz\right| \leq 2\pi M \frac{c\cdot\log M}{M^2}$

$\left(|f(z)| < \frac{c}{|z|^2}\right.$ for some constant $\left. c\right)$ Therefore $\displaystyle\lim_{M\to\infty}\int_{C_M} f(z)\log z\, dz = 0$.

$$\left| \int_{C_\varepsilon} f(z) \log z \, dz \right| \leq \pi \cdot \varepsilon \max_{z \in C_\varepsilon} |f(z)| \, |\log(z)| < c \, \varepsilon \cdot |\log \varepsilon| \text{ for some}$$

constant c (f is continuous at 0.)

So $\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z) \log z \, dz = 0.$

Then $2\pi i \sum_{i=1}^{s} \text{Res}\left(f(z)\log z, z_i\right) = \lim_{\substack{\varepsilon \to 0 \\ M \to \infty}} \int_{K_{\varepsilon, M}} f(z) \log z \, dz = \int_0^\infty f(x) \log x \, dx - \int_0^\infty f(x) \log x \, dx$

$$- 2\pi i \int_0^\infty f(x) \, dx .$$

So $\int_0^\infty f(x) \, dx = - \sum_{i=1}^{s} \text{Res}\left(f(z) \log z, z_i\right).$ $\blacksquare$

Remarks: 1 We might have taken $f(z)$ to be a function as in the first theorem rather than a rational function with the given properties.

2 If there are some poles on the non-negative reals, then we may translate everything and take the integral accordingly.

Example: $\int_0^\infty \frac{dx}{1+x^3}$ Poles are $z_1 = e^{i\pi/3}$, $z_2 = e^{i\pi}$, $z_3 = e^{i5\pi/3}$.

$\text{Res}\left(\frac{\log z}{1+z^3}, z_1\right) = \frac{\log e^{i\pi/3}}{3\,(e^{i\pi/3})^2} = -\frac{i\pi}{9}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ — $\int_0^\infty \frac{dx}{1+x^3} = \frac{2}{9}\pi\sqrt{3}.$

$\text{Res}\left(\frac{\log z}{1+z^3}, z_2\right) = \cdots = \frac{i\pi}{3}$, $\text{Res}\left(\frac{\log z}{1+z^3}, z_3\right) = \cdots = -\frac{5\pi i}{9}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

A variant / generalization : Q a nonconstant polynomial & $\alpha \in (0,1)$.

Consider $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{Q(x)}\,dx$. Let $K_{\varepsilon,M}$ be as above, then for z inside $K_{\varepsilon,M}$

we have $z^{\alpha-1} = \exp((\alpha-1)\log z)$

For $z = i\varepsilon + x$ on I_1 we have $z^{\alpha-1} \to x^{\alpha-1}$ as $\varepsilon \to 0$, and

for $z = -i\varepsilon + x$ on I_2 we have $z^{\alpha-1} \to x^{\alpha-1} e^{2\pi i(\alpha-1)}$

The integral along C_M & C_ε again approach to 0. Hence.

$$\left(1 - e^{2\pi i(\alpha-1)}\right) \int_0^\infty \frac{x^{\alpha-1}}{P(x)}\,dx = 2\pi i \sum_{i=1}^{s} \operatorname{Res}\left(\frac{z^{\alpha-1}}{P(z)}, z_i\right).$$

<u>Example</u>: $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \cdots = \pi \qquad \left(\alpha = \frac{1}{2}\right).$

Now we consider $\displaystyle\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta$ where R is a rational function in two variables.

$(dz = ie^{i\theta}d\theta)$

Write $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$. Putting $z = e^{i\theta}$,

$\displaystyle\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta$ becomes $\displaystyle\int_C R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{1}{iz}\,dz$, where

C is the unit circle. Now given R, the latter integral could be calculated in terms of residues of poles inside C.

Example: $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ $\quad \cos\theta = \dfrac{z+\frac{1}{z}}{2}$, $d\theta = \dfrac{dz}{iz}$.

$$\| $$

$$\frac{2}{i}\int_C \frac{dz}{z^2+4z+1} = \frac{2}{i}\, 2\pi i \;\mathrm{Res}\left(\frac{1}{z^2+4z+1}, \sqrt{3}-2\right) = \frac{2}{3}\pi\sqrt{3}.$$

(Roots of z^2+4z+1 are $\sqrt{3}-2$ and $-\sqrt{3}-2$; only $\sqrt{3}-2$ is inside C)

— o —

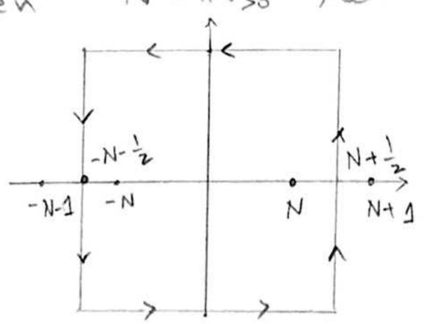
$\boxed{\text{Lecture 18}}$

Next we'll calculate infinite sums. Suppose f has finitely many poles throughout the whole $\mathbb{C}$, (none on $\mathbb{R}$). We shall calculate $\sum_{n=-\infty}^{\infty} f(n)$ in terms of residues of a function $g(z)$ whose residues contain $\{f(n) : n \in \mathbb{Z}\}$.

$g(z) = f(z)\,\pi\cot\pi z$ works, because each $n\in\mathbb{Z}$ is a pole and

$$\mathrm{Res}\left(f(z)\pi\cot\pi z, n\right) = \frac{f(z)\,\pi\cos\pi z}{\pi\cos\pi z}\Bigg|_{z=n} = f(n).$$ (Note that n can't be a pole of f.)

Given $N \in \mathbb{N}_{>0}$, let C_N be the square below:



Note that C_N contains $-N, -N+1, \dots, N-1, N$ and for large enough N, it also contains all the poles of f.

Consider z on the right vertical side of C_N:

$$\cot\pi z = i\,\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1} = i\,\frac{e^{\pi i - 2\pi y}+1}{e^{\pi i - 2\pi y}-1}$$

(where $y = \mathrm{Im}(z)$).

$$\| $$

So $|\cot\pi z| < 1$. $\qquad i\,\dfrac{1-e^{-2\pi y}}{-1-e^{-2\pi y}}$

For z on the other vertical line segment has the same property.

For z on the horizontal sides, one may show that $|\cot\pi z| < 2$

$$\text{So } \left| \int_{C_N} f(z)\, \pi \cot \pi z \right| \leq (8N+4)\, 2\pi \max_{z\in C_N} |f(z)| \leq c \underbrace{\max_{z\in C_N} |z f(z)|}_{\text{in terms of } N}$$

If we assume $|f(z)| \leq \dfrac{c^*}{|z|^2}$ for some constant c^*, then $\max|z f(z)|$ is bounded independently of N. Hence $\lim_{N\to\infty} \int_{C_N} f(z)\pi\cot\pi z\, dz = 0$.

$$\text{Therefore: } 0 = \lim_{N\to\infty} \int_{C_N} f(z)\pi \cot(\pi z)\, dz = \lim_{N\to\infty} \left(2\pi i \sum_{n=-N}^{N} f(n) + 2\pi i \sum_{i=1}^{s} \operatorname{Res}(f(z)\pi\cot\pi z, z_i) \right)$$

where $z_1, \dots, z_s$ are the poles of f (in C_N).

$$\text{Thus } \sum_{n=-\infty}^{\infty} f(n) = - \sum_{i=1}^{s} \operatorname{Res}\left(f(z)\pi \cot(\pi z), z_i\right).$$

<u>Example</u>: $$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2} \operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2}, 0 \right)$$

$$\cot z = \frac{1}{z} - \frac{1}{3} z^3 - \frac{1}{45} z^3 - \dots \quad \text{So} \quad \frac{\pi \cot \pi z}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3z} - \frac{\pi^4 z}{45} - \dots$$

$$\text{Hence } \operatorname{Res}\left(\frac{\pi\cot(\pi z)}{z^2}, 0 \right) = \frac{-\pi^2}{3} \quad \& \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Now we consider alternating sums: $\sum_{n=-\infty}^{\infty} (-1)^n f(n)$. This time, we use $\operatorname{cosec} \pi z$ in the place of $\operatorname{cotan} \pi z$:

$$\operatorname{Res}\left(f(z)\, \pi \frac{1}{\sin \pi z}, n \right) = \left(\frac{f(z) \cdot \pi}{\pi \cos \pi z} \right) \Big|_{z=n} = (-1)^n f(z).$$

Going through very similar arguments, we get the following:

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = - \sum_{i=1}^{s} \operatorname{Res}\left(f(z)\, \pi \operatorname{cosec} \pi z, z_i \right)$$ where $z_1, \dots, z_s$ are the poles of f.

Example: $\sum (-1)^n \frac{1}{n^2} = \dots = -\frac{\pi^2}{12}$.

— o —

$\binom{n}{k}$ = coefficient of z^k in $(z+1)^n = \frac{1}{2\pi i} \int_C \frac{(z+1)^n}{z^{k+1}}$ (by Generalized Cauchy Int. formula.)

where C is any simple closed curve around 0.

So for instance, $\binom{2n}{n} = \frac{1}{2\pi i} \int_{|z|=1} \frac{(z+1)^{2n}}{z^{n+1}} \leq 4^n$ $\left(\left| \int_C \frac{(z+1)^{2n}}{z^{n+1}} \right| \leq \frac{|(1+1)^{2n}|}{1^{n+1}} \cdot 2\pi \right.$; by ML-ineq

$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}$ converges and is less than $\frac{1}{1-\frac{4}{5}} = 5$. However, we'll calculate its exact value now.

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C \frac{(1+z)^{2n}}{(5z)^n z} \, dz$$ for any simple closed curve C around 0.

Let's take C to be $|z|=1$ again. Then for $z \in C$ we have $\left| \frac{(z+1)^2}{5z} \right| \leq \frac{4}{5}$. Hence $\sum_{n=0}^{\infty} \frac{(z+1)^{2n}}{(5z)^n}$ converges uniformly on C to $\frac{5z}{-(z^2-3z+1)}$

Therefore, $\displaystyle \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_C \frac{(1+z)^{2n}}{(5z)^n}\,\frac{1}{z}\,dz = \frac{5}{2\pi i}\int_C \frac{dz}{-z^2+3z-1}$

$$= 5\ \operatorname{Res}\left(\frac{1}{-z^2+3z-1},\ \frac{3-\sqrt{5}}{2}\right) = \sqrt{5}.$$

pole inside C

Let's calculate $\displaystyle\sum_{k=0}^{n}\binom{n}{k}^2$.

On the one hand $\binom{n}{k}$ is the coefficient of z^k in $(z+1)^n$, on the other hand it is the coefficient of z^{-k} in $(1+\frac{1}{z})^n$. Then $\displaystyle\sum_{k=0}^{n}\binom{n}{k}^2 = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{k}$ is the constant coefficient of $(z+1)^n(1+\frac{1}{z})^n$.

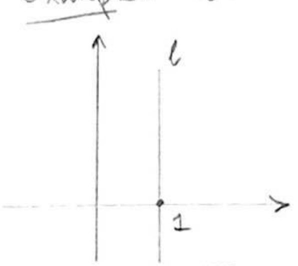
It can be calculated as $\displaystyle\frac{1}{2\pi i}\int_C (1+z)^n\left(1+\frac{1}{z}\right)^n\frac{1}{z}\,dz$

This is $\displaystyle\frac{1}{2\pi i}\int_C \frac{(1+z)^{2n}}{z^{n+1}}\,dz$, which is indeed $\binom{2n}{n}$.

(C is any simple closed curve.)

Now we'll consider some (complex) integrals along a curve that are easier to calculate using another curve.

Example: let ℓ be the line given by $z(t) = 1+it \quad t\in\mathbb{R}$.



We'd like to calculate $\displaystyle\int_\ell \frac{e^z\,dz}{(z+2)^3}$.

This will be similar to calculating definite integrals.

So let C_R be given by $1+Re^{it}$ $t\in[0,2\pi]$ & ℓ_R be the line segment given by $1+it$ $t\in[-1,1]$.

Then
$$\int_{C_R+\ell_R} = 2\pi i \operatorname{Res}\left(\frac{e^z}{(z+2)^3}, -2\right) \quad \text{for large enough } R.$$

Since e^z is bounded on the left of ℓ, we have $\lim_{R\to\infty}\int_{C_R}\frac{e^z}{(z+2)^3}\,dz = 0$.

$$\text{So} \quad \int_\ell \frac{e^z}{(z+2)^3}\,dz = \lim_{R\to\infty}\int_{\ell_R}\frac{e^z}{(z+2)^3}\,dz = 2\pi i \operatorname{Res}\left(\frac{e^z}{(z+2)^3}, -2\right)$$

$$\frac{e^z}{(z+2)^3} = \frac{1}{e^2}\,\frac{\sum_{k=0}^{\infty}\frac{1}{k!}(z+2)^k}{(z+2)^3} = \frac{1}{e^2}\sum_{k=0}^{\infty}\frac{1}{k!}(z+2)^{k-3} \qquad k=2$$

$$\operatorname{Res}\left(\frac{e^z}{(z+2)^3}, -2\right) = \frac{1}{e^2 2!} = \frac{1}{2e^2}.$$

$$\text{So} \quad \int_\ell \frac{e^z}{(z+2)^3}\,dz = 2\pi i\,\frac{1}{2e^2} = \frac{\pi i}{e^2}$$

$\boxed{\text{Lecture 19}}$

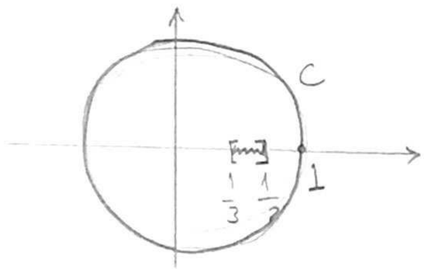
Example: $\displaystyle\int_C \frac{dz}{\sqrt{6z^2-5z+1}} = ?$ where C is the unit circle.

But first: what does $\sqrt{6z^2-5z+1}$ mean?

Let's start with $\sqrt{z}$: It is defined as $\exp(\frac{1}{2}\log z)$, but $\log z$ needs to be defined in a special kind of domain. Let's take $\mathbb{C}\setminus(-\infty,0]$.

Zeros of $6z^2-5z+1$ are $\frac{1}{3}$ and $\frac{1}{2}$. So $\sqrt{6z^2-5z+1}$ is defined and analytic on $\mathbb{C}\setminus(-\infty,\frac{1}{2}]$.

But we may show that it is actually defined and continuous on $(-\infty, \frac{1}{3})$. So $\sqrt{6z^2-5z+1}$ is analytic on $\mathbb{C} \setminus [\frac{1}{3}, \frac{1}{2}]$.



So $\displaystyle\int_C \frac{dz}{\sqrt{6z^2-5z+1}} = \int_{C_R} \frac{dz}{\sqrt{6z^2-5z+1}}$ where C_R is the circle centered at 0 and of radius $R>1$ (WHY?)

Write $\sqrt{6z^2-5z+1} = \sqrt{6}\, z + \varepsilon(z)$. Note that for $|z| \to \infty$ we have $\frac{\varepsilon(z)}{z} \to 0$ (WHY?)

Then
$$\int_{C_R} \frac{dz}{\sqrt{6z^2-5z+1}} - \int_{C_R} \frac{dz}{\sqrt{6}\, z} = -\frac{1}{6} \int_{C_R} \frac{\varepsilon(z)}{z\left(z + \frac{\varepsilon(z)}{\sqrt{6}}\right)}\, dz \longrightarrow 0 \quad \text{for } R \to \infty$$
by ML-ineq.

Hence
$$\int_{C_R} \frac{dz}{\sqrt{6z^2-5z+1}} = \int_{C_R} \frac{dz}{\sqrt{6}\, z} = \frac{2\pi i}{\sqrt{6}}.$$

HW: Read section 12.2 in Bak-Newman.

VII. CONFORMAL MAPPINGS

In informal terms a map is conformal if "it preserves angles". But what angles?

Below curves are smooth and $z'(t) \neq 0$ for every t (in its domain)

Definition: Let C_1 and C_2 be curves intersecting at z_0 with tangent lines l_1 and l_2 at z_0. Then the angle between C_1 and C_2 at z_0 is the angle between l_1 and l_2 (counterclockwise)

Suppose that C is given by $z(t) = x(t) + iy(t)$ $t \in [a,b]$, say $z_0 = z(t_0)$. Then the direction of the tangent line at z_0 is $z'(t_0) = x'(t_0) + iy'(t_0)$
Hence the tangent line makes the angle $\mathrm{Arg}(z'(t_0))$ with the real axis.
So if C_1, C_2 are given by z_1 and z_2, and $z_0 = z_1(t_0) = z_2(t_0)$, then the angle between C_1 and C_2 at z_0 is $\mathrm{Arg}(z_2'(t_0)) - \mathrm{Arg}(z_1'(t_0))$ (mod 2π)

Definition: Let f be a (continuous) function defined around z_0. If for any C_1, C_2 intersecting at z_0, the angle between C_1 and C_2 is the same as the angle between $f(C_1)$ and $f(C_2)$ at $f(z_0)$, then we say f is conformal at z_0.

Suppose f is analytic around z_0 and a curve C passing through z_0 is given by $z(t)$. Let $D = f(C)$. So D is given by $f(z(t))$.
So $\frac{\partial}{\partial t} f(z(t)) \Big|_{t=t_0} = f'(z(t_0))\, z'(t_0) = f'(z_0) \cdot z'(t_0)$, and the direction of the tangent line at $f(z_0)$ is $\mathrm{Arg}(f'(z_0)) + \mathrm{Arg}(z'(t_0))$. Therefore if we have two curves C_1, C_2 intersecting at z_0, then the angle between C_1 & C_2 at z_0 is the same as the angle between $f(C_1)$ & $f(C_2)$ at $f(z_0)$, unless $f'(z_0) = 0$. So we have the following:

Theorem. Let f be analytic (on a domain). Then f is conformal at any point z_0 (from the domain) with $f'(z_0) \neq 0$.

The condition that $f'(z_0) \neq 0$ is crucial. Consider $f(z) = z^2$, an entire function with $f'(0) = 0$. Also let $z_1(t) = t$ & $z_2(t) = it$ for $t \in [-1,1]$.

Then the angle between C_1 & C_2 is $\pi/2$. But if we take $D_1 = f(C_1)$ and $D_2 = f(C_2)$, then the angle between at 0 D_1 & D_2 is 0. (WHY?)

This has something to do with f being 1-1

Definition: We say f is locally 1-1 at z_0 if there is $\delta > 0$ s.t $f \upharpoonright D(z_0, \delta)$ is 1.1.

(Note that $f(z) = z^2$ is locally 1.1 at every nonzero z_0, but not locally 1.1 at $z_0 = 0$.)

Theorem: Let f be analytic around z_0 and suppose that $f'(z_0) \neq 0$. Then f is locally 1.1 at z_0.

Proof: Let $\alpha = f(z_0)$, and take $\delta' > 0$ s.t $f(z) - \alpha$ has no other zeros in $D(z_0, \delta')$. Then let C be the circle centered at z_0 and of radius δ'. Then

$$1 = \text{number of zeros of } f(z) - \alpha \text{ in } C = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z) - \alpha} \, dz$$

$$= \frac{1}{2\pi i} \int_{f(C)} \frac{w}{w - \alpha} \, dw = w(f(C), \alpha)$$

Since $w(f(C), -)$ is locally constant, we have $\frac{1}{2\pi i} \int_{f(C)} \frac{dw}{w - \beta} = 1$

for every $\beta \in D(\alpha, \varepsilon)$ for $\varepsilon > 0$ small enough.

By continuity of f, there is $\delta < \delta'$ s.t $D(z_0, \delta) \subseteq f^{-1}(D(\alpha, \varepsilon))$

It follows that $1 = \frac{1}{2\pi i} \int_{f(C)} \frac{dw}{w - f(z_1)}$ for any $z_1 \in D(z_0, \delta)$.

So $1 = \int_C \frac{f'(z)}{f(z)-f(z_1)} dz$ for any $z_1 \in D(z_0, \delta)$. So $f(z)-f(z_1)$ has a single zero in $D(z_0, \delta)$ for every z_1. ■

Examples: ① $f(z)=e^z$ is entire and $f'(z_0) \neq 0$ for every z_0. So it is conformal and locally 1.1.

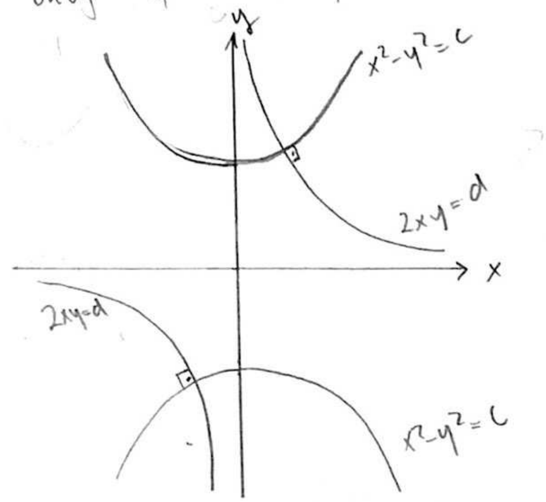
Let C_1 and C_2 respectively be x-coordinate axis and y-coordinate axis.

So C_1 is $z_1(t) = t$ and C_2 is $z_2(t) = it$ for $t \in (-\infty, \infty)$.

So their images under e^z are e^t and e^{it}. Hence $D_1 = f(C_1)$ is still the x-axis and $D_2 = f(C_2)$ is the unit circle. They intersect at $f(0) = 1$ orthogonally.

Actually any horizontal line is mapped to a line passing through 0, and and vertical line is mapped to a circle centered at 0. Hence they are still orthogonal to each other at the intersection points.

② $f(z) = z^2$. As observed above f is confrmal and locally 1-1 except at 0. Consider lines $u=c$, $v=d$ in the image (f is onto.) They are orthogonal. So their preimages should be as well. $Re(f(x+iy))=c$ if and only if $(x^2-y^2)=c$. Similarly $Im(f(x+iy))=d$ iff $2xy=d$



Lecture 20

Definition: let f be defined on D_1 with image D_2. We say f maps D_1 onto D_2 in a k-to-1 way if for every $\alpha \in D_2$, there are exactly k many (counting with multiplicity) $z \in D_1$ with $f(z) = \alpha$.

Lemma: $f(z) = z^k$ is a k-to-1 map of $D(0,\delta)$ onto $D(0,\delta^k)$. Also f magnifies angles at 0 by k.

Proof: Fix $\theta \in [0, 2\pi)$. Then $\{ r e^{i\theta} : r \in \mathbb{R}^{\geq 0} \}$ is sent to $\{ r^k e^{ik\theta} : r \in \mathbb{R}^{\geq 0} \}$ by f. So angles are magnified by k. It's also clear that $f(z) = \alpha$ has exactly k many solutions. ∎

Theorem: let f be non-constant analytic around z_0 with $f'(z_0) = 0$. Then there is $k \in \mathbb{N}_{>0}$ and $\delta > 0$ s.t. f is k-to-1 mapping on $D(z_0, \delta)$. This k can be determined as the smallest positive ℓ with $f^{(\ell)}(z_0) \neq 0$.

Proof: Without loss of generality $f(z_0) = 0$. Write down the Taylor series of f around z_0:

$$f(z) = a_k (z - z_0)^k + a_{k+1} (z - z_0)^{k+1} + \cdots = (z - z_0)^k \left(a_k + a_{k+1}(z - z_0) + \cdots \right)$$

(Note that $a_k = \dfrac{f^{(k)}(z_0)}{k!}$.)

Let $g(z) = a_k + a_{k+1}(z - z_0) + \cdots$. Then for some $\delta > 0$, there is $(h(z))^k = g(z)$ for $z \in D(z_0, \delta)$. $(g(z_0) \neq 0)$ Here h is analytic on $D(z_0, \delta)$.

So $f(z) = \left((z - z_0) h(z) \right)^k$. Let $\ell(z) = (z - z_0) h(z)$. Then $\ell(z_0) = 0$ and $\ell'(z_0) = h(z_0) \neq 0$. So ℓ is conformal and locally 1-1 at z_0.

Since $f = m \circ \ell$ where $m(z) = z^k$, we have the desired result. ∎

Corollary: let $f: D \to \mathbb{C}$ be analytic and 1-1. Then f is conformal in D and $f^{-1}: f(D) \to \mathbb{C}$ is also conformal (in $f(D)$)

A map f as above will be called a conformal mapping.

Note that earlier we have only defined being "conformal at a point" and that definition had nothing to do with analiticity.

Note also that the full exp is not a conformal mapping, but it is when we restrict it to a smaller domain.

Definition: Two regions D_1 and D_2 are called conformally equivalent if there is $f: D_1 \to \mathbb{C}$ a conformal mapping s.t $f(D_1) = D_2$.

This is indeed an equivalence relation. (WHY?)

let's list some conformal mappings from simpler ones to more complicated ones.

$z \mapsto z$ identity is of course conformal. We may make it more complicated by multiplying by a (positive) real number α. This map $z \mapsto \alpha z$ dilates (magnifies) by α. It's clearly conformal.

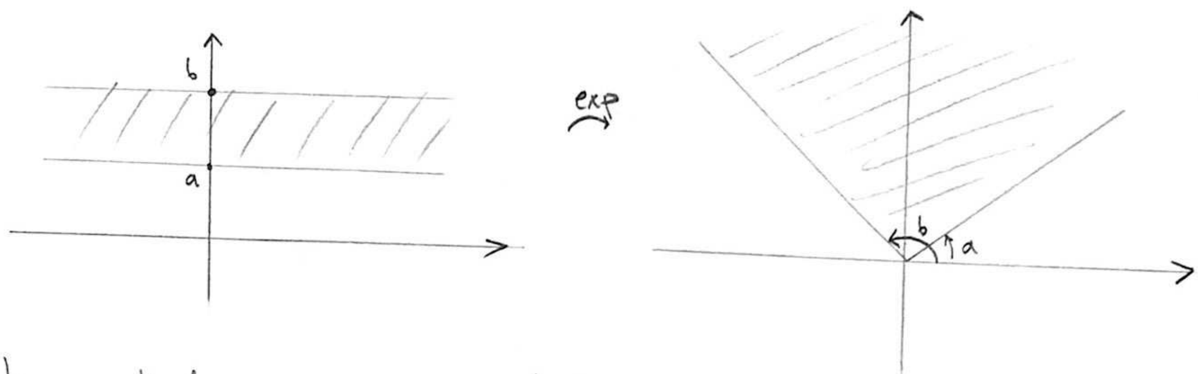
Next multiply by $e^{i\theta}$ for $\theta \in [0, 2\pi)$: $z \mapsto e^{i\theta} z$. This rotates by the angle θ; so it's conformal.

We may also translate by any $\beta \in \mathbb{C}$: $z \mapsto z + \beta$. This is also clearly conformal.

Putting these together we get any linear map: $z \mapsto az + b$.

As mentioned above $z \mapsto \exp(z)$ is not conformal when defined on all of $\mathbb{C}$. However, if we restrict it to any strip $\{z : a < \operatorname{Im}(z) < b\}$ where $b - a \leq \pi$. $(a, b \in \mathbb{R})$

Any line $t+ia$ is mapped by exp to the ray $e^{t}\cdot e^{i\theta}$.
So exp maps the strip $\{z: a<\mathrm{Im}(z)<b\}$ to $\{z: a<\mathrm{Arg}(z)<b\}$.

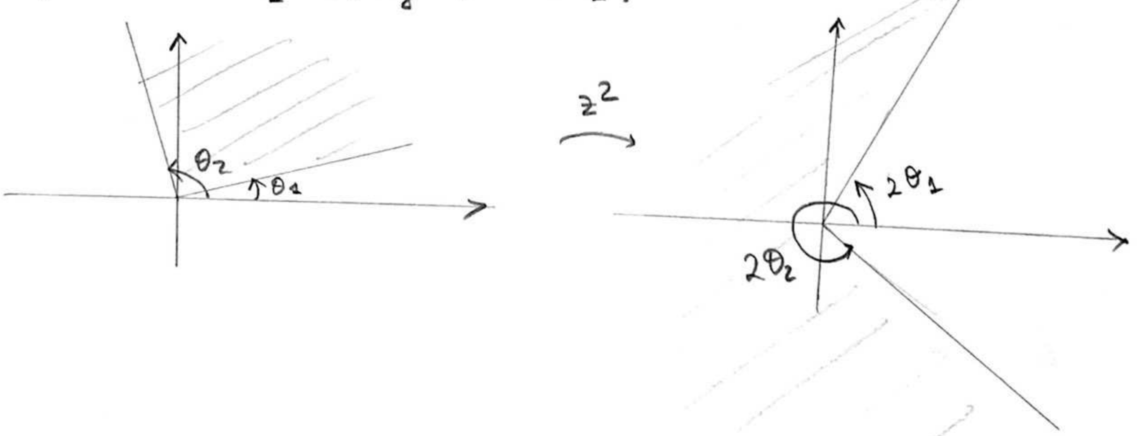


In particular, exp maps $\{z: 0<z<\pi\}$ to the upper half plane.

Now let's consider $z \longmapsto z^{a}$ $(a\in\mathbb{R}^{>0})$ By defintion $z^{a}=\exp(a\log z)$.
Choose $\log z$ to be the branch on $\mathbb{C}\setminus\{z: \mathrm{Re}\, z\leq 0\}$ with real values on $\mathbb{R}^{>0}$.

This maps $S=\{z\in\mathbb{C}: \theta_1<\mathrm{Arg}\, z<\theta_2\}$ with $\theta_2-\theta_1\leq \frac{2\pi}{a}$ to

$T=\{z\in\mathbb{C}: a\theta_1<\mathrm{Arg}\, z<a\theta_2\}$.



In particular, $z\mapsto z^2$ maps upper half plane to $\mathbb{C}\setminus\mathbb{R}^{>0}$.

Or $z\mapsto z^{1/2}$ maps $\{z: \mathrm{Re}\, z>0\}$ to $\{z: -\frac{\pi}{4}<\mathrm{Arg}\, z<\frac{\pi}{4}\}$.

$\boxed{\text{Lecture 21}}$

Next consider $z\longmapsto \frac{1}{z}$ on $\mathbb{C}\setminus\{0\}$. This is clearly a conformal mapping. Take for instance a line C given by $t+i2t$ $t\in(-\infty,\infty)$.

Then $f(t+i2t)=\dfrac{1}{t+i2t}=\dfrac{t-i2t}{t^2+(i2t)^2}=\dfrac{t}{t^2-4t^2}+i\dfrac{2t}{-t^2+4t^2}=-\dfrac{t}{3t^2}+i\dfrac{2t}{3t^2}$

This is again a line through the origin, but "going towards the opposite direction". (what does this mean?)

If C is the circle re^{it} $t \in [0, 2\pi]$, then $f(re^{it}) = \frac{1}{r} e^{-it}$.

Again this is a circle going towards opposite direction; in this case this means clockwise. In particular, inside of C is mapped to the outside of $f(C)$.

let's prove these in general:

Proposition: let $f(z) = \frac{1}{z}$ and let C be a line or a circle. Then $f(C)$ is again a line or a circle.

Proof: We'll use the following:

Fact: let $a, b \in \mathbb{C}$ distinct, and $\alpha \in \mathbb{R}^{>0}$. Then the set of points z satisfying $\frac{|z-a|}{|z-b|} = \alpha$ in a circle if $\alpha \neq 1$, and it's a line if $\alpha = 1$.

(The advantage of this for us is that it gives a uniform equation for lines and circles. Please check this fact.)

We may assume $a \neq 0$ and $b \neq 0$. (WHY?)

Then
$$\left| \frac{\frac{1}{z} - \frac{1}{a}}{\frac{1}{z} - \frac{1}{b}} \right| = \frac{|z-a| / |a||z|}{|z-b| / |b||z|} = \alpha \cdot \left|\frac{b}{a}\right|.$$

So the set of points $w = \frac{1}{z}$ is on the line or circle given by
$$\left| \frac{w - \frac{1}{a}}{w - \frac{1}{b}} \right| = \alpha \left| \frac{b}{a} \right|. \quad \blacksquare$$

HW: Go through the details to see when a line maps to circle, etc.

Now we consider Möbius mappings (bilinear mappings):

$f(z) = \dfrac{az+b}{cz+d}$ where $ad-bc \neq 0$. This is defined on $\mathbb{C} \setminus \{-\frac{d}{c}\}$.

$$f'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \quad \text{for } z \neq -\frac{d}{c}.$$

So $f'(z) \neq 0$ for any $z \in \mathbb{C} \setminus \{-\frac{d}{c}\}$, and hence locally 1-1 at all such z. However, it is 1-1 globally. (Check this.)

Therefore f is a conformal mapping.

Note that $\lim_{z \to \infty} f(z) = \frac{a}{c}$. So nothing maps to $\frac{a}{c}$. However any $w \in \mathbb{C} \setminus \{\frac{a}{c}\}$ has a preimage: $f\left(\dfrac{dw-b}{-c+a}\right) = w$. Indeed,

$f^{-1}(w) = \dfrac{dw-b}{-c+a}$ is the inverse of f.

So f is a conformal mapping of $\mathbb{C} \setminus \{-\frac{d}{c}\}$ onto $\mathbb{C} \setminus \{\frac{a}{c}\}$.

(If we consider f to be defined on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, then it is an automorphism of it by sending $-\frac{c}{d}$ to ∞)

Note that $f(z) = \frac{1}{c}\left(a - (ad-bc)\dfrac{1}{cz+d}\right)$. So letting $f_1(z) = cz+d$,

$f_2(z) = \dfrac{1}{z}$ and $f_3(z) = -\delta z + \dfrac{a}{c}$ $(\delta = ad-bc)$, we get $f = f_3 \circ f_2 \circ f_1$

Hence, by earlier work we get that Möbius transformations send lines and circles to lines and circles.

Note that for $0<a<1$, $f_1(z)=\frac{1}{z}$ sends the circle centered at a and of radius a to the line $x=\frac{1}{a}$ (WHY?)

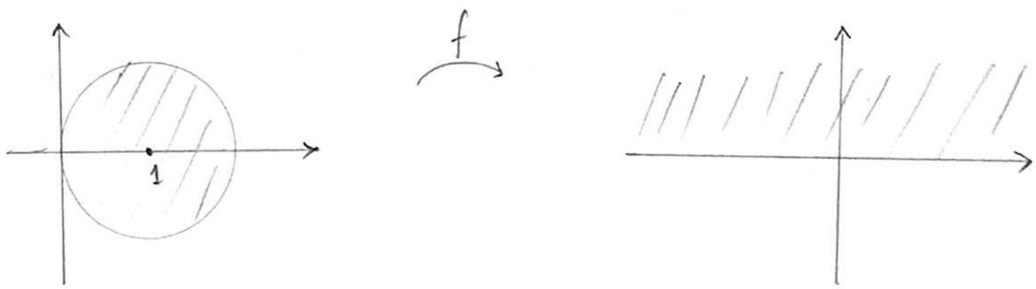
We may write $\mathbb{D}(1,1)$ as a union of circles as above. Hence

$f_1(D(1,1))$ is $\{z : \operatorname{Re}(z) > 1\}$.

Now let $f_2(z) = z-1$. So $f_2(\{z : \operatorname{Re}(z) > 1\})$ is $\{z : \operatorname{Re}(z) > 0\}$.

Finally, letting $f_3(z) = \underbrace{e^{i\frac{\pi}{2}}}_{=\,i} z$, we get $(f_3 \circ f_2 \circ f_1)(D(1,1)) =$ upper half plane.

So $f = f_3 \circ f_2 \circ f_1$ is a conformal equivalence of $D(1,1)$ and the upper half plane.



It's easy to see that for any three points p, q, r on the Riemann sphere, there is a Möbius transformation f sending p to 0, q to 1 and r to the north pole (∞).
So Möbius transformations are 3-transitive. (?) (We'll do this in more detail later.)

<u>Definition</u>: A conformal $f : D \to\!\!\!\to D$ is called an <u>automorphism</u> of D.
(D is a region; f is 1-1 by conformity and we assume it's surjective as well.)

Note that the set of automorphisms $\operatorname{Aut}(D)$ of D becomes a group with composition.

Lemma: Let D_1 and D_2 be conformally equivalent via f.

(i) If h is also a conformal equivalence between D_1 and D_2, then $h = g \circ f$ for some automorphism g of D_2.

(ii) Automorphisms of D_1 are of the form $f^{-1} \circ g \circ f$ where g is an automorphism of D_2.

Proof: (i) Just note that g defined as $f^{-1} \circ h$ is in $Aut(D_2)$.
(ii) Let $h \in Aut(D_1)$. Then $f \circ h : D_1 \to D_2$ conf. eq. So by (i), $f \circ h = g \circ f$ for some $g \in Aut(D_2)$. So $h = f^{-1} \circ g \circ f$. ■

This lemma can be seen as follows: If $f : D_1 \to D_2$ is a conformal mapping, then $Aut(D_2) \to Aut(D_1)$ is a group isomorphism.

$$g \longmapsto f^{-1} \circ g \circ f$$

Below $D = D(0,1)$ is the unit disc. We'd like to determine $Aut(D)$.

Lemma: Let $f \in Aut(D)$ with $f(0) = 0$. Then $f(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi)$.

Proof: Recall Schwarz Lemma: $f : D \to D$, $f(0) = 0$, 1-1. Then $|f(z)| \leq |z|$.
This holds for both f and f^{-1} (f is not necessarily onto in SL.)
But this means $|f(z)| = |z|$ for all $z \in D$. So $f(z) = e^{i\theta} z$, using the second part of Schwarz Lemma. ■

Now the general case.

Theorem: Elements of $Aut(D)$ are of the form $f(z) = e^{i\theta} \dfrac{z - \alpha}{1 - \bar{\alpha} z}$ for $\alpha \in D$ and $\theta \in [0, 2\pi)$.

Proof: Consider $g(z) = \dfrac{z - \alpha}{1 - \bar{\alpha} z}$ (for some fixed $\alpha \in D$).

Let $z \in \bar{D} \setminus D$. Then $|g(z)|^2 = g(z) \cdot \overline{g(z)} = \dfrac{z-\alpha}{1-\bar{\alpha}z} \dfrac{\bar{z}-\bar{\alpha}}{1-\alpha\bar{z}} = \cdots = 1$.

Then by Max. Mod. Prin., g maps D onto itself; hence $g \in Aut(D)$.

Then $f(z) = e^{i\theta} g(z) \in Aut(D)$ for all $\theta \in [0, 2\pi)$ as well.

Now we need to show that any $f \in Aut(D)$ is of this form. Suppose $f^{-1}(0) = \alpha$. Since g as above sends α to 0, we have that $f \circ g^{-1} \in Aut(D)$ with $(f \circ g^{-1})(0) = 0$. So $(f \circ g^{-1})(z) = e^{i\theta} z$ for some θ.

Then $f(z) = e^{i\theta} \dfrac{z-\alpha}{1-\bar{\alpha}z}$. ∎

$\boxed{\text{Lecture 22}}$

Next let $\mathbb{H}$ be the upper half plane.

<u>Theorem</u>: Conformal equivalences of $\mathbb{H}$ and D are of the form

$$h(z) = e^{i\theta} \frac{z-\alpha}{z-\bar{\alpha}} \quad \text{for some } \alpha \in \mathbb{H}.$$

<u>Proof</u>: For $z \in \mathbb{R}$ we have $|z-\alpha| = |z-\bar{\alpha}|$. Hence $\dfrac{z-\alpha}{z-\bar{\alpha}}$ maps the real line to the boundary of D. Since $\dfrac{z-\alpha}{z-\bar{\alpha}}$ maps α to 0, we get that it maps $\mathbb{H}$ onto D. Hence $e^{i\theta}\dfrac{z-\alpha}{z-\bar{\alpha}}$ is a conformal mapping of $\mathbb{H}$ onto D. Since each such conformal map is of the form $g \circ f$ where $f(z) = e^{i\theta}\left(\dfrac{z-\alpha}{z-\bar{\alpha}}\right)$ and $g \in Aut(D)$, we get the result. ∎

<u>Theorem</u>: Elements of $Aut(\mathbb{H})$ are of the form $\dfrac{az+b}{cz+d}$ where $a,b,c,d \in \mathbb{R}$ with $ad-bc > 0$.

<u>Proof</u>: Let $f(z) = \dfrac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{R}$, $ad-bc>0$. Then f maps real line to itself and $f(i) = \dfrac{ac+bd}{c^2+d^2} + i\,\dfrac{ad-bc}{c^2+d^2}$. So $f(i) \in \mathbb{H}$, and hence $f \in Aut(\mathbb{H})$.

In order to show that these are all the automorphisms, let $f(z) = \dfrac{z-i}{z+i}$

be a conformal mapping of $\mathbb{H}$ onto $\mathbb{D}$. Then an automorphism of $\mathbb{H}$ is of the form $f^{-1} \circ g \circ f$ where $g \in \mathrm{Aut}(\mathbb{D})$. Some calculations conclude the proof. ■

Theorem: Each non-identity Möbius transformation has at most 2 fixed pts.

Proof: Let $f(z) = \dfrac{az+b}{cz+d}$ be a Möbius transformation.

Note that $f(z) = z$ iff $cz^2 + dz = az + b$ iff $cz^2 + (d-a)z - b = 0$. This polynomial has at most two zeros unless $c = b = 0$ and $a = d$. But in that case $f(z) = z$. ■

As mentioned before, if we take distinct $p, q, r \in \mathbb{C}$, then there is a Möbius transformation $f(z)$ with $f(p) = \infty$, $f(q) = 0$, and $f(r) = 1$. Here is that transformation:

$$f(z) = \frac{(z-q)(r-p)}{(z-p)(r-q)}.$$

If $g(z)$ is another such transformation, then $f \circ g^{-1}$ fixes $\infty, 0, 1$. Hence $g = f$. So there is unique such Möbius transformation.

The cross-ratio of distinct z_1, z_2, z_3, z_4 is

$$(z_1, z_2, z_3, z_4) := \frac{z_4 - z_2}{z_4 - z_1} \, \frac{z_3 - z_1}{z_3 - z_2}.$$

(So this number is $f(z_4)$ where f maps z_1, z_2, z_3 to $\infty, 0, 1$.)

It's easy to show that if f is a Möbius transformation, then

$$(f(z_1), f(z_2), f(z_3), f(z_4)) = (z_1, z_2, z_3, z_4).$$

(HW).

Take z_1, z_2, z_3 and w_1, w_2, w_3; and let f and g be Möbius transformations sending z_1, z_2, z_3 and w_1, w_2, w_3 to $\infty, 0, 1$.

Then $h = g^{-1} \circ f$ sends z_1, z_2, z_3 to w_1, w_2, w_3. So for each $z \in \bar{\mathbb{C}}$ we have

$$f(z) = (z_1, z_2, z_3, z) = (h(z_1), h(z_2), h(z_3), h(z))$$
$$= (w_1, w_2, w_3, h(z))$$

It follows that

$$\frac{h(z) - w_2}{h(z) - w_1} \; \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_2}{z - z_1} \; \frac{z_3 - z_1}{z_3 - z_2}.$$

Actually, this equality determines h. (WHY?)

— ∘ —

Next, we'd like to map half strips to the upper half plane.

We'll show that $\sin z$ maps $\{z : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2},\ \operatorname{Im} z > 0\}$ onto $\mathbb{H}$.

Recall the following: $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$,

$$\sinh z = \frac{e^{z} - e^{-z}}{2}.$$

Note that $\cos(it) = \cos(-it) = \dfrac{e^{t} - e^{-t}}{2} = \cosh(t)$, and

$$\sin(it) = \frac{e^{-t} - e^{t}}{2i} = i \, \frac{e^{t} - e^{-t}}{2} = i \sinh(t).$$

Then $\sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy) = \sin x \cosh(y) + i \cos x \sinh(y)$.

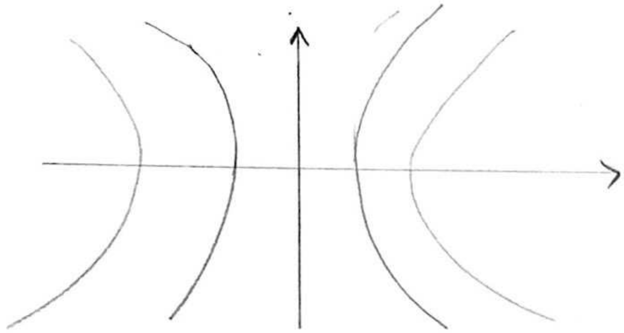
The advantage of writing $\sin z$ this way is that all functions involved are real functions. In particular, $u(x,y) = \sin x \cosh(y)$ & $v(x,y) = \cos x \sinh(y)$

Note that $\cosh(y) - \sinh(y) = e^{-y}$. So for large y, $\cosh(y)$ and $\sinh(y)$ have very close values.

Consider a vertical line l_c given as $x = c$. The image of l_c under sin consists of $\underbrace{\sin c \cosh(y)}_{u(c,y)} + i \underbrace{\cos c \sinh(y)}_{v(c,y)}$

We know $\cosh(y)^2 - \sinh(y)^2 = 1$. This means that $\frac{u(c,y)^2}{\sin^2 c} - \frac{v(c,y)^2}{\cos^2 c} = 1$.

So $(u(x,y), v(x,y))$ is on the hyperbola $\frac{1}{\sin^2 c} X^2 - \frac{1}{\cos^2 c} Y^2 = 1$.

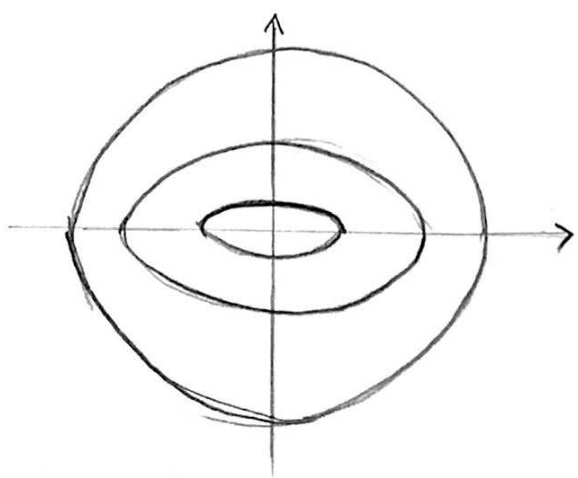


($c \in \mathbb{Z}\pi \Rightarrow$ y-axis.)

($c = \frac{\pi}{2}$?)

Now conside l_c' given as $y = d$. The image of l_c' under sin consists of $\sin(x)\cosh(d) + i \cos x \sinh(d)$. This time we have

$\left(\frac{u(x,d)}{\cosh(d)}\right)^2 + \left(\frac{v(x,d)}{\sinh(d)}\right)^2 = 1$. So $\frac{1}{\cosh(d)^2} u^2 + \frac{1}{\sinh(d)^2} v^2 = 1$ is an ellipse. (For large values of d, this is almost a circle)



Now consider a rectangle

$$R_d = \{ z : -\tfrac{\pi}{2} < \operatorname{Re}(z) < \tfrac{\pi}{2},\ 0 < \operatorname{Im} z < d \}$$

sin maps R_d to inside of an ellipse.

Hence we see that sin maps

$\{ z : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}, \operatorname{Im}(z) > 0 \}$ to $\mathbb{H}$ in a conformal way

So "$\sin^{-1} z$" maps $\mathbb{H}$ onto the half strip. However, what's $\sin^{-1} z$?

$\sin z$ is entire, but $\sin^{-1} z$ need not be.

We may define it as $\sin^{-1} z = \int_0^z \frac{d\varsigma}{\sqrt{1-\varsigma^2}}$. This certainly isn't analytic

at ± 1. The function $\frac{1}{\sqrt{1-\varsigma^2}}$ is analytic on $\mathbb{C} \setminus X$ where

$X = \{1+iy : y \leq 0\} \cup \{-1+iy : y \leq 0\}$. (HW) Then $\int_0^z \frac{d\varsigma}{\sqrt{1-\varsigma^2}}$ is

also analytic on $\mathbb{C} \setminus X$. Observe that $\lim_{\substack{z \to 1 \\ (z\to -1)}} \int_0^z \frac{d\varsigma}{\sqrt{1-\varsigma^2}} = \frac{\pi}{2}$. (WHY?)

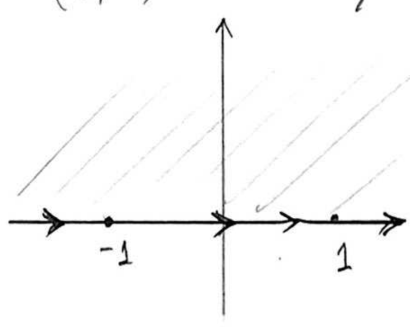
Hence $\sin^{-1} z$ is analytic on $\mathbb{H}$ and cont. on $\mathbb{H} \cup \mathbb{R}$.

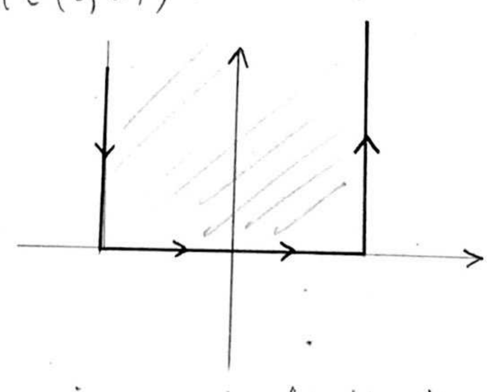
It's then not hard to see that $\sin^{-1} z$ maps

$(-\infty, -1)$ to $-\frac{\pi}{2} + it$ $(t \in (-\infty, 0))$,

$[-1, 1]$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and

$(1, \infty)$ to $\frac{\pi}{2} + it$ $(t \in (0, \infty))$.





The top of $\mathbb{R}$ (i.e $\mathbb{H}$) is mapped "inside the strip".

Our aim to map $\mathbb{H}$ into the inside of a given polygon.

For instance, for a rectangle we'll try to change the half strip as above into a rectangle.

Let f be analytic (on some region). Suppose f' has constant argument on a ray ℓ (given by $a + te^{i\alpha}$ $(t>0)$); say $\mathrm{Arg}(f'(z)) = \theta$.

Then for $z \in \ell$ we have

$$f(z) - f(a) = \int_a^z f'(\zeta)\,d\zeta = \int_0^z |f'(\zeta)|\,e^{i\theta}\,d\zeta = \underset{\text{Argument}}{e^{i\theta}} \underbrace{\int_0^z |f'(\zeta)|\,d\zeta}_{\in \mathbb{R}^{>0}}$$

So f maps ℓ to another ray $(f(a) + se^{i(\alpha+\theta)}$ $s>0)$.

Consider $a \in \mathbb{R}$. Then $\mathrm{Arg}(z-a) = \begin{cases} \pi & : z < a \\ 0 & : z > a \end{cases}$ (for $z \in \mathbb{R} \setminus \{a\}$)

So $\mathrm{Arg}(\sqrt{z-a}) = \begin{cases} \pi/2 & : z < a \\ 0 & : z > a \end{cases}$

Now let $a < b < c < d$ be real, and suppose we could define f such that $f'(z) = \dfrac{1}{\sqrt{(z-a)(z-b)(z-c)(z-d)}}$

Then: $\mathrm{Arg}(f'(z)) = \begin{cases} 0 & \text{for } z < a \\ \pi/2 & \text{for } a < z < b \\ \pi & \text{for } b < z < c \\ \frac{3\pi}{2} & \text{for } c < z < d \\ 0 & \text{for } d < z \end{cases}$

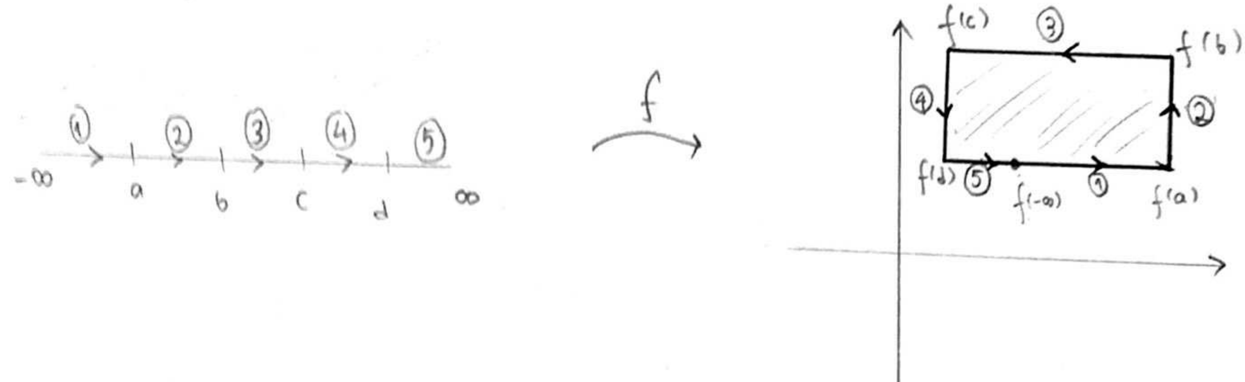
So define $f(z) = \displaystyle\int_0^z f'(\zeta)\,d\zeta = \int_0^z \frac{d\zeta}{\sqrt{(\zeta-a)(\zeta-b)(\zeta-c)(\zeta-d)}}$

As in the case of $\sin^{-1} z$, we may show that f is analytic on $\mathbb{H}$ and cont. on $\mathbb{H} \cup \mathbb{R}$.

Using integration techniques from before we may show that

$$\int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{(\zeta-a)(\zeta-b)(\zeta-c)(\zeta-d)}} = 0 \quad (\text{HW: do this})$$

So "$f(-\infty) = f(\infty)$" & we have the following:



So f maps $\mathbb{H}$ to inside of the rectangle above.

One might want to choose a, b, c, d in a nicer way. A nice one is $-\frac{1}{k} < -1 < 1 < \frac{1}{k}$. ($k \in (0,1)$). In this case, f is given by an "elliptic integral":

$$f(z) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}.$$

Finally, we overview the case of a general convex polygon.

Let $\theta_1, \theta_2, \dots, \theta_n$ be the angles. Then $\sum_{i=1}^{n} \theta_i \in 2\pi\mathbb{Z}$. (WHY?)

Let $\alpha_i = \frac{\theta_i}{\pi}$ for $i = 1, \dots, n$ & choose $a_1 < \dots < a_n$ in $\mathbb{R}$. Then we'd like to choose f so that $f'(z) = (z-a_1)^{-\alpha_1} \cdots (z-a_n)^{-\alpha_n}$

Thus $f(z) = \int_0^z \frac{d\zeta}{(\zeta-a_1)^{\alpha_1} \cdots (\zeta-a_n)^{\alpha_n}}$. (Check details such as analyticity, points at $\pm\infty$ & so on)