Nudging to reconstruct pressure field from noisy measurement

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Abstract

We want to investigate a simple data assimilation algorithm of nudging to reconstruct the pressure field from noisy measurements of the pressure gradient. This method can be considered a reduced version (thus may not be optimal) of the Kalman filter for parameter estimation in addition to other interprations.

1 Introduction

Measuring pressure or acoustic field non-intrusively is difficult. One way to achieve so is to use PIV/PTV to measure the velocity field $\boldsymbol{u}$ and in turn the pressure gradient can be directly evaluated using Navier Stokes equations $\nabla p = f(\boldsymbol{u})$. However, integrating pressure gradient ∇p for pressure p is not trivial especially when the experimental measurement of u is contaminated by error or noise, which will propagate to the pressure gradient. We propose a data assimilation method to solve this problem.

2 Nudging as a fixed point iteration

We now consider a minimum problem of reconstructing pressure p from noisy ∇p measurement. Let the pressure field p be the state (or the parameter) to be estimated, and the pressure gradient measured from experiments is the observation. We assume the pressure at one location is known as the Dirichlet condition (note, this location does not have to be on the boundary).

We design the nudging algorithm as follows:

$$\hat{p}_{k+1} = \hat{p}_k + K \begin{pmatrix} \widetilde{\nabla p} - D\hat{p}_k \\ P - \Pi \hat{p}_k \end{pmatrix} = \hat{p}_k + K(y_k - C\hat{p}_k), \tag{1}$$

where D is a (discrete) differential operator representing the gradient (i.e., $\nabla p = Dp$, considering the noise, $\widetilde{\nabla p} = \nabla p + \epsilon_{\nabla p} = Dp + \epsilon_{\nabla p}$). P is the Dirichlet condition, and Π is a function sampling $\hat{p}_k$ at the location of the Dirichlet condition, and y_k is the vector of observation.

The true pressure field should be unique and is

$$p_{k+1} = p_k. \tag{2}$$

Defining the discrepancy between the estimates and the true value as the error $e_k = \hat{p}_k - p_k$ and comparing (3) and (2) leads to

$$e_{k+1} = (I - KC)e_k + K\epsilon_{\nabla p}, \tag{3}$$

If K is chosen so that the spectral radius of $(I - KC)$ is smaller than unity, but K is not too big to over-amplify the error from the experiments, this observer should asymptotically converge.

3 Variational interpretation and optimal gain

Consider a cost function of

$$J = -(\hat{p}_{k+1} - \hat{p}_k)^T(\hat{p}_{k+1} - \hat{p}_k) + (C\hat{p}_k - y_k)^T R^{-1}(C\hat{p}_k - y_k), \tag{4}$$

where R is the covariance matrix that measures the accuracy of the observation. The first term in J drives the solution of p_{k+1} to be unique (when the algorithm converges) and the second term measures the weighted energy of the discrepancy between the observation and the prediction.

Minimizing J by computing $\delta J/\delta p_{k+1}$ and set it be zero leads to

$$p_{k+1} = p_k + C^T R^{-1}(Cp_k - y_k)^T = p_k + K(Cp_k - y_k)^T, \tag{5}$$

which recovers (1), and $K = C^T R^{-1}$ is the (optimal) gain for nudging.

References