Section 8.1 PS 4

Q4 (3a) A country uses as currency coins with values of 1 peso, 2 pesos, 5 pesos, and 10 pesos and bills with values of 5 pesos, 10 pesos, 20 pesos, 50 pesos, and 100 pesos. Find a recurrence relation for the number of ways to pay a bill of n pesos if the order in which the coins and bills are paid matters.

Let a_n denote the number of ways to pay n pesos

We'll clasify the cases according to the last coins/bills used to pay n pesos.

$$a_n = \underbrace{a_{n-1}}_{\text{if the last one is coins of 1 peso}} + a_{n-2} + 2a_{n-5} + \underbrace{2a_{n-10}}_{\text{if the last one is coins of 10 pesos or bills of 10 pesos}} + a_{n-20} + a_{n-50} + a_{n-100}$$

Q8 (Q6) a. Find a recurrance relation for the number of bit strings of length n that contain three consecutive 0s.

Let a_n denote the # of bit strings of length n that contain 3 consecutive 0's.

- a_n
  - ending with 1
  - ending with 0
    - ending with 10
    - ending with 00
      - ending with 100
      - ending with 000

$\underbrace{\_\_\_}_{n-1}1$: a_{n-1}

$\underbrace{\_\_\_}_{n-2}10$: a_{n-2}

$\underbrace{\_\_\_}_{n-3}100$: a_{n-3}

$\underbrace{\_\_\_}_{n-3}000$: 2^{n-3}

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3} \quad n \geq 3$$

b. What are the initial conditions?

$a_1 = 0$ $\quad a_2 = 0$ $\quad a_3 = 1 \rightarrow 000$

c. How many bit strings of length 7 contain 3 consequtive 0's?

$a_4 = a_3 + a_2 + a_1 + 2 = 3$

$a_5 = a_4 + a_3 + a_2 + 2^2 = 3 + 1 + 0 + 4 = 8$

$a_6 = a_5 + a_4 + a_3 + 2^3 = 8 + 3 + 1 + 8 = 20$

$a_7 = a_6 + a_5 + a_4 + 2^4 = 20 + 8 + 3 + 16 = 47$

Q9 (7) a. Find a recurrance relation for the number of bit strings of length n that do not contain 3 consecutive 0's.

a_n = # of bit strings of length n that do not contain 3 cons. 0's.

a_n:
- ending with 1 $\quad \underbrace{\_\_\_\_}_{n-1}1 \quad a_{n-1}$
- ending with 10 $\quad \underbrace{\_\_\_\_}_{n-2}10 \quad a_{n-2}$
- ending with 100 $\quad \underbrace{\_\_\_\_}_{n-3}100 \quad a_{n-3}$
- ~~ending with 000~~

Thus $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ $\quad n \geq 3$

b. What are the initial conditions?

$a_1 = 2$ $\quad (0, 1)$

$a_2 = 4$ $\quad (00, 01, 10, 11)$

$a_3 = 7$ $\quad 2^3 - |\{000\}|$

c. How many bit strings of length 7 that do not contain 3 consecutive 0's

$a_4 = a_3 + a_2 + a_1 = 7 + 4 + 2 = 13$

$a_5 = a_4 + a_3 + a_2 = 24$

$a_6 = a_5 + a_4 + a_3 = 24 + 13 + 7 = 44$

$a_7 = a_6 + a_5 + a_4 = 44 + 24 + 13 = 81$

Q20 (14) A bus driver pays all tolls, using only nickels (5 cent) and dimes (10 cent) by throwing one coin at a time into the mechanical toll collector.

a. Find a recurrence relation for the number of different ways the bus driver can pay a toll of n cents (where the order in which the coins are used matters.)

Let a_n = # ways to pay a toll of n cents.

→ first pay a nickel (5 cent) ; a_{n-5} diff. ways to pay remaining n−5 cents.
→ first pay a dime (10 cent) ; a_{n-10} diff ways to pay " n−10 cents

$$a_n = a_{n-5} + a_{n-10} \quad n \geq 10$$

b. In how many different ways can the driver pay a toll of 45 cents?

$a_k = 0$ if $5 \nmid k$

$a_5 = 1$ $\quad a_{10} = 2$

$a_{15} = a_{10} + a_5 = 3$

$a_{20} = a_{15} + a_{10} = 3 + 2 = 5$

$a_{25} = a_{20} + a_{15} = 5 + 3 = 8$

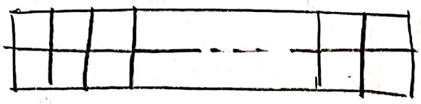
$a_{30} = a_{25} + a_{20} = 8 + 5 = 13$

$a_{35} = a_{30} + a_{25} = 13 + 8 = 21$

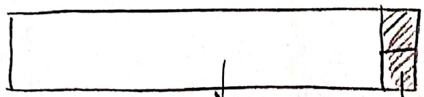
$a_{40} = a_{35} + a_{30} = 21 + 13 = 34$

$a_{45} = a_{40} + a_{35} = 34 + 21 = 55$.

Q26 (18) a. Find a recurrence relation for the number of ways to completely cover a $2 \times n$ board with 1×2 dominoes. (Hint: Consider seperately the coverings where the position in the top right corner of the board is covered by a domino positioned vertically) n

2 

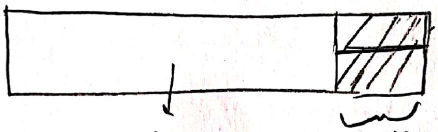
Let a_n be the number of ways to cover a $2 \times n$ board.



$2 \times (n-1)$ 2×1

last one is a vertical domino

remanings can be covered in a_{n-1} ways



$2.(n-2)$ 2×2

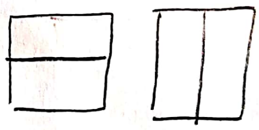
two horizontal dominoes in the last 2 positions. we have to cover remaining $2.(n-2)$ board in a_{n-2} ways

$a_n = a_{n-1} + a_{n-2}$ $n \geq 3$

b. What are the initial conditions

For $n=1$ we have 2×1 board & there is one way 

$a_1 = 1$.

For $n=2$ we have 2×2 board & there is 2way 

$a_2 = 2$

c. How many ways are there to cover a 2×17 board?

$a_3 = a_2 + a_1 = 3$

$a_4 = a_3 + a_2 = 5$

$a_5 = 8 \rightarrow a_6 = 13 \rightarrow a_7 = 21$ $a_8 = 34$ $a_9 = 55$ $a_{10} = 89$

$a_{11} = 144$ $a_{12} = 233$ $a_{13} = 377$ $a_{14} = 610$

$a_{15} = 987$ $a_{16} = 1597$ $a_{17} = 2584$

Q28(20) Show that Fibonacci numbers satisfy the recurrece relation $f_n = 5f_{n-4} + 3f_{n-5}$ for $n = 5, 6, 7, \ldots$ together with the initial conditions $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2$, and $f_4 = 3$. Use this recurrence relation to show that f_{5n} is divisible by 5, for $n = 1, 2, 3, \ldots$

$$\begin{aligned}
f_n &= f_{n-1} + f_{n-2} \quad \text{by defn of Fibonacci numbers.} \\
&= (f_{n-2} + f_{n-3}) + f_{n-2} \\
&= 2f_{n-2} + f_{n-3} \\
&= 2(f_{n-3} + f_{n-4}) + f_{n-3} \\
&= 3f_{n-3} + 2f_{n-4} \\
&= 3(f_{n-4} + f_{n-5}) + 2f_{n-4} \\
&= 5f_{n-4} + 3f_{n-5}
\end{aligned}$$

Now we will prove f_{5n} is divisible by 5. by induction

Basis step. for $n=1$ $f_5 = 5$ is divisible by 5

Inductive step: Assume f_{5k} is divisible by 5 for some $k \in \mathbb{Z}^+$

Then $f_{5(k+1)} = f_{5k+5} \underset{\substack{\downarrow \\ \text{by above} \\ \text{equation}}}{=} 5.f_{5k+1} + \underbrace{(3f_{5k})}_{\substack{\text{divisible} \\ \text{by 5 by I.H}}}$ is divisible by 5.

Thus $\forall n \geq 1$ f_{5n} is divisible by 5.

Q11 a. Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

Let a_n denote the number of ways climbing n stairs.

a_n → First move is to climb one stair in a time → For the remaining $n-1$ stairs, there is a_{n-1} way

a_n → First move is to climb two stairs in a time → For the remaining $n-2$ stairs, there is a_{n-2} way

Thus $a_n = a_{n-1} + a_{n-2}$ $n \geq 3$

b. What are the initial conditions

$a_1 = 1$ $a_2 = 2$ (2 stairs in a time or 1 stair + 1 stair)

c. In how many ways can this person climb a flight of eight stairs?

$a_3 = a_2 + a_1 = 3$ $a_4 = 5$ $a_5 = 8$ $a_6 = 13$ $a_7 = 21$ $a_8 = \underline{34}$

Q13 a.) Find a recurrence relation for the number of ternary strings of length n that do not contain two consecutive 0s.

Ternary string: A string that contains only 0's, 1's and 2's.

a_n = # ternary strings of length n that do not contain two consecutive 0's.

a ternary string of length n →
- 2 ___ : a_{n-1}
- 1 ___ : a_{n-1}
- 01 ___ : a_{n-2}
- 02 ___ : a_{n-2}
- 00 ___ ✗ → it contains 2 consecutive 0's

$a_n = 2a_{n-1} + 2a_{n-2}$

b. What are the initial conditions?

$a_1 = 3 = \# \{0, 1, 2\}$

$a_2 = 8 = 3^2 - |\{00\}|$

c. $a_6 = ?$

$a_3 = 2(a_2 + a_1) = 22$ $\quad a_4 = 60$ $\quad a_5 = 164$ $\quad a_6 = 224 \cdot 2 = 448$

Q24 Find a recurrence relation for the number of bit sequences of length n with an even number of 0s.

Let a_n = # bit sequences of length n with an even number of 0's.

Take a bit sequence of length n

$0\underbrace{\_\_\_\_\_\_}_{n-1}$ $\qquad$ $1\underbrace{\_\_\_\_\_\_}_{n-1}$

It contains an even number of zeros if the remaining bit sequence of length (n-1) contains odd number of 0's

which is $\underbrace{2^{n-1}}_{\text{total bit strings of length } n-1} - \underbrace{a_{n-1}}_{\text{even number of zeros}}$

It contains an even number of zeros if the remaining bit sequence of length (n-1) contains even number of 0's

which is a_{n-1}

Then $a_n = 2^{n-1} - a_{n-1} + a_{n-1} = 2^{n-1}$

Section 6.1

Q 36 (20) How many functions are there from the set $\{1, 2, ..., n\}$, where n is a positive integer, to the set $\{0, 1\}$?

There are n elts in the domain and there is a choice of 2 function values for each elts → Thus there are 2^n functions in total.

Q 37 (21) a. that are one-to-one?

If $n \geq 3$ there is no function that is one-to-one.

if $n = 1$ there are 2 functions

$f: \{1\} \to \{0, 1\}$
$1 \to 0$

$f: \{1\} \to \{0, 1\}$
$1 \to 1$

If $n = 2$ there are 2 functions that is 1-1 which are

$f_1: \{1, 2\} \to \{0, 1\}$
$1 \to 0$
$2 \to 1$

$f_2: \{1, 2\} \to \{0, 1\}$
$1 \to 1$
$2 \to 0$

b. that assign 0 to both 1 and n

$\{1, 2, ... n\} \to \{0, 1\}$
$1 \to 0$
$n \to 0$

There are $n-2$ elts remaining in the domain, each has 2 choice for function value. Thus there are 2^{n-2} functions satisfying the condition

c. that assign 1 to exactly one of the positive integers less than n.

We can assign 1 for an element of the set $\{1, \ldots, n-1\}$ and assign 0 for the others with $n-1$ choices.

we can assign 1 or 0 to n → 2 choices.

Hence there are $2(n-1)$ functions that satisfy the condition in total

Q47(31) In how many ways can a photographer at a wedding arrange 6 people in a row, including the bride and groom, if

a the bride must be next to the groom.

B G · · · ·
1 2 3 4 5

think of bride and groom as 1 person

$5! \cdot 2 = 240$
↓
bride and groom can change places.

b The bride is not next to the groom

all arrangments − bride next to groom $= 6! - 240 = 720 - 240 = 480$

c. The bride is positioned somewhere to the left of the groom
In half of the arrangments, the bride is to the left of the groom, and to the right in the other half.
$\frac{720}{2} = 360$