Lecture 9
Quasi-linear PDEs
Classification, characteristic curves

Many important problems in mathematical physics reduce to second order quasi-linear PDEs of the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + f(x,y,u,u_x,u_y) = 0. \qquad (1)$$

Depending on the coefficients a, b, c these equations are classified into three categories:

$$\begin{aligned} &\text{If} \quad b^2 - ac < 0, \qquad \text{it is called elliptic,}\\ &\text{If} \quad b^2 - ac = 0, \qquad \text{it is called parabolic,}\\ &\text{If} \quad b^2 - ac > 0, \qquad \text{it is called hyperbolic.} \end{aligned}$$

Let $d = b^2 - ac$. Consider the hyperbolic type. We introduce new variables

$$\xi = \xi(x,y), \qquad \eta = \eta(x,y), \qquad \xi \in C^2, \qquad \eta \in C^2, \qquad D(\frac{\xi, \eta}{x, y}) \neq 0.$$

For the solution $v(\xi,\eta) = u(x,y)$ we obtain a new equation

$$A\frac{\partial^2 v}{\partial \xi^2} + 2B\frac{\partial^2 v}{\partial \xi \partial \eta} + C\frac{\partial^2 v}{\partial \eta^2} + F(\xi,\eta,v,v_\xi,v_\eta) = 0,$$

where

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2,$$

$$B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y,$$

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2.$$

Let us require that $\xi(x,y)$ and $\eta(x,y)$ are chosen in such a way that $A = 0$ and $C = 0$, that is, $\xi(x,y)$ and $\eta(x,y)$ satisfy

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0,$$

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0.$$

Both equations are equivalent to

$$\xi_x - \lambda_1(x,y)\xi_y = 0, \qquad \eta_x - \lambda_2(x,y)\eta_y = 0,$$

where

$$\lambda_1(x,y) = \frac{-b+\sqrt{d}}{a}, \qquad \lambda_2(x,y) = \frac{-b-\sqrt{d}}{a}.$$

Thus, if $d > 0$, we obtain a canonical form of the hyperbolic category

$$\frac{\partial^2 v}{\partial\xi\partial\eta} + \hat{F}(\xi,\eta,v,v_\xi,v_\eta) = 0. \tag{2}$$

Using standard change of variables

$$s = \xi + \eta, \qquad t = \xi - \eta$$

yields another well-known canonical form of the hyperbolic category

$$\frac{\partial^2 v_1}{\partial s^2} - \frac{\partial^2 v_1}{\partial t^2} + F_1(s,t,v_1,v_{1s},v_{1t}) = 0, \tag{3}$$

where solution is $v_1(s,t) = v(\xi,\eta)$.

Let us summarise what we have said. In this analysis, equation

$$a\omega_x^2 + 2b\omega_x\omega_y + c\omega_y^2 = 0$$

is called characteristic equation. If $d > 0$, it has two real solutions $\xi(x,y)$ and $\eta(x,y)$, satisfying

$$\xi_x - \lambda_1(x,y)\xi_y = 0, \qquad \eta_x - \lambda_2(x,y)\eta_y = 0.$$

These solutions are called characteristics. In the plane (x,y), these two functions define two families of characteristic curves $y = y_1(x)$ and $y = y_2(x)$ determined by the equations

$$\xi(x,y) = C_1, \qquad \eta(x,y) = C_2.$$

The functions $y = y_1(x)$ and $y = y_2(x)$ satisfy the ODEs

$$y' = \lambda_1(x, y), \qquad y' = \lambda_2(x, y).$$

This means that $\xi(x, y)$ and $\eta(x, y)$ are the integrals of these ODEs. However, the most important fact is that they allow us to transform the initial quasi-linear equation to the canonical form of hyperbolic type. An example of the hyperbolic type is the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x, t). \tag{4}$$

What happens if $d = 0$? In this case we have

$$\lambda_1 = \lambda_2 = \frac{-b}{a}.$$

Thus, we have only one characteristic solution satisfying

$$\xi_x + \frac{b}{a}\xi_y = 0.$$

The second characteristic can be chosen in the form $\eta = x$. As a result, we obtain

$$A = 0, \qquad B = a\xi_x + b\xi_y = 0, \qquad C = a.$$

Hence,

$$a\frac{\partial^2 v}{\partial \eta^2} + F(\xi, \eta, v, v_\xi, v_\eta) = 0.$$

Dividing the last equation by a, we come to the canonical form for the parabolic category

$$\frac{\partial^2 v}{\partial \eta^2} + \tilde{F}(\xi, \eta, v, v_\xi, v_\eta) = 0.$$

An example of the parabolic type is the heat equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(x, t). \tag{5}$$

Let us consider the last case $d < 0$ corresponding to the elliptic category. In this situation we have two complex characteristic solutions $\omega(x, y)$ and $\omega^*(x, y)$ (* means complex conjugation), satisfying

$$\omega_x + \lambda_1(x, y)\omega_y = 0, \qquad \omega^*_x + \lambda_2(x, y)\omega^*_y = 0,$$

as $\lambda^*_1(x, y) = \lambda_2(x, y)$. Using $\omega(x, y)$ and $\omega^*(x, y)$, we choose two real characteristic solutions

$$\xi(x, y) = \frac{\omega(x, y) + \omega^*(x, y)}{2}, \qquad \eta(x, y) = \frac{\omega(x, y) - \omega^*(x, y)}{2i}.$$

It is clear that complex function $\omega(x, y)$ must satisfy

$$a\omega_x^2 + 2b\omega_x\omega_y + c\omega_y^2 = 0.$$

Separately equating the real and imaginary part of this equation to zero, we obtain

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2,$$

$$a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 0.$$

Hence, we have that $A = C$ and $B = 0$. The corresponding canonical form for the elliptic category looks like this

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} + \tilde{F}(\xi, \eta, v, v_\xi, v_\eta) = 0.$$

An example is the Poisson's equation

$$\Delta u = -f(x, t), \tag{6}$$

where the Laplace operator is given by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}.$$