UNIVERSITY OF SOUTHAMPTON
SEMESTER 1 Examination 2001/2002
MA367 Numerical Methods

Duration: 120 mins

Full marks may be obtained for complete answers to FOUR questions.

Only the best Four answers will be taken into account.

Use SEPARATE answer books for each question.

Formula Sheet FS/ENGIII/367/01 will be provided.

Programmable calculator MAY be used. Performing calculations, please, retain four decimal places after coma

1. Solve the 3×3 system $Ax = f$, where $x = (x_1, x_2, x_3)^T$,

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 1 & -1 & 0 \\ 2 & -2 & 1 \end{pmatrix}, \qquad f = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

(i) Check that the system has a unique solution (calculate $\det A$)

(ii) Using Gaussian elimination, obtain a system $Ux = y$, where U is the upper triangular matrix. Find the low triangular matrix L so that L and U give you Crout's LU decomposition of A

(iii) check the result $A = LU$ and find x

(iiii) check that $Ly = f$.

2. Consider a 3×3 system $Ax = f$. We assume that matrix A is symmetric and positive definite, that is, its eigenvalues are all real and positive, so $0 < \lambda_1 < \lambda_2 < \lambda_3$. It is clear that the initial system is equivalent to $x = (I - \alpha A)x + \alpha f$, where I is the identity 3×3 matrix, and α is a positive number. Consider iteration process $x^{(n+1)} = (I - \alpha A)x^{(n)} + \alpha f$. Show that

(i) if $y^{(n)} = x - x^{(n)}$, then $y^{(n} = Ry^{(0)}$, where x is the exact solution and matrix R is given by $R = I - \alpha A$

(ii) if $||R|| < 1$, then $||x - x^{(n)}|| \to 0$ as $n \to +\infty$

(iii) eigenvalues of R are $(1 - \alpha\lambda_1, 1 - \alpha\lambda_2, 1 - \alpha\lambda_3)$

(iiii) if $\alpha = 2/(\lambda_1 + \lambda_n)$, the iteration process $x^{(n+1)} = (I - \alpha A)x^{(n)} + \alpha f$ is convergent. Use the fact that if all absolute values of eigenvalues of R are less than 1 then $||R|| < 1$.

3. Consider the following integral

$$\int_0^{\pi/2} \cos^2(x)dx.$$

Evaluate the integral

(i) analytically

(ii) using the rectangular formula $\int_a^b f(x)dx = h\sum_{i=1}^4 f(a+h(i-1/2))$, $h=(b-a)/4$ and find the error, calculate error analytically

(iii) derive the Simpson's formula for $n=4$

(iiii) using the Simpson's formula for $n=4$, evaluate the integral and find the error, calculate error analytically

4. Consider the following three steps Adams-Bashforth formula:

$$y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1}),$$

which solves the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

(i) Derive this formula using the method of undetermined coefficients,

(ii) rewrite out this formula in the general form of multistep method

$$a_k y_n + a_{k-1} y_{n-1} + ... + a_0 y_{n-k} = h[b_k f_n + b_{k-1} f_{n-1} + ... + b_0 f_{n-k}]$$

along with polynomials $p(z) = a_k z^k + a_{k-1} z^{k-1} + ... + a_0$ and $q(z) = b_k z^k + b_{k-1} z^{k-1} + ... + b_0$, determine whether the method is explicit or implicit,

(iii) determine the order of this method (order of local truncation error)

(iiii) show that this method is convergent because of stability and consistency conditions hold

5. Consider the boundary value problem

$$-u'' = f(x), \qquad u'(0) = 0, \quad u'(\pi) = 0.$$

Solve the problem using Ritz method.

(i) Derive a formula for the Ritz quadratic functional $J(u) =< Au, u > -2 < f, u >$, where $Au = f$ is the differential equation given above

(ii) For the case $f(x) = \cos(x)$, find expression for $J(u) = J(c_1, c_2, ..., c_n)$, where

$$u(x) = \sum_{m=1}^{n} c_i \frac{\cos(mx)}{m}.$$

Note that the sequence of elements $\cos x$, $\cos(2x)/2$,...,$\cos(nx)/n$ satisfy the boundary conditions.

(iii) Find the Ritz system

$$\frac{\partial}{\partial c_m} J(c_1, c_2, ..., c_n) = 0,$$

$m = 1, 2, ..., n$, find a solution

(iiii) Find a solution of the boundary value problem with $f(x) = \cos(x)$ analytically and check the results.

6. Consider the initial boundary value problem for the heat equation

$$u_{xx} = u_t, \qquad 0 < x < \pi, \quad t > 0$$

with the boundary conditions

$$u(0,t) = 0, \qquad u(\pi, t) = 0,$$

and the initial condition $u(x,0) = \sin x$.

(i) Replace this differential problem by a discretized version of it using finite differences for all derivatives and obtain explicit algorithm. For numerical solution use the grid points $(x_i, t_j) = (ih, jk)$, $i = 0, 1, ..., n+1$, $j = 0, 1, ...$

(ii) Suppose that the difference equation is written in the form

$$U^{(j+1)} = AU^{(j)},$$

where $U^{(j)} = (u_{1j}, u_{2j}, u_{3j})^T$, that is , $n = 3$. Determine the initial vector $U^{(0)}$ and the matrix A.

(iii) Find the eigenvalues of the matrix A. To do this represent matrix A in the form $A = I - sB$, where I is the identity matrix, and

$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

(iiii) Obtain the sufficient condition for the explicit algorithm to be stable ensuring that absolute values of all three eigenvalues of the matrix A are less than 1.