Lecture 7
Partial Differential Equations (PDE)

1. Introduction to PDE

Many important problems in mathematical physics reduce to second order quasi-linear PDEs of the form

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f(x, u, \nabla u) = 0. \tag{1}$$

Depending on the constant coefficients A, B, C these equations are classified into three categories:

$$\begin{array}{lll} \text{If} & B^2 - AC < 0, & \text{it is called elliptic,} \\ \text{If} & B^2 - AC = 0, & \text{it is called parabolic,} \\ \text{If} & B^2 - AC > 0, & \text{it is called hyperbolic.} \end{array}$$

In the next section, we will develop the heat equation (parabolic type)

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(x, t), \tag{2}$$

where u is the temperature, which depends on x coordinate and time t, D is thermal diffusivity. Another example is the wave equation (hyperbolic type)

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + f(x, t), \tag{3}$$

which describes small vibrations of a string, and c is the speed of propagating waves. In the multidimensional coordinate space these two equations involve the Laplace's operator

$$\frac{\partial u}{\partial t} = D\Delta u + f(x, t), \qquad \frac{\partial^2 u}{\partial t^2} = c^2\Delta u + f(x, t),$$

which in case of 2D coordinate space is given by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}.$$

Without time dependence both equations reduce to Poisson's equation

$$\Delta u = -f(x, t), \tag{4}$$

where u is a function of (x, y). Corresponding homogeneous equation is called Laplace's equation

$$\Delta u = 0. \tag{5}$$

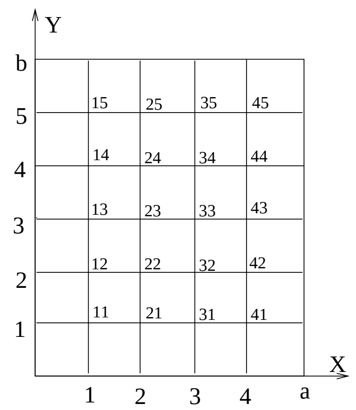

Figure 1: Dirichlet problem on a rectangle $0 < x < a, \quad 0 < y < b$

In next section we pay particular attention to the last equation in case of 2D space. A concrete physical problem involving (5) would contain also the specification of a region Ω in the x, y plane where the solution u is sought, and where the boundary conditions places on u, that is,

$$u\Big|_{\partial\Omega} = \phi, \qquad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = \psi, \tag{6}$$

the Dirichlet or Neumann boundary conditions, respectively, where $\partial\Omega$ is the boundary of Ω domain.

2. Dirichlet Boundary Value Problem

The Dirichlet problem is summarised by

$$\begin{cases} u_{xx} + u_{yy} = f(x, y), & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = \phi, & \text{on} \quad \partial\Omega. \end{cases} \tag{7}$$

This problem has a unique solution if the domain Ω has a smooth boundary and f and ϕ are continuous functions. We are going to solve this problem numerically using the method of finite-differences To illustrate this method, we shall consider the Dirichlet problem on a rectangle $0 < x < a, \quad 0 < y < b$ (see Fig. 1).

This approach to the numerical solution of the problem (7) employs finite-difference approximations to the derivatives

$$y''(x) = \frac{1}{h^2}[y(x+h) + y(x-h) - 2y(x)] - \frac{h^2}{12} y^{(4)}(\xi). \tag{8}$$

First, a network of grid points is established in the rectangle $\bar{\Omega}$:

$$(x_i, y_j) = (ih_1, jh_2), \qquad 0 \le i \le n+1, \qquad 0 \le j \le m+1,$$

$$h_1 = \frac{a}{n+1}, \qquad h_2 = \frac{b}{m+1}.$$

Next, the differential equation in (7) at the mesh point (x_i, y_j) is replaced by its finite-difference analogue at that point, which is

$$\frac{1}{h_1^2}[u_{i-1,j} + u_{i+1,j} - 2u_{i,j}] + \frac{1}{h_2^2}[u_{i,j-1} + u_{i,j+1} - 2u_{i,j}] = f_{ij},$$

or

$$u_{i-1,j} + u_{i+1,j} - 2u_{i,j} + \alpha[u_{i,j-1} + u_{i,j+1} - 2u_{i,j}] = h^2 f_{ij}, \tag{9}$$

where $\alpha = (h_1/h_2)^2$ and $h = h_1$. Here $u_{i,j}$ as a solution of the discrete problem (the finite-difference equation) is intended to approximate $u(x_i, y_j)$ solution of the continuous problem. The values of $u_{i,j}$ are known when $i = 0$ or $n+1$ and when $j = 0$ or $m+1$, since these are the prescribed boundary values in the problem. This means that we shall be solving a non-homogeneous system of linear equations with unknowns $u_{i,j}$ and $1 \le i \le n$, $1 \le j \le m$. To take a simple case, let $n = 2$ and $m = 3$. Thus, we obtain the 6×6 system

$$\begin{cases} u_{01} - 2u_{11} + u_{21} + \alpha[u_{10} - 2u_{11} + u_{12}] = h^2 f_{11}, \\ u_{02} - 2u_{12} + u_{22} + \alpha[u_{11} - 2u_{12} + u_{13}] = h^2 f_{12}, \\ u_{03} - 2u_{13} + u_{23} + \alpha[u_{12} - 2u_{13} + u_{14}] = h^2 f_{13}, \\ u_{11} - 2u_{21} + u_{31} + \alpha[u_{20} - 2u_{21} + u_{22}] = h^2 f_{21}, \\ u_{12} - 2u_{22} + u_{32} + \alpha[u_{21} - 2u_{22} + u_{23}] = h^2 f_{22}, \\ u_{13} - 2u_{23} + u_{33} + \alpha[u_{22} - 2u_{23} + u_{24}] = h^2 f_{23}. \end{cases} \tag{10}$$

The unknown quantities in this problem can be ordered in many ways. We select the one known as the natural ordering

$$u = [u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}]^T.$$

The system has the form $Au = F$

$$A = \begin{pmatrix} -2(1+\alpha) & \alpha & 0 & 1 & 0 & 0 \\ \alpha & -2(1+\alpha) & \alpha & 0 & 1 & 0 \\ 0 & \alpha & -2(1+\alpha) & 0 & 0 & 1 \\ 1 & 0 & 0 & -2(1+\alpha) & \alpha & 0 \\ 0 & 1 & 0 & \alpha & -2(1+\alpha) & \alpha \\ 0 & 0 & 1 & 0 & \alpha & -2(1+\alpha) \end{pmatrix},$$

$$F = \begin{pmatrix} h^2 f_{11} - u_{01} - \alpha u_{10} \\ h^2 f_{12} - u_{02} \\ h^2 f_{13} - u_{03} - \alpha u_{14} \\ h^2 f_{21} - u_{31} - \alpha u_{20} \\ h^2 f_{22} - u_{32} \\ h^2 f_{23} - u_{33} - \alpha u_{24} \end{pmatrix},$$

where in the expressions of the components of the vector F only the boundary values of $u_{i,j}$ are present. In general case the $n \times m$ system will be sparse because each equation will contain at most fine unknowns. Iterative procedure such as the Gauss-Seidel iterative method can be quite effective in this situation. The known boundary values are

$$u_{0,j} = \phi(x_0, y_j) = \phi(0, y_j), \qquad u_{i,0} = \phi(x_i, y_0) = \phi(x_i, 0),$$

$$u_{n+1,j} = \phi(x_{n+1}, y_j) = \phi(a, y_j), \qquad u_{i,m+1} = \phi(x_i, y_{m+1}) = \phi(x_i, b).$$

Thus in the algorithm, the full array of elements $u_{i,j}$ contains $(n+1)(m+1)$ elements.
The following version of the iteration method is very effective in practice of numerical analysis:

$$(\frac{2}{h_1^2} + \frac{2}{h_2^2})u_{ij} = \frac{u_{i-1,j} + u_{i+1,j}}{h_1^2} + \frac{u_{i,j-1} + u_{i,j+1}}{h_2^2} - f_{ij}.$$

This equation is used for updating u_{ij}. When this equation is used, the value obtained from the right-hand side replaces the old value of u_{ij}. It is important that only interior grid points will be involved in this updating. For the initial iteration for the interior grid points zeros can be taken. Note that the matrix A is not stored in the computer memory because the iterative method does not require this.
In the error analysis we introduce the error

$$e_{ij} = u_{ij} - \hat{u}_{ij},$$

where $\hat{u}_{ij} = u(x_i, y_j)$ are the values of the exact solution. Substituting

$$u_{ij} = e_{ij} + \hat{u}_{ij}$$

into the difference equation, we obtain

$$\frac{1}{h_1^2}[e_{i-1,j} + e_{i+1,j} - 2e_{i,j}] + \frac{1}{h_2^2}[e_{i,j-1} + e_{i,j+1} - 2e_{i,j}] =$$

$$= f_{ij} - \frac{1}{h_1^2}[\hat{u}_{i-1,j} + \hat{u}_{i+1,j} - 2\hat{u}_{i,j}] - \frac{1}{h_2^2}[\hat{u}_{i,j-1} + \hat{u}_{i,j+1} - 2\hat{u}_{i,j}].$$

Using the finite-difference approximation formula (8) yields

$$\frac{1}{h_1^2}[e_{i-1,j}+e_{i+1,j}-2e_{i,j}]+\frac{1}{h_2^2}[e_{i,j-1}+e_{i,j+1}-2e_{i,j}]=$$

$$=f_{ij}-[u_{xx}(x_i,y_j)+\frac{h_1^2}{12}u_{xxxx}(\xi_i,y_i)]-[u_{yy}(x_i,y_j)+\frac{h_2^2}{12}u_{yyyy}(\mathbf{x}_i,\zeta_j)]=$$

$$=-\frac{h_1^2}{12}u_{xxxx}(\xi_i,y_i)-\frac{h_2^2}{12}u_{yyyy}(\mathbf{x}_i,\zeta_j)$$

as

$$u_{xx}(x_i,y_j)+u_{yy}(x_i,y_j)=f_{ij}.$$

Hence, the error satisfies the difference equation

$$e_{i-1,j}+e_{i+1,j}-2e_{i,j}+\alpha[e_{i,j-1}+e_{i,j+1}-2e_{i,j}]=$$

$$=-\frac{h_1^4}{12}u_{xxxx}(\xi_i,y_i)-\frac{h_1^2h_2^2}{12}u_{yyyy}(\mathbf{x}_i,\zeta_j),$$

and zero boundary conditions. Analysing this equality and using well-known principle of maximum for elliptic PDE it is possible to prove that

$$||e_{i,j}||\leq \mathrm{const}h_1h_2.$$

This means that in the limit $n\to+\infty$ and $m\to+\infty$ the norm of the error tends to zero.

3. Ritz method

Ritz, Galerkin, Square Least methods are used widely on problems in which it is required to determine an unknown function. Of course, boundary value problems for differential equations are in this category. Suppose we are confronted with a problem of the form

$$Au(x) = f(x) \tag{11}$$

in which L is the linear operator

$$Au = -div\Big(p(x)\nabla u(x)\Big) + q(x)u = f(x), \quad x = (x_1, ..., x_n) \in \Omega, \quad \Omega \subset \mathbf{R}^n, \quad n = 2, 3,$$

$f(x)$ is a given function, and $u(x)$ is is the function to be determined from the equation and boundary Dirichlet condition Dirichlet,

$$u|_{\partial\Omega} = 0.$$

We assume that $p(x) \geq p_0 > 0$, $q(x) \geq 0$, $p \in C^1(\Omega)$, $q \in C(\Omega)$ for $x \in \Omega$, and $f \in L_2(\Omega)$. The operator is defined for the functions u that belong to $D_A = C^2(\Omega) \cap C^1(\bar{\Omega})$ and $Au \in L_2(\Omega)$. This means that operator A is symmetric and positive in the real Hilbert space $H = L_2(\Omega)$ since, using the integration by parts, we have

$$< Au, u >= \int_\Omega Au \cdot u dx = \int_\Omega (-div(p\nabla u)u + qu^2)dx =$$

$$= \int_\Omega (p|\nabla u|^2 dx + qu^2)dx \geq 0$$

for arbitrary $u \in D_A$. The quantity $< Au, u >$ is called the energy of the element u relative the operator A, and provides the energy norm of the element

$$\|u\|_A^2 =< Au, u >$$

In fact the domain of definition of A is the Sobolev space $W_2^1(\Omega)$. It is dense with respect to $H = L_2(\Omega)$. Thus $D_A = W_2^1(\Omega) = H_a$, and it is a Hilbert space that is called energy space $H_A = W_2^1(\Omega)$. It is possible to prove that the operator A is positive definite in $H_A = W_2^1(\Omega)$, namely,

$$< Au, u >\geq a^2\|u\|^2, \quad a > 0, \tag{12}$$

or

$$\|u\|_A \geq a\|u\|, \tag{13}$$

where $\|u\|$ is the norm in $L_2(\Omega)$.

Consider a linear quadratic functional

$$J(u) =< Au, u > -2 < f, u >= \int_\Omega (p|\nabla u|^2 + qu^2)dx - 2\int_\Omega fudx. \tag{14}$$

A theorem states that the functional is continuous and bounded from below. If the equation $Au(x) = f(x)$ has a solution u_0, then this solution minimize the functional $J(u)$. Conversely, if there exists an element u_0 that minimizes the functional $J(u)$, this element satisfies the equation $Au(x) = f(x)$. The proof is based on the possibility to reduce the functional $J(u)$ to the form

$$J(u) =< A(u - u_0), u - u_0 > - < Au_0, u_0 >= \|u - u_0\|_A^2 - \|u_0\|_A^2,$$

as $Au_0 = f$. Analysing of the function with arbitrary $v \in D_a$

$$J(u_0(x) + tv(x)) =< Au_0, u_0 > +2t < Au_0, v > +t^2 < Av, v >=$$

$$-2 < f, u_0 > -2t < f, v >,$$

where $u_0 \in H_a$ is the minimum of $J(u)$. Equating the derivative of $J(u_0(x)+tv(x))$ with respect to t to zero, we obtain

$$< Au_0 - f, v >= 0, \quad \forall v \in D_A. \tag{15}$$

This equation agrees with the Gakerkin method. Thus, we have $Au_0 = f$. The second derivative taken with respect to t leads to

$$\frac{d^2 J}{dt^2} = 2 < Av, v >= 2\|v\|_A^2 > 0$$

for $v \neq 0$. This is due to the minimum of of $J(u)$.

The basic idea of Ritz method consists in replacing the boundary value problem for $Au(x) = f(x)$ by the problem of minimizing the functional $J(u)$. Suppose we select linear independent basis functions $\varphi_1, \varphi_2, ..., \varphi_n, ...$ in $L_2(\Omega)$. Hence, we seek solution in the form

$$u_n(x) = \sum_{m=1}^{n} c_m \varphi_m(x), \tag{16}$$

where $c_1, c_2, ..., c_n$ are unknown constants. Substituting this sum into the functional $J(u) =< Au, u > -2 < f, u >$, we obtain

$$J(u_n) = J(c_1, c_2, ..., c_n) = \sum_{m,k=1}^{n} c_m c_k < A\varphi_m, \varphi_k > -2\sum_{m=1}^{n} c_m < f, \varphi_m >, \tag{17}$$

which is quadratic form with respect to the unknown constants $c_1, c_2, ..., c_n$. Looking for a minimum element of $J(u_n) = J(c_1, c_2, ..., c_n)$, we require that

$$\frac{\partial}{\partial c_m} J(c_1, c_2, ..., c_n) = 0, \qquad m = 1, 2, ..., n.$$

Thus, we come to the linear $n \times n$ system of algebraic equations for the unknown constants $c_1, c_2, ..., c_n$

$$\sum_{m=1}^{n} c_m < A\varphi_m, \varphi_k >=< f, \varphi_k >, \qquad k = 1, 2, ..., n, \tag{18}$$

$$< A\varphi_m, \varphi_k >= \int_\Omega (p\nabla\varphi_m \cdot \nabla\varphi_k + q\varphi_m\varphi_k)dx, \qquad < f, \varphi_k >= \int_\Omega f\varphi_k dx,$$

which is called Ritz system. After we have found the unknown constants $c_1, c_2, ..., c_n$, the expression (29) gives us an approximate solution.

It is worth remarking that for Galirkin method that is more general than Ritz method we have

$$< Au_0 - f, \psi_k >=< A(\sum_{m=1}^{n} c_m\varphi_m) - f, \psi_k >, \quad k = 1, 2, ..., n, \tag{19}$$

$$\sum_{m=1}^{n} c_m < A\varphi_m, \psi_k >=< f, \psi_k >, \qquad k = 1, 2, ..., n, \tag{20}$$

$$< A\varphi_m, \psi_k >= \int_\Omega (p\nabla\varphi_m \cdot \nabla\psi_k + q\varphi_m\psi_k)dx, \qquad < f, \psi_k >= \int_\Omega f\psi_k dx,$$

where the set of linear independent functions $\psi_k \in H_a$ are called test fumctions and in general differ from the set of linear independent basis functions $\varphi_k \in H_a$. If both sets are identical then Galerkin procedure turns into the Ritz method approximation.

According to Shwartz inequality and using (13) we could derive some useful estimates, namely, we have

$$| < Au_0, v > | = | < f, v > | \le \|f\|\|u\| \le \frac{1}{a}\|f\|\|u\|_A,$$

as we assume that $Au_0 = f$. Let $v = u_0$, then we obtain a very important estimate

$$\|u\|_A \le \frac{1}{a}\|f\|. \tag{21}$$

If we trying to solve the problem of finding approximation $Au_n = f_n$, where

$$u_n = \sum_{m=1}^{n} c_m\varphi_m, , \tag{22}$$

then we have for $A(u_0 - u_n) = f - f_n$

$$\|u_0 - u_n\|_A \le \frac{1}{a}\|f - f_n\| = \frac{1}{a}\|f - Au_n\|. \tag{23}$$

This means that if $\lim_{n\to\infty} Au_n = f$, then $\lim_{n\to\infty} u_n = u_0$ in the sense of the energy norm $\|u\|_A$.

Consider the following simple example for 1D case:

$$-u'' = \cos x, \qquad u(0) = 0, \qquad u(\pi) = 0.$$

This boundary value problem has the exact solution $u = \cos x + 2x/\pi - 1$. Let us solve this problem using Ritz method. First we choose the basis functions

$$\sin 2x, \quad \sin 4x, ..., \quad \sin 2nx$$

as the solution is symmetric with respect to $x = \pi/2$. Hence, we seek solution in the form

$$u_n(x) = \sum_{m=1}^{n} c_m \sin 2mx,$$

For the matrix elements $< Lu_m, u_k >$ in the system (18) we have

$$< A\varphi_m, \varphi_k >= \int_0^\pi 4m^2 \sin 2mx \sin 2kx dx = \begin{cases} 0, & \text{if} \quad m \neq k, \\ 2\pi m^2, & \text{if} \quad m = k. \end{cases}$$

Then,

$$< f, \varphi_m >= \int_0^\pi \cos x \sin 2m dx = \frac{1}{2}\int_0^\pi \sin(2m-1)x dx +$$
$$\frac{1}{2}\int_0^\pi \sin(2m+1)x dx = \frac{4m}{4m^2-1}.$$

Hence, the Ritz system is given by

$$\begin{pmatrix} 2\pi 1^2 c_1 & 0 & ... & 0 \\ 0 & 2\pi 2^2 c_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & 0 & 2\pi n^2 c_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ ... \\ c_n \end{pmatrix} = \begin{pmatrix} \frac{4}{4\cdot 1^2-1} \\ \frac{4\cdot 2}{4\cdot 2^2-1} \\ ... \\ \frac{4n}{4\cdot n^2-1} \end{pmatrix}$$

Thus, we obtain

$$c_m = \frac{2}{\pi m(4m^2-1)},$$

and approximation to the solution is given by

$$u_n = \frac{2}{\pi}\sum_{m=1}^{n} \frac{\sin 2mx}{m(4m^2-1)}.$$

In the case $n \to +\infty$, we obtain the infinite Fourier series which converges to the exact solution $u = \cos x + 2x/\pi - 1$ in the real Hilbert space $L_2[a, b]$.

3. Finite Element Method

Consider the following Dirichlet problem in $\Omega \subset \mathbf{R}^2$

$$-\Delta u = f(M), \quad M(x,y) \in \Omega, \tag{24}$$

$$u|_S = g(M), \quad M(x,y) \in S, \tag{25}$$

where the boundary $S = \partial\Omega$ is sufficiently smooth. We separate the problem in two problems $u(M) = u^{(1)} + u_2$, where

$$-\Delta u^{(1)} = f(M), \qquad u^{(1)}|_S = 0, \tag{26}$$

and

$$-\Delta u^{(2)} = 0, \qquad u^{(2)}|_S = g(M). \tag{27}$$

Below we assume that $w = u^{(2)}$ The problem (26) with homogeneous boundary condition is to soled by means of Ritz or Galerkin method described above. Here we concentrate on solving the Laplace-Dirichlet problem (27) using the Finite element Method (FEM). Using the integration by parts, we obtain important integral relations

$$0 =< -\Delta w, \psi >= -\int_\Omega \Delta w \psi dx = \int_\Omega \nabla w \cdot \nabla \psi dx - \int_S \psi \frac{\partial w}{\partial n} ds = \tag{28}$$
$$= \int_\Omega \nabla w \cdot \nabla \psi dx - \int_S g \frac{\partial \psi}{\partial n} ds,$$

where we assumed that $\Delta\psi = 0$. Thus, Galerkin method for $A = -\Delta$ in this case could be written as follows

$$w_n(x) = \sum_{m=1}^{n} c_m \varphi_m(x), \tag{29}$$

$$\sum_{m=1}^{n} c_m \int_\Omega (\nabla\varphi_m \cdot \nabla\psi_k) dx = \int_S g \frac{\partial \psi_k}{\partial n} ds, \qquad k = 1, 2, ..., n, \tag{30}$$

what could be derived from Ritz quadratic ‘functional

$$J(w) = \int_\Omega |\nabla w|^2 dx - 2\int_S g \frac{\partial w}{\partial n} ds. \tag{31}$$

Below we assume that $\psi_k \equiv \varphi_k$ and according to (28)

$$\int_\Omega \nabla w \cdot \nabla \psi dx - \int_S \psi \frac{\partial w}{\partial n} ds = 0,$$

we going to use (30) in the following form, namely

$$\sum_{m=1}^{n} c_m \int_\Omega (\nabla\varphi_m \cdot \nabla\varphi_k) dx = \sum_{m=1}^{n} c_m \int_S \varphi_k \frac{\partial \varphi_m}{\partial n} ds, \qquad k = 1, 2, ..., n. \tag{32}$$

Let us solve the problem

$$-\Delta w = 0, \qquad M(x,y) \in \Omega, \qquad w|_S = g(M), \qquad M(x,y) \in S, \tag{33}$$

using approximation of FEM for a very simple case of square $0 < x, y < 2$ $(a = 2)$ shown in Fig. 2. As a result of triangulation we have four elements - triangles $\{T_m\}_{m=1}^4$ and five nodes $\{E_k\}_{k=1}^5$. Every triangle has a three vertices such that in total we have a set on vertices $\{N_j^{(m)}\}$ for $m = 1, ..., 4$ and $j = 1, 2, 3$ (see Fog. 2). For every triangle we define three polynomials

$$P_j^{(m)}(x,y) = a_j^{(m)} + b_j^{(m)} x + c_j^{(m)} y$$

in such a way that

$$P_j^{(m)}(N_i^{(m)}) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

where $i = 1, 2, 3$.

Thus, for the triangle T_1 we obtain

$$P_1^{(1)} = 1 - \frac{1}{2}x - \frac{1}{2}y, \quad P_2^{(1)} = \frac{1}{2}x - \frac{1}{2}y, \quad P_3^{(1)} = y, \tag{34}$$

as

$$\begin{cases} a_1^{(1)}1 + b_1^{(1)}0 + c_1^{(1)}0 = 1, \\ a_1^{(1)}1 + b_1^{(1)}2 + c_1^{(1)}0 = 0, \\ a_1^{(1)}1 + b_1^{(1)}1 + c_1^{(1)}1 = 0, \end{cases} \quad \begin{cases} a_2^{(1)}1 + b_2^{(1)}0 + c_2^{(1)}0 = 0, \\ a_2^{(1)}1 + b_2^{(1)}2 + c_2^{(1)}0 = 1, \\ a_2^{(1)}1 + b_2^{(1)}1 + c_2^{(1)}1 = 0, \end{cases}$$

$$\begin{cases} a_3^{(1)}1 + b_3^{(1)}0 + c_3^{(1)}0 = 0, \\ a_3^{(1)}1 + b_3^{(1)}2 + c_3^{(1)}0 = 0, \\ a_3^{(1)}1 + b_3^{(1)}1 + c_3^{(1)}1 = 1, \end{cases}$$

Similarly to the triangle T_1, for the the triangle $T_{2,3,4,}$ we shall have

$$P_1^{(2)} = \frac{1}{2}x - \frac{1}{2}y, \quad P_2^{(2)} = -1 + \frac{1}{2}x + \frac{1}{2}y, \quad P_3^{(2)} = 2 - x, \tag{35}$$

for the triangle T_2,

$$P_1^{(3)} = -1 + \frac{1}{2}x + \frac{1}{2}y, \quad P_2^{(3)} = -\frac{1}{2}x + \frac{1}{2}y, \quad P_3^{(3)} = 2 - y, \tag{36}$$

for the triangle T_3,

$$P_1^{(4)} = -\frac{1}{2}x + \frac{1}{2}y, \quad P_2^{(4)} = 1 - \frac{1}{2}x - \frac{1}{2}y, \quad P_3^{(4)} = x, \tag{37}$$

finally, for the triangle T_4.

Let us introduce the following basis (shape) functions $\varphi_k(M)$, $k = 1,2,3,4,5$, associated with the corresponding node E_k. Namely, we have

$$\varphi_1(M) = \begin{cases} P_1^{(1)}(M), & M \in T_1, \\ P_2^{(4)}(M), & M \in T_4, \\ \quad 0, & M \in T_2 \cup T_3, \end{cases} \quad \varphi_2(M) = \begin{cases} P_2^{(1)}(M), & M \in T_1, \\ P_1^{(2)}(M), & M \in T_2, \\ \quad 0, & M \in T_4 \cup T_3, \end{cases} \tag{38}$$

$$\varphi_3(M) = \begin{cases} P_2^{(2)}(M), & M \in T_2, \\ P_3^{(3)}(M), & M \in T_1, \\ \quad 0, & M \in T_1 \cup T_4, \end{cases} \quad \varphi_4(M) = \begin{cases} P_2^{(3)}(M), & M \in T_3, \\ P_1^{(4)}(M), & M \in T_4, \\ \quad 0, & M \in T_1 \cup T_2, \end{cases} \tag{39}$$

$$\varphi_5(M) = \begin{cases} P_3^{(1)}(M), & M \in T_1, \\ P_3^{(2)}(M), & M \in T_2, \\ P_3^{(3)}(M), & M \in T_3, \\ P_3^{(4)}(M), & M \in T_4. \end{cases} \tag{40}$$

Our approximation is given by

$$w_n(x) = \sum_{m=1}^{5} c_m \varphi_m(x), \tag{41}$$

and using the boundary condition $w|_S = g(M), \quad M(x,y) \in S$, at the nodes $E_{1,2,3,4,}$, we obtain

$$c_1 = g(E_1), \quad c_2 = g(E_2), \quad c_3 = g(E_3), \quad c_4 = g(E_4). \tag{42}$$

The final unknown c_5 is to found from the only one of the relations (32), namely

$$\sum_{m=1}^{5} c_m \int_\Omega (\nabla\varphi_m \cdot \nabla\varphi_5) dx = \sum_{m=1}^{5} c_m \int_S \varphi_5 \frac{\partial \varphi_m}{\partial n} ds. \tag{43}$$

Having evaluated the coefficients of this relation

$$\int_\Omega (\nabla\varphi_k \cdot \nabla\varphi_5) dx = -1, \qquad k = 1,2,3,4, \tag{44}$$

$$\int_\Omega |\nabla\varphi_5|^2 dx = 4, \qquad k = 1,2,3,4, \tag{45}$$

$$\int_S \varphi_5 \frac{\partial \varphi_m}{\partial n} ds = 0, \qquad k = 1,2,3,4,5, \tag{46}$$

we obtain from (43) that c_5 is the mean value with respect to the known corner values $w(E_k)$ with $k = 1,2,3,4$

$$c_5 = \frac{c_1 + c_2 + c_3 + c_4}{4} = \frac{1}{4} \sum_{k=1}^{4} g(E_k). \tag{47}$$

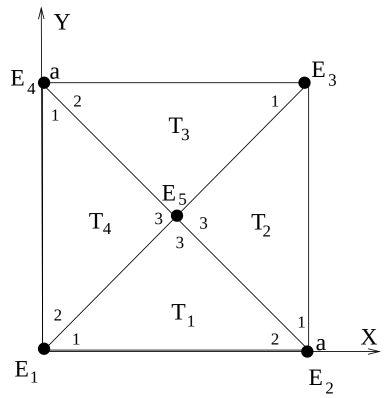

Figure 2: Application of FEM to solving Dirichlet problem for a square $0 < x < a, \quad 0 < y < a$

It is worth remarking that this perfect result could be obtained independently from the famous theorem about harmonic functions stating that for every harmonic function w in a domain Ω the following curvilinear integral along its boundary with normal derivative of w is always zero, namely

$$\int_S \frac{\partial w}{\partial n} ds = 0. \tag{48}$$

If $S = U_{m=1}^4 S_m$, then

$$0 = \int_S \frac{\partial w}{\partial n} ds = \sum_{m=1}^{5} c_m \int_{S_m} \frac{\partial \varphi_m}{\partial n} ds = 2c_1 + 2c_2 + 2c_3 + 2c_4 - 8c_5. \tag{49}$$

Thus, again we derive the result (47).